X16/Shiu/BRS Carrier-Slice Estimate
Focused External Review Packet

revised packet after first assessment

1 Review request

Please check the divisor-sum step used in the BRS singular TC1 branch. This is not a request to verify the whole Goldbach proof. The requested review is the following local question:

> Does the stated Shiu/AP divisor-correlation argument prove the X16C core carrier-slice estimate, and hence the BRS carrier-slice estimate, with the claimed logarithmic losses and without circular dependence on C1 or TTH?

The point of the packet is to make this carrier-slice step reviewable independently of the larger proof architecture.

2 Minimal context

The proof has a singular TC1 branch in which a terminal B1-origin coarea test could concentrate on a short additive image of a marked carrier. The BRS lemma says that this situation is not a genuine terminal GoodAWACK residual: either it is already routed to an existing tag, or the short-image subcell is a strict Edge contribution. The only analytic input needed for this conclusion is the X16 carrier-slice estimate.

The local chain being reviewed is

$$\mathrm{BRS} + \mathrm{X16BRS} + \mathrm{X16C} \Longrightarrow \mathrm{TTH}.$$

Here TTH is the structural consequence that every unrouted active B1-origin coarea test has near-global length

$$H \geq X(\log X)^{-B}.$$

That near-global range is then used elsewhere with the Davenport/AP input. This packet asks only about the X16/Shiu/BRS part.

3 BRS carrier-slice statement

Let $\mathcal{B}$ be a fixed-depth typed B1 dyadic block. Let C be a B1 carrier of dyadic height X_C, and let I be an additive interval. Put

$$Y_{16} := \max\{|I \cap \mathbb{Z}|,\ X_C(\log N)^{-B_{16}}\}.$$

The required carrier-slice estimate is

$$\mathrm{Mass}_{\mathcal{B}}(C \in I) \ll N(\log N)^{C_{16}} \frac{Y_{16}}{X_C} + N^{1-\rho_{16}}(\log N)^{C_{16}}. \qquad \text{(X16-BRS)}$$

Here $C_{16} = C_{16}(J_0)$, $\rho_{16} = \rho_{16}(J_0) > 0$, and in the active proof-source file one may take, after harmless enlargement,

$$C_{16} = 100J_0^2 + 100, \qquad \rho_{16} = \frac{1}{10^6 J_0^4}.$$

The allowed BRS carriers are:

1. grouped product carriers;
2. Goldbach complementary carriers $N - P$;
3. quotient carriers s from a recorded equation $L = ds$;
4. controlled divisor quotients L/d_0 with $d_0 \leq (\log N)^C$.

The reduction from these four carriers to grouped product carriers is in X16BRS. Quotient carriers require the divisor to have already been tagged or controlled before BRS is invoked.

4 X16C core analytic model

For a grouped product carrier P with height X_P, the B1 equation is reduced to a fixed-depth divisor-correlation model. If $P = p$, if u is the complementary product on the same B1 side, and if the opposite side is forced to have product

$$Q = N - pu,$$

then the tuple mass is bounded by

$$(\log N)^{O_{J_0}(1)} \sum_{p \in I_{16}^{\#}} \tau_{K_1}(p) \sum_{u \asymp U} \tau_{K_2}(u) \tau_{K_3}(N - pu) \mathbf{1}_{N-pu>0}, \tag{1}$$

with $K_i \leq 2J_0$, $X_P U \asymp N$, and $|I_{16}^{\#}| \asymp Y_{16}$. The target estimate for the double sum in (1) is

$$\begin{aligned} &\sum_{p \in I_{16}^{\#}} \tau_{K_1}(p) \sum_{u \asymp U} \tau_{K_2}(u) \tau_{K_3}(N - pu) \mathbf{1}_{N-pu>0} \\ &\qquad \ll Y_{16} U (\log N)^{O_{J_0}(1)} + N^{1-\rho_{16}} (\log N)^{O_{J_0}(1)}. \end{aligned} \tag{2}$$

Since $X_P U \asymp N$, (2) gives (X16-BRS) for product carriers. The complementary, quotient, and controlled quotient carriers are then reduced to this case in X16BRS.

This correlation is the key point. The proof does not use the false shortcut of bounding only $\sum_{p \in I} \tau_K(p)$. It keeps the moving complementary divisor factor $\tau_{K_3}(N - pu)$.

5 X16C checking table

Lemma X16C records the Shiu/AP route as the following review checklist:

- carrier fixing: a fixed product $p = P(a_i : i \in S)$ has at most $\tau_{K_1}(p)$ factorizations;
- same-side complement: the complementary product u has at most $\tau_{K_2}(u)$ factorizations;
- opposite side: the Goldbach complement $N - pu$ is kept as $\tau_{K_3}(N - pu)$, not averaged away;

- fixed p: $N - pu$ is treated in one residue class modulo p;
- fixed u: $N - up$ is treated in one residue class modulo u in the large-carrier orientation;
- Cauchy–Schwarz is followed by Shiu/AP for squared divisor weights;
- $\sum_{u \sim U} \tau_K(u)^2$ is controlled by the standard fixed-divisor second moment;
- non-coprime AP classes are handled by separating local gcd factors;
- CRT and quotient restrictions are tagged and cost only polylogarithmic loss;
- if $Y_{16}U \le N^{1-\rho_{16}}$, the residual small-volume case is handled by the trivial divisor bound.

Thus Shiu/AP is applied only after the relevant arithmetic progression and carrier orientation have been fixed.

6 Shiu/AP input

The analytic input is Shiu's Brun–Titchmarsh theorem for multiplicative functions in arithmetic progressions:

P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions*, J. Reine Angew. Math. 313 (1980), 161–170, DOI `10.1515/crll.1980.313.161`.

The fixed divisor-function second moment used in the Cauchy–Schwarz step is cited as:

G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Graduate Studies in Mathematics 163, American Mathematical Society, 3rd ed., 2015, Ch. II.5, Theorem 5.

The proof uses the standard fixed-divisor-function corollary: for fixed K, A, $f(n) = \tau_K(n)^A$ is admissible in AP intervals, including the case $f(n) = \tau_K(n)^2$. The precise form used in X16C is the following.

For fixed $K, A \ge 1$ and $0 < \delta < 1/10$, put $f(n) = \tau_K(n)^A$. If $J \subset [1, N]$ is an interval of length H, $N^\delta \le H \le N$, and $q \le H^{1-\delta}$, then for every residue class a mod q

$$\sum_{\substack{n \in J \\ n \equiv a \pmod q}} f(n) \ll \left(\frac{H}{q} + 1\right) (\log N)^{C_{\mathrm{SH}}} \mathcal{E}_{q,a}. \tag{SH}$$

Lemma (Local factor averaging). The factor $\mathcal{E}_{q,a}$ is a local non-coprime-class cost supported on primes dividing (a, q). It is harmless on average in the following precise sense. For every X16 carrier interval $I_{16}^{\#} \subset [X/2, 3X]$ with $|I_{16}^{\#}| \gg X(\log N)^{-B_{16}}$,

$$\sum_{c \in I_{16}^{\#}} \tau_{K_0}(c)^A \mathcal{E}_{c,N}^{1/2} \ll |I_{16}^{\#}|(\log N)^{C_{\mathrm{loc}}}. \tag{LFA}$$

The same estimate holds over a full dyadic interval $c \asymp X$. This dyadic version is used in Case 2 when $c = u$.

This is the only place where non-coprime AP classes enter the X16C estimate.

The proof of (LFA) is as follows. For a non-coprime class write $g = (a, q)$, $a = ga_1$, $q = gq_1$, with $(a_1, q_1) = 1$. The summand becomes $f(gn_1)$ in a coprime class modulo q_1, and submultiplicativity gives $f(gn_1) \ll f(g)f(n_1)$. Thus the local cost is bounded by a fixed divisor power of $g = (a, q)$ and by a harmless Euler factor over primes dividing q. In X16C the relevant residue is $a = N$, so the cost is dominated by a fixed divisor power of (c, N). Averaging the multiplicative function

$$c \mapsto \tau_{K_0}(c)^A \tau_M((c, N))^B$$

over the carrier interval gives (LFA) by Shiu's ordinary interval theorem; if the carrier scale is polylogarithmic, the same bound is trivial after increasing the logarithmic exponent.

The requested check is whether this Shiu/AP input applies to the sums in (1) in the two range splits below.

7 Bilinear estimate

Let

$$S = \sum_{p \in I_{16}^{\#}} \tau_{K_1}(p) \sum_{u \asymp U} \tau_{K_2}(u) \tau_{K_3}(N - pu) \mathbf{1}_{N - pu > 0}.$$

Case 1: $X_P \le N^{1-\delta}$

Fix $p \in I_{16}^{\#}$. The positive values $N - pu$, with $u \asymp U$, belong to the residue class $N \bmod p$. They are contained in an interval $J_p \subset [1, N]$, and we enlarge monotonically to length $H_p = N$. Since $X_P U \asymp N$,

$$\frac{H_p}{p} + 1 \asymp \frac{N}{p} + 1 \asymp U.$$

The modulus condition is explicit:

$$p \le N^{1-\delta} = H_p^{1-\delta} \le H_p^{1-\delta/2}.$$

Cauchy–Schwarz gives

$$\sum_{u \asymp U} \tau_{K_2}(u)\tau_{K_3}(N - pu)\mathbf{1}_{N-pu>0}$$
$$\le \left(\sum_{u \asymp U} \tau_{K_2}(u)^2\right)^{1/2} \left(\sum_{u \asymp U} \tau_{K_3}(N - pu)^2 \mathbf{1}_{N-pu>0}\right)^{1/2}.$$

The first factor uses the standard second moment

$$\sum_{u \asymp U} \tau_K(u)^2 \ll_K U(\log 2U)^{K^2-1}.$$

The second factor is controlled by (SH), applied to $f = \tau_{K_3}^2$ in the residue class $N \bmod p$:

$$\sum_{u \asymp U} \tau_{K_3}(N - pu)^2 \mathbf{1}_{N-pu>0} \ll U(\log N)^{O_{J_0}(1)} \mathcal{E}_{p,N}.$$

Therefore, for fixed p,

$$\sum_{u \asymp U} \tau_{K_2}(u)\tau_{K_3}(N - pu)\mathbf{1}_{N-pu>0} \ll U(\log N)^{O_{J_0}(1)} \mathcal{E}_{p,N}^{1/2}.$$

Now (LFA) gives

$$\sum_{p \in I_{16}^{\#}} \tau_{K_1}(p)\mathcal{E}_{p,N}^{1/2} \ll Y_{16}(\log N)^{O_{J_0}(1)}.$$

Consequently

$$S \ll Y_{16} U(\log N)^{O_{J_0}(1)}.$$

Case 2: $X_P > N^{1-\delta}$

Then $U \ll N^{\delta}$. If
$$Y_{16}U \le N^{1-\rho_{16}},$$
the trivial divisor bound gives the required power-saving term:
$$S \ll N^{\varepsilon} Y_{16} U (\log N)^{O_{J_0}(1)} \le N^{1-\rho_{16}+\varepsilon}(\log N)^{O_{J_0}(1)}.$$

Taking $\varepsilon = \rho_{16}/2$, and then relabelling the resulting positive saving after the initial harmless shrinkage of ρ_{16}, gives the second term in (2). Explicitly, with $\rho_{16}' = \rho_{16}/2$,
$$N^{1-\rho_{16}+\varepsilon} \le N^{1-\rho_{16}'},$$
and then ρ_{16}' is renamed as ρ_{16}.

Assume now that $Y_{16}U > N^{1-\rho_{16}}$. Fix $u \asymp U$. As $p \in I_{16}^{\#}$, the values $N-up$ lie in the residue class $N \bmod u$ and in an interval $J_u \subset [1,N]$ of length
$$H_u \asymp uY_{16}.$$

Since $u \asymp U$, the non-small-volume assumption gives
$$H_u \gg UY_{16} > N^{1-\rho_{16}}.$$

The required Shiu modulus condition follows from
$$\delta < (1-\rho_{16})(1-\delta/2). \tag{3}$$

Indeed, $u \asymp U \ll N^{\delta}$, while $H_u \gg N^{1-\rho_{16}}$; for large N, (3) implies
$$u \le H_u^{1-\delta/2}.$$

For the displayed choices $\delta = 1/(20J_0^2)$ and $\rho_{16} = 1/(10^6 J_0^4)$, (3) holds for every $J_0 \ge 1$.

Cauchy–Schwarz in the p-variable gives
$$\begin{aligned}
&\sum_{p\in I_{16}^{\#}} \tau_{K_1}(p)\tau_{K_3}(N-up)\mathbf{1}_{N-up>0} \\
&\quad \le \left(\sum_{p\in I_{16}^{\#}} \tau_{K_1}(p)^2\right)^{1/2} \left(\sum_{p\in I_{16}^{\#}} \tau_{K_3}(N-up)^2\mathbf{1}_{N-up>0}\right)^{1/2}.
\end{aligned}$$

The first factor is
$$\ll Y_{16}^{1/2}(\log N)^{O_{J_0}(1)},$$
using the $q = 1$ divisor-function interval estimate and the floor $Y_{16} \geq X_P(\log N)^{-B_{16}}$. The second factor is controlled by (SH), applied modulo u to the residue class $N \bmod u$:
$$\sum_{p\in I_{16}^{\#}} \tau_{K_3}(N-up)^2 \mathbf{1}_{N-up>0} \ll Y_{16}(\log N)^{O_{J_0}(1)}\mathcal{E}_{u,N}.$$
Thus
$$\sum_{p\in I_{16}^{\#}} \tau_{K_1}(p)\tau_{K_3}(N-up)\mathbf{1}_{N-up>0} \ll Y_{16}(\log N)^{O_{J_0}(1)}\mathcal{E}_{u,N}^{1/2}.$$
Summing this bound with weight $\tau_{K_2}(u)$ and using the dyadic form of (LFA) gives
$$S \ll Y_{16}U(\log N)^{O_{J_0}(1)} + N^{1-\rho_{16}}(\log N)^{O_{J_0}(1)}.$$
This proves (2), hence X16C.

8 Noncircularity and parameters

The X16 floor is a monotone upper-bound device, not an appeal to TTH. If the actual interval is shorter than $X_C(\log N)^{-B_{16}}$, it is enlarged only for the purpose of an upper bound. The added contribution is exactly of size
$$N(\log N)^{C_{16}}\frac{X_C(\log N)^{-B_{16}}}{X_C} = N(\log N)^{C_{16}-B_{16}},$$
up to fixed C1/B1 coefficient losses, and the parameter register chooses
$$B_{16} > C_0 + C_1 + C_{16} + 20.$$
After BRS is proved, TTH chooses B_κ large enough to dominate
$$B_{16} + C_0 + C_1 + C_{16} + \rho_{16}^{-1} + 20.$$
Thus X16C is used before TTH and should not depend on TTH.

This packet directly verifies the product-carrier X16C estimate. The full BRS carrier-slice estimate additionally uses X16BRS to reduce complementary, quotient, and controlled quotient carriers to this product-carrier model.

9 Expected positive outcome

A positive review can be short:

> The BRS carrier-slice estimate follows from the stated Shiu/AP divisor-correlation argument. The AP moduli, divisor weights, local factors, Cauchy–Schwarz loss, and X16 floor are compatible with the claimed constants, and the argument is not circular in C1 or TTH.

10 Failure modes to flag

Please flag the first point at which any of the following fails:

1. a BRS carrier is not covered by the X16BRS reduction;
2. the divisor-correlation majorant (1) does not dominate the actual B1 tuple mass;
3. Shiu's theorem does not apply to the AP modulus/interval range used in one of the cases;
4. the non-coprime AP residue classes create local factors not absorbed by the stated averaging;
5. the Cauchy–Schwarz square-root step loses more than recorded;
6. the trivial saving branch does not cover the complementary large-carrier range;
7. the proof of X16C uses C1 or TTH circularly.