Stability-Induced Discreteness from the Second Variation of Action

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A common route to discreteness in physics is to postulate a Hilbert-space operator and then solve its eigenvalue problem. Here a different, stability-based route is formulated. The starting point is a variational stability principle: a physically realized stationary state is required not only to satisfy the stationarity condition $\delta S = 0$, but also to be stable with respect to the second variation, $\delta^2 S \geq 0$, understood as a positive stability form. Under standard assumptions on the second variation — symmetry, closedness, lower semiboundedness, and coercivity after a shift — this form defines a self-adjoint stability operator. If the physical boundary conditions make the relevant embedding compact, the stability operator has compact resolvent and therefore a discrete spectrum. In this sense, discreteness is not introduced as an independent quantum postulate; it arises as a spectral consequence of the second variation together with stability and boundary conditions. The result is stated as a theorem and proved using the representation theorem for closed semibounded forms and compactness of the Sobolev embedding. The physical meaning is clarified by distinguishing three levels of discreteness: discrete stability modes, discreteness of action variables, and quantum-type energy quantization. Periodic classical systems, including bounded orbital motion, naturally give discrete stability modes, whereas quantization of energies requires an additional minimal action scale and a single-valued phase condition.

I. INTRODUCTION

Discreteness appears in many areas of physics. In quantum mechanics it is usually obtained from an operator eigenvalue problem, for example

$$\hat{H}\psi_n = E_n \psi_n. \quad (1)$$

This formulation is extremely successful, but it begins with an already established operator structure. The question addressed here is more elementary: can a discrete spectrum of stationary states or stable perturbations arise before the quantum postulates, from a variational condition of stability?

The guiding idea is that stationarity alone is not sufficient for physical realization. A stationary trajectory or field configuration can be unstable. Therefore a selection rule is needed. The stability principle considered in this paper is

$$\delta S = 0, \qquad \delta^2 S \geq 0. \quad (2)$$

The first condition gives the usual Euler-Lagrange or Hamilton equations [1–3]. The second condition says that the stationary state must be a local minimum of a suitable stability functional. The notation S is used in a broad variational sense: it may denote an action, a generalized action, a thermodynamic potential, an entropy-related functional with a chosen sign convention, or a Lyapunov functional [4, 5]. If S is literally thermodynamic entropy of an isolated system, the usual maximum-entropy convention would give the opposite sign for the entropy itself. Here the sign is chosen so that stable states are minima of the stability functional.

The main result is that the second variation defines a quadratic stability form. Under standard analytic assumptions this form determines a self-adjoint operator [6–8]. When the relevant physical boundary conditions give compactness, the stability operator has a discrete spectrum. Thus the mechanism is

$$\delta S = 0 \Rightarrow \text{stationarity}, \qquad \delta^2 S \geq 0 \Rightarrow \hat{L}_{\mathrm{st}} \geq 0, \quad (3)$$

$\hat{L}_{\mathrm{st}}$ with compact resolvent $\Rightarrow$ discrete stability spectrum.

This statement is deliberately more modest than claiming that all classical energies become discrete. The theorem gives discreteness of stability modes. Energy quantization requires an additional relation between the stability spectrum, action variables, and a minimal action scale such as $\hbar$.

The paper is organized as follows. Section II formulates the stability principle and its relation to the second variation. Section III states and proves the theorem. Section IV explains the role of cyclicity and periodic boundary conditions. Section V discusses the Kepler problem and the distinction between discreteness of modes and discreteness of orbital energies. Section VI shows how a quantum-type rule follows when a minimal action scale is added. The conclusion summarizes the scope and limitations of the result.

II. STABILITY PRINCIPLE AND SECOND VARIATION

Let $S[u]$ be a sufficiently smooth functional on a space of admissible configurations. Let u_0 be a stationary point:

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$$\delta S[u_0] = 0. \quad (5)$$

For a small admissible variation η, write

$$u = u_0 + \eta. \quad (6)$$

The Taylor expansion of S has the form

$$S[u_0+\eta] = S[u_0]+\delta S[u_0](\eta)+\frac{1}{2}\delta^2 S[u_0](\eta,\eta)+O(\|\eta\|^3). \quad (7)$$

Because of Eq. (5), the linear term vanishes and local stability is governed by the second variation:

$$S[u_0 + \eta] - S[u_0] = \frac{1}{2}\delta^2 S[u_0](\eta,\eta) + O(\|\eta\|^3). \quad (8)$$

The stability condition is

$$\delta^2 S[u_0](\eta,\eta) \geq 0 \quad \text{for all admissible } \eta. \quad (9)$$

Introduce the bilinear form

$$a(\eta,\xi) = \delta^2 S[u_0](\eta,\xi). \quad (10)$$

If this form is closed, symmetric, and lower semibounded, it is represented by a self-adjoint operator [6, 7]. This is the step at which the second variation becomes a spectral object.

The principle can therefore be interpreted as a selection rule:

The role of Eq. (9) is not simply to keep a trajectory bounded. It supplies a positive quadratic form and hence a stability operator.

III. THEOREM: DISCRETENESS FROM THE SECOND VARIATION

We now formulate the central mathematical statement. The result is a standard consequence of the spectral theory of closed semibounded forms [6–8], but its physical interpretation here is that the second variation of the action generates the operator whose spectrum labels stable modes.

Theorem. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Let

$$\mathcal{V} = H_0^1(\Omega) \quad (12)$$

or another Hilbert space encoding the physical boundary conditions, and let

$$H = L^2(\Omega). \quad (13)$$

Let

$$S : \mathcal{V} \to \mathbb{R} \quad (14)$$

be a twice Frechet differentiable functional, and let $u_0 \in \mathcal{V}$ be a stationary point:

$$\delta S[u_0] = 0. \quad (15)$$

Assume that the second variation defines a symmetric, closed, lower-semibounded bilinear form

$$a(\eta,\xi) = \delta^2 S[u_0](\eta,\xi), \qquad \eta,\xi \in \mathcal{V}, \quad (16)$$

and that the stability condition is satisfied:

$$a(\eta,\eta) = \delta^2 S[u_0](\eta,\eta) \geq 0, \qquad \forall \eta \in \mathcal{V}. \quad (17)$$

Assume also ellipticity/coercivity of the shifted form: there exist $\alpha > 0$ and $c > 0$ such that

$$a(\eta,\eta) + \alpha\|\eta\|_L^2 \geq c\|\eta\|_H^2, \qquad \forall \eta \in \mathcal{V}. \quad (18)$$

Then there exists a self-adjoint operator

$$\hat{L}_{\rm st} \geq 0 \quad (19)$$

in H associated with the form a. Moreover, $\hat{L}_{\rm st}$ has compact resolvent and its spectrum is discrete:

$$0 \leq \mu_1 \leq \mu_2 \leq \dots, \qquad \mu_n \to \infty. \quad (20)$$

The eigenfunctions $\{\eta_n\}$ form an orthonormal basis in $L^2(\Omega)$. Thus every admissible stable perturbation can be expanded as

$$\eta = \sum_{n=1}^{\infty} c_n \eta_n, \qquad \hat{L}_{\rm st}\eta_n = \mu_n \eta_n. \quad (21)$$

Consequently, the stationary states selected by the stability principle are labelled, up to degeneracies and zero modes, by a discrete stability spectrum.

Proof. Since u_0 is stationary, Eq. (7) gives

$$S[u_0 + \eta] - S[u_0] = \frac{1}{2}a(\eta,\eta) + O(\|\eta\|^3). \quad (22)$$

The stability condition gives $a(\eta,\eta) \geq 0$, so a is a positive stability form.

By assumption, $a(\cdot,\cdot)$ is symmetric, closed, and lower semibounded. The Friedrichs-Kato representation theorem therefore gives a unique self-adjoint operator $\hat{L}_{\rm st}$ such that

$$a(\eta, \xi) = \langle \hat{L}_{\rm st}\eta, \xi\rangle_{L^2}, \qquad \eta \in \mathcal{D}(\hat{L}_{\rm st}), \quad \xi \in \mathcal{V}. \quad (23)$$

Since $a(\eta, \eta) \geq 0$, the associated operator satisfies $\hat{L}_{\rm st} \geq 0$.

The shifted coercivity condition, Eq. (18), implies existence and continuity of the weak solution of

$$(\hat{L}_{\rm st} + \alpha I)u = f. \quad (24)$$

Equivalently,

$$(\hat{L}_{\rm st} + \alpha I)^{-1} : L^2(\Omega) \to H_0^1(\Omega) \quad (25)$$

is bounded. Since Ω is bounded, the Rellich-Kondrachov theorem gives the compact embedding [9]

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega). \quad (26)$$

Therefore the composition

$$L^2(\Omega) \xrightarrow{(\hat{L}_{\rm st}+\alpha I)^{-1}} H_0^1(\Omega) \hookrightarrow L^2(\Omega) \quad (27)$$

is a compact operator in $L^2(\Omega)$. Hence the resolvent is compact.

A self-adjoint operator with compact resolvent has a purely point spectrum with finite multiplicities and no finite accumulation point except infinity [7]. Therefore Eq. (20) holds, and the eigenfunctions form an orthonormal basis in $L^2(\Omega)$. The expansion (21) follows. The theorem is proved.

The theorem identifies the precise mathematical place where discreteness appears. It does not follow from non-negativity of $\delta^2 S$ alone. It follows from the combination of a variational principle, positivity of the stability form, self-adjoint representation, and compactness generated by physical boundary conditions.

IV. PERIODICITY AND CYCLIC BOUNDARY CONDITIONS

A particularly important case occurs when the stationary state is periodic:

$$u_0(t + T) = u_0(t). \quad (28)$$

Then admissible perturbations should preserve the periodic boundary condition:

$$\eta(t + T) = \eta(t), \qquad \dot{\eta}(t + T) = \dot{\eta}(t). \quad (29)$$

The interval $[0, T]$ with identified endpoints is compact. This compactness leads to a discrete spectrum of the corresponding stability operator.

The simplest example is the positive operator

$$\hat{L}_{\rm st} = -\frac{d^2}{dt^2} + \Omega^2 \quad (30)$$

on periodic functions. The eigenvalue problem is

$$\left(-\frac{d^2}{dt^2} + \Omega^2\right)\eta_n(t) = \mu_n \eta_n(t), \qquad \eta_n(t+T) = \eta_n(t). \quad (31)$$

The eigenfunctions are Fourier modes,

$$\eta_n(t) = e^{i2\pi nt/T}, \qquad n \in \mathbb{Z}, \quad (32)$$

and the eigenvalues are

$$\mu_n = \left(\frac{2\pi n}{T}\right)^2 + \Omega^2. \quad (33)$$

Thus periodicity supplies a discrete label n, while the second variation supplies the stability operator whose eigenvalues determine the stable modes.

This distinction is important. Fourier decomposition alone gives a discrete representation of periodic functions, but the coefficients remain continuous. The stability principle adds the physical content: the modes are not merely mathematical harmonics; they are eigenmodes of the operator generated by the second variation.

V. KEPLER MOTION AND THE MACROSCOPIC LIMIT

For a test body of mass m in the gravitational field of a central mass M, the Lagrangian is

$$L = m\left(\frac{v^2}{2} + \frac{GM}{r}\right). \quad (34)$$

The action is proportional to m:

$$S = m\int\left(\frac{v^2}{2} + \frac{GM}{r}\right)dt. \quad (35)$$

Therefore the specific action S/m gives the same stationary trajectories as S:

$$\delta S = 0 \iff \delta(S/m) = 0. \quad (36)$$

This is consistent with the mass-independence of test-body motion in Newtonian gravity.

The stationary condition gives the inverse-square equation of motion and hence the Kepler laws [1–3]. However, the Kepler problem contains a continuous family of bound orbits. The orbital energy is

$$E=-\frac{GM}{2A}, \quad (37)$$

where A is the semi-major axis. The specific energy is

$$\epsilon=\frac{E}{m}=-\frac{GM}{2A}. \quad (38)$$

If A is allowed to vary continuously, then both E and E/m vary continuously.

The stability principle plus periodicity does imply a discrete spectrum of perturbation modes around a fixed orbit. It does not by itself quantize A or E. This is why one must distinguish

discrete stability modes $\neq$ discrete orbital energies.

For a fixed periodic orbit, the small perturbations have a discrete mode expansion. But the background orbit itself can still be chosen from a continuous family unless an additional action-selection rule is imposed.

In action-angle language [2, 3], the Kepler problem has an action variable of the form

$$\Lambda=m\sqrt{GM\Lambda}. \quad (40)$$

The corresponding specific action is

$$\lambda=\frac{\Lambda}{m}=\sqrt{GM\Lambda}. \quad (41)$$

Here λ denotes a non-inertial or specific orbital action variable, not an eigenvalue. If one postulates an additional discrete rule

$$\lambda_n=n\lambda_0, \quad (42)$$

then Eq. (41) gives

$$A_n=\frac{\lambda_0^2}{GM}n^2, \quad (43)$$

and hence

$$\epsilon_n=-\frac{G^2M^2}{2\lambda_0^2}\frac{1}{n^2}. \quad (44)$$

This is a possible macroscopic analogy, but Eq. (42) is an additional assumption. It is not a consequence of the Kepler laws alone.

Empirical orbital patterns such as Titius-Bode-like spacings may be discussed only as phenomenological illustrations of stability, cyclicity, and resonance selection. They are not used here as proof of the theorem. A resonance condition

$$\frac{T_{n+1}}{T_n}=q \quad (45)$$

combined with Kepler's third law gives

$$\frac{A_{n+1}}{A_n}=q^{2/3}, \quad (46)$$

which can generate geometric orbital patterns. Such patterns may resemble empirical spacing rules, but the rigorous result of this paper remains the discreteness of stability modes from the second variation.

A. Illustrative Comparison with Planetary Orbital Spacing

Although the resonance condition

$$\frac{T_{n+1}}{T_n}=q \quad (47)$$

and the resulting relation

$$\frac{A_{n+1}}{A_n}=q^{2/3} \quad (48)$$

do not constitute a derivation of the empirical Titius-Bode law, they naturally generate a geometric sequence of orbital radii. It is therefore instructive to compare the resulting pattern with the observed Solar System.

As an illustration, consider

$$A_n=A_0r^n, \qquad A_0=0.353\,\mathrm{AU}, \qquad r=1.733. \quad (49)$$

The resulting sequence is compared with the observed semimajor axes of the planets and with Ceres representing the asteroid belt.

The agreement is not exact and should not be interpreted as a proof of the Titius-Bode law. Rather, it demonstrates that the stability principle, supplemented by cyclicity or resonance conditions, naturally produces orbital spacing patterns of the same geometric type as those observed in the Solar System.

This example should therefore be viewed as a phenomenological illustration of the stability-selection mechanism rather than as a derivation of planetary dynamics.

VI. MINIMAL ACTION SCALE AND QUANTUM-TYPE DISCRETENESS

The theorem gives a discrete stability spectrum. To obtain quantum-type energy levels one needs an additional phase or action condition. Suppose that a state is represented in the form

TABLE I. Observed planetary semimajor axes and the geometric sequence generated by Eq. (48).

Body	Observed A (AU)	Model A_n (AU)	Relative error (%)
Mercury	0.387	0.353	-8.7
Venus	0.723	0.612	-15.3
Earth	1.000	1.061	+6.1
Mars	1.524	1.839	+20.7
Ceres	2.767	3.186	+15.1
Jupiter	5.203	5.520	+6.1
Saturn	9.537	9.566	+0.3
Uranus	19.191	16.575	-13.6
Neptune	30.070	28.721	-4.5

$$\psi = R \exp\left(\frac{iS}{Y_{\text{min}}}\right), \quad (50)$$

where Y_{min} is a minimal action scale. Single-valuedness of the phase gives

$$\Delta\left(\frac{S}{Y_{\text{min}}}\right) = 2\pi n, \qquad n \in \mathbb{Z}, \quad (51)$$

and therefore

$$\Delta S = 2\pi n Y_{\text{min}}. \quad (52)$$

For a closed orbit this becomes

$$\oint p dq = 2\pi n Y_{\text{min}}. \quad (53)$$

When $Y_{\text{min}} = \hbar$, Eq. (53) is the usual semiclassical quantization rule [10, 11].

This observation clarifies the difference between atomic and planetary systems. In an atomic system, $\hbar$ is a physically relevant universal action scale, and Eq. (53) leads to observable energy discreteness. In a planetary system, the orbital action is enormous compared with $\hbar$

$$J \sim mV\sqrt{GMA} \gg \hbar. \quad (54)$$

Thus any quantum spacing would be practically unobservable in the macroscopic limit. The stability principle still supplies discrete stability modes, but it does not automatically make planetary energies discrete.

VII. DISCUSSION

The stability-induced mechanism proposed here has three logically distinct layers.

First, stationarity:

$$\delta S = 0 \quad (55)$$

gives the classical equations of motion [1–3]. In celestial mechanics this reproduces Keplerian motion in the two-body approximation.

Second, variational stability:

$$\delta^2 S \geq 0 \quad (56)$$

turns the second variation into a positive quadratic form. Under the assumptions of the theorem this form defines a self-adjoint stability operator. Boundary conditions and compactness yield a discrete spectrum of stable modes.

Third, energy quantization requires an additional action-scale condition such as Eq. (53). Without such a scale, a classical system can have discrete perturbation modes and still possess continuous orbital parameters.

This hierarchy avoids a common confusion. Periodicity alone allows a Fourier series. It does not discretize all physical quantities. The stability principle adds the operator generated by the second variation. A minimal action scale adds quantum-type discreteness of action variables and, in suitable systems, energy levels.

VIII. CONCLUSION

A stability-based route to discreteness has been formulated. The central statement is that the second variation of a stability functional defines a quadratic form. If the form is positive, closed, and semibounded, and if the boundary conditions provide compactness, the associated self-adjoint stability operator has compact resolvent and a discrete spectrum. Thus discreteness arises as a consequence of variational stability and physical boundary conditions.

The result does not claim that all classical energies are automatically quantized. For periodic classical systems it gives a discrete spectrum of stability modes. Energy discreteness requires an additional minimal action scale and phase condition [12–14]. This distinction allows the same principle to include the classical macroscopic limit and the quantum-like discrete limit within a single variational framework.

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DATA AVAILABILITY

No numerical data were generated or analyzed in this work.

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