CKP/X10 Smooth-Weight Matching Packet

targeted external-review packet

June 2, 2026

Review request

Please check the CKP branch only. The question is whether the CKP nonzero frequency contribution produced by the internal reduction fits the hypotheses of the stated Duke–Friedlander–Iwaniec Kloosterman-fraction input, called X10 in the proof ledger, after the smooth-weight derivative verification in CKPD.

This is not a request to review the full Goldbach proof.

Target claim

Target claim. The CKP terminal contribution satisfies

$$R_{\mathrm{CKP}}(N) = M_{\mathrm{CKP}}(N) + o(N),$$

where $M_{\mathrm{CKP}}(N)$ is the canonical local projection admitted by H4.

The nonzero-frequency CKP contribution is $o(N)$ by X10. The zero-frequency part is local and is passed to H4 through B1LD.

Logical chain

$$G1a + G2a + G3a + CKPD + G4a/X10 + X10ER + B1LD \Longrightarrow G8a.$$

More explicitly:

1. G1a gives the exact gcd splitting $u = ga$, $u' = gq$.
2. G2a performs the smooth AP/Fourier expansion.
3. G3a converts the nonzero-frequency CKP terms to bilinear Kloosterman-fraction sums with two-variable smooth weights $W_{g,h}(a, q)$.

4. CKPD proves the derivative bounds required to insert these weights into X10.

5. G4a applies X10 on the central balanced nonzero-frequency range.

6. X10ER records that high-frequency, small-conductor, large-g, boundary, and short-volume CKP ranges are routed to C1P/C1A/C1 or H4 before X10 is invoked.

7. G8a combines the nonzero-frequency estimate with the zero-frequency local term.

Expanded block descriptions

The symbols in the chain mean the following local assertions.

G1a: gcd splitting

Starting from a tagged CKP atom of the schematic form

$$\sum_{uy+u'y'=N} \alpha(u)\alpha'(u')\beta(y)\beta'(y')W_U(u)W_{U'}(u')W_Y(y)W_{Y'}(y'),$$

G1a splits

$$u = ga, \qquad u' = gq, \qquad (a,q) = 1.$$

The equation becomes

$$ay + qy' = N_g, \qquad N_g = N/g.$$

Layers with $g \nmid N$ are empty. Large or unbalanced g-layers are routed outside X10. The central CKP branch retains balanced layers with inherited divisor-bounded coefficient sequences.

G2a: smooth AP/Fourier expansion

For fixed g, a, q, the equation

$$ay + qy' = N_g$$

is reduced to the congruence

$$y \equiv N_g\bar{a} \pmod{q}.$$

G2a applies a smooth AP/Fourier expansion to the y-fibre. The frequency $h = 0$ produces the local term. Frequencies $h \neq 0$ produce oscillatory terms with phase

$$e\left(\frac{hN_g\overline{a}}{q}\right).$$

G3a: weighted Kloosterman-fraction form

G3a keeps the Fourier fibre as a genuine two-variable smooth weight rather than separating it artificially. The nonzero-frequency layer has the form

$$\mathcal{O}_{g,h} = \sum_{\substack{a\sim A_g,\ q\sim Q_g \\ (a,q)=1}} \alpha_g(a)\gamma_g(q)W_{g,h}(a,q)e\left(\frac{hN_g\overline{a}}{q}\right).$$

This is the exact signed phase sent to X10. The external DFI integer parameter is

$$r = |h|N_g.$$

For $h < 0$, we apply the same estimate to the conjugate phase, so the positive external integer parameter is still $r = |h|N_g$.

X10/DFI theorem statement used in this packet

The external theorem invoked by X10 is used in the following dyadic smooth-weight form. Let $M, Q \geq 1$, $r \geq 1$, and let α_m, β_q be arbitrary complex sequences supported on $m \asymp M$, $q \asymp Q$. Let $F(m,q)$ be supported on the same dyadic box, satisfy $|F(m,q)| \leq 1$, and obey

$$\partial_m^i \partial_q^j F(m,q) \ll \eta^{i+j} M^{-i} Q^{-j}, \qquad 0 \leq i, j \leq 2. \tag{DFI-wt}$$

Then, for every $\varepsilon > 0$,

$$\sum_{\substack{m\asymp M,\ q\asymp Q \\ (m,q)=1}} \alpha_m \beta_q F(m,q) e\left(\frac{r\overline{m}}{q}\right) \ll_\varepsilon \eta^2 \|\alpha\|_2 \|\beta\|_2 (r+MQ)^{3/8}(M+Q)^{11/48+\varepsilon}. \tag{DFI-X10}$$

The CKP application has $\eta \leq (\log N)^C$, so the η^2 factor is part of the polylogarithmic loss budget.

CKPD: exact weight, normalization, and derivative check

CKPD proves that the actual nonseparated Fourier fibre weight satisfies the DFI/X10 smooth-weight hypotheses. The central layer has

$$a \asymp A_g, \qquad q \asymp Q_g, \qquad y \asymp Y, \qquad y' \asymp Y',$$

with central support

$$A_g \asymp Q_g, \qquad Y \asymp Y', \qquad \frac{Y}{Q_g} \asymp g. \tag{CS}$$

The remaining fibre variable is

$$z(a,q,y) = \frac{N_g - ay}{q}.$$

Let $\omega_A, \omega_Q, W_Y, W_{Y'}$ be the smooth dyadic cutoffs from the fixed CKP tag, with

$$\omega_A^{(r)} \ll_r A_g^{-r}, \qquad \omega_Q^{(r)} \ll_r Q_g^{-r}, \qquad W_Y^{(r)} \ll_r Y^{-r}, \qquad W_{Y'}^{(r)} \ll_r (Y')^{-r}.$$

The smooth fibre is

$$\Phi_{a,q}(y) = \omega_A(a)\omega_Q(q)W_Y(y)W_{Y'}(z(a,q,y)). \tag{Phi}$$

For $h \neq 0$, the actual two-variable weight entering the Kloosterman-fraction sum is

$$\mathcal{W}_{g,h}(a,q) = \frac{1}{q}\int_{\mathbb{R}} \Phi_{a,q}(y) e\left(-\frac{hy}{q}\right) dy. \tag{W}$$

Thus the nonzero-frequency layer is

$$\mathcal{O}_{g,h} = \sum_{\substack{a\sim A_g,\ q\sim Q_g \\ (a,q)=1}} \alpha_g(a)\gamma_{g,h}(q)\mathcal{W}_{g,h}(a,q) e\left(\frac{hN_g\bar{a}}{q}\right). \tag{CKP-X10}$$

The amplitude used to normalize the DFI weight is

$$\mathcal{A}_{g,h,R} = (\log N)^{C_*} g(1+|h|g)^{-R}, \tag{Amp}$$

and the normalized DFI weight is

$$\widetilde{W}_{g,h}(a,q) = \mathcal{A}_{g,h,R}^{-1}\mathcal{W}_{g,h}(a,q). \tag{Wtilde}$$

The factor $\mathcal{A}_{g,h,R}$ is kept outside the DFI weight and is included in the final g, h-summation below.

The derivative check uses the exact dependence $z(a, q, y) = (N_g - ay)/q$, not a separated surrogate. On the central support,

$$\partial_a z = -\frac{y}{q}, \qquad \partial_q z = -\frac{z}{q}.$$

Since $y \asymp Y$, $z \asymp Y'$, $A_g \asymp Q_g$, and $Y \asymp Y'$, each a-derivative of $W_{Y'}(z)$ contributes

$$W'_{Y'}(z)\frac{y}{q} \ll (Y')^{-1}\frac{Y}{Q_g} \ll A_g^{-1},$$

and each q-derivative contributes $O(Q_g^{-1})$. Repeated and mixed derivatives follow by Faa di Bruno, giving for fixed i, j

$$\partial_a^i \partial_q^j \Phi_{a,q}(y) \ll A_g^{-i} Q_g^{-j} \mathbf{1}_{y \asymp Y}. \tag{Phi-der}$$

Differentiating the Fourier integral in (W), derivatives falling on $\Phi_{a,q}$ give the expected $A_g^{-i}Q_g^{-j}$ factors, derivatives falling on q^{-1} give powers of Q_g^{-1}, and derivatives falling on the phase cost at most powers of $1+|h|g$. If $|h|g \le 1$, the trivial estimate $q^{-1}\int |\Phi_{a,q}(y)|dy \ll Y/Q_g \asymp g$, and its differentiated version, gives the desired bound because $1 + |h|g \asymp 1$. If $|h|g > 1$, integration by parts in y gives, for any fixed B, i, j,

$$\partial_a^i \partial_q^j \mathcal{W}_{g,h}(a, q) \ll A_g^{-i} Q_g^{-j} g(1 + |h|g)^{-B+i+j}. \tag{W-der}$$

In the central DFI range $|h|g \le (\log N)^{B_{\mathrm{HF}}}$, choose R in (Amp) larger than the derivative order and the summation losses. Then

$$\widetilde{W}_{g,h}(a, q) \ll 1, \qquad \partial_a^i \partial_q^j \widetilde{W}_{g,h}(a, q) \ll (\log N)^C A_g^{-i} Q_g^{-j} \quad (0 \le i, j \le 2). \tag{DFI-wt}$$

This is the smooth-weight hypothesis required by X10.

G4a/X10: DFI estimate

G4a invokes the X10 external input, a DFI bilinear Kloosterman-fraction estimate in the dyadic form

$$\sum_{\substack{m \asymp M,\ q \asymp Q \\ (m,q)=1}} \alpha_m \beta_q F(m, q) e\left(\frac{r\overline{m}}{q}\right)$$

with the substitution

$$m = a, \qquad M = A_g, \qquad r = |h|N_g, \qquad Q = Q_g.$$

For $h < 0$, this is the same application to the conjugate phase. Boundary, high-frequency, small-conductor, and large-g ranges are excluded from the X10 application and routed by the existing X10ER, C1P/C1A/C1, G2a, and G8a interfaces.

B1LD and G8a: zero frequency and recombination

The $h = 0$ term is not estimated by X10. B1LD identifies the finite-convolution local densities inherited from B1 with the local model used by H4. G8a then combines:

1. zero-frequency local projection;
2. central nonzero-frequency cancellation by G3a/G4a/X10/CKPD;
3. boundary, high-frequency, small-conductor, and large-g routing through X10ER and C1P/C1A/C1;

to prove

$$R_{\mathrm{CKP}}(N) = M_{\mathrm{CKP}}(N) + o(N).$$

Central DFI loss accounting

In the central balanced range write

$$A_g \asymp Q_g \asymp S_g, \qquad S_g = \frac{N^{1/2+O(\eta_0)}}{g}.$$

The coefficient norms are

$$\|\alpha_g\|_2 \ll A_g^{1/2}(\log N)^C, \qquad \|\gamma_{g,h}\|_2 \ll Q_g^{1/2}(\log N)^C.$$

The DFI estimate is applied with

$$m = a, \qquad n = q, \qquad r = |h|N_g, \qquad M = A_g, \qquad Q = Q_g.$$

Because

$$\frac{|h|N_g}{A_gQ_g} \asymp |h|g \le (\log N)^{B_{\mathrm{HF}}},$$

the DFI factor $(r+MQ)^{3/8}$ contributes only a polylogarithmic loss beyond $(A_g Q_g)^{3/8}$. Including the amplitude factor (Amp), one obtains

$$|\mathcal{O}_{g,h}| \ll (\log N)^C g(1+|h|g)^{-A}\|\alpha_g\|_2\|\gamma_{g,h}\|_2(S_g^2)^{3/8}(2S_g)^{11/48+\varepsilon}.$$

Thus

$$|\mathcal{O}_{g,h}| \ll (\log N)^C g(1+|h|g)^{-A} S_g^{1+3/4+11/48+\varepsilon},$$

and since $1+3/4+11/48 = 95/48$,

$$|\mathcal{O}_{g,h}| \ll N^{95/96+O(\eta_0)+\varepsilon}(\log N)^C g^{-47/48}(1+|h|g)^{-A}. \qquad \text{(Layer)}$$

The h-sum is absolutely convergent for fixed large A:

$$\sum_{h\neq 0}(1+|h|g)^{-A} \ll 1.$$

Also $g \mid N$ by the G1a splitting, hence the number of g-layers is divisor-bounded:

$$\#\{g : g \mid N\} \ll_\varepsilon N^\varepsilon.$$

Therefore

$$\sum_{g|N}\sum_{h\neq 0}|\mathcal{O}_{g,h}| \ll N^{95/96+O(\eta_0)+o(1)} = o(N),$$

after choosing η_0 and the DFI ε so that

$$O(\eta_0)+\varepsilon+o(1) < \frac{1}{96}.$$

Excluded-range decomposition

The DFI theorem is invoked only on the central nonzero-frequency range. The complete CKP nonzero-frequency decomposition is

$$\mathrm{CKP}_{h\neq 0} = \mathrm{CentralDFI} \sqcup \mathrm{HighFreq} \sqcup \mathrm{SmallConductor} \sqcup \mathrm{LargeG} \sqcup \mathrm{Boundary/Short}.$$

The routing is:

- CentralDFI: central balance, $|h|g \le (\log N)^{B_{\mathrm{HF}}}$, and non-small conductor; routed to X10 after CKPD.
- HighFreq: $|h|g > (\log N)^{B_{\mathrm{HF}}}$; routed by G2a/X10ER to C1P/C1A/C1.
- SmallConductor: $q/(q, hN_g) \le (\log N)^B$; routed to C1P/C1A/C1.

- LargeG: $g > N^{\eta_0}$ or outside CKP balance; routed by G1a/G8a/X10ER to C1P/C1A/C1.
- Boundary/Short: boundary support or short volume; routed to C1P/C1A/C1 or H4 as appropriate.
- Zero frequency: $h = 0$; routed to the B1LD/H4 local term, not X10.

Thus this packet directly verifies the central DFI matching. The full CKP conclusion additionally uses the listed X10ER/C1P/C1A/C1 routing of excluded ranges and the B1LD/H4 treatment of the zero frequency.

Precise point to verify

The central technical assertion is that

$$W_{g,h}(a,q),$$

as produced by G3a, satisfies the smoothness, support, derivative, modulus, coprimality, and parameter restrictions required by the X10 Kloosterman-fraction theorem statement.

Please especially check:

1. whether the variables a, q in G3a match the variables in X10;
2. whether all arithmetic moduli and coprimality restrictions are compatible;
3. whether CKPD differentiates the actual weight entering X10, not a simplified surrogate;
4. whether summation over g, h and dyadic parameters preserves the claimed $o(N)$ saving;
5. whether the zero-frequency term is correctly excluded from the X10 estimate and routed to H4.

Minimal source map

The detailed proof-source files are:

1. `../../../manuscript_md/appendices/appendix_c_ckp_x10.md`

2. `../../../Lemmas/ckp_x10_smooth_weight_derivative_appendix_ltx.md`
3. `../../../External/x_10_verification_ltx.md`
4. `../../../Lemmas/g_8_a_ltx.md`

The full technical chain is:

$$G1a, \quad G2a, \quad G3a, \quad CKPD, \quad G4a, \quad G8a, \quad B1LD, \quad X10.$$

Checklist

X10 hypotheses

- The theorem statement used as X10 is stated precisely enough for this packet.
- The CKP variables a, q match the X10 variables.
- The summation ranges satisfy the X10 size restrictions.
- The moduli and coprimality assumptions match X10.
- The nonzero-frequency parameter h is in the required range.

Smooth weights

- CKPD differentiates the actual weight $W_{g,h}(a, q)$.
- The support of $W_{g,h}$ is compatible with X10.
- All first and higher derivative bounds required by X10 are proved.
- Boundary cutoffs do not create uncontrolled derivative losses.
- Summing over dyadic partitions preserves the claimed saving.

Local term

- The zero-frequency term is not incorrectly estimated by X10.
- The zero-frequency term is correctly identified as a canonical local term.
- B1LD/H4 compatibility is sufficient for the CKP local contribution.

Expected outcome

A positive review would say:

> The CKP nonzero-frequency reduction fits the stated X10 hypotheses after CKPD, and no missing derivative/modulus/range condition is visible.

A negative review should identify the first failed hypothesis or range.