GoodAWACK Finite Grammar Theorem Package

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May 2026

Abstract

This note isolates the finite combinatorial part of the GoodAWACK closure. It proves that actual terminal GoodAWACK skeletons are generated by a finite B1/B3/F3/F4/E5-clean routing grammar, that this grammar preserves a no-untagged-rank-drop invariant, and therefore that no terminal GoodAWACK skeleton contains an untagged rank-dropping affine regrouping. As a consequence the FreeAffineHighTC residual is empty, and the HighTC-GoodAWACK branch is routed to already handled terminal classes or contributes zero.

Contents

1 Scope

The theorem concerns actual terminal GoodAWACK skeletons. An actual skeleton is a descendant of the proof-tree construction

$$B1 \longrightarrow B3 \longrightarrow F3/F4,$$

with E5 used only as clean content stability for records already generated by that construction. The theorem does not classify arbitrary bounded affine systems written down after the fact.

The goal is the structural implication

$$\boxed{\text{actual terminal GoodAWACK skeleton} \Longrightarrow \text{no untagged rank-dropping AFF occurrence.}} \tag{1.1}$$

The analytic TC1 route is not part of this package. This package handles the finite grammar and HighTC structural closure.

2 Routing Records and Terminal Skeletons

An actual routing record is a tuple

$$\mathfrak{r} = (V, \mathcal{C}, \mathcal{L}, \mathcal{Q}, \tau, \mathrm{orig}, W),$$

where:

1. V is the finite list of active variables inherited from B1;
2. $\mathcal{C}$ records dyadic, congruence, content, gcd and divisor restrictions;
3. $\mathcal{L}$ is the finite list of visible affine forms;
4. $\mathcal{Q}$ records fixed-divisor, quotient and local tags;
5. τ is the current routing tag;
6. orig records the origin of every rank-changing operation;
7. W is the bounded or polylogarithmic weight data.

A terminal GoodAWACK skeleton is the terminal record

$$\mathfrak{S} = (\mathcal{B}, \Gamma, \mathfrak{r}, \Lambda_{\mathfrak{S}}, \Omega_{\mathfrak{S}}, \mathcal{L}_{\mathfrak{S}}, \mathcal{M}_{\mathfrak{S}}, \mathrm{orig}_{\mathfrak{S}}, \mathcal{W}_{\mathfrak{S}})$$

produced by this routing record when the terminal label is GoodAWACK.

For terminal affine forms $L_\rho(z) = \ell_\rho \cdot z + c_\rho$, the TC1/HighTC test uses the tensors

$$Q_\rho = \ell_\rho \otimes \ell_\rho.$$

After terminality these vectors and tensors are fixed routing data. Later analytic operations may test or restrict the fixed terminal object, but may not replace its terminal vector list and treat the replacement as a new terminal GoodAWACK skeleton.

3 Allowed Grammar

Define the finite GoodAWACK grammar $\mathcal{G}_{\mathrm{GA}}$ as follows. A state is a tuple

$$\mathfrak{s} = (V, \mathcal{L}, \mathcal{C}, \mathcal{Q}, \mathcal{T}, \mathcal{O}),$$

where $\mathcal{O}$ is the origin record for rank-changing operations.

The start states are exactly the B1/B3 grouped product cells. The transition set consists of the following operations.

Transition	Allowed effect	Required tag or outcome
B1 start-state creation	creates product variables, dyadic cells and weights	start-state origin
B3 finite grouping	selects one finite grouped candidate	B3 grouping origin
controlled CRT restriction	finite-index full-rank restriction or incompatible fibre	CRT or empty
fixed-divisor quotient	quotient by a recorded fixed divisor	FixedDiv
variable quotient residual	quotient/divisor residual selected by F4	VarQuot or routed tag
square-divisor routing	Edge, controlled divisibility, or empty state	Edge or CRT
grouping selection/elimination	finite candidate selection or removal	B3/F3 grouping origin
local/diagonal/gcd detection	equality, proportionality, repeated form, or forced local relation	LocalDiag
CKP-balanced detection	balanced bilinear Kloosterman-fraction structure	CKP
strict saving or boundary detection	C1 Edge, short-volume, boundary, high-frequency or small-conductor case	Edge
full-rank affine transport	injective on active and terminal tensor-test spans	no rank-drop tag needed
tagged rank-dropping transport	rank drop inherited from an earlier routing origin	inherited allowed tag
terminal labelling	labels Edge, CKP, GoodAWACK, LocalDiag or LongAP/Local	terminal label
post-terminal analytic operation	estimates or tests a fixed terminal object	PostTerminalNonGenerator

There is no transition whose output is an untagged rank-dropping affine regrouping.

4 Grammar Completeness

Theorem 1 (Completeness of the GoodAWACK routing grammar)**.** *Every actual-generated operation that can generate or modify an actual terminal GoodAWACK affine skeleton is one of the transitions of $\mathcal{G}_{\mathrm{GA}}$.*

Proof. B1 supplies only fixed-depth Heath–Brown product variables, dyadic cells and exact convolution weights. B3 supplies only a finite list of product-grouping start states. Neither layer introduces a free affine rank drop.

F3 and F4 perform the complete pre-terminal routing. The generic F3 operations are controlled CRT absorption, the F4 large-divisor decision, square-divisor routing, finite grouping selection or elimination, terminal LocalDiag detection, terminal Edge detection, and terminal class labelling. F4 exhausts ordinary divisor and quotient decisions. Any quotient, local-dependence, CKP, Edge, impossible, or continuation outcome receives a recorded tag before terminality.

E5 is applied only to records already produced by B1/B3/F3/F4. In the clean interface it is either full-rank on the active affine span and on the terminal tensor-test span, or else it inherits an already recorded rank-drop origin. Thus E5 is content stability, not an independent terminal generator.

After terminal labelling, Cauchy–Schwarz, cube expansions, Fourier expansions, coarea tests, TC1 testing, BRS/X16 estimates, Davenport/AP estimates and local projection arguments act only on the fixed terminal object. They cannot produce a new B1/B3/F3/F4 descendant and cannot replace the terminal tensor-test vectors.

These cases exhaust the actual B1-origin construction. Hence every operation that generates a terminal skeleton belongs to $\mathcal{G}_{\mathrm{GA}}$. □

5 Finite Grammar Invariant

The allowed rank-drop tags are

$$\begin{gathered}\text{Fix/Proj}, \quad \text{CRT}, \quad \text{FixedDiv}, \quad \text{VarQuot}, \\ \text{LocalDiag}, \quad \text{CKP}, \quad \text{Edge}, \quad \text{PostTerminalNonGenerator}.\end{gathered} \tag{5.1}$$

Theorem 2 (No-untagged-rank-drop invariant)**.** *For every state reachable in $\mathcal{G}_{\mathrm{GA}}$, every rank-dropping affine operation visible in that state carries one of the tags in (5.1). Consequently, a reachable terminal GoodAWACK skeleton contains no untagged rank-dropping AFF occurrence.*

Proof. We argue by induction on the length of the grammar derivation.

At length zero, the state is a B1/B3 grouped product cell. It contains product variables, dyadic restrictions and grouped candidates, but no rank-dropping affine operation. The invariant is therefore true.

Assume the invariant for a reachable state and apply one transition. If the transition is fixing or projection, the origin tag is Fix/Proj. If it is controlled CRT restriction, then either the fibre is incompatible and disappears, or the operation has the tag CRT. A fixed-divisor quotient has the tag FixedDiv. A variable quotient residual has the F4 tag VarQuot, unless it routes to LocalDiag, CKP, Edge, LongAP/Local or empty support.

If the transition detects a local, diagonal, gcd, repeated-form or proportional relation, the state leaves GoodAWACK with tag LocalDiag. If it detects balanced multiplicative structure, the state leaves GoodAWACK with tag CKP. If it detects strict saving, short volume, square-divisor saving, high frequency, small conductor or a boundary case, the state leaves GoodAWACK with tag Edge.

A full-rank affine transport has no rank drop. A rank-dropping E5 transport is allowed only when the rank drop is inherited from an already recorded routing origin, so the induction hypothesis supplies the tag. Terminal labelling is a label and performs no coordinate operation.

Finally, a post-terminal analytic operation is allowed only after the terminal tensor-test vectors are fixed. It can restrict a sum or introduce testing variables, but it cannot change the terminal routing record. Hence any rank drop that occurs in a post-terminal analytic calculation is tagged PostTerminalNonGenerator and cannot generate a new terminal GoodAWACK skeleton.

Thus every transition preserves the invariant. The theorem follows. □

6 No Untagged AFF in Actual GoodAWACK

Corollary 1 (No untagged rank-dropping AFF)**.** *Let $\mathfrak{S}$ be an actual terminal GoodAWACK skeleton. Then every rank-dropping affine occurrence in $\mathfrak{S}$ is tagged by one of (5.1).*

Proof. By the grammar-completeness theorem, $\mathfrak{S}$ is reachable in $\mathcal{G}_{\mathrm{GA}}$. By the invariant theorem, every rank-dropping affine operation in any reachable state is tagged. Hence an untagged rank-dropping AFF occurrence cannot occur in $\mathfrak{S}$. □

7 Formal Free-Affine Witnesses

The formal 4AP-like family

$$Y_i = x + ir, \qquad 0 \le i \le 3,$$

with coefficient vectors $\ell_i = (1, i)$, satisfies a quadratic tensor relation

$$\ell_0 \odot \ell_0 - 3\ell_1 \odot \ell_1 + 3\ell_2 \odot \ell_2 - \ell_3 \odot \ell_3 = 0. \tag{7.1}$$

This family is useful as an interface witness: it shows that a broad phrase such as "bounded affine regrouping" would be too large if it were read without the actual routing origin.

It is not, by itself, an actual terminal GoodAWACK obstruction. If it arises from a full-rank transport, the TC1/HighTC classification is invariant under that transport. If it arises from a tagged rank drop, the resulting HighTC certificate is origin-degenerate. If it arises only from an untagged rank-dropping affine parametrization, it is not generated by $\mathcal{G}_{\mathrm{GA}}$ and is not an actual terminal GoodAWACK skeleton.

8 HighTC-GoodAWACK Closure

HGO2R routes every origin-degenerate HighTC certificate to one of

$$\text{CKP}, \qquad \text{LocalDiag}, \qquad \text{Edge}, \qquad \text{Impossible}. \tag{8.1}$$

Those outputs are handled outside GoodAWACK by the CKP, local, Edge and zero branches.

The only remaining HighTC residual is FreeAffineHighTC: a high-true-complexity certificate generated by a rank-dropping affine operation with no allowed origin tag.

Theorem 3 (FreeAffineHighTC is empty)**.** *For actual terminal GoodAWACK skeletons,*

$$\boxed{R_{\mathrm{FreeAffineHighTC}}(N) = 0.}$$

Proof. Suppose a FreeAffineHighTC skeleton existed. By definition its HighTC tensor relation would have to be produced by an untagged rank-dropping affine operation; otherwise HGO2R would classify it as origin-degenerate and route it to one of (8.1).

But by the preceding corollary, no actual terminal GoodAWACK skeleton contains an untagged rank-dropping AFF occurrence. Therefore the assumed FreeAffineHighTC skeleton is not an actual B1-origin terminal GoodAWACK descendant. Hence the FreeAffineHighTC contribution is empty. □

Corollary 2 (HighTC-GoodAWACK closure)**.** *Every HighTC-GoodAWACK terminal contribution is either routed by HGO2R to CKP, LocalDiag, Edge or Impossible, or belongs to the empty FreeAffineHighTC class. Thus the HighTC-GoodAWACK contribution contributes no unresolved GoodAWACK mass.*

9 Role of the E10S Reproducibility Record

The conceptual proof is the grammar-completeness theorem plus the induction invariant above. E10S-MECH is a reproducibility record for the source layer: it lists the proof files checked, their hashes, the search terms, and the occurrence-to-class map for all rank-affecting language in the Branch B / GoodAWACK proof-source files.

The E10S record is not a proof by search. Its function is narrower: it makes the finite source-layer verification reproducible and records when the check must be refreshed. If a Branch B / GoodAWACK source file changes, a new rank-changing term is introduced, or the F3/F4 routing grammar changes, the hash record and occurrence classification must be updated before the E10S record is cited again.

10 Output for E10L

The package supplies the structural input

$$\mathrm{E10Y} + \mathrm{E10X} + \mathrm{E10M} + \mathrm{E10K} \Longrightarrow R_{\mathrm{FreeAffineHighTC}}(N) = 0.$$

Together with HGO2R, this closes the HighTC-GoodAWACK branch. E10L combines this HighTC closure with the separate TC1 near-global-or-routed theorem TNG and the near-global X9L-GT input to prove

$$R_{\mathrm{GoodAWACK}}(N) = o(N).$$

11 Logical Dependencies

Internal dependencies: B1, B3, F3, F3A, F3T, F4 and E5; BGS and HGO2R; BAOC and E10G–E10J as the reduction chain isolating FreeAffineHighTC; E10Y, E10S-MECH, E10M, E10X and E10K; C1, G8a and H4 as destinations for routed outputs.

External dependencies: none. This package is combinatorial and structural.

Children served: E10L and the GoodAWACK theorem.