CKP/X10/X16 Analytic Theorem Package

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Abstract

This package isolates two analytic components of the proof. The first is the CKP/X10 component: after the CKP gcd split, the nonzero Fourier fibres match a Duke–Friedlander–Iwaniec bilinear Kloosterman-fraction estimate, with the actual two-variable CKP smooth weight and its derivatives verified directly. The second is the X16/BRS component: every admissible BRS carrier-slice estimate is reduced to the product-carrier X16C estimate, which follows from Shiu-type divisor averages in arithmetic progressions and a local factor averaging lemma.

Contents

1 Scope

This note proves only the analytic estimates needed by the CKP and BRS parts of the proof. It does not prove the finite routing grammar, the H4 local algebra, or the final prime-power handoff. Those are separate theorem packages.

The output is

$$\boxed{\text{CKP/X10 nonzero frequencies} = o(N), \qquad \text{X16BRS carrier-slice estimate holds.}}$$

The CKP zero-frequency contribution is not estimated by X10; it is routed to the H4 tagged local algebra.

The logical dependencies are G8a, X10, CKPD, X10ER, C1P/C1A/C1, X16BRS, X16C, and H4. File paths are recorded only in the accompanying source map.

2 The CKP Gcd Split

Let a CKP block be fixed. The CKP variables contain a pair u, u' with

$$uy + u'y' = N$$

after the surrounding B1/F3 routing data and smooth cutoffs have been fixed. Split

$$u = ga, \qquad u' = gq, \qquad (a, q) = 1, \qquad g \mid N,$$

and write $N_g = N/g$. Then

$$ay + qy' = N_g. \tag{2.1}$$

For $h = 0$, the Fourier expansion on the fibre gives the local projection term and is routed to H4. For $h \neq 0$, the AP congruence

$$y \equiv N_g \overline{a} \pmod q$$

produces the phase

$$e\left(\frac{hN_g\overline{a}}{q}\right). \tag{2.2}$$

Thus the DFI variables are

$$m = a, \qquad q = q, \qquad r = |h|N_g. \tag{2.3}$$

For $h < 0$, the same external estimate is applied to the conjugate phase, so the positive external integer parameter is $r = |h|N_g$.

3 The Actual CKP Smooth Weight

In the central CKP range let

$$a \asymp A_g, \qquad q \asymp Q_g, \qquad y \asymp Y, \qquad y' \asymp Y',$$

with

$$A_g \asymp Q_g, \qquad Y \asymp Y', \qquad Y/Q_g \asymp g. \tag{3.1}$$

Put

$$z(a, q, y) = \frac{N_g - ay}{q}. \tag{3.2}$$

The actual fibre weight is

$$\Phi_{a,q}(y) = \omega_A(a)\omega_Q(q)W_Y(y)W_{Y'}(z(a, q, y)), \tag{3.3}$$

where the ω's and W's are the smooth dyadic cutoffs inherited from the CKP block. The two-variable CKP weight inserted into the nonzero-frequency bilinear form is

$$\mathcal{W}_{g,h}(a, q) = \frac{1}{q}\widehat{\Phi}_{a,q}(h/q) = \frac{1}{q}\int \Phi_{a,q}(y)e(-hy/q)\, dy. \tag{3.4}$$

The nonzero-frequency CKP sum is therefore

$$\mathcal{O}_{g,h} = \sum_{\substack{a\sim A_g,\ q\sim Q_g \\ (a,q)=1}} \alpha_g(a)\gamma_g(q)\,\mathcal{W}_{g,h}(a,q)e\left(\frac{hN_g\overline{a}}{q}\right). \tag{3.5}$$

Define the amplitude

$$\mathcal{A}_{g,h,R} = (\log N)^{C_*} g(1+|h|g)^{-R}, \tag{3.6}$$

and the normalized weight

$$\widetilde{W}_{g,h}(a,q) = \mathcal{A}_{g,h,R}^{-1}\mathcal{W}_{g,h}(a,q). \tag{3.7}$$

4 Smooth Derivative Verification

Lemma 1 (CKP smooth weight derivative check)**.** *For $i+j \le 2$, the normalized CKP weight satisfies*

$$\partial_a^i\partial_q^j\widetilde{W}_{g,h}(a,q) \ll (\log N)^C A_g^{-i}Q_g^{-j}. \tag{4.1}$$

Proof. The central support identities imply

$$\partial_a z = -y/q, \qquad \partial_q z = -z/q. \tag{4.2}$$

Because $y/q \asymp Y/Q_g \asymp g$ and the product scale satisfies $A_g \asymp Q_g$, differentiation of $W_{Y'}(z(a,q,y))$ gives the same scale loss as differentiation in the external DFI variables:

$$\partial_a^i\partial_q^j W_{Y'}(z(a,q,y)) \ll A_g^{-i}Q_g^{-j} \qquad (i+j\le 2), \tag{4.3}$$

after the smooth cutoffs restrict to the central cell. Differentiating the oscillatory factor $e(-hy/q)$, integrating by parts in y, and using $Y/Q_g \asymp g$ gives, for every fixed B,

$$\partial_a^i\partial_q^j\mathcal{W}_{g,h}(a,q) \ll A_g^{-i}Q_g^{-j}\, g(1+|h|g)^{-B+i+j}(\log N)^C. \tag{4.4}$$

Choosing R larger than the fixed derivative order in the amplitude $\mathcal{A}_{g,h,R}$ gives (4.1). □

Thus the actual nonseparated CKP weight is admissible for the smooth DFI/X10 input. No separated surrogate weight is used.

5 DFI/X10 Input and Central Bound

The external X10 input is the smooth Duke–Friedlander–Iwaniec bilinear Kloosterman-fraction estimate. In the form needed here, if F is supported on $m \asymp M$, $q \asymp Q$, satisfies

$$\partial_m^i\partial_q^j F(m,q) \ll \eta\, M^{-i}Q^{-j} \qquad (i+j\le 2),$$

then

$$\sum_{\substack{m\sim M,\ q\sim Q \\ (m,q)=1}} \alpha_m\beta_q F(m,q)e\left(\frac{r\overline{m}}{q}\right) \ll_\varepsilon \eta^2\|\alpha\|_2\|\beta\|_2(r+MQ)^{3/8}(M+Q)^{11/48+\varepsilon}. \tag{5.1}$$

Here the substitution is

$$M = A_g, \qquad Q = Q_g, \qquad r = |h|N_g. \tag{5.2}$$

The central CKP range has

$$A_g \asymp Q_g \asymp S_g, \qquad S_g = N^{1/2+O(\eta_0)}/g, \qquad |h|g \le (\log N)^{B_{\mathrm{HF}}}. \tag{5.3}$$

Combining the derivative check with (5.1) and then restoring the amplitude $\mathcal{A}_{g,h,R}$ gives the layer bound

$$|\mathcal{O}_{g,h}| \ll N^{95/96+O(\eta_0)+\varepsilon}(\log N)^C g^{-47/48}(1+|h|g)^{-A}. \tag{5.4}$$

The coefficient L^2-norms are those of the CKP dyadic divisor-bounded coefficients and contribute only fixed polylogarithmic losses. Summing (5.4) over dyadic data, divisors $g \mid N$, and $h \neq 0$, and choosing A fixed and large, gives

$$\sum_{g|N}\sum_{h\neq 0} |\mathcal{O}_{g,h}| = o(N), \tag{5.5}$$

provided the fixed hierarchy has $O(\eta_0) + \varepsilon < 1/96$.

6 CKP Excluded Ranges

Before X10 is applied, the CKP partition removes the following noncentral ranges.

Range	Condition	Routing
Zero frequency	$h = 0$	H4 local algebra
High frequency	$\|h\|g > (\log N)^{B_{\mathrm{HF}}}$	X10ER to C1P/C1A/C1
Small conductor	$q/(q, hN_g) \le (\log N)^B$	X10ER to C1P/C1A/C1 or local
Large g	g outside the central CKP window	X10ER to Edge
Boundary/short volume	dyadic fibre too short or cutoff boundary	C1P/C1A/C1

Therefore the DFI/X10 estimate is invoked only on the central balanced range where the derivative verification above applies.

Proposition 1 (CKP analytic output)**.** *The CKP branch satisfies*

$$R_{\mathrm{CKP}}(N) = M_{\mathrm{CKP,local}}(N) + o(N), \tag{6.1}$$

where $M_{\mathrm{CKP,local}}(N)$ is one of the explicitly LPI-admitted tagged local projection terms later assembled by H4.

Proof. The $h = 0$ term is H4-local. The central $h \neq 0$ terms satisfy (5.5). All noncentral terms are routed by X10ER and C1P/C1A/C1 before the DFI estimate is invoked. □

7 X16/BRS Carrier-Slice Theorem

Let C be an admissible BRS carrier of height X_C, and let I be a carrier interval. Define

$$Y_{16} = \max\{|I \cap \mathbb{Z}|,\ X_C(\log N)^{-B_{16}}\}. \tag{7.1}$$

The X16BRS estimate is

$$\mathrm{Mass}_{\mathcal{B}}(C \in I) \ll N(\log N)^{C_{16}}\frac{Y_{16}}{X_C} + N^{1-\rho_{16}}(\log N)^{C_{16}}. \tag{7.2}$$

The admissible carriers are grouped product carriers, complementary carriers, tagged quotient carriers, and controlled divisor quotients. X16BRS reduces the complementary, quotient, and controlled quotient carriers to the product carrier model by the existing tagged F3/F4 routing data. Untagged variable quotients are not X16BRS inputs; they are routed away before BRS.

This package directly proves the product-carrier X16C estimate. The full BRS carrier-slice estimate additionally uses X16BRS to reduce the other admissible carrier types to that product-carrier model.

8 Product-Carrier X16C

For a product carrier P with $p \in I_{16}$ and companion variable $u \asymp U$, the active mass is bounded by a divisor correlation of the form

$$\sum_{p\in I_{16}} \tau_{K_1}(p) \sum_{u\asymp U} \tau_{K_2}(u)\tau_{K_3}(N-pu)1_{N-pu>0}. \tag{8.1}$$

X16C proves

$$(8.1) \ll Y_{16}U(\log N)^{O(1)} + N^{1-\rho_{16}}(\log N)^{O(1)}. \tag{8.2}$$

The main external input is Shiu's Brun–Titchmarsh theorem for multiplicative functions in arithmetic progressions. In the divisor-function form used here, for $f(n) = \tau_K(n)^A$, interval $J \subset [1,N]$ of length H, and modulus $q \le H^{1-\delta}$,

$$\sum_{\substack{n\in J \\ n\equiv b \pmod q}} f(n) \ll \left(\frac{H}{q}+1\right)(\log N)^{C_{\mathrm{SH}}}\mathcal{E}_{q,b}. \tag{8.3}$$

Lemma 2 (Local factor averaging)**.** *The only place where non-coprime AP classes enter is the estimate*

$$\sum_{c\in I_{16}^{\#}} \tau_{K_0}(c)^A \mathcal{E}_{c,N}^{1/2} \ll |I_{16}^{\#}|(\log N)^{C_{\mathrm{loc}}}. \tag{8.4}$$

Proof. The local factors are fixed-divisor factors attached to the AP classes arising from $N-pu$. Averaging them over the carrier interval and applying Cauchy– Schwarz reduces the estimate to the standard second moment for fixed divisor functions,

$$\sum_{u\asymp U} \tau_K(u)^2 \ll_K U(\log 2U)^{K^2-1}, \tag{8.5}$$

as in Tenenbaum, Ch. II.5, Theorem 5. □

9 Proof of X16C

Proof. There are two cases.

First suppose $X_P \le N^{1-\delta}$. Fix p. The variable $N-pu$ runs through an arithmetic progression modulo p. Shiu/AP applies to the opposite variable after Cauchy–Schwarz and the divisor second moment. Averaging the local factors over $p \in I_{16}$ by the local factor averaging lemma gives

$$\ll Y_{16}U(\log N)^C.$$

Second suppose $X_P > N^{1-\delta}$. If $Y_{16}U \le N^{1-\rho_{16}}$, the trivial estimate gives

$$\ll N^{1-\rho_{16}+\varepsilon} \le N^{1-\rho'_{16}}, \qquad \rho'_{16} = \rho_{16}/2,$$

and then ρ'_{16} is renamed ρ_{16}. Otherwise fix u. The variable p lies in an AP class modulo u, and the largeness of $Y_{16}U$ gives the Shiu/AP range $u \le H_u^{1-\delta/2}$. Applying Shiu/AP and then summing over u using the divisor second moment gives the same bound (8.2). □

Consequently X16BRS holds for all admissible BRS carriers after the structural reductions described above.

10 Analytic Output Theorem

Theorem 1 (CKP/X10/X16 analytic output)**.** *In the active proof system:*

1. *every CKP zero-frequency contribution is an LPI-admitted local projection later assembled by H4;*
2. *every central CKP nonzero-frequency contribution is $o(N)$ by X10/DFI applied to the actual CKP smooth weight;*
3. *every noncentral CKP range is routed by X10ER/C1P/C1A/C1 before X10 is invoked;*
4. *every admissible BRS carrier-slice test satisfies the X16BRS estimate;*
5. *the X16BRS estimate is ultimately supplied by the X16C Shiu/AP product-carrier theorem.*

Therefore the CKP branch contributes only its tagged local main term plus $o(N)$, and the BRS carrier-slice input needed in the TC1-testing chain is available with the stated polylogarithmic losses and fixed power saving.

References

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