On Algebraic Integral For Dummies

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Abstract

We explain the essence of the integral axiom and provide examples of integrating elementary functions using the new integration method.

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Introduction

With his famous painting Black Square, Kazimir Malevich urged artists to embrace simplicity and to avoid multiplying entities beyond necessity — as is well known, Malevich's black square symbolizes economy. Like Malevich, with this paper I want to urge mathematicians toward economy of meaning and the pursuit of simplicity — extra complexity distances them from "ordinary people", while the universe is beautiful in its simplicity, and practice shows that any mathematical concept can be explained to a child if one understands it well enough.

The integration method I propose in this paper is simple and economical. I am writing it in response to requests from several people who told me that the first paper [1] was too complicated and that the practical application of the theory of the algebraic integral was not obvious. I decided to answer these requests with this paper, in order to bring the matter to some logical conclusion. In this paper, I will explain in simple words the essence of the theory of the algebraic integral and show how to integrate functions using a new method that is much simpler than the classical one. The substitution method currently used to solve integrals is complicated — its application is an art, mastering which causes considerable difficulties for students. Moreover, using this method assumes that we can replace a function with an independent variable — students usually have a rather vague idea of the nature of this operation, but in reality, justifying the method requires invoking concepts from topology and differential geometry — such as limit, continuity, and smoothness. The new integration method proposed in this paper uses pure algebra, without appealing even to the concept of a limit — either explicitly or implicitly. Furthermore, in the conclusion, I will suggest some other, most obvious applications of the algebraic integral.

1 What is the integral axiom?

In the first part of the paper, we will show that the integral is not a function of a single variable, but an operation $f \triangleright g = \int f\,dg$. While the method of change of variables is nothing other than the distributivity of composition over that operation:

$$(f \triangleright g) \circ h = (f \circ h) \triangleright (g \circ h)$$

Let's begin.

Let $\langle R, \cdot, \circ \rangle$ be a near-ring. That is, a ring with only right distributivity. That is, R such a structure that

$$\forall f, g, h \in R : (f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$$

Let $d \colon R \to R$ be a function satisfying the chain rule, that is

$$\forall f, g \in R : d(f \circ g) = \big(d(f) \circ g\big) \cdot d(g)$$

Let $\int \colon R \to R$ be a function opposite to d. Let's call this function the **antiderivative**. Then

Theorem. $\forall f, g, h \in R :$

$$\Big(\int f \cdot d(g)\Big) \circ h = \int (f \circ h)\, d(g \circ h)$$

Lemma. $\forall f, g \in R :$

$$\int (f) \circ \int (g) = \int \Big(f \circ \Big(\int g\Big) \cdot g\Big)$$

Proof. For any $x, y \in R$, chain rule gives:

$$d(x \circ y) = d(x) \circ y \cdot d(y)$$

Substitute $x = \int f$, $y = \int g$:

$$d\Big(\int f \circ \int g\Big) = \Big(f \circ \int g\Big) \cdot g$$

Applying $\int$ to both sides yields

$$\int f \circ \int g = \int \big(f \circ (\int g) \cdot g\big) \quad \square$$

Proof of Theorem:

$$\big(\int f\, d(g)\big) \circ h = \big(\int [f \cdot d(g)]\big) \circ \int d(h)$$

Applying Lemma:

$$= \int \big(\big[f \cdot d(g)\big] \circ \big(\int d(h)\big) \cdot d(h)\big)$$

$$= \int \big(\big[f \cdot d(g)\big] \circ h \cdot d(h)\big)$$

By right distributivity in R:

$$= \int \big(\big[f \circ h\big] \cdot \big[d(g) \circ h\big] \cdot d(h)\big)$$

And reduce the expression by the chain rule for d:

$$= \int \big(\big[f \circ h\big] \cdot d(g \circ h)\big)$$

Or in other words:

$$(f \triangleright g) \circ h = (f \circ h) \triangleright (g \circ h)$$

— We call it "integral axiom". ■

There is also a converse statement: informally speaking, for any distributive operation, there exists a corresponding function that satisfies the chain rule. For details see Theorem 2 in "On Algebraic Integral" [1].

2 Axioms

Before moving on to practical examples of solving integrals using the algebraic method, I will recall the axioms of a **square rring with exponent**, which you can find in Definitions 4.11, 4.10 and 4.4 in [1].

Definition. In particular, the ring of diffeomorphisms $\langle C^1, \cdot, \circ\rangle$ together with the antiderivative as an algebraic integral forms a **square rring with a bilinear integral** $\langle R, +, \cdot, \circ, \triangleright, 0, 1, \mathrm{x}, \mathrm{e}\rangle$, that is $\langle R, \cdot, \circ\rangle$ is a near-ring such that R is vector space over the field F and $\forall f, g, h \in R$, $\forall \alpha, \beta \in F$:

1. $1 \triangleright h = h$ (identity axiom)
2. $(f \cdot g) \triangleright h = f \triangleright (g \triangleright h)$ (associativity)
3. $(f \triangleright h) \circ g = (f \circ g) \triangleright (h \circ g)$ (integral axiom)
4. $(\alpha f + \beta g) \triangleright h = \alpha(f \triangleright h) + \beta(g \triangleright h)$ (linearity in the first argument)
5. $f \triangleright (\alpha g + \beta h) = \alpha(f \triangleright g) + \beta(f \triangleright h)$ (linearity in the second argument)

and the integral $\triangleright$ has an exponential element $\mathrm{e} \in R$ such that $\mathrm{e} \triangleright \mathrm{x} = \mathrm{e}$. In addition we may assume that the exponential element is multiplicative:

$$\mathrm{e} \circ (f + g) = (\mathrm{e} \circ f) \cdot (\mathrm{e} \circ g)$$

- to have the Leibniz rule by Theorem 6 from [1].

Remark. It is easy to see that $d(\mathrm{x}) = 1$:

$$d(\mathrm{x}) = d(\mathrm{x} \circ \mathrm{x}) = d(\mathrm{x}) \circ \mathrm{x} \cdot d(\mathrm{x}) = d(\mathrm{x}) \cdot d(\mathrm{x})$$

- therefore we simply write $\int f$ instead of the usual notation $\int f\, dx$. Note that composition has priority over the multiplication.

Remark. It is easy to see that the identity axiom and associativity also hold for the antiderivative. Denoting $d(h) = t$, we get:

1. $\int 1\, dh = \int 1 \cdot t = h$
2. $\int f\, d(\int g\, dh) = \int f\, d(\int g \cdot t) = \int fgt = \int (fg)\, dh$

3 Examples

We will now integrate several elementary functions using the integral axiom, to demonstrate how the theorem simplifies integration by turning it into a purely algebraic process.

Example 1. $\mathrm{x} \triangleright \ln = \mathrm{x}$.

Denoting by ln the compositional inverse of the exponential element, $\mathrm{e} \circ \ln = \ln \circ \mathrm{e} = \mathrm{x}$, let's solve:

$$\mathrm{x} \triangleright \ln = (\mathrm{e} \circ \ln) \triangleright (\mathrm{x} \circ \ln) = (\mathrm{e} \triangleright \mathrm{x}) \circ \ln$$

- by integral axiom

$$= \mathrm{e} \circ \ln = \mathrm{x} \quad \square$$

Example 2. $\frac{1}{x} \triangleright x = \ln$:

$$\frac{1}{x} \triangleright x = \frac{1}{\mathrm{x}} \triangleright (\mathrm{x} \triangleright \ln)$$

- by Example 1

$$= (\frac{1}{\mathrm{x}} \cdot \mathrm{x}) \triangleright \ln = 1 \triangleright \ln = \ln$$

-by associativity and identity axioms $\square$

Example 3.

$$\frac{1}{\mathrm{x} \cdot \ln} \triangleright \mathrm{x} = \frac{1}{\ln} \triangleright (\frac{1}{\mathrm{x}} \triangleright \mathrm{x}) = \frac{1}{\ln} \triangleright \ln$$

- by associativity axiom and Example 2

$$= (\frac{1}{\mathrm{x}} \circ \ln) \triangleright (\mathrm{x} \circ \ln) = (\frac{1}{\mathrm{x}} \triangleright \mathrm{x}) \circ \ln$$

- apply the integral axiom.

$$= \ln(\ln(\mathrm{x})) \quad \square$$

Example 4.

$$\mathrm{x}^2(\mathrm{x}^3+1)^{\frac{1}{3}} \triangleright \mathrm{x} = (\mathrm{x}^3+1)^{\frac{1}{3}} \triangleright (\mathrm{x}^2 \triangleright \mathrm{x}) = (\mathrm{x}^3+1)^{\frac{1}{3}} \triangleright \frac{1}{3}\mathrm{x}^3$$

- using associativity and integrating x^2

$$= \frac{1}{3}[\mathrm{x}^{\frac{1}{3}} \circ (\mathrm{x}^3+1) \triangleright (\mathrm{x}^3+1)] = \frac{1}{3}[(\mathrm{x}^{\frac{1}{3}} \triangleright \mathrm{x}) \circ (\mathrm{x}^3+1)]$$

- use linearity in the second argument and the axiom of integral. Note that composition has priority over the integral.

$$= \frac{1}{3}[(\frac{3}{4}\mathrm{x}^{\frac{4}{3}}) \circ (\mathrm{x}^3+1)] = \frac{1}{4}(\mathrm{x}^3+1)^{\frac{4}{3}} \quad \square$$

Example 5.

$$\frac{\sin}{1+\cos^2} \triangleright x = \frac{1}{1+\cos^2} \triangleright (\sin \triangleright x) = \frac{1}{1+\cos^2} \triangleright (-\cos)$$

$$= -([\frac{1}{1+\mathrm{x}^2} \circ \cos] \triangleright [\mathrm{x} \circ \cos]) = -(\frac{1}{1+\mathrm{x}^2} \triangleright \mathrm{x}) \circ \cos$$

- the integral axiom

$$= -\arctan(\cos(\mathrm{x})) \quad \square$$

Remark.
In the case when the exponential element is multiplicative, i.e.

$$\mathrm{e}(f+g) = \mathrm{e}(f) \cdot \mathrm{e}(g)$$

this is equivalent to the additivity of the logarithm. Given that, and as we established above, $d(\mathrm{x}) = 1$, d is uniquely determined on $\alpha\mathrm{x}^n$ for $n \in \mathbb{N}$ and $\alpha \in F$:

$$d(\alpha\mathrm{x}^n) = \alpha d(\mathrm{e} \circ (n\ln)) = \alpha[d(\mathrm{e}) \circ (n\ln) \cdot d(n\ln)] = \frac{\alpha n\mathrm{x}^n}{\mathrm{x}}$$

- you can use the Leibniz rule by Theorem 6 from [1] to obtain the same result by induction. Using the compositional inverses of $\alpha\mathrm{x}^n$ and the chain rule, we uniquely extend d to $\alpha\mathrm{x}^q$, where $q \in \mathbb{Q}$. Hence, by the converse part of Theorem 2 in [1], we obtain the integrals of the form $\alpha\mathrm{x}^q \triangleright \mathrm{x}$.

Sine and cosine, in turn, may be defined as elements satisfying the rules: $d(\sin) = \cos$, $d(\cos) = -\sin$, $\sin(0) = 0$, $\cos(0) = 1$ — it is well known that these conditions uniquely determine sine and cosine over the real numbers; the remaining trigonometric functions are defined in terms of these.

Alternatively, in Examples 4 and 5 one may simply regard R as the ring of diffeomorphisms with the standard antiderivative.

4 Theorem 2. On Exponential Element of a Bilinear Integral

The following are equivalent for any $f, g \in (R \setminus F)$:

- $\mathrm{e} \circ (f + g) = (\mathrm{e} \circ f) \cdot (\mathrm{e} \circ g)$ (Exponential Multiplicativity)
- $f \cdot g = f \triangleright g + g \triangleright f$ (Integration by parts rule)

Proof. Assume the multiplicativity of the exponential. It is easy to verify that it is equivalent to the additivity of the logarithm:

$$\ln \circ (f \cdot g) = \ln \circ f + \ln \circ g$$

Applying the exponential to both sides:

$$f \cdot g = \mathrm{e} \circ (\ln \circ f + \ln \circ g)$$

By the definition of the exponential:

$$= (\mathrm{e} \triangleright \mathrm{x}) \circ (\ln \circ f + \ln \circ g)$$

And applying the integral axiom:

$$= [\mathrm{e} \circ (\ln \circ f + \ln \circ g)] \triangleright [\mathrm{x} \circ (\ln \circ f + \ln \circ g)]$$

Looking at the second step of the proof:

$$= (f \cdot g) \triangleright (\ln \circ f + \ln \circ g)$$

And applying linearity in the second argument:

$$= fg \triangleright (\ln \circ f) + fg \triangleright (\ln \circ g)$$

Now apply associativity:

$$= g \triangleright (f \triangleright \ln \circ f) + f \triangleright (g \triangleright \ln \circ g)$$

And again the integral axiom:

$$= g \triangleright ([\mathrm{x} \triangleright \ln] \circ f) + f \triangleright ([\mathrm{x} \triangleright \ln] \circ g)$$

By Example 1,

$$= g \triangleright f + f \triangleright g \quad \square$$

It is easy to verify that all implications in the proof are equivalences. Therefore, the converse follows similarly, in reverse order. ■

Example 6. Assuming the multiplicativity of the exponential and $\mathbb{Q} \subset F$,

$$\mathrm{x}^n \triangleright \mathrm{x} = \frac{\mathrm{x}^{n+1}}{n+1}$$

Induction base:

By theorem 2,

$$\mathrm{x} \triangleright \mathrm{x} = \mathrm{x}^2 - \mathrm{x} \triangleright \mathrm{x}$$

Induction step:

$$\mathrm{x}^n \triangleright \mathrm{x} = \mathrm{x} \triangleright (\mathrm{x}^{n-1} \triangleright \mathrm{x})$$

— by associativity. And by the induction hypothesis:

$$= \mathrm{x} \triangleright \frac{\mathrm{x}^n}{n} = \frac{\mathrm{x}^{n+1}}{n} - \frac{1}{n}(\mathrm{x}^n \triangleright \mathrm{x})$$

— by theorem 2 and linearity in the first argument. So,

$$\frac{n+1}{n}(\mathrm{x}^n \triangleright \mathrm{x}) = \frac{\mathrm{x}^{n+1}}{n} \quad \square$$

Conclusion

Simplicity of finding integrals demonstrated in Examples 1–5 suggests that the proposed algebraic method of integration can be turned into a fast and efficient symbolic integration algorithm for computer algebra systems and symbolic libraries such as MATLAB and SymPy. It remains an open question whether the method can accelerate numerical integration — which, in theory, could lead to the optimization and acceleration of neural networks and other mathematical systems.

References

[1] N. Umerov *On Algebraic Integral*. Zenodo, 2025. DOI: 10.5281/zenodo.17376700