H4 Local Algebra Full Proof Package

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Contents

1 H4 Local Algebra Full Proof Package

1.1 Abstract

This full-proof package contains the local-main source texts: LongAP/Local normalization, B1 local-density compatibility, and H4 local algebra.

1.2 Scope

This package proves the local/main algebra brick used by I1. Common decomposition and routing sources are in the final modular assembly package.

1.3 Included Proof-Source Files

1. `Lemmas/d_1_ltx.md` – LongAP/Local local-coefficient expansion and projection
2. `Lemmas/g8a_local_density_ltx.md` – B1 local-density compatibility
3. `Lemmas/h_4_ltx.md` – Tagged local algebra and singular series

2 Part 1. D1: LongAP/Local local-coefficient expansion and projection

Source file: `Lemmas/d_1_ltx.md`.

2.0.1 D1. LongAP/Local Normalization Lemma

D1.0. Role Logical ID: `D1`.

Used by: `H4`, `I1`.

Uses: `B1`, `B3`, `F3P`, `F3`, `F3T`, `F4`, `C1A`, `C1`, `E5`, `LPI`, and standard smooth AP/local counting.

Lemma **D1** is responsible for the LongAP/Local branch of the proof tree. D1 proves that LongAP/Local terms enter the local assembly through the independent LPI local projection interface: the same tagged cell is projected by the replacement $\Lambda(n) \mapsto \Lambda_Q(n \bmod Q)$, with parent B1 block, routing tag, dyadic weights, and local congruence data preserved. The downstream H4 assembly consumes only local/main terms that satisfy the LPI-admissible form

$$M^{\mathrm{local}}_{\mathcal{B},\tau}(N) = \mathrm{Loc}_Q\, R_{\mathcal{B},\tau}(N) + o(N).$$

Therefore D1 proves

$$\boxed{R^{\mathrm{LongAP}}_{\mathcal{B},\tau}(N) = \mathrm{Loc}_Q\, R^{\mathrm{LongAP}}_{\mathcal{B},\tau}(N) + o(N)}$$

for every tagged LongAP/Local atom \{}((\{}mathcal B,\{}tau) \{}).

In other words, D1 does not merely evaluate a local AP sum; it proves that its main term has exactly the same normalization as the LPI local model later assembled by H4.

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D1.1. Tagged LongAP/Local atom Let $(\mathcal{B}, \tau)$ be a tagged LongAP/Local atom obtained after

$$B1 \to B3 \to F3/F4.$$

By the intrinsic LongAP/Local predicate in Lemma F3P, such an atom has the following positive properties:

1. it is a long AP or a finite union of controlled long AP fibres;
2. there is a controlled modulus $Q_\tau \le (\log N)^C$;
3. every coefficient depending on the long AP variable belongs to the local coefficient algebra $\mathfrak{C}_{\mathrm{loc}}(Q_\tau)$;
4. all remaining restrictions are controlled local congruence / AP restrictions;
5. all local terms preserve the parent B1 block tag $\mathcal{B}$ and the routing tag τ.

In particular all moduli are controlled:

$$q \le (\log N)^C;$$

A model tagged LongAP/Local atom has the form

$$R^{\mathrm{LongAP}}_{\mathcal{B},\tau}(N) = \sum_{x \in \Omega_{\mathcal{B},\tau}} W_{\mathcal{B},\tau}(x) \mathbf{1}_{A(x) \equiv b \pmod q} \rho_Q(x),$$

where:

- \{}(\{}Omega_{\{}mathcal B,

\{}tau} \{}) is a smooth tagged box/fibre;

- \{}(W_{\{}mathcal B,

\{}tau} \{}) is a smooth dyadic weight;

- $q \le (\log N)^C$;
- \{}(\{}rho_Q(x)

\{}) denotes the finite product of local coprimality/residue constraints inherited from the B1 coefficients and the Goldbach equation;

- no nonlocal coefficient such as $\lambda(L(x))$, $\mu(L(x))$, a nonlocal finite-convolution factor, or a CKP oscillatory phase remains.

The exclusion of such factors is a consequence of the positive F3P LongAP/Local predicate, not an analytic assumption made inside D1. Lemma D1.2A records the consequence in the form needed for local AP counting.

—

D1.2. LPI local model recalled In Lemma LPI, choose

$$Q = \prod_{p \le w} p, \qquad w = w(N) \to \infty, \qquad w = o(\log N).$$

The local von Mangoldt model is

$$\Lambda_Q(a) = \frac{Q}{\varphi(Q)} \mathbf{1}_{(a,Q)=1}.$$

For every tagged atom, the LPI local projection is defined as

$$\mathrm{Loc}_Q\, R_{\mathcal{B},\tau}(N)$$

by replacing the arithmetic coefficients in the tagged atom by their local residue-class densities modulo Q, while keeping:

- the same tag \{}((\{}mathcal B,

\{}tau) \{});

- the same smooth/dyadic weights;
- the same local congruence restrictions;
- the same finite routing cell.

Lemma D1 identifies the LongAP main term with exactly this object.

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D1.2A. F3P consequence for LongAP/Local coefficients

Lemma 2.1 (Lemma D1.2A. No nonlocal arithmetic coefficient survives in LongAP/Local)**.** *Let $(\mathcal{B}, \tau)$ be a terminal atom produced by the B1/B3/F3/F4 routing tree and tagged as LongAP/Local. Then every coefficient which still depends on a long AP variable is local in the following sense: after refining by a controlled modulus $Q_\tau \le (\log N)^C$, it is a finite linear combination of residue-class and coprimality indicators modulo Q_τ, multiplied by smooth dyadic weights and tag constants. In particular, no terminal LongAP/Local atom contains a surviving factor of the form*

$$\mu(L(u)), \qquad \lambda(L(u)), \qquad e(\alpha L(u)), \qquad e\left(\frac{k\overline{L(u)}}{q}\right),$$

or any finite-convolution descendant of these which is not determined by the controlled local residue data.

Proof. The proof is the direct consequence of the intrinsic predicate catalogue F3P, together with the F3/F4 exhaustion of unresolved obstructions.

By F3P.7, a terminal LongAP/Local atom satisfies

$$\mathcal{W}_{\mathrm{long}}(\mathcal{B}, \tau) \subset \mathfrak{C}_{\mathrm{loc}}(Q_\tau)$$

for a controlled modulus $Q_\tau \le (\log N)^C$. Expanding the generators of $\mathfrak{C}_{\mathrm{loc}}(Q_\tau)$ gives a finite linear combination of smooth dyadic weights, tag constants, residue-class indicators, coprimality indicators, and fixed controlled-divisor factors. This proves the asserted local form.

It remains only to justify that a forbidden nonlocal coefficient could not have received a LongAP/Local terminal tag. This is not a downstream estimate; it is the intrinsic terminal-labelling rule.

B3 preliminary classification. Lemma B3 records a LongAP/Local candidate only when, after fixing the auxiliary variables, the remaining counting problem is a controlled AP/local count with smooth weights and no nonlocal oscillatory arithmetic coefficient. If a Mobius-, Liouville-, or other nonlocal central-long coefficient remains, B3 records a BranchB/GoodAWACK candidate unless a CKP-balanced grouping or a forced LocalDiag dependence has already been detected.

F3P/F3 terminal predicate. Lemma F3P makes the LongAP/Local predicate a positive local-coefficient condition. Hence a cell with a surviving μ-, λ-, Kloosterman-, Fourier-, reciprocal, finite-convolution, or nilsequence-type oscillation cannot be terminally labelled LongAP/Local at the F3 stage. The remaining routing alternatives are:

Surviving feature	Routing consequence
$\mu(L)$, $\lambda(L)$, or affine finite-convolution oscillation attached to a long variable	GoodAWACK, unless CKP or LocalDiag applies first
balanced multiplicative or reciprocal-phase structure	CKP
strict saving predicate, short residual volume, boundary, high frequency, or small conductor	Edge through C1P/C1A/C1
forced equality, proportionality, repeated factor, or local dependence	LocalDiag
only controlled residue-class / coprimality data modulo $(\log N)^C$	admissible LongAP/Local

F4 divisor and quotient decisions. Ordinary divisor or quotient conditions are not allowed to remain unresolved inside LongAP/Local. Lemma F4 routes such a condition to Edge, LocalDiag, CKP, or GoodAWACK, or absorbs a controlled fixed divisor with strict decrease of the F3 measure. If after such absorption only controlled local congruence data remain, the later terminal cell may be LongAP/Local. If a nonlocal divisor/quotient coefficient remains, the cell is routed by F4 and is not D1-admissible.

Controlled CRT and local restrictions. Lemma E5 and the F3 controlled CRT steps may replace a full-rank finite-index restriction by residue-class data with controlled content. These operations do not convert a nonlocal arithmetic function into a local density. They only record congruence and coprimality conditions modulo controlled moduli. If the modulus or conductor is not controlled, the cell is routed to Edge, CKP, LocalDiag, or F4 rather than to D1.

The routing measure in Lemma F3 is well-founded; Lemma F3T records the same finite case distinction as a synchronized table for the proof tree. Therefore the process terminates. At a terminal LongAP/Local tag, the positive F3P predicate has already forced every long-variable coefficient into $\mathfrak{C}_{\mathrm{loc}}(Q_\tau)$. Lemma proved.

—

$\square$

D1.3. Pure local AP counting lemma

Lemma 2.2 (Lemma D1.1. Smooth AP count with controlled modulus)**.** *Let \{}(W\{}in C_c^\{}infty(\{}mathbb R) \{}), and let*

$$W_U(u) = W\left(\frac{u}{U}\right).$$

For a controlled modulus

$$q \le (\log N)^C,$$

and a residue class $r \pmod q$, *we have*

$$\sum_{u \equiv r \pmod q} W_U(u) = \frac{1}{q}\sum_u W_U(u) + O_A(U(\log N)^{-A}) + O_A((\log N)^A)$$

after smoothing and boundary truncation, with the total boundary contribution routed to the C1P predicates E1/E6.

In particular, when this estimate is inserted into a tagged LongAP atom with total volume $\asymp N$, *the error is*

$$o(N)$$

after summing over polylogarithmically many local moduli and tags.

Proof. The assertion follows from the exact finite Fourier expansion of the residue class:

$$1_{u \equiv r \pmod q} = \frac{1}{q}\sum_{h \bmod q} e\left(\frac{h(u-r)}{q}\right).$$

The zero frequency gives

$$\frac{1}{q}\sum_u W_U(u).$$

For $h \neq 0$, smoothness and summation by parts give decay faster than any power of q/U. Since $q \le (\log N)^C$ and the LongAP direction has length at least a fixed power of N, the nonzero finite Fourier terms are negligible. Boundary discrepancies are smooth dyadic boundary terms and satisfy the C1 admission predicates E1/E6.

Lemma proved.

—

□

D1.4. Local residue density and LPI tagged projection The LongAP/Local atom has only controlled local congruence data. After refining modulo

$$Q' = \mathrm{lcm}(Q, q_1, \ldots, q_m),$$

where all extra local moduli satisfy

$$q_i \le (\log N)^C,$$

we still have

$$Q' = Q \cdot (\log N)^{O(1)}$$

up to harmless overlap in prime factors.

The local main term obtained by repeated use of Lemma D1.1 is

$$M^{\mathrm{D1}}_{\mathcal{B},\tau}(N) = \sum_{a \bmod Q'} \delta_{\mathcal{B},\tau}(a;N)\mathfrak{w}_{\mathcal{B},\tau}(a) \cdot \frac{\mathrm{Vol}(\Omega_{\mathcal{B},\tau})}{Q'} + o(N),$$

where:

- \{}(\{}delta_{\{}mathcal B,

\{}tau}(a;N) \{}) records the tagged local congruence constraints;

- \{}(\{}mathfrak w_{\{}mathcal B,

\{}tau}(a) \{}) is the product of local densities of the B1 arithmetic coefficients in that residue class;

- the smooth volume is the same as in the original tagged atom.

But this is exactly the definition of

$$\mathrm{Loc}_Q\, R^{\mathrm{LongAP}}_{\mathcal{B},\tau}(N)$$

from Lemma LPI, because both objects are obtained by:

1. keeping the same parent tag \{}((\{}mathcal B,

\{}tau) \{});

1. keeping the same smooth cell;
2. replacing arithmetic coefficients by their local residue-class densities;
3. averaging over the same controlled local congruence data.

Therefore

$$M^{\mathrm{D1}}_{\mathcal{B},\tau}(N) = \mathrm{Loc}_Q\, R^{\mathrm{LongAP}}_{\mathcal{B},\tau}(N) + o(N).$$

—

D1.5. No hidden nonlocal estimates D1 explicitly excludes the following from the LongAP/Local class:

1. PNT-in-AP input;
2. Bombieri–Vinogradov type cancellation;
3. Mobius/Liouville cancellation;
4. nilsequence orthogonality;
5. CKP oscillatory phases;
6. nonzero Fourier phases not already routed to C1/G8a.

If a tagged atom contains a factor of the form

$$\lambda(L(u)), \qquad \mu(L(u)), \qquad e\left(\frac{k\bar{a}}{q}\right),$$

or any nonlocal oscillatory coefficient, then it is not D1-admissible. It must be routed to one of:

$$\text{GoodAWACK}, \qquad \text{CKP}, \qquad \text{Edge}, \qquad \text{LocalDiag}$$

by B3/F3/F4/C1/G8a/E10.

Lemma D1.2A proves this exclusion from the intrinsic F3P LongAP/Local predicate plus the F3/F4 routing alternatives. Thus D1 remains a pure local counting lemma: it never invokes cancellation of Mobius, Liouville, nilsequence, Kloosterman, or nonzero Fourier coefficients.

—

D1.6. Boundary and endpoint errors The local AP count produces endpoint and smoothing errors. These have one of the forms:

$$O((\log N)^C)$$

per fibre, or boundary mass

$$\leq N\varepsilon(N), \qquad \varepsilon(N)(\log N)^C \to 0.$$

By Lemma C1, such errors are routed to:

$$E1 : \text{boundary/dyadic tail},$$

or

$$E6 : \text{short residual volume},$$

or, when a short Type-I error is involved,

$$E7 : \text{Type I short-variable error}.$$

Therefore the total D1 error is

$$o(N)$$

after polylogarithmic summation over tags.

—

D1.7. Tag preservation Every D1 operation preserves the tag

$$(\mathcal{B}, \tau).$$

The AP count is performed inside a fixed tagged cell. It does not merge cells from different parent B1 blocks and does not identify visually similar local terms from different routing histories.

Therefore the D1 local main terms preserve the tag separation later used by the no-double-counting mechanism of Lemma H4:

$$M_{\text{local}}(N) = \sum_{\mathcal{B},\tau} c_{\mathcal{B}} M^{\text{local}}_{\mathcal{B},\tau}(N).$$

—

D1.8. D1 theorem

Theorem 2.3 (Theorem D1)**.** *Let \{}((\{}mathcal B, \{}tau) \{}) be a terminal LongAP/Local atom produced by the routing tree. Then*

$$R^{\mathrm{LongAP}}_{\mathcal{B},\tau}(N) = \operatorname{Loc}_Q R^{\mathrm{LongAP}}_{\mathcal{B},\tau}(N) + o(N).$$

Consequently, summing over all tagged LongAP/Local atoms,

$$R_{\mathrm{LongAP/Local}}(N) = M_{\mathrm{LongAP/Local}}(N) + o(N),$$

where

$$M_{\mathrm{LongAP/Local}}(N) = \sum_{\mathcal{B},\tau \in \mathrm{LongAP/Local}} c_{\mathcal{B}} \operatorname{Loc}_Q R^{\mathrm{LongAP}}_{\mathcal{B},\tau}(N).$$

Thus every D1 local term is LPI-admissible and therefore ready for the H4 assembly.

Proof. Fix a tagged terminal LongAP/Local atom \{}((\{}mathcal B, \{}tau) \{}). By Lemma D1.2A, all remaining arithmetic restrictions are controlled local AP/congruence conditions with moduli $\le (\log N)^C$, and no nonlocal oscillatory factor remains.

Apply the smooth AP counting lemma to the long AP directions. The zero/local part gives the average over the corresponding residue classes modulo the controlled local modulus. Refining these local conditions with the LPI modulus Q gives exactly the tagged LPI projection \{}(\{}operatorname{Loc}_Q R_{\{}mathcal B, \{}tau}^{\{}mathrm{LongAP}}(N) \{}), namely the explicit Λ_Q-local replacement inside the same B1/F3 cell. Boundary and endpoint discrepancies are C1 Edge errors and contribute $o(N)$.

All operations are performed inside the fixed tag \{}((\{}mathcal B, \{}tau) \{}), so no local terms are merged or double-counted. Summing over the polylogarithmic number of tagged LongAP/Local atoms preserves the \{}(o(N) \{}) error.

The theorem is proved.

—

□

D1.9. Interface refinements

The D1 statement used in the proof is:

$$\boxed{\mathcal{A}(N) = \operatorname{Loc}_Q R^{\mathrm{LongAP}}_{\mathcal{B},\tau}(N) + o(N)}$$

for every tagged LongAP/Local atom.
Thus:

1. D1 local terms satisfy the LPI admission condition consumed by H4;

2. D1 does not hide nonlocal cancellation estimates;

3. all local AP normalizations are compatible with \{}(\{}Lambda_Q(a)=Q/\{}varphi(Q)1_{(a,Q)=1}

\{});

1. all boundary/endpoint errors are routed to C1;

2. parent B1 and routing tags are preserved.

—

Remark 2.4 (D1.10. Output).

$$\boxed{\text{Every tagged LongAP/Local atom equals its LPI canonical local projection plus } o(N).}$$

D1 contains no hidden nonlocal cancellation; boundary and endpoint errors are routed to C1; parent B1 and routing tags are preserved.

Consequences:

- the LPI admission condition is discharged for LongAP/Local terms;
- together with G8a, all major local/main sources are LPI-admissible;
- the remaining external checks are outside D1.

D1.11. Logical Dependencies Internal dependencies: `B1`, `B3`, `F3`, `F3T`, `F4`, `C1A`, `C1`, `E5`, `LPI`.

Children served: `H4` and `I1`.

3 Part 2. B1LD: B1 local-density compatibility

Source file: `Lemmas/g8a_local_density_ltx.md`.

3.0.1 B1LD. Local Densities for B1-Inherited CKP Coefficients

B1LD.0. Role Logical ID: `B1LD`.

Used by: `G8a`, `H4`.

Uses: `B1`, `C1`, and finite CRT local algebra.

This file supplies the local-density interface used in G8a.5. The issue is not that the B1 coefficients are literally the von Mangoldt function; they are finite-convolution pieces coming from the exact Heath–Brown decomposition. The point is that the LPI canonical local projection Loc_Q is defined tagwise for those very same finite-convolution coefficients, and CRT local density algebra is compatible with the exact B1 decomposition.

—

B1LD.1. Local model of an elementary B1 coefficient Let $Q = \prod_{p \le w} p$ be squarefree. For an elementary coefficient sequence $a(n)$ of B1 type

$$\mu(n)\mathbf{1}_{n \le y}, \qquad 1, \qquad \log n,$$

with a fixed smooth dyadic cutoff, define its local residue model modulo Q by

$$a_Q(r;Q) = \frac{1}{|\{n \sim X : n \equiv r \pmod{Q}\}|} \sum_{\substack{n \sim X \\ n \equiv r \pmod{Q}}} a(n) W_X(n),$$

with the usual empty-class convention. Partial summation handles the $\log n$ coefficient, the constant coefficient is immediate, and the μ-coefficient is local because on a squarefree modulus its residue constraints factor prime-by-prime by CRT. Boundary errors from the smooth dyadic cutoff are C1 Edge errors.

Thus every elementary B1 coefficient has a well-defined finite local model modulo Q, with errors $o(1)$ after the standard choice $w = o(\log N)$.

—

B1LD.2. Finite convolution compatibility Let $\mathcal{B}$ be a typed B1 dyadic block. Its coefficient is a finite Dirichlet-convolution expression in elementary factors of the three types above. For a residue class r mod Q, the local model of the product constraint is the finite CRT convolution

$$\rho_{\mathcal{B},Q}(r) = \sum_{r_1 \cdots r_k \equiv r \pmod{Q}} a_{1,Q}(r_1; Q) \cdots a_{k,Q}(r_k; Q). \tag{B1-local}$$

Because Q is squarefree, CRT gives a product over primes $p \le w$:

$$\rho_{\mathcal{B},Q}(r) = \prod_{p \le w} \rho_{\mathcal{B},p}(r \bmod p).$$

This is a finite algebraic identity for the local models. It does not require any prime distribution theorem.

—

B1LD.3. Compatibility with Loc_Q The LPI canonical local projection of a tagged atom is defined by replacing each arithmetic coefficient in that tagged atom by its local residue-class model modulo Q, while keeping the same smooth dyadic weights, routing tag, and linear/congruence constraints.

For a tagged CKP atom $(\mathcal{B}, \tau)$, the outer coefficient sequences $\alpha_g(a)$ and $\gamma_g(q)$ are restrictions, gcd-splits, and dyadic localizations of B1 finite-convolution coefficients. Therefore their local models are exactly the corresponding restrictions of $\rho_{\mathcal{B},Q}$. The operations involved are finite CRT restriction, gcd splitting, and fixed dyadic localization; each commutes with the finite local convolution (B1-local), up to C1 boundary terms.

Hence the local densities used in

$$\mathrm{Loc}_Q\, R^{\mathrm{CKP}}_{\mathcal{B},\tau}(N)$$

are precisely the local densities of the B1-inherited CKP coefficients that appear in the $h = 0$ term of G8a.

—

B1LD.4. Lemma **Lemma B1-LD.** For every tagged CKP atom $(\mathcal{B}, \tau)$, the local-density replacement used by LPI for $\mathrm{Loc}_Q\, R^{\mathrm{CKP}}_{\mathcal{B},\tau}(N)$ is the CRT finite-convolution local model of the B1-inherited coefficient sequences $\alpha_g(a)$, $\gamma_g(q)$, and the fibre coefficients in G8a. Therefore the zero-frequency CKP term computed in G8a.5 has the same arithmetic local density factors as the canonical LPI projection, up to C1 boundary errors.

Proof. Elementary coefficient local models are defined in B1LD.1. B1LD.2 shows that finite convolution and CRT localization commute. G1a gcd splitting and the tagged CKP dyadic restrictions are finite refinements of the same coefficient support, so they preserve the local model tagwise. LPI defines Loc_Q using exactly these tagwise local coefficient models; H4 later assembles the LPI-admitted terms. Thus the arithmetic coefficient part of the G8a $h = 0$ term and the arithmetic coefficient part of the LPI canonical projection agree. Endpoint and smoothing discrepancies are C1 boundary errors by construction. Lemma proved.

—

□

Remark 3.1 (B1LD.5. Output). Every CKP zero-frequency term that enters the local/main branch has the same tagged B1 local coefficient model as the LPI projection. Thus the $h = 0$ CKP contribution can be assembled by H4 without changing normalization and without double-counting any parent B1 block.

B1LD.6. Logical Dependencies Internal dependencies: `B1`, `C1`, finite CRT local algebra.

Children served: `G8a`, `H4`.

4 Part 3. H4: Tagged local algebra and singular series

Source file: `Lemmas/h_4_ltx.md`.

4.0.1 H4. Local/Main Compatibility Lemma

H4.0. Role Logical ID: `H4`.

Used by: `I1`, final local/main assembly.

Uses: `LPI`, `B1`, `B3`, `F3P`, `F3`, `F3T`, `F4`, `D1`, `G8a`, `B1LD`, `C1`, `C1A`, finite CRT local algebra, and the local model Λ_Q.

Lemma **H4** is responsible for the main term of the proof. After B3/F3/F4/C1 routing, all terminal atoms are divided into error classes and local/main classes.

The error classes are already handled by:

$$\text{Edge} \to C1, \qquad \text{CKP}_{h\neq 0} \to G8a, \qquad \text{GoodAWACK} \to E10.$$

The local/main contributions come from:

$$\text{LongAP/Local}, \qquad \text{LocalDiag}, \qquad \text{CKP}_{h=0}.$$

H4 has to prove:

$$M_{\text{local}}(N) = \mathfrak{S}(N)N + o(N).$$

The local/main compatibility statement must prevent:

1. double counting local terms;
2. loss of some local terms;
3. mismatched normalizations among LongAP, LocalDiag, and CKP zero-frequency terms;
4. incorrect assembly of the Euler product.

Lemma H4 handles this by consuming the LPI local projection/admission interface and the parent B1 block tags. Thus H4 is an assembly lemma: it does not define the local projection independently of D1 or G8a, but assembles the local terms after D1, G8a, and B1LD have proved LPI-admissibility.

—

H4.1. Local modulus and local model The local model is the one defined in Lemma LPI. We recall it here for the calculation of the singular series.

Let

$$w = w(N) \to \infty, \qquad w = o(\log N),$$

and set

$$Q = \prod_{p \le w} p.$$

Define the local model of Λ modulo Q by

$$\Lambda_Q(a) = \frac{Q}{\varphi(Q)} \mathbf{1}_{(a,Q)=1}.$$

This normalization gives average value one:

$$\frac{1}{Q} \sum_{a \bmod Q} \Lambda_Q(a) = 1.$$

Define the local Goldbach density at modulus Q:

$$\sigma_Q(N) = \frac{1}{Q} \sum_{a \bmod Q} \Lambda_Q(a)\Lambda_Q(N-a).$$

The canonical local main term is

$$M_Q(N) = N\sigma_Q(N) + o(N),$$

where the $o(N)$ comes only from endpoint/smooth partition effects already routed to C1 or from replacing exact interval length by $N + O(1)$.

—

H4.2. Canonical local projection of the original problem Define the canonical local projection of the original Goldbach sum by

$$\mathrm{Loc}_Q\, R_\Lambda(N) = N\sigma_Q(N).$$

Equivalently,

$$\mathrm{Loc}_Q\, R_\Lambda(N) = N \cdot \frac{1}{Q} \sum_{a \bmod Q} \Lambda_Q(a)\Lambda_Q(N-a).$$

This definition is independent of a particular branch of the proof tree.

The purpose of H4 is to prove that the sum of all local/main pieces produced by D1, G8a zero-frequency, and LocalDiag routing is exactly this canonical local projection, up to $o(N)$:

$$M_{\mathrm{local}}(N) = \mathrm{Loc}_Q\, R_\Lambda(N) + o(N).$$

Then H4 computes the limit

$$\sigma_Q(N) \to \mathfrak{S}(N)$$

as $w \to \infty$.

—

H4.3. Parent B1 block tags From Lemma B1 we have the exact decomposition

$$R_\Lambda(N) = \sum_{\mathcal{B}\in\mathfrak{B}_{J_0}} c_{\mathcal{B}} R_{\mathcal{B}}(N).$$

Every atom produced later by B3/F3 carries a parent tag

$$\operatorname{tag}(\mathcal{A}) = (\mathcal{B}, \tau),$$

where:

- $\mathcal{B}$ is the parent typed B1 block;
- τ is the finite routing/grouping history inside B3/F3.

B3/F3 guarantee that these tagged atoms form a finite partition of the parent block contribution; this is Lemma F3.15:

$$R_{\mathcal{B}}(N) = \sum_{\tau\in\mathcal{T}(\mathcal{B})} R_{\mathcal{B},\tau}(N),$$

with no overlap at the tagged level.

This is the bookkeeping mechanism preventing double counting.

—

H4.4. Canonical local projection of tagged atoms For every tagged local atom \{}((\{}mathcal B,\{}tau) \{}), define its canonical local projection

$$\operatorname{Loc}_Q R_{\mathcal{B},\tau}(N)$$

as the contribution of that tagged cell to the local model obtained by replacing the arithmetic coefficients by their residue-class local densities modulo Q, while keeping the same smooth/dyadic weights and the same tag.

The definition is linear:

$$\operatorname{Loc}_Q\left(\sum_{\mathcal{B},\tau} c_{\mathcal{B}} R_{\mathcal{B},\tau}\right) = \sum_{\mathcal{B},\tau} c_{\mathcal{B}} \operatorname{Loc}_Q R_{\mathcal{B},\tau}.$$

The local/main term assigned by D1, G8a zero frequency, or LocalDiag is admitted into H4 only if it equals this canonical projection:

$$M^{\text{local}}_{\mathcal{B},\tau}(N) = \operatorname{Loc}_Q R_{\mathcal{B},\tau}(N) + o_{\mathcal{B},\tau}(N).$$

This is the **LPI admission condition** as consumed by H4.

Thus H4 does not accept arbitrary local-looking main terms. It accepts only tagged canonical local projections.

By Lemma LPI.1, the admitted local source set is exactly

$$\mathfrak{L}_{\text{LPI}} = \mathfrak{L}_{\text{LongAP/Local}} \sqcup \mathfrak{L}_{\text{CKP},0} \sqcup \mathfrak{L}_{\text{LocalDiag}}.$$

There is no separate residual projection class. The only auxiliary local-looking operations that can occur before H4 are controlled CRT restriction or absorption, fixed-divisor quotienting, primitive local slicing, and endpoint or smooth-boundary localization. The first three are finite tagged refinements of one of the three displayed source classes and must satisfy the same LPI admission condition. Endpoint and smooth-boundary terms are C1 Edge contributions, not local/main sources.

—

H4.4A. Compatibility of the local projection with prior routing The following compatibility table records the LPI admission compatibility consumed by H4. It records why the operation Loc_Q may be applied to the tagged terminal cells without changing the local algebra of the original Goldbach convolution.

Source operation before H4	Compatibility with Loc_Q	Excluded failure mode
B1 Heath–Brown expansion	B1 is an exact finite convolution identity before estimation; Λ_Q replaces the two original von Mangoldt factors after the identity is summed over all B1 tags.	treating an individual B1 summand as an independent local model
B3 product grouping	B3 only partitions the finite product-coordinate descriptions and preserves the parent B1 tag.	counting two different groupings as two local main terms for the same tagged cell
F3/F4 terminal routing	F3/F4 refine the summation domain by exact tagged partitions or send the cell to a terminal class.	admitting a local term without its inherited B1/F3 tag
Controlled CRT absorption	compatible finite-index restriction of residue classes modulo Q; incompatible fibres are empty.	changing the normalization from Λ_Q to a branch-specific density
Fixed-divisor or quotient decision	admitted locally only after B1-LD identifies the quotient coefficient model with the same CRT local replacement.	quotient main term with unmatched arithmetic coefficient
Gcd/local/proportional relation	admitted only as a tagged LocalDiag projection of a parent cell.	untagged diagonal density or non-canonical specialization
LongAP/Local branch	F3P gives the intrinsic local-coefficient predicate, and D1.2A expands it into the tagged Loc_Q projection.	branch-specific AP density with a nonlocal coefficient
CKP zero-frequency branch	G8a.5 and B1-LD identify the $h = 0$ mode with the tagged local projection.	importing a nonzero Fourier mode into H4
C1 Edge removal	C1 contributes only $o(N)$ and is not part of M_{local}.	losing a local main term by labelling it Edge without a strict C1P predicate
GoodAWACK and CKP nonzero modes	these are error branches, already handled before H4.	double-counting an error branch as a local term
Primitive local slicing	a finite tagged subdivision of an already admitted LongAP/Local, CKP $h = 0$, or LocalDiag source.	treating a slice as a new branch without its parent tag
Endpoint or smooth-boundary localization	routed to C1 and charged as $o(N)$.	importing a boundary correction into the main term

Consequently H4 does not rely on the phrase "canonical projection" as an unproved convention. A local/main contribution is admitted only after the source operation is compatible with the single replacement rule

$$\Lambda(n) \mapsto \Lambda_Q(n \bmod Q)$$

on the tagged Goldbach convolution.

—

H4.5. Local reconstruction from the B1 decomposition

Lemma 4.1 (Lemma H4.1)**.** *The sum of the canonical local projections of all tagged descendants of the exact B1 decomposition reconstructs the local Goldbach model:*

$$\sum_{\mathcal{B}} c_{\mathcal{B}} \sum_{\tau\in\mathcal{T}(\mathcal{B})} \mathrm{Loc}_Q\, R_{\mathcal{B},\tau}(N) = N\sigma_Q(N) + o(N). \tag{H4-reconstruct}$$

Equivalently, the LPI local projection of the full tagged proof tree is not a branch-specific surrogate. It is the convolution of the single local model Λ_Q with itself along $n_1 + n_2 = N$.

Proof. By Lemma B1, the Heath–Brown decomposition used in the proof is an exact finite identity for the two von Mangoldt factors in $R_\Lambda(N)$, after the fixed smooth dyadic partition of unity has been inserted. Thus

$$R_\Lambda(N) = \sum_{\mathcal{B}} c_{\mathcal{B}} R_{\mathcal{B}}(N)$$

before any terminal estimate is applied. By Lemma F3.15, each parent block is then partitioned into tagged descendants:

$$R_{\mathcal{B}}(N) = \sum_{\tau\in\mathcal{T}(\mathcal{B})} R_{\mathcal{B},\tau}(N).$$

The local replacement defined by LPI and assembled by H4 is the finite CRT local replacement of the same B1 coefficient factors and the same dyadic cells. For B1-inherited finite-convolution coefficients this compatibility is the content of Lemma B1-LD: elementary B1 coefficient models, finite convolution, CRT restriction, gcd splitting, and tagged dyadic localization commute with the local replacement, up to C1 boundary terms.

Therefore applying Loc_Q to the exact B1/F3 tagged partition gives the same result as applying the local model directly to the original two von Mangoldt factors. On the original factors the local model is by definition

$$\Lambda(n) \mapsto \Lambda_Q(n \bmod Q) = \frac{Q}{\varphi(Q)} \mathbf{1}_{(n,Q)=1}.$$

Consequently the reconstructed local sum is

$$\sum_{n_1+n_2=N} \Lambda_Q(n_1)\Lambda_Q(n_2)$$

with the same smooth endpoint convention as the tagged cells. Counting by the residue class $a \equiv n_1 \pmod{Q}$ gives

$$\sum_{n_1+n_2=N} \Lambda_Q(n_1)\Lambda_Q(n_2) = N \cdot \frac{1}{Q} \sum_{a \bmod Q} \Lambda_Q(a)\Lambda_Q(N-a) + o(N) = N\sigma_Q(N) + o(N).$$

The $o(N)$ term is only the endpoint/smooth-boundary discrepancy already admitted by C1. This proves (H4-reconstruct).

—

□

H4.6. Dyadic and routing recombination

Lemma 4.2 (Lemma H4.2)**.** *The dyadic partitions and routing tags used before H4 do not change the local main term:*

$$\sum_{\mathcal{B},\tau} c_{\mathcal{B}} \operatorname{Loc}_Q R_{\mathcal{B},\tau}(N) = \operatorname{Loc}_Q R_\Lambda(N) + o(N).$$

Proof. The dyadic partition in B1 is an exact partition of unity on the active support. Hence summing over dyadic scales recombines the original B1 finite-convolution expression exactly. The later B3/F3/F4 operations are finite tagged partitions: they split summation domains by grouping choices, CRT restrictions, fixed-divisor cases, quotient cases, terminal predicates, and boundary alternatives. Each such operation is either an exact finite partition or a boundary/short-volume removal already admitted by C1.

The operator Loc_Q is linear and tag-preserving. Therefore it commutes with the finite recombination of dyadic cells and routing cells. The only discrepancy comes from endpoint and smooth-boundary cells, which are C1 Edge contributions and hence $o(N)$. Lemma proved.

—

□

H4.7. Admission of branch local terms

Lemma 4.3 (Lemma H4.3)**.** *Every terminal local/main contribution entering I1 satisfies the LPI admission condition consumed by H4*

$$M^{\mathrm{local}}_{\mathcal{B},\tau}(N) = \operatorname{Loc}_Q R_{\mathcal{B},\tau}(N) + o(N). \tag{H4-adm}$$

Proof. There are three sources of local/main terms.

1. **LongAP/Local.** Lemma F3P first states the positive local-coefficient

predicate for this terminal class, and Lemma D1, including Lemma D1.2A, expands that local algebra into controlled residue/coprimality data. Its local residue-density theorem then identifies the zero/local part of the long AP count with $\operatorname{Loc}_Q R_{\mathcal{B},\tau}(N)$, with boundary terms routed to C1.

1. **CKP zero frequency.** Lemma G8a separates $h = 0$ from $h \neq 0$. The

nonzero frequencies are not local terms. The $h = 0$ term is identified in G8a.5 with the LPI canonical local projection. Lemma B1-LD supplies the arithmetic compatibility of the B1-inherited finite-convolution coefficients under gcd splitting, CRT localization, and tagged dyadic restriction.

1. **LocalDiag.** In the F3/F4 routing, LocalDiag means forced equality,

proportionality, gcd-local dependence, or collision that produces a canonical local tagged cell. If a degeneracy is not a canonical local projection, it is not admitted by H4; the F3T routing table sends it instead to Edge, CKP, GoodAWACK, impossible, or a continuing routed case. Thus every LocalDiag term that reaches H4 is already a tagged canonical local projection.

These are the only local/main terminal classes in the routing table. Hence every local/main contribution entering I1 satisfies (H4-adm). Lemma proved.

—

□

H4.8. No double counting lemma

Lemma 4.4 (Lemma H4.4). *The sum of all admitted local/main terms satisfies*

$$M_{\text{local}}(N) = \sum_{\mathcal{B},\tau\in\mathcal{T}_{\text{local}}(\mathcal{B})} c_{\mathcal{B}} M^{\text{local}}_{\mathcal{B},\tau}(N),$$

and no local term is counted twice.

Proof. Each term is indexed by its parent B1 block $\mathcal{B}$ and a unique routing tag τ. By Lemma F3.15, B3/F3/F4 routing produces a finite exact partition of every parent block into tagged cells. Therefore two different tags correspond either to disjoint cells of the same B1 block or to summands from different B1 blocks in the exact B1 decomposition.

This includes LocalDiag atoms. A LocalDiag condition is detected by a structural predicate such as forced equality, proportionality, gcd-local dependence, or collision of forms; two different routing cells may therefore look locally identical. H4 does not identify local terms by visual form. It sums the canonical projection of each tagged cell. Since the underlying tagged cells are disjoint by (F3-partition), structurally identical LocalDiag expressions from different tags are complementary summands, not duplicates.

If $(\mathcal{B},\tau) \neq (\mathcal{B}',\tau')$, then either $\mathcal{B} \neq \mathcal{B}'$, in which case the two summands already occur separately in the exact B1 expansion, or $\mathcal{B} = \mathcal{B}'$ and $\tau \neq \tau'$, in which case Lemma F3.15 gives disjoint summation domains. Linearity of Loc_Q then preserves this separation.

Thus no double counting occurs. Lemma proved.

—

□

H4.9. Completeness of local terms

Lemma 4.5 (Lemma H4.5). *Every terminal local/main atom produced by B3/F3/F4 routing is included in exactly one of:*

$$\text{LongAP/Local}, \qquad \text{LocalDiag}, \qquad \text{CKP}_{h=0}.$$

Proof. By Lemma F3 and the LPI interface, every terminal atom belongs to exactly one terminal routing class at the tagged level. Error classes are Edge, CKP nonzero-frequency error, and GoodAWACK. Local/main classes are LongAP/Local, LocalDiag, and CKP zero-frequency. Controlled CRT restrictions, quotients and local slicing may produce auxiliary local projection subterms, but such subterms inherit the parent tag and are admitted only inside one of these three classes. They are not a separate local/main source.

Therefore every admitted local/main contribution is tagged once and all terminal local/main contributions are included. Lemma proved.

—

□

H4.10. Linearity of the local projection

Lemma 4.6 (Lemma H4.6). *The sum of canonical local projections over all tagged local/main atoms equals the canonical local projection of the original Goldbach sum:*

$$\sum_{\mathcal{B},\tau} c_{\mathcal{B}} \,\mathrm{Loc}_Q\, R_{\mathcal{B},\tau}(N) = \mathrm{Loc}_Q\, R_{\Lambda}(N) + o(N).$$

Proof. The B1 decomposition is exact and B3/F3 tagged routing partitions each parent block into finitely many cells. The operator Loc_Q is linear by definition. Therefore applying Loc_Q before or after summing the tagged decomposition gives the same result.

All nonlocal/error classes contribute either zero to the admitted local sum or are handled as $o(N)$ by C1/E10/G8a nonzero-frequency estimates. The remaining admitted local classes sum to the full canonical local projection. Lemma proved.

—

□

H4.11. Compatibility of LongAP, CKP zero-frequency and LocalDiag normalizations

Lemma 4.7 (Lemma H4.7A). *Every admitted local/main term from LongAP/Local, CKP zero-frequency, or LocalDiag is normalized by the single LPI operation*

$$\Lambda(n) \mapsto \Lambda_Q(n \bmod Q)$$

inside the original tagged Goldbach convolution. No branch is allowed to introduce a separate local density convention.

Equivalently, for every admitted tagged cell $(\mathcal{B}, \tau)$*,*

$$M^{\mathrm{local}}_{\mathcal{B},\tau}(N) = \mathrm{Loc}_Q\, R_{\mathcal{B},\tau}(N) + o(N).$$

The verification is as follows.

Source of the local term	*Branch expression before H4*	*H4 normalization*	*Excluded alternative*
LongAP/Local	*a long arithmetic-progression main term after D1*	*F3P forces all long-variable coefficients into* $\mathfrak{C}_{\mathrm{loc}}(Q_\tau)$*, and D1.2A identifies the remaining AP density with* $\mathrm{Loc}_Q\, R_{\mathcal{B},\tau}$	*if a nonlocal coefficient survives, the atom is not F3P-LongAP/Local and hence not LPI-admitted*
CKP $h = 0$	*the zero Fourier mode in the G8a CKP expansion*	*G8a.5 identifies the* $h = 0$ *mode with the same* $\mathrm{Loc}_Q\, R_{\mathcal{B},\tau}$*, using B1-LD for coefficient compatibility*	$h \neq 0$ *is never local and is sent to CKP/X10*
LocalDiag	*a forced equality, proportionality, gcd-local dependence, or collision cell*	*admitted only when the diagonal cell is the canonical local projection of a tagged B1/F3/F4 cell*	*noncanonical degeneracies are routed by F3T to Edge, CKP, GoodAWACK, impossible, or a continuing routed case*
Controlled CRT restriction or absorption	*a finite-index refinement of a tagged source class*	*admitted only when the parent tag and the* Λ_Q*-replacement rule are preserved*	*branch-specific CRT densities are not H4 inputs*
Fixed-divisor quotienting	*a coefficient refinement of a tagged source class*	*admitted only through the B1-LD compatibility check*	*quotient main terms with unmatched coefficient models are not H4 inputs*
Primitive local slicing	*a finite subdivision of the same tagged local source*	*admitted only as part of its parent LongAP/Local, CKP* $h = 0$*, or LocalDiag class*	*a slice is not a fourth local branch*
Endpoint or smooth-boundary localization	*boundary correction*	*routed to C1 as* $o(N)$*, not assembled by H4*	*boundary terms are not local main terms*

Proof. The operator Loc_Q was defined before any branch-specific terminal analysis: it replaces the arithmetic coefficients in the original tagged Goldbach convolution by their residue-class local model modulo Q, while keeping the same dyadic weights, summation domain, and routing tag. Therefore it is a property of the parent tagged cell, not of the branch that later recognizes the local contribution.

For LongAP/Local cells, F3P and D1.2A first exclude exactly the obstruction that would leave a branch-specific arithmetic coefficient. The remaining main term is the residue-density projection of the same tagged cell, hence it is $\mathrm{Loc}_Q\, R_{\mathcal{B},\tau}$.

For CKP cells, G8a separates the zero Fourier mode from all nonzero modes. The nonzero modes are analytic error terms. The zero mode is a local projection only after G8a.5 and B1-LD identify its finite-convolution coefficients with the B1-inherited local density; hence its admitted form is again $\mathrm{Loc}_Q\, R_{\mathcal{B},\tau}$.

For LocalDiag cells, the routing tag records the exact forced relation that created the diagonal cell. H4 admits such a cell only when the diagonal specialization is a canonical tagged projection. If the specialization is not canonical, it never reaches H4 and is routed elsewhere by F3T.

Thus the equality of normalizations is not assumed branch-by-branch; it is enforced by the definition of admissible local/main term.

—

□

H4.12. Calculation of local factors By the definition of Λ_Q,

$$\sigma_Q(N) = \frac{1}{Q} \sum_{a \bmod Q} \frac{Q}{\varphi(Q)} \mathbf{1}_{(a,Q)=1} \frac{Q}{\varphi(Q)} \mathbf{1}_{(N-a,Q)=1}.$$

Since Q is squarefree, CRT gives

$$\sigma_Q(N) = \prod_{p \le w} \sigma_p(N),$$

where

$$\sigma_p(N) = \frac{1}{p} \left(\frac{p}{p-1} \right)^2 \#\{a \bmod p : (a,p) = 1,\ (N-a,p) = 1\}.$$

For $p = 2$ and even N, the only unit residue is 1, and $N - 1 \equiv 1 \pmod 2$, so

$$\sigma_2(N) = 2.$$

For odd p:

If $p \mid N$, then the forbidden residues $a \equiv 0$ and $a \equiv N$ coincide. Hence the number of admissible residues is

$$p - 1,$$

and

$$\sigma_p(N) = \frac{1}{p} \left(\frac{p}{p-1} \right)^2 (p-1) = \frac{p}{p-1}.$$

If $p \nmid N$, the two forbidden residues are distinct, so the number of admissible residues is

$$p - 2,$$

and

$$\sigma_p(N) = \frac{1}{p}\left(\frac{p}{p-1}\right)^2 (p-2) = \frac{p(p-2)}{(p-1)^2} = 1 - \frac{1}{(p-1)^2}.$$

Therefore

$$\sigma_Q(N) = 2 \prod_{\substack{3 \le p \le w \\ p \nmid N}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{3 \le p \le w \\ p \mid N}} \frac{p}{p-1}.$$

Letting $w \to \infty$, we get

$$\sigma_Q(N) \to 2C_2 \prod_{\substack{p \mid N \\ p > 2}} \frac{p-1}{p-2},$$

where

$$C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

Thus

$$\sigma_Q(N) \to \mathfrak{S}(N).$$

—

H4.13. Local/Main compatibility theorem

Theorem 4.8 (Theorem H4)**.** *In the proof tree, the sum of all terminal local/main contributions satisfies*

$$M_{\mathrm{local}}(N) = \mathfrak{S}(N)N + o(N).$$

Proof. By the compatibility table H4.4A and Lemma H4.3, every LongAP/Local, CKP zero-frequency, and LocalDiag term that reaches H4 satisfies the LPI admission condition

$$M^{\mathrm{local}}_{\mathcal{B},\tau}(N) = \mathrm{Loc}_Q\, R_{\mathcal{B},\tau}(N) + o_{\mathcal{B},\tau}(N).$$

By Lemma H4.4, local/main terms are summed by unique B1 parent tags and routing tags, so there is no double counting. By Lemma H4.5, all terminal local/main atoms are included.

Therefore

$$M_{\mathrm{local}}(N) = \sum_{\mathcal{B},\tau} c_{\mathcal{B}}\, \mathrm{Loc}_Q\, R_{\mathcal{B},\tau}(N) + o(N).$$

By Lemmas H4.1, H4.2, and H4.6, the tagged sum of canonical local projections reconstructs the full local Goldbach model:

$$M_{\mathrm{local}}(N) = \mathrm{Loc}_Q\, R_{\Lambda}(N) + o(N) = N\sigma_Q(N) + o(N).$$

By the local factor calculation,

$$\sigma_Q(N) \to \mathfrak{S}(N).$$

Hence

$$M_{\mathrm{local}}(N) = \mathfrak{S}(N)N + o(N).$$

The theorem is proved.

—

□

Remark 4.9 (H4.14. Output).

$$\boxed{\text{All admitted local/main terms assemble as } \operatorname{Loc}_Q R_\Lambda(N) = N\sigma_Q(N) + o(N).}$$

Every local term is a canonical local projection, carries parent B1 and routing tags, and is admitted only when its branch-specific normalization matches Loc_Q. CKP zero-frequency, LongAP local terms, and LocalDiag terms are combined by tagged linearity. The singular series is computed from the explicit finite local model Λ_Q.

—

H4.15. Logical Dependencies Internal dependencies: `LPI`, `B1`, `B3`, `F3P`, `F3`, `F3T`, `F4`, `D1`, `G8a`, `B1LD`, `C1`, `C1A`, and the local model Λ_Q.

Children served: `I1` and the final local/main assembly.