A Conditional and Computational Program for Hard-Edge PF_3 Positivity of de Bruijn Moment Jensen Coefficients

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Abstract

We develop a finite-order positivity program for hard-edge Jensen–Toeplitz minors associated with the de Bruijn moment sequence. The main focus is the 3×3 solid Toeplitz layer, denoted here as the hard-edge PF_3 problem. The central algebraic reduction is the identity

$$N_{3,q} = 2v_q^3 - v_q^4 - (1 - v_q)^2 E_q, \qquad E_q = v_q \Delta^2 v_q - (\Delta_- v_q)(\Delta_+ v_q),$$

together with the factorization

$$x_q = \rho_{d,q} \tau_q, \qquad \tau_q = \frac{m_{q-1} m_{q+1}}{m_q^2}.$$

Under an effective Tau-Weak hypothesis controlling $\tau_q - 1$ and its first two discrete differences, the PF_3 inequality follows outside a finite certificate region. We also present certificate-readiness diagnostics, including finite rectangle checks, fixed-small-q Sturm diagnostics, fixed-small-h profiles, and artificial tau-box interval evaluations.

This manuscript does not claim a proof of the Riemann hypothesis. It records a conditional and computationally supported PF_3 route and explains why a direct all-order extension through normalized Toda/Gamma ratios is not presently available. High-order finite-d solid-minor diagnostics show that direct all-r numerical scanning is unstable and cannot by itself provide a PF_∞ mechanism. Thus the contribution is deliberately finite-order: a structured route toward a rigorous PF_3 theorem, plus a clear separation from the unresolved PF_∞ problem.

Scope of the paper. The goal of this manuscript is finite-order and deliberately modest: to isolate a tractable PF_3 mechanism and reduce the remaining proof obligations to an effective Tau-Weak estimate and a finite certificate. The numerical high-order diagnostics in Section 21 are included to prevent overinterpretation: they show that the normalized Toda/Gamma ratio is a useful diagnostic, but not a direct PF_∞ proof mechanism.

Contents

1 Introduction

The Riemann Ξ-function is an even entire function whose zeros encode the nontrivial zeros of the Riemann zeta function. The Riemann hypothesis states that all zeros of Ξ are real. Classical work of Pólya, Schur, Aissen, Schoenberg, Whitney, and others connects real-rootedness of entire functions and polynomial sections with total positivity and Pólya frequency sequences.

In this setting, the function

$$G(w) = \Xi(i\sqrt{w})$$

is a natural hard-edge object: under the Riemann hypothesis, the zeros of G lie on the nonpositive real axis. Total positivity of the associated coefficient sequences is therefore a natural way of organizing finite-dimensional shadows of the Riemann hypothesis.

The present paper studies the coefficient arrays

$$A_{d,k} = \binom{d}{k} \frac{k!}{(2k)!} m_k,$$

where

$$m_k = \int_0^\infty \Phi(u) u^{2k}\, du.$$

Here Φ is the de Bruijn kernel appearing in the Fourier representation of Ξ. The hard-edge Jensen polynomial is

$$J_d(X) = \sum_{k=0}^{d} A_{d,k} X^k.$$

The full PF_∞ property is far stronger than any fixed finite-order positivity statement. In the appropriate limiting hard-edge Jensen setting, a genuine PF_∞ theorem would be connected to the Riemann hypothesis. This manuscript does not prove or claim PF_∞. Instead, it studies a first nontrivial finite layer, namely PF_3.

For fixed d, define the Toeplitz solid minors

$$D_{r,q}^{(d)} = \det[A_{q+j-i}]_{i,j=0}^{r-1}.$$

For $r = 3$, the valid interior range is

$$2 \le q \le d-2,$$

because the determinant involves

$$A_{q-2}, A_{q-1}, A_q, A_{q+1}, A_{q+2}.$$

The target PF_3 assertion is

$$D_{3,q}^{(d)} > 0 \qquad (2 \le q \le d-2).$$

The main contribution of this manuscript is a clean reduction of this PF_3 condition to effective estimates for the moment-ratio sequence

$$\tau_q = \frac{m_{q-1} m_{q+1}}{m_q^2}.$$

The exact algebraic structure is simple. Define

$$x_q = \frac{A_{q-1}A_{q+1}}{A_q^2}, \qquad v_q = 1 - x_q.$$

Then the normalized 3×3 determinant

$$N_{3,q}^{(d)} = \frac{D_{3,q}^{(d)}}{A_q^3}$$

satisfies

$$N_{3,q} = v_q^2 - x_q^2 v_{q-1} v_{q+1} = 2v_q^3 - v_q^4 - (1-v_q)^2 E_q,$$

where

$$E_q = v_{q-1}v_{q+1} - v_q^2.$$

Thus $N_{3,q} > 0$ follows if v_q is positive and sufficiently smooth in q.

The key factorization is

$$x_q = \rho_{d,q}\tau_q,$$

where

$$\rho_{d,q} = \frac{d-q}{d-q+1}\frac{(2q-1)(2q)}{(2q+1)(2q+2)}$$

is explicit, while

$$\tau_q = \frac{m_{q-1}m_{q+1}}{m_q^2}$$

contains the analytic information of the de Bruijn moments. The positivity problem is therefore separated into:

(i) rational estimates for $\rho_{d,q}$;

(ii) effective analytic estimates for τ_q;

(iii) a certificate architecture for the remaining finite-pair and finite-family regimes.

The analytic input required is deliberately weak. It is enough to prove that for all sufficiently large q,

$$0 \le \tau_q - 1 \le \frac{C_0}{q\log q},$$

$$|\tau_{q+1} - \tau_q| \le \frac{C_1}{q^2\log q},$$

and

$$|\Delta^2\tau_q| \le \frac{C_2}{q^3\log q}.$$

The expected sharper asymptotic is

$$\tau_q - 1 = \frac{2}{q\log q}\left[1 + \frac{\log\log q + \log(\pi/2) - 1}{\log q} + O\left(\frac{(\log\log q)^2}{(\log q)^2}\right)\right].$$

However, the full second-order expansion is not needed for the conditional PF_3 reduction.

Informal main result. The paper proves an exact algebraic reduction of the hard-edge PF_3 inequality to the positivity and second-difference control of $v_q = 1 - \rho_{d,q}\tau_q$. Conditional on Tau-Weak and on a rigorous finite certificate for the remaining bounded regimes, all valid 3×3 solid Toeplitz minors are positive. The result is therefore a conditional PF_3 theorem plus a certificate architecture, not an unconditional PF_3 theorem and not a PF_∞ theorem.

The rest of the paper is organized as follows. Section 2 recalls the hard-edge coefficient array and the normalized determinant hierarchy. Section 3 derives the exact $r = 3$ identity. Section 4 gives the positivity criterion. Section 5 isolates the factorization $x_q = \rho_{d,q}\tau_q$. Section 6 states the Tau-Weak moment-ratio input. Section 7 records the analytic roadmap needed to make that input effective. Section 8 proves the first saddle-location and curvature estimates. Section 9 proves the tail separation estimate. Section 10 sets up the differentiable Laplace remainder framework. Section 11 proves the central non-differentiated Laplace correction, Section 12 controls its complement, Section 13 proves the first differentiated correction, and Section 14 records the uniform $j \le 4$ differentiated mechanism. Section 15 derives Tau-Weak from these differentiable estimates. Section 16 gives a unified leading positivity framework for the bulk, left-edge, and right-edge regimes. Section 17 outlines the proposed saddle-point route to Tau-Weak. Section 18 describes the certificate architecture. Section 20 assembles the conditional PF_3 theorem, and Section 22 records the logical status and remaining gaps.

2 Hard-edge Jensen coefficients and Toeplitz minors

Let $\Phi(u)$ denote the de Bruijn kernel associated with the Riemann Ξ-function. We use the moment notation

$$m_k = \int_0^\infty \Phi(u)u^{2k}\, du.$$

The coefficients of the hard-edge Jensen polynomial are

$$A_{d,k} = \binom{d}{k}\frac{k!}{(2k)!}m_k.$$

We suppress the dependence on d when no confusion can occur and write simply A_k.

Definition 2.1 (Hard-edge Jensen polynomial)**.** *For $d \geq 0$, define*

$$J_d(X) = \sum_{k=0}^{d} A_{d,k}X^k.$$

Definition 2.2 (Toeplitz solid minors)**.** *For $r \geq 1$, define*

$$D_{r,q}^{(d)} = \det[A_{q+j-i}]_{i,j=0}^{r-1}.$$

For $r = 3$,

$$D_{3,q}^{(d)} = \det\begin{pmatrix} A_q & A_{q+1} & A_{q+2} \\ A_{q-1} & A_q & A_{q+1} \\ A_{q-2} & A_{q-1} & A_q \end{pmatrix}.$$

The PF_r condition for a finite coefficient sequence requires the nonnegativity of all minors of order at most r in the associated Toeplitz matrix. In this paper we isolate the solid minors of order 3, which are a necessary part of the PF_3 condition.

Remark 2.3. *The full PF_∞ condition is much stronger. Proving $D_{3,q}^{(d)} > 0$ for all valid d, q does not imply the Riemann hypothesis. It is only a finite-order total positivity layer.*

3 Normalized $r = 3$ determinant identity

Define

$$x_q = \frac{A_{q-1}A_{q+1}}{A_q^2}$$

and

$$v_q = 1 - x_q.$$

Then

$$D_{2,q}^{(d)} = A_q^2 - A_{q-1}A_{q+1} = A_q^2 v_q.$$

Thus

$$N_{2,q} := \frac{D_{2,q}^{(d)}}{A_q^2} = v_q.$$

Lemma 3.1 (Normalized $r = 3$ identity)**.** *Let*

$$N_{3,q}^{(d)} = \frac{D_{3,q}^{(d)}}{A_q^3}.$$

Then

$$N_{3,q} = v_q^2 - x_q^2 v_{q-1}v_{q+1}.$$

Equivalently,

$$N_{3,q} = v_q^2 - (1 - v_q)^2 v_{q-1}v_{q+1}.$$

Proof. The Desnanot–Jacobi identity for adjacent Toeplitz minors gives

$$D_{3,q}A_q = D_{2,q}^2 - D_{2,q-1}D_{2,q+1}.$$

Since

$$D_{2,q} = A_q^2 v_q,$$

we have

$$D_{2,q}^2 = A_q^4 v_q^2.$$

Also,

$$D_{2,q-1} = A_{q-1}^2 v_{q-1}, \qquad D_{2,q+1} = A_{q+1}^2 v_{q+1}.$$

Thus

$$D_{2,q-1}D_{2,q+1} = A_{q-1}^2 A_{q+1}^2 v_{q-1}v_{q+1} = A_q^4 x_q^2 v_{q-1}v_{q+1}.$$

Therefore

$$D_{3,q}A_q = A_q^4 \left(v_q^2 - x_q^2 v_{q-1}v_{q+1}\right).$$

Dividing by A_q^4 gives

$$\frac{D_{3,q}}{A_q^3} = v_q^2 - x_q^2 v_{q-1}v_{q+1}.$$

Finally, $x_q = 1 - v_q$. □

Now define the multiplicative defect

$$E_q = v_{q-1}v_{q+1} - v_q^2.$$

Then

$$v_{q-1}v_{q+1} = v_q^2 + E_q.$$

Hence

$$\begin{aligned} N_{3,q} &= v_q^2 - (1-v_q)^2(v_q^2 + E_q) \\ &= v_q^2\{1 - (1-v_q)^2\} - (1-v_q)^2 E_q \\ &= v_q^2(2v_q - v_q^2) - (1-v_q)^2 E_q. \end{aligned}$$

Therefore

$$\boxed{N_{3,q} = 2v_q^3 - v_q^4 - (1-v_q)^2 E_q.}$$

4 A positivity criterion

We express E_q using finite differences. Let

$$\Delta_+ v_q = v_{q+1} - v_q,$$

$$\Delta_- v_q = v_q - v_{q-1},$$

and

$$\Delta^2 v_q = v_{q+1} - 2v_q + v_{q-1}.$$

Lemma 4.1 (Defect identity)**.**

$$E_q = v_q \Delta^2 v_q - (\Delta_- v_q)(\Delta_+ v_q).$$

Consequently,

$$|E_q| \le v_q|\Delta^2 v_q| + |\Delta_- v_q|\,|\Delta_+ v_q|.$$

Proof. Since

$$v_{q+1} = v_q + \Delta_+ v_q$$

and

$$v_{q-1} = v_q - \Delta_- v_q,$$

we compute

$$\begin{aligned} E_q &= (v_q - \Delta_- v_q)(v_q + \Delta_+ v_q) - v_q^2 \\ &= v_q(\Delta_+ v_q - \Delta_- v_q) - (\Delta_- v_q)(\Delta_+ v_q). \end{aligned}$$

But

$$\Delta_+ v_q - \Delta_- v_q = \Delta^2 v_q.$$

The absolute value bound follows immediately. □

Lemma 4.2 (Scale positivity criterion)**.** *Suppose that for some scale $M \geq 1$,*

$$v_q \geq \frac{a}{M},$$

$$|\Delta_\pm v_q| \leq \frac{A}{M^2},$$

and

$$|\Delta^2 v_q| \leq \frac{B}{M^3}.$$

Assume also that $v_q \leq C/M$. Then

$$|E_q| \leq \frac{CB + A^2}{M^4}.$$

In particular, for sufficiently large M,

$$N_{3,q} > 0.$$

Proof. By the defect identity,

$$|E_q| \leq v_q |\Delta^2 v_q| + |\Delta_- v_q|\,|\Delta_+ v_q|.$$

Using the assumed bounds,

$$|E_q| \leq \frac{C}{M}\frac{B}{M^3} + \frac{A^2}{M^4} = \frac{CB + A^2}{M^4}.$$

On the other hand,

$$2v_q^3 - v_q^4 = v_q^3(2 - v_q).$$

For large M, $v_q \leq 1$, and therefore

$$2v_q^3 - v_q^4 \geq v_q^3 \geq \frac{a^3}{M^3}.$$

Thus

$$(1 - v_q)^2 |E_q| \leq |E_q| = O(M^{-4}),$$

while

$$2v_q^3 - v_q^4 = \Omega(M^{-3}).$$

For M sufficiently large,

$$(1 - v_q)^2 |E_q| < 2v_q^3 - v_q^4,$$

and hence $N_{3,q} > 0$. □

5 Factorization of x_q

Write

$$A_{d,q} = b_{d,q} m_q,$$

where

$$b_{d,q} = \binom{d}{q} \frac{q!}{(2q)!}.$$

Then

$$x_q = \frac{A_{q-1} A_{q+1}}{A_q^2} = \frac{b_{d,q-1} b_{d,q+1}}{b_{d,q}^2} \frac{m_{q-1} m_{q+1}}{m_q^2}.$$

Define

$$\rho_{d,q} = \frac{b_{d,q-1} b_{d,q+1}}{b_{d,q}^2}$$

and

$$\tau_q = \frac{m_{q-1} m_{q+1}}{m_q^2}.$$

Then

$$x_q = \rho_{d,q} \tau_q.$$

Lemma 5.1 (Explicit base factor). *For $1 \leq q \leq d-1$,*

$$\rho_{d,q} = \frac{d-q}{d-q+1} \frac{(2q-1)(2q)}{(2q+1)(2q+2)}.$$

Proof. Since

$$b_{d,q} = \binom{d}{q} \frac{q!}{(2q)!} = \frac{d!}{(d-q)!} \frac{1}{(2q)!},$$

we have

$$\frac{b_{d,q-1}}{b_{d,q}} = \frac{(d-q)!}{(d-q+1)!} \frac{(2q)!}{(2q-2)!} = \frac{(2q-1)(2q)}{d-q+1},$$

and

$$\frac{b_{d,q+1}}{b_{d,q}} = \frac{(d-q)!}{(d-q-1)!} \frac{(2q)!}{(2q+2)!} = \frac{d-q}{(2q+1)(2q+2)}.$$

Multiplying gives

$$\rho_{d,q} = \frac{d-q}{d-q+1} \frac{(2q-1)(2q)}{(2q+1)(2q+2)}.$$

□

Thus

$$v_q = 1 - x_q = 1 - \rho_{d,q} \tau_q.$$

Equivalently,

$$\boxed{v_q = (1 - \rho_{d,q}) - \rho_{d,q}(\tau_q - 1).}$$

6 The Tau-Weak moment-ratio input

The main analytic input is the following.

Assumption 6.1 (Tau-Weak)**.** *There exist constants $Q_0, C_0, C_1, C_2 > 0$ such that for all $q \geq Q_0$,*

$$0 \leq \tau_q - 1 \leq \frac{C_0}{q \log q},$$

$$|\tau_{q+1} - \tau_q| \leq \frac{C_1}{q^2 \log q},$$

and

$$|\Delta^2 \tau_q| \leq \frac{C_2}{q^3 \log q}.$$

Remark 6.2 (Unconditional lower bound)**.** *The lower bound $\tau_q \geq 1$ is unconditional. Indeed, if*

$$M(z) = \int_0^\infty \Phi(u) u^{2z} \, du, \qquad s(z) = \log M(z),$$

then

$$s''(z) = 4 \operatorname{Var}_z(\log U) \geq 0.$$

Hence s is convex and

$$\log \tau_q = s(q-1) - 2s(q) + s(q+1) \geq 0.$$

Thus the analytic burden in Tau-Weak lies in the upper bound for $\tau_q - 1$ and in the first and second discrete difference estimates.

Remark 6.3. *This is the central unproved analytic input in the present manuscript. Numerical and saddle-point evidence suggest the sharper expansion*

$$\tau_q - 1 = \frac{2}{q \log q} \left[1 + \frac{\log \log q + \log(\pi/2) - 1}{\log q} + O\left(\frac{(\log \log q)^2}{(\log q)^2} \right) \right],$$

but the conditional PF_3 reduction only requires the weaker estimates in Tau-Weak.

7 Effective Tau-Weak proof roadmap

This section records the analytic tasks required to turn the Tau-Weak input of Section 6 into a theorem. It is not used as a completed proof in the present manuscript. Its role is to isolate the remaining estimates in a form that can be attacked independently.

Recall that

$$\tau_q = \frac{m_{q-1} m_{q+1}}{m_q^2}, \qquad s(z) = \log M(z),$$

where $M(z)$ denotes a smooth interpolation of the even de Bruijn moments. The desired Tau-Weak bounds are

$$0 \leq \tau_q - 1 \leq \frac{C_0}{q \log q},$$

$$|\Delta \tau_q| \leq \frac{C_1}{q^2 \log q}, \qquad |\Delta^2 \tau_q| \leq \frac{C_2}{q^3 \log q}.$$

7.1 Task T0: unconditional lower bound

The lower bound

$$\tau_q \geq 1$$

should be separated from the harder upper and difference bounds. Formally, if

$$s''(z) \geq 0,$$

then the discrete midpoint identity

$$\log \tau_q = s(q-1) + s(q+1) - 2s(q) = \int_{-1}^{1} (1-|t|)s''(q+t)\,dt$$

implies

$$\log \tau_q \geq 0.$$

Thus the lower side of Tau-Weak is a log-convexity statement for the moment interpolation. The main analytic burden is the upper bound and the first two difference bounds.

7.2 Task T1: effective saddle location

The phase model used in the saddle analysis is

$$F_z(u) = 2z \log u + \frac{9}{2}u - \pi e^{2u}.$$

Let u_z be the saddle point defined by

$$F_z'(u_z) = 0.$$

The first effective task is to prove explicit bounds of the form

$$c_1 \log z \leq u_z \leq c_2 \log z$$

for all sufficiently large z, together with sharper asymptotics

$$u_z = \frac{1}{2}\log z - \frac{1}{2}\log\log z + O(1).$$

One also needs effective derivative bounds for u_z, for example

$$u_z^{(j)} = O_j\left(\frac{1}{z^j}\right)$$

in the range needed to differentiate the saddle expansion up to fourth order.

7.3 Task T2: curvature and local Gaussian scale

Define

$$a_z = -F_z''(u_z).$$

The required scale is

$$a_z \asymp \frac{z}{\log z}.$$

More precisely, the proof should produce effective constants such that

$$c_3 \frac{z}{\log z} \leq a_z \leq c_4 \frac{z}{\log z}$$

for all large z. Higher derivatives at the saddle should satisfy bounds of the schematic form For $k \geq 3$, the exponential part contributes at scale $e^{2u_z} \asymp z/u_z$, so the useful bound is

$$F_z^{(k)}(u_z) = O_k\left(\frac{z}{u_z}\right) = O_k\left(\frac{z}{\log z}\right).$$

The normalized Gaussian coefficients then satisfy

$$\frac{F_z^{(k)}(u_z)}{a_z^{k/2}} = O_k\left(\left(\frac{z}{\log z}\right)^{1-k/2}\right), \qquad k \geq 3.$$

These are the bounds needed for the local Laplace expansion.

7.4 Task T3: tail separation of the de Bruijn kernel

The de Bruijn kernel expansion has the form

$$\Phi(u) = \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2}\right) e^{-\pi n^2 e^{2u}}.$$

The $n = 1$ term should dominate near the main saddle. The required statement is an effective tail separation estimate: for every fixed $A > 0$,

$$\sum_{n\geq 2} \int_0^{\infty} \left|\left(2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2}\right) e^{-\pi n^2 e^{2u}}\right| u^{2z}\, du = O_A(z^{-A})$$

relative to the $n = 1$ main contribution, uniformly with the differentiated bounds needed later. This is one of the key bookkeeping tasks: it must be proved with constants stable under z-differentiation.

7.5 Task T4: differentiable Laplace expansion

Let

$$S(z) = F_z(u_z) + \log(2\pi^2) + \frac{1}{2}\log\frac{2\pi}{a_z}.$$

The central analytic estimate should be

$$s^{(j)}(z) - S^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 0 \leq j \leq 4.$$

This is the main analytic bottleneck. The estimate must control not only the Laplace remainder but also its first four derivatives with respect to z. A proof should avoid informal differentiation of asymptotic symbols; it should either provide an explicitly differentiable remainder formula or use a dominated-differentiation framework with effective majorants.

7.6 Task T5: derivation of Tau-Weak from $s^{(j)}$-bounds

Once T4 is proved, Tau-Weak should follow by finite-difference estimates. The identity

$$\log \tau_q = \int_{-1}^{1} (1 - |t|) s''(q + t)\, dt$$

gives

$$\log \tau_q \ll \frac{1}{q \log q}.$$

Since $\log \tau_q \geq 0$, this gives

$$0 \leq \tau_q - 1 \ll \frac{1}{q \log q}.$$

Similarly, first and second finite differences of $\log \tau_q$ are controlled by third and fourth derivatives of s, giving

$$|\Delta \log \tau_q| \ll \frac{1}{q^2 \log q}, \qquad |\Delta^2 \log \tau_q| \ll \frac{1}{q^3 \log q}.$$

The passage from logarithmic differences to differences of τ_q then uses

$$\tau_q = 1 + O\left(\frac{1}{q \log q}\right).$$

7.7 Deliverable for an unconditional version

An unconditional PF_3 theorem would require the following deliverables:

$$\boxed{\text{D1: an effective proof of T0–T5 for all } q \geq Q_0,}$$

$$\boxed{\text{D2: explicit constants } C_0, C_1, C_2, Q_0,}$$

$$\boxed{\text{D3: a rigorous interval/ball certificate for the remaining finite region.}}$$

The present manuscript supplies the algebraic reduction and the certificate architecture. After Sections 8 and 9, the main remaining non-computational task is the differentiable Laplace remainder T4, followed by the finite-difference conversion T5.

8 First effective saddle estimates

This section proves the first two analytic tasks from Section 7. These estimates do not yet prove Tau-Weak, but they supply the effective saddle geometry needed for the later differentiable Laplace expansion.

Throughout this section let

$$F_z(u) = 2z \log u + \frac{9}{2}u - \pi e^{2u}, \qquad z \geq z_0,$$

and let u_z denote the unique critical point of F_z on $(0, \infty)$.

Lemma 8.1 (Existence and uniqueness of the saddle)**.** *For all sufficiently large z, the equation*

$$F_z'(u) = 0$$

has a unique solution $u_z > 0$. Moreover,

$$\frac{1}{3} \log z \leq u_z \leq \log z$$

for all sufficiently large z.

Proof. We have

$$F_z'(u) = \frac{2z}{u} + \frac{9}{2} - 2\pi e^{2u}.$$

Equivalently, define

$$H_z(u) = 2\pi e^{2u} - \frac{9}{2} - \frac{2z}{u}.$$

Then
$$H_z'(u) = 4\pi e^{2u} + \frac{2z}{u^2} > 0,$$
so H_z is strictly increasing on $(0, \infty)$. Since
$$\lim_{u \downarrow 0} H_z(u) = -\infty, \qquad \lim_{u \to \infty} H_z(u) = +\infty,$$
there is exactly one zero.

It remains to localize it. At $u = (1/3)\log z$,
$$2\pi e^{2u} = 2\pi z^{2/3}, \qquad \frac{2z}{u} = \frac{6z}{\log z}.$$
For large z,
$$2\pi z^{2/3} - \frac{9}{2} - \frac{6z}{\log z} < 0,$$
so $H_z((1/3)\log z) < 0$. At $u = \log z$,
$$2\pi e^{2u} = 2\pi z^2, \qquad \frac{2z}{u} = \frac{2z}{\log z},$$
and hence $H_z(\log z) > 0$ for large z. Since H_z is strictly increasing, the unique zero lies between these two points. □

Lemma 8.2 (First asymptotic location)**.** *As $z \to \infty$,*
$$u_z = \frac{1}{2}\log z - \frac{1}{2}\log\log z + O(1).$$
More precisely,
$$2u_z + \log u_z + \log \pi = \log z + O\left(\frac{\log z}{z}\right).$$

Proof. The saddle equation is
$$\frac{2z}{u_z} + \frac{9}{2} = 2\pi e^{2u_z}.$$
Multiplying by $u_z/2$, we obtain
$$z + \frac{9}{4}u_z = \pi u_z e^{2u_z}.$$
Taking logarithms gives
$$2u_z + \log u_z + \log \pi = \log z + \log\left(1 + \frac{9u_z}{4z}\right).$$
By Lemma 8.1, $u_z \le \log z$ for large z, so
$$\log\left(1 + \frac{9u_z}{4z}\right) = O\left(\frac{\log z}{z}\right).$$
Therefore
$$2u_z + \log u_z + \log \pi = \log z + O\left(\frac{\log z}{z}\right).$$
Using again $u_z \asymp \log z$, this implies
$$2u_z = \log z - \log\log z + O(1),$$
which is the asserted first asymptotic location. □

Lemma 8.3 (Derivative bounds for the saddle)**.** *For each fixed integer $j \geq 1$,*

$$u_z^{(j)} = O_j(z^{-j}).$$

In particular,

$$u_z' = O(z^{-1}), \qquad u_z'' = O(z^{-2}).$$

Proof. Let

$$H(z, u) = 2\pi e^{2u} - \frac{9}{2} - \frac{2z}{u}.$$

The saddle is defined by

$$H(z, u_z) = 0.$$

We have

$$H_u(z, u) = 4\pi e^{2u} + \frac{2z}{u^2}.$$

At $u = u_z$, Lemma 8.1 gives $u_z \asymp \log z$, and the saddle equation gives $e^{2u_z} \asymp z/u_z$. Hence

$$H_u(z, u_z) \asymp \frac{z}{\log z}.$$

Also

$$H_z(z, u) = -\frac{2}{u}.$$

Implicit differentiation gives

$$u_z' = -\frac{H_z(z, u_z)}{H_u(z, u_z)} = O\left(\frac{1/\log z}{z/\log z}\right) = O(z^{-1}).$$

Further differentiations of $H(z, u_z) = 0$ express $u_z^{(j)}$ as a finite sum of terms involving derivatives of H, lower derivatives of u_z, and powers of H_u^{-1}. Since derivatives of H at the saddle are bounded by powers of u_z^{-1} and by $e^{2u_z} \asymp z/u_z$, the same induction gives

$$u_z^{(j)} = O_j(z^{-j}).$$

This proves the claim. The induction is standard; the important point is that each z-differentiation gains one power of z^{-1}, while logarithmic factors are harmless for the stated bound. $\square$

Lemma 8.4 (Curvature scale)**.** *Let*

$$a_z = -F_z''(u_z).$$

Then

$$a_z = \frac{4z}{u_z} + \frac{2z}{u_z^2} + 9,$$

and therefore

$$a_z \asymp \frac{z}{\log z}.$$

More precisely, for all sufficiently large z,

$$c\frac{z}{\log z} \leq a_z \leq C\frac{z}{\log z}$$

with absolute constants $c, C > 0$.

Proof. We compute

$$F_z''(u) = -\frac{2z}{u^2} - 4\pi e^{2u}.$$

Thus

$$a_z = \frac{2z}{u_z^2} + 4\pi e^{2u_z}.$$

From the saddle equation,

$$2\pi e^{2u_z} = \frac{2z}{u_z} + \frac{9}{2}.$$

Multiplying by 2, we get

$$4\pi e^{2u_z} = \frac{4z}{u_z} + 9.$$

Therefore

$$a_z = \frac{4z}{u_z} + \frac{2z}{u_z^2} + 9.$$

Since $u_z \asymp \log z$, the leading term is $4z/u_z$, and hence

$$a_z \asymp \frac{z}{\log z}.$$

□

Lemma 8.5 (Normalized higher derivative scale)**.** *For each fixed $k \geq 3$,*

$$F_z^{(k)}(u_z) = O_k\left(\frac{z}{\log z}\right).$$

Consequently,

$$\frac{F_z^{(k)}(u_z)}{a_z^{k/2}} = O_k\left(\left(\frac{z}{\log z}\right)^{1-k/2}\right).$$

In particular,

$$\frac{F_z^{(3)}(u_z)}{a_z^{3/2}} = O\left(\sqrt{\frac{\log z}{z}}\right), \qquad \frac{F_z^{(4)}(u_z)}{a_z^{2}} = O\left(\frac{\log z}{z}\right).$$

Proof. For $k \geq 3$, the derivatives of $2z \log u$ contribute $O_k(z/u_z^k)$, while the derivatives of $-\pi e^{2u}$ contribute $O_k(e^{2u_z})$. The linear term $(9/2)u$ contributes only to the first derivative. Since

$$e^{2u_z} \asymp \frac{z}{u_z}, \qquad u_z \asymp \log z,$$

we have

$$F_z^{(k)}(u_z) = O_k\left(\frac{z}{u_z}\right) = O_k\left(\frac{z}{\log z}\right).$$

By Lemma 8.4,

$$a_z \asymp \frac{z}{\log z}.$$

Dividing by $a_z^{k/2}$ gives

$$\frac{F_z^{(k)}(u_z)}{a_z^{k/2}} = O_k\left(\left(\frac{z}{\log z}\right)^{1-k/2}\right).$$

The stated $k = 3$ and $k = 4$ estimates follow immediately. □

Remark 8.6. *The estimates in this section establish the saddle location and Gaussian scale required for the proposed Tau-Weak route. They do not control the differentiated Laplace remainder or the $n \geq 2$ tail. Those are the remaining tasks T3 and T4 in Section 7.*

9 Effective tail separation for the de Bruijn kernel

This section proves the tail-separation task T3 from Section 7, in a form sufficient for the later differentiable Laplace analysis. The goal is to show that the $n \geq 2$ terms in the de Bruijn kernel are smaller than the $n = 1$ saddle contribution by an arbitrary negative power of z.

Write

$$\Phi(u) = \sum_{n=1}^{\infty} \Phi_n(u),$$

where

$$\Phi_n(u) = \left(2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2}\right) e^{-\pi n^2 e^{2u}}.$$

For $t \geq 0$, define the phase

$$P_{z,t}(u) = 2z \log u + \frac{9}{2}u - \pi e^{2t} e^{2u}, \qquad u > 0.$$

Thus the first part of the n-th term corresponds to $t = \log n$. Let

$$m_z(t) = \sup_{u>0} P_{z,t}(u),$$

and write $u_{z,t}$ for the unique maximizer. For $t = 0$, $u_{z,0} = u_z$ is the saddle from Section 8.

Lemma 9.1 (Monotone saddle shift)**.** *For each fixed z and $t \geq 0$, the phase $P_{z,t}$ has a unique maximizer $u_{z,t}$. Moreover,*

$$0 < u_{z,t} \leq u_z.$$

Proof. The critical point equation is

$$\frac{2z}{u} + \frac{9}{2} = 2\pi e^{2t} e^{2u}.$$

Equivalently,

$$H_{z,t}(u) = 2\pi e^{2t} e^{2u} - \frac{9}{2} - \frac{2z}{u} = 0.$$

Since

$$\partial_u H_{z,t}(u) = 4\pi e^{2t} e^{2u} + \frac{2z}{u^2} > 0,$$

there is exactly one zero. At $u = u_z$, using the $t = 0$ saddle equation,

$$H_{z,t}(u_z) = 2\pi e^{2u_z}(e^{2t} - 1) \geq 0.$$

Since $H_{z,t}$ is strictly increasing and tends to $-\infty$ as $u \downarrow 0$, the zero $u_{z,t}$ satisfies $u_{z,t} \leq u_z$. $\square$

Lemma 9.2 (Saddle-value gap)**.** *For all sufficiently large z and all $t \geq 0$,*

$$m_z(t) \leq m_z(0) - \frac{2z}{u_z} t.$$

Consequently, for all sufficiently large z and all integers $n \geq 2$,

$$n^4 e^{m_z(\log n)} \leq e^{m_z(0)} n^{-z/u_z}.$$

Proof. By the envelope theorem,

$$m_z'(t) = \frac{\partial}{\partial t} P_{z,t}(u_{z,t}) = -2\pi e^{2t} e^{2u_{z,t}}.$$

Using the critical point equation at $u_{z,t}$,

$$2\pi e^{2t} e^{2u_{z,t}} = \frac{2z}{u_{z,t}} + \frac{9}{2}.$$

Therefore

$$m_z'(t) = -\frac{2z}{u_{z,t}} - \frac{9}{2}.$$

By Lemma 9.1, $u_{z,t} \le u_z$, so

$$m_z'(t) \le -\frac{2z}{u_z}.$$

Integrating from 0 to t gives

$$m_z(t) - m_z(0) \le -\frac{2z}{u_z} t.$$

Now put $t = \log n$. Since $u_z \asymp \log z$, for large z one has $z/u_z \ge 4$. Hence

$$n^4 e^{m_z(\log n)} \le e^{m_z(0)} n^{4-2z/u_z} \le e^{m_z(0)} n^{-z/u_z}.$$

□

Lemma 9.3 (Off-center gap). *Let $v = u_{z,t}$. For all sufficiently large z, uniformly in $t \ge 0$,*

$$P_{z,t}(v) - P_{z,t}(u) \ge cz$$

whenever $0 < u \le v/2$ or $u \ge 2v$, where $c > 0$ is an absolute constant.

Proof. Since $P_{z,t}$ increases on $(0, v)$ and decreases on (v, ∞), it is enough to check $u = v/2$ and $u = 2v$.

Put

$$A = \pi e^{2t}.$$

The saddle equation gives

$$A e^{2v} = \frac{z}{v} + \frac{9}{4}.$$

For the left side,

$$\begin{aligned} P_{z,t}(v) - P_{z,t}(v/2) &= 2z \log 2 + \frac{9}{4} v - A(e^{2v} - e^{v}) \\ &= 2z \log 2 + \frac{9}{4} v - \left(\frac{z}{v} + \frac{9}{4} \right) (1 - e^{-v}). \end{aligned}$$

Since $1 - e^{-v} \le v$, this gives

$$P_{z,t}(v) - P_{z,t}(v/2) \ge (2 \log 2 - 1) z.$$

For the right side,

$$\begin{aligned} P_{z,t}(v) - P_{z,t}(2v) &= -2z \log 2 - \frac{9}{2} v + A(e^{4v} - e^{2v}) \\ &= -2z \log 2 - \frac{9}{2} v + \left(\frac{z}{v} + \frac{9}{4} \right) (e^{2v} - 1). \end{aligned}$$

Since $e^{2v} - 1 \ge 2v$, it follows that

$$P_{z,t}(v) - P_{z,t}(2v) \ge (2 - 2\log 2) z.$$

Taking

$$c < \min\{2 \log 2 - 1,\ 2 - 2 \log 2\}$$

proves the claim. □

Lemma 9.4 (A crude concave-phase integral bound)**.** *For each fixed integer $J \geq 0$, there exist constants $C_J, B_J > 0$ such that, for all sufficiently large z and all $t \geq 0$,*

$$\int_0^\infty (1 + |\log u|)^J e^{P_{z,t}(u)}\, du \leq C_J z^{B_J} (1+t)^J e^{m_z(t)}.$$

Proof. Let $v = u_{z,t}$. We first note that

$$|\log v| \leq C(1 + t + \log z).$$

Indeed, $v \leq u_z \leq \log z$ gives the upper logarithmic bound. For the lower bound, the critical point equation and $v \leq \log z$ imply, for all large z,

$$\frac{2z}{v} \leq 2\pi e^{2t} e^{2v} \leq 2\pi e^{2t} z^2,$$

hence

$$v \geq c\, e^{-2t} z^{-1}.$$

Thus $|\log v| \leq C(1 + t + \log z)$.

Split the integral into

$$(0, v/2] \cup [v/2, 2v] \cup [2v, \infty).$$

On the middle interval,

$$(1 + |\log u|)^J \leq C_J (1 + t + \log z)^J \leq C_J z (1+t)^J$$

after increasing the polynomial power in z. The interval has length $O(v) \leq O(\log z)$, and $P_{z,t}(u) \leq m_z(t)$. Therefore the middle contribution is at most

$$C_J z^{B_J} (1+t)^J e^{m_z(t)}.$$

On the left interval, Lemma 9.3 gives

$$P_{z,t}(u) \leq m_z(t) - cz.$$

Moreover,

$$\int_0^{v/2} (1 + |\log u|)^J\, du \leq C_J (1 + t + \log z)^J (1 + v) \leq C_J z^{B_J} (1+t)^J.$$

Hence the left contribution is bounded by

$$C_J z^{B_J} (1+t)^J e^{m_z(t) - cz},$$

which is stronger than required.

For the right interval, decompose into dyadic intervals

$$I_\ell = [2^\ell v, 2^{\ell+1} v], \qquad \ell \geq 1.$$

Since $P_{z,t}$ is decreasing on $[v, \infty)$,

$$\int_{I_\ell} (1 + |\log u|)^J e^{P_{z,t}(u)}\, du \leq |I_\ell| \sup_{u \in I_\ell} (1 + |\log u|)^J e^{P_{z,t}(2^\ell v)}.$$

The same calculation as in Lemma 9.3, with 2 replaced by 2^ℓ, shows that the phase loss grows at least like a positive multiple of

$$z\, 2^\ell$$

for large ℓ, and is already $\gg z$ at $\ell = 1$. Therefore the dyadic sum is dominated by its first terms and is bounded by

$$C_J z^{B_J} (1+t)^J e^{m_z(t)}.$$

Combining the three regions proves the lemma. □

Proposition 9.5 (Tail separation, non-differentiated form). *For every $A > 0$, there exists $C_A > 0$ such that, for all sufficiently large z,*

$$\sum_{n\geq 2} \int_0^\infty |\Phi_n(u)| u^{2z} \, du \leq C_A z^{-A} e^{F_z(u_z)}.$$

More precisely, the right side may be replaced by

$$C_A z^{-A} e^{F_z(u_z)} a_z^{-1/2}$$

after inserting the standard local lower scale for the $n = 1$ saddle contribution.

Proof. For $n \geq 1$ and $u \geq 0$,

$$n^2 e^{5u/2} \leq n^4 e^{9u/2},$$

because $n^2 e^{2u} \geq 1$. Hence

$$|\Phi_n(u)| \leq C n^4 e^{9u/2} e^{-\pi n^2 e^{2u}}.$$

Therefore

$$\int_0^\infty |\Phi_n(u)| u^{2z} \, du \leq C n^4 \int_0^\infty e^{P_{z,\log n}(u)} \, du.$$

By Lemma 9.4 with $J = 0$,

$$\int_0^\infty e^{P_{z,\log n}(u)} \, du \leq C z^B e^{m_z(\log n)}.$$

Using Lemma 9.2,

$$n^4 e^{m_z(\log n)} \leq e^{m_z(0)} n^{-z/u_z}.$$

Thus

$$\sum_{n\geq 2} \int_0^\infty |\Phi_n(u)| u^{2z} \, du \leq C z^B e^{m_z(0)} \sum_{n\geq 2} n^{-z/u_z}.$$

Since $u_z \asymp \log z$, the exponent $z/u_z \to \infty$. Therefore, for every $A > 0$,

$$z^B \sum_{n\geq 2} n^{-z/u_z} = O_A(z^{-A}).$$

Since $m_z(0) = F_z(u_z)$, the claimed estimate follows. □

Proposition 9.6 (Tail separation with z-derivatives). *For every fixed integer $J \geq 0$ and every $A > 0$,*

$$\sum_{n\geq 2} \int_0^\infty |\Phi_n(u)| (1 + |\log u|)^J u^{2z} \, du \leq C_{A,J} z^{-A} e^{F_z(u_z)}.$$

Consequently, the same tail separation holds after differentiating the moment integral with respect to z any fixed number of times. In particular, for $0 \leq j \leq 4$,

$$\sum_{n\geq 2} \left| \frac{d^j}{dz^j} \int_0^\infty \Phi_n(u) u^{2z} \, du \right| \leq C_{A,j} z^{-A} e^{F_z(u_z)}.$$

Proof. The weighted estimate is proved exactly as Proposition 9.5, using Lemma 9.4 with the chosen J. The additional factor

$$(1 + \log n)^J$$

arising from $t = \log n$ is absorbed by the super-polynomial decay of

$$\sum_{n\geq 2} n^{-z/u_z}.$$

For differentiation, note that

$$\frac{d^j}{dz^j} u^{2z} = (2\log u)^j u^{2z}.$$

The interchange of differentiation and integration is justified by dominated convergence on compact z-ranges, using the super-exponential decay

$$e^{-\pi n^2 e^{2u}}$$

as $u \to \infty$ and the integrability of $u^{2z}|\log u|^j$ near $u = 0$ for $z > 0$. Thus the derivative estimate follows from the weighted estimate with $J = j$. $\square$

Remark 9.7. *The estimates above close T3 at the level required for the Tau-Weak program: the $n \geq 2$ part of the de Bruijn kernel is negligible, even after finitely many z-derivatives, compared with the $n = 1$ saddle scale. The remaining analytic bottleneck is T4: proving a differentiable Laplace expansion for the $n = 1$ contribution itself through four z-derivatives.*

10 Differentiable Laplace remainder framework

This section begins the analytic task T4 from Section 7. The purpose is to isolate the $n = 1$ contribution in an exact normalized form and to reduce the differentiable Laplace remainder estimate to a single domination lemma. Unlike Sections 8 and 9, the full T4 estimate is not yet claimed as proved here.

10.1 The $n = 1$ contribution

For the $n = 1$ term of the de Bruijn kernel,

$$\Phi_1(u) = \left(2\pi^2 e^{9u/2} - 3\pi e^{5u/2}\right) e^{-\pi e^{2u}},$$

write

$$\Phi_1(u)u^{2z} = 2\pi^2 A(u)e^{F_z(u)},$$

where

$$F_z(u) = 2z\log u + \frac{9}{2}u - \pi e^{2u},$$

and

$$A(u) = 1 - \frac{3}{2\pi}e^{-2u}.$$

Thus the main $n = 1$ moment is

$$M_1(z) = 2\pi^2 \int_0^\infty A(u)e^{F_z(u)}\, du.$$

Let u_z be the saddle of F_z, and let

$$a_z = -F_z''(u_z).$$

Make the exact change of variables

$$u = u_z + \frac{y}{\sqrt{a_z}}, \qquad du = \frac{dy}{\sqrt{a_z}}.$$

The range $u > 0$ becomes

$$y > -\sqrt{a_z}u_z.$$

Hence

$$M_1(z) = 2\pi^2 e^{F_z(u_z)} a_z^{-1/2} \int_{-\sqrt{a_z}u_z}^{\infty} A\left(u_z + \frac{y}{\sqrt{a_z}}\right) \exp\left(F_z\left(u_z + \frac{y}{\sqrt{a_z}}\right) - F_z(u_z)\right) dy.$$

Define

$$R_z(y) = F_z\left(u_z + \frac{y}{\sqrt{a_z}}\right) - F_z(u_z) + \frac{1}{2}y^2,$$

and

$$B_z(y) = \mathbf{1}_{\{y > -\sqrt{a_z}u_z\}} A\left(u_z + \frac{y}{\sqrt{a_z}}\right) e^{R_z(y)}.$$

Then

$$M_1(z) = 2\pi^2 e^{F_z(u_z)} a_z^{-1/2} \int_{\mathbb{R}} e^{-y^2/2} B_z(y)\, dy.$$

The Gaussian main term is therefore

$$S(z) = F_z(u_z) + \log(2\pi^2) + \frac{1}{2}\log\frac{2\pi}{a_z}.$$

Write

$$\eta(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-y^2/2}(B_z(y) - 1)\, dy.$$

Then the exact identity is

$$M_1(z) = e^{S(z)}(1 + \eta(z)).$$

Remark 10.1. *The purpose of T4 is to prove*

$$\eta^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 0 \le j \le 4.$$

Together with the tail separation of Section 9, this would imply

$$s^{(j)}(z) - S^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 0 \le j \le 4.$$

10.2 Local expansion of $R_z(y)$

By Taylor expansion at the saddle,

$$F_z\left(u_z + \frac{y}{\sqrt{a_z}}\right) - F_z(u_z) = -\frac{1}{2}y^2 + \sum_{k=3}^{K} \frac{F_z^{(k)}(u_z)}{k!\,a_z^{k/2}} y^k + \mathcal{R}_{K,z}(y).$$

Therefore

$$R_z(y) = \sum_{k=3}^{K} \frac{F_z^{(k)}(u_z)}{k!\,a_z^{k/2}} y^k + \mathcal{R}_{K,z}(y).$$

By Lemma 8.5,

$$\frac{F_z^{(3)}(u_z)}{a_z^{3/2}} = O\left(\sqrt{\frac{\log z}{z}}\right), \qquad \frac{F_z^{(4)}(u_z)}{a_z^{2}} = O\left(\frac{\log z}{z}\right).$$

Thus the first local odd term is of size

$$O\left(\sqrt{\frac{\log z}{z}}\right) y^3.$$

Its Gaussian integral vanishes by parity at first order, so the first nonzero contribution to $\eta(z)$ is expected at size

$$O\left(\frac{\log z}{z}\right).$$

10.3 The domination lemma needed for T4

The remaining difficulty is not the formal expansion but the differentiated domination. The following lemma is the precise target needed to close T4.

Conditional Lemma 10.2 (T4 domination lemma)**.** *For each fixed $j \in \{0, 1, 2, 3, 4\}$, there exist constants $C_j, K_j > 0$ such that, for all sufficiently large z,*

$$\left|\partial_z^j B_z(y)\right| \le C_j z^{-j}(1 + |y|)^{K_j} e^{y^2/8}$$

for all $y \in \mathbb{R}$. Moreover,

$$\left|\partial_z^j (B_z(y) - 1)\right| \le C_j \frac{\log z}{z^{1+j}}(1 + |y|)^{K_j} e^{y^2/8}$$

on the central region $|y| \le a_z^{1/7}$, after subtracting the Gaussian main term.

Remark 10.3. *The second displayed estimate is schematic and must be made precise: the odd y^3 term is not pointwise of order $\log z/z$; rather, it is of order $(\log z/z)^{1/2} y^3$ and cancels after Gaussian integration. A rigorous proof should either keep the odd terms explicitly and integrate them, or define a corrected B_z^{even} after subtracting odd Taylor contributions.*

Conditional Proposition 10.4 (Differentiable Laplace remainder target)**.** *Assume the T4 domination lemma and the corresponding Taylor expansion with integrable differentiated remainders. Then*

$$\eta^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 0 \le j \le 4.$$

Consequently, after combining with the T3 tail estimate,

$$s^{(j)}(z) - S^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 0 \le j \le 4.$$

Proof. The identity

$$M_1(z) = e^{S(z)}(1 + \eta(z))$$

is exact. Under the domination lemma, differentiation may be moved under the Gaussian integral:

$$\eta^{(j)}(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-y^2/2} \partial_z^j (B_z(y) - 1)\, dy.$$

The integrable majorant $e^{-y^2/2} e^{y^2/8} = e^{-3y^2/8}$ controls the differentiated integrals. The Taylor expansion of R_z, together with the parity cancellation of the first odd term and the bounds from Section 8, gives

$$\eta^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right).$$

Finally,

$$\log M_1(z) = S(z) + \log(1 + \eta(z)),$$

and the same bounds follow for the derivatives of $\log M_1 - S$. Section 9 shows that the $n \ge 2$ tail is $O_A(z^{-A})$, with the same stability under finitely many z-derivatives. Hence the same estimate holds for $s(z) = \log M(z)$. $\square$

10.4 Immediate next proof target

The next concrete step is to replace Lemma 10.2 by a proved statement. The main work is to control the derivatives of

$$u_z, \qquad a_z, \qquad A\left(u_z + \frac{y}{\sqrt{a_z}}\right), \qquad R_z(y),$$

uniformly in the central Gaussian region and then to show that the complement contributes $O_A(z^{-A})$.

11 Central-region expansion for the $n = 1$ Laplace term

This section proves the first concrete part of T4: the central Gaussian contribution to the $n = 1$ Laplace remainder is of the expected size for $j = 0$. The differentiated estimates are not yet completed here, but the calculation identifies the parity cancellation and the correct scale of the first nonzero correction.

Set

$$Y_z = a_z^{1/7}.$$

The exponent $1/7$ is chosen only for convenience. Any fixed $\delta < 1/6$ would work. The strict inequality $\delta < 1/6$ ensures that the cubic term in the normalized Taylor expansion is $o(1)$ uniformly on $|y| \le a_z^{\delta}$.

Throughout this section,

$$h = \frac{y}{\sqrt{a_z}}, \qquad u = u_z + h.$$

Recall

$$R_z(y) = F_z(u_z + h) - F_z(u_z) + \frac{1}{2}y^2,$$

and

$$B_z(y) = A(u_z + h)e^{R_z(y)}$$

on the central region, where the indicator condition $u_z + h > 0$ is automatically satisfied for large z.

Lemma 11.1 (Central domain and amplitude)**.** *For $|y| \le Y_z$ and sufficiently large z, one has $u_z + h > 0$, and*

$$A(u_z + h) = 1 + O\left(\frac{\log z}{z}\right),$$

uniformly in the central region.

Proof. Since

$$|h| \le a_z^{1/7-1/2} = a_z^{-5/14} \to 0$$

and $u_z \asymp \log z$, we have $u_z + h > 0$ for large z. Also

$$A(u) = 1 - \frac{3}{2\pi}e^{-2u}.$$

By the saddle equation,

$$e^{2u_z} \asymp \frac{z}{u_z},$$

so

$$e^{-2u_z} = O\left(\frac{\log z}{z}\right).$$

Since $h = o(1)$,

$$e^{-2(u_z+h)} = e^{-2u_z}e^{-2h} = O\left(\frac{\log z}{z}\right),$$

uniformly for $|y| \le Y_z$. The claim follows. □

Lemma 11.2 (Central Taylor expansion)**.** *For $|y| \le Y_z$,*

$$R_z(y) = \alpha_3(z)y^3 + \alpha_4(z)y^4 + O\left(a_z^{-3/2}(1 + |y|)^5\right),$$

where

$$\alpha_3(z) = \frac{F_z^{(3)}(u_z)}{6a_z^{3/2}}, \qquad \alpha_4(z) = \frac{F_z^{(4)}(u_z)}{24a_z^2},$$

and

$$\alpha_3(z) = O(a_z^{-1/2}), \qquad \alpha_4(z) = O(a_z^{-1}).$$

In particular,

$$R_z(y) = O(a_z^{-1/14})$$

uniformly for $|y| \le Y_z$.

Proof. Taylor's theorem gives

$$F_z(u_z + h) - F_z(u_z) = \frac{1}{2} F_z''(u_z) h^2 + \frac{F_z^{(3)}(u_z)}{6} h^3 + \frac{F_z^{(4)}(u_z)}{24} h^4 + O\left(\sup_\xi |F_z^{(5)}(\xi)| \, |h|^5 \right),$$

where ξ lies between u_z and $u_z + h$. Since $F_z'(u_z) = 0$ and $F_z''(u_z) = -a_z$, the quadratic term is $-y^2/2$, which is exactly cancelled in the definition of $R_z(y)$.

For $|h| = o(1)$, the same derivative scale used in Lemma 8.5 gives

$$F_z^{(k)}(\xi) = O_k(a_z), \qquad k \ge 3,$$

because $a_z \asymp z/\log z$. Thus the fifth-order remainder is

$$O\left(a_z |h|^5\right) = O\left(a_z^{-3/2} |y|^5\right).$$

The coefficient estimates

$$\alpha_3 = O(a_z^{-1/2}), \qquad \alpha_4 = O(a_z^{-1})$$

follow from $F_z^{(k)}(u_z) = O_k(a_z)$ for $k \ge 3$. Finally, for $|y| \le a_z^{1/7}$,

$$|\alpha_3 y^3| \ll a_z^{-1/2} a_z^{3/7} = a_z^{-1/14},$$

and the remaining terms are smaller. This proves the claim. □

Lemma 11.3 (Central exponential expansion)**.** *For* $|y| \le Y_z$,

$$e^{R_z(y)} = 1 + \alpha_3 y^3 + \alpha_4 y^4 + \frac{1}{2} \alpha_3^2 y^6 + O\left(a_z^{-3/2}(1 + |y|)^9\right).$$

Proof. By Lemma 11.2, $R_z(y) = o(1)$ uniformly on $|y| \le Y_z$. Therefore

$$e^{R_z} = 1 + R_z + \frac{1}{2} R_z^2 + O(R_z^3).$$

Write

$$R_z = \alpha_3 y^3 + \alpha_4 y^4 + \rho_z(y),$$

where

$$\rho_z(y) = O(a_z^{-3/2}(1 + |y|)^5).$$

Since $\alpha_3 = O(a_z^{-1/2})$ and $\alpha_4 = O(a_z^{-1})$, the only quadratic term of size a_z^{-1} is

$$\frac{1}{2} \alpha_3^2 y^6.$$

All other terms are absorbed into

$$O\left(a_z^{-3/2}(1 + |y|)^9\right).$$

This proves the expansion. □

Proposition 11.4 (Central $j = 0$ Laplace correction)**.** *The central contribution to the normalized $n = 1$ Laplace remainder satisfies*

$$\int_{|y|\le Y_z} e^{-y^2/2}(B_z(y) - 1)\, dy = O\left(\frac{\log z}{z}\right).$$

Equivalently,

$$\frac{1}{\sqrt{2\pi}} \int_{|y|\le Y_z} e^{-y^2/2}(B_z(y) - 1)\, dy = O\left(\frac{\log z}{z}\right).$$

Proof. On the central region,

$$B_z(y) = A(u_z + h)e^{R_z(y)}.$$

By Lemma 11.1,

$$A(u_z + h) = 1 + O\left(\frac{\log z}{z}\right).$$

Thus the amplitude contributes

$$O\left(\frac{\log z}{z}\right)$$

after integration against the Gaussian weight.

It remains to estimate the exponential part. By Lemma 11.3,

$$e^{R_z(y)} - 1 = \alpha_3 y^3 + \alpha_4 y^4 + \frac{1}{2}\alpha_3^2 y^6 + O\left(a_z^{-3/2}(1 + |y|)^9\right).$$

The cubic term integrates to zero by parity:

$$\int_{|y|\le Y_z} e^{-y^2/2} y^3\, dy = 0.$$

The remaining displayed terms contribute

$$O(a_z^{-1}) = O\left(\frac{\log z}{z}\right),$$

because

$$\alpha_4 = O(a_z^{-1}), \qquad \alpha_3^2 = O(a_z^{-1}).$$

The remainder contributes $O(a_z^{-3/2})$, since

$$\int_{\mathbb{R}} e^{-y^2/2}(1 + |y|)^9\, dy < \infty.$$

Combining the amplitude, even Taylor terms, and remainder gives

$$\int_{|y|\le Y_z} e^{-y^2/2}(B_z(y) - 1)\, dy = O(a_z^{-1}) + O\left(\frac{\log z}{z}\right) = O\left(\frac{\log z}{z}\right).$$

□

Remark 11.5. *Proposition 11.4 proves only the central, non-differentiated part of T4. The remaining T4 tasks are:*

(i) control the complement $|y| > Y_z$,

(ii) prove the same expansion after z-differentiation up to order 4,

and

(iii) combine these estimates with the T3 tail separation.

12 Complement estimate for the $n = 1$ Laplace term

This section completes the non-differentiated $j = 0$ Laplace estimate by controlling the complement of the central region used in Section 11. Recall

$$Y_z = a_z^{1/7}.$$

We prove that the contribution of $|y| > Y_z$ is smaller than any fixed negative power of z, and hence is negligible compared with the central $O(\log z/z)$ correction.

Throughout this section,

$$u = u_z + \frac{y}{\sqrt{a_z}}, \qquad h = \frac{y}{\sqrt{a_z}}.$$

On the $n = 1$ branch,

$$e^{-y^2/2} B_z(y) = \mathbf{1}_{\{u>0\}}\, A(u) \exp(F_z(u) - F_z(u_z)).$$

Since $u > 0$,

$$|A(u)| = \left| 1 - \frac{3}{2\pi} e^{-2u} \right| \le C.$$

Lemma 12.1 (Quadratic loss near the saddle)**.** *There exists $c > 0$ such that, for all sufficiently large z, if*

$$\frac{1}{2} u_z \le u_z + h \le 2u_z,$$

then

$$F_z(u_z + h) - F_z(u_z) \le -c \frac{y^2}{\log z}, \qquad y = h\sqrt{a_z}.$$

Proof. For $w \in [u_z/2, 2u_z]$,

$$-F_z''(w) = \frac{2z}{w^2} + 4\pi e^{2w} \ge \frac{2z}{w^2} \ge c \frac{z}{(\log z)^2},$$

because $u_z \asymp \log z$. Since $F_z'(u_z) = 0$, Taylor's theorem with integral remainder gives

$$F_z(u_z + h) - F_z(u_z) = \int_0^h (h - s) F_z''(u_z + s)\, ds \le -c \frac{z}{(\log z)^2} h^2.$$

Using

$$h = \frac{y}{\sqrt{a_z}}, \qquad a_z \asymp \frac{z}{\log z},$$

we obtain

$$\frac{z}{(\log z)^2} h^2 = \frac{z}{(\log z)^2} \frac{y^2}{a_z} \asymp \frac{y^2}{\log z}.$$

This proves the claim. □

Lemma 12.2 (Far-region loss)**.** *There exists $c > 0$ such that, for all sufficiently large z,*

$$F_z(u) - F_z(u_z) \le -cz$$

whenever $0 < u \le u_z/2$ or $u \ge 2u_z$.

Proof. This is Lemma 9.3 with $t = 0$, applied to the phase $P_{z,0} = F_z$ and its maximizer u_z. □

Proposition 12.3 (Complement bound, $j = 0$)**.** *For every $A > 0$,*

$$\int_{\substack{|y|>Y_z \\ u_z+y/\sqrt{a_z}>0}} e^{-y^2/2}|B_z(y)|\,dy = O_A(z^{-A}).$$

Consequently,

$$\int_{\substack{|y|>Y_z \\ u_z+y/\sqrt{a_z}>0}} e^{-y^2/2}(B_z(y) - 1_{\{|y|\le Y_z\}})\,dy = O_A(z^{-A})$$

in the sense needed for the normalized $n = 1$ Laplace remainder.

Proof. Since $|A(u)| \le C$,

$$e^{-y^2/2}|B_z(y)| \le C\exp(F_z(u) - F_z(u_z)).$$

Split the region $u > 0$ into

$$\frac{1}{2}u_z \le u \le 2u_z$$

and its complement.

On the local part, Lemma 12.1 gives

$$e^{-y^2/2}|B_z(y)| \le C\exp\left(-c\frac{y^2}{\log z}\right).$$

Therefore

$$\int_{\substack{|y|>Y_z \\ u_z/2\le u\le 2u_z}} e^{-y^2/2}|B_z(y)|\,dy \le C\int_{|y|>Y_z} \exp\left(-c\frac{y^2}{\log z}\right)\,dy.$$

Since

$$Y_z^2 = a_z^{2/7} \asymp \left(\frac{z}{\log z}\right)^{2/7},$$

we have

$$\frac{Y_z^2}{\log z} \asymp \frac{z^{2/7}}{(\log z)^{9/7}} \to \infty$$

faster than any multiple of $\log z$. Hence this local complement is $O_A(z^{-A})$ for every $A > 0$.

On the far part, Lemma 12.2 gives

$$e^{-y^2/2}|B_z(y)| \le Ce^{-cz}.$$

The left far range has finite length $O(\sqrt{a_z}u_z)$, and the right far range has super-exponential decay inherited from the term $-\pi e^{2u}$ in $F_z(u)$. Thus the far contribution is also $O_A(z^{-A})$.

Combining the local and far estimates proves the first assertion. The second assertion is simply the corresponding complement contribution in the decomposition of the normalized Gaussian integral. □

[Full $j = 0$ $n = 1$ Laplace correction] For the $n = 1$ contribution,

$$\eta(z) = O\left(\frac{\log z}{z}\right)$$

at the non-differentiated level $j = 0$, up to the remaining issue that the z-differentiated estimates have not yet been proved.

Proof. The central contribution is

$$O\left(\frac{\log z}{z}\right)$$

by Proposition 11.4. The complement is $O_A(z^{-A})$ by Proposition 12.3. Taking, for instance, $A = 2$, gives

$$\eta(z) = O\left(\frac{\log z}{z}\right).$$

□

Remark 12.4. *Sections 11 and 12 prove the $j = 0$ version of the $n = 1$ Laplace remainder. The remaining T4 work is the differentiated version:*

$$\eta^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 1 \le j \le 4.$$

13 First differentiated Laplace correction

This section proves the first differentiated version of the $n = 1$ Laplace correction:

$$\eta'(z) = O\left(\frac{\log z}{z^2}\right).$$

The proof follows the same central/complement decomposition as Sections 11 and 12. The new point is that the relevant Taylor coefficients gain one additional factor z^{-1} after z-differentiation.

13.1 Derivative scale of the saddle coefficients

Recall

$$a_z = -F_z''(u_z) = \frac{4z}{u_z} + \frac{2z}{u_z^2} + 9.$$

Lemma 13.1 (Derivative scale of a_z)**.** *One has*

$$a_z' = O\left(\frac{a_z}{z}\right).$$

Proof. Differentiate

$$a_z = \frac{4z}{u_z} + \frac{2z}{u_z^2} + 9.$$

Using

$$u_z \asymp \log z, \qquad u_z' = O(z^{-1}),$$

we obtain

$$a_z' = \frac{4}{u_z} - \frac{4zu_z'}{u_z^2} + \frac{2}{u_z^2} - \frac{4zu_z'}{u_z^3} = O\left(\frac{1}{\log z}\right).$$

Since

$$a_z \asymp \frac{z}{\log z},$$

this is

$$a_z' = O(a_z/z).$$

□

Lemma 13.2 (Derivative scale of normalized Taylor coefficients)**.** *For*

$$\alpha_k(z) = \frac{F_z^{(k)}(u_z)}{k! a_z^{k/2}}, \qquad k \geq 3,$$

one has

$$\alpha_k'(z) = O_k\left(\frac{a_z^{1-k/2}}{z}\right).$$

In particular,

$$\alpha_3'(z) = O\left(\frac{a_z^{-1/2}}{z}\right), \qquad \alpha_4'(z) = O\left(\frac{a_z^{-1}}{z}\right).$$

Proof. For $k \geq 3$,

$$F_z^{(k)}(u_z) = O_k(a_z).$$

Differentiating with respect to z,

$$\frac{d}{dz} F_z^{(k)}(u_z) = \partial_z F_z^{(k)}(u_z) + F_z^{(k+1)}(u_z) u_z'.$$

The first term comes only from $2z \log u$, hence is $O_k(u_z^{-k})$. The second term is

$$O_k(a_z) O(z^{-1}) = O_k(a_z/z).$$

Since $u_z^{-k} = O(1/\log z) = O(a_z/z)$, we get

$$\frac{d}{dz} F_z^{(k)}(u_z) = O_k(a_z/z).$$

Now differentiate

$$\alpha_k(z) = \frac{F_z^{(k)}(u_z)}{k! a_z^{k/2}}.$$

Using Lemma 13.1,

$$\alpha_k' = O_k\left(\frac{a_z/z}{a_z^{k/2}}\right) + O_k\left(a_z\, a_z^{-k/2} \frac{a_z'}{a_z}\right) = O_k\left(\frac{a_z^{1-k/2}}{z}\right).$$

□

Lemma 13.3 (Derivative scale of the amplitude)**.** *For* $|y| \leq Y_z = a_z^{1/7}$*, with*

$$h = \frac{y}{\sqrt{a_z}},$$

one has

$$\partial_z A(u_z + h) = O\left(\frac{\log z}{z^2}\right)$$

uniformly in the central region.

Proof. Since

$$A(u) = 1 - \frac{3}{2\pi} e^{-2u},$$

we have

$$A'(u) = O(e^{-2u}).$$

On the central region,

$$e^{-2(u_z+h)} = O\left(\frac{\log z}{z}\right).$$

Moreover,

$$\partial_z(u_z+h) = u_z' - \frac{1}{2} y a_z^{-3/2} a_z' = O(z^{-1}) + O\left(a_z^{1/7} a_z^{-3/2} \frac{a_z}{z}\right) = O(z^{-1}).$$

Therefore

$$\partial_z A(u_z+h) = A'(u_z+h)\partial_z(u_z+h) = O\left(\frac{\log z}{z}\right) O(z^{-1}) = O\left(\frac{\log z}{z^2}\right).$$

□

13.2 Central $j=1$ estimate

Lemma 13.4 (Differentiated central expansion). *For $|y| \le Y_z$,*

$$\partial_z\left(e^{R_z(y)} - 1\right) = \alpha_3'(z) y^3 + \alpha_4'(z) y^4 + \alpha_3(z)\alpha_3'(z) y^6 + O\left(\frac{a_z^{-3/2}}{z}(1+|y|)^9\right).$$

Proof. From Lemma 11.3,

$$e^{R_z(y)} - 1 = \alpha_3 y^3 + \alpha_4 y^4 + \frac{1}{2}\alpha_3^2 y^6 + O\left(a_z^{-3/2}(1+|y|)^9\right).$$

Differentiating the displayed coefficients gives

$$\alpha_3' y^3 + \alpha_4' y^4 + \alpha_3 \alpha_3' y^6.$$

The derivative of the remainder has the stated size because the coefficients in the Taylor remainder are built from normalized derivatives of F_z, and each z-derivative gains one factor z^{-1}, as in Lemma 13.2. This proves the claimed expansion. □

Proposition 13.5 (Central $j=1$ Laplace correction). *The central contribution satisfies*

$$\frac{d}{dz}\int_{|y|\le Y_z} e^{-y^2/2}(B_z(y)-1)\,dy = O\left(\frac{\log z}{z^2}\right).$$

Proof. Differentiate the moving-boundary integral:

$$\frac{d}{dz}\int_{-Y_z}^{Y_z} f_z(y)\,dy = \int_{-Y_z}^{Y_z} \partial_z f_z(y)\,dy + Y_z'\big(f_z(Y_z) + f_z(-Y_z)\big),$$

where

$$f_z(y) = e^{-y^2/2}(B_z(y)-1).$$

The boundary term is $O_A(z^{-A})$ for every $A>0$, because

$$e^{-Y_z^2/2} = \exp(-a_z^{2/7}/2)$$

is super-polynomially small and Y_z' grows at most polynomially.

It remains to estimate the integral of $\partial_z f_z$. On the central region,

$$B_z(y) = A(u_z+h)e^{R_z(y)}.$$

By Lemma 13.3, the differentiated amplitude contributes

$$O\left(\frac{\log z}{z^2}\right)$$

after Gaussian integration.

For the exponential part, Lemma 13.4 gives

$$\partial_z(e^{R_z}-1)=\alpha_3'y^3+\alpha_4'y^4+\alpha_3\alpha_3'y^6+O\left(\frac{a_z^{-3/2}}{z}(1+|y|)^9\right).$$

The cubic term integrates to zero by parity:

$$\int_{-Y_z}^{Y_z} e^{-y^2/2}y^3\,dy=0.$$

The remaining terms contribute

$$O(\alpha_4')+O(\alpha_3\alpha_3')+O\left(\frac{a_z^{-3/2}}{z}\right).$$

By Lemma 13.2,

$$\alpha_4'=O(a_z^{-1}/z),\qquad \alpha_3\alpha_3'=O(a_z^{-1}/z).$$

Since

$$a_z^{-1}/z=O\left(\frac{\log z}{z^2}\right),$$

the central contribution is

$$O\left(\frac{\log z}{z^2}\right).$$

□

13.3 Complement for $j=1$

Proposition 13.6 (Complement bound, $j=1$)**.** *For every $A>0$,*

$$\frac{d}{dz}\int_{\substack{|y|>Y_z\\ u_z+y/\sqrt{a_z}>0}} e^{-y^2/2}B_z(y)\,dy=O_A(z^{-A}).$$

Proof. The complement proof is the differentiated version of Proposition 12.3. Differentiating the integrand produces only polynomial factors in z and y, coming from derivatives of

$$u_z,\qquad a_z,\qquad A(u_z+y/\sqrt{a_z}),\qquad F_z(u_z+y/\sqrt{a_z})-F_z(u_z).$$

The local complement still has the exponential loss

$$\exp\left(-c\frac{y^2}{\log z}\right),\qquad |y|>Y_z,$$

and the far region still has loss e^{-cz}. These losses dominate all polynomial factors. Boundary terms from differentiating Y_z are also $O_A(z^{-A})$, since

$$e^{-Y_z^2/2}=\exp(-a_z^{2/7}/2)$$

is super-polynomially small. Thus the differentiated complement is $O_A(z^{-A})$. □

[First differentiated $n=1$ Laplace correction] For the $n=1$ contribution,

$$\eta'(z) = O\left(\frac{\log z}{z^2}\right).$$

Proof. The central contribution is handled by Proposition 13.5, and the complement by Proposition 13.6. $\square$

Remark 13.7. *This proves the first differentiated estimate in T4. The remaining differentiated estimates require the same mechanism for $j=2,3,4$: each z-derivative of the normalized Taylor coefficients should gain another factor of z^{-1}, while the odd terms continue to cancel after Gaussian integration.*

14 Uniform differentiated central estimates up to order four

This section records the uniform mechanism behind the $j=1$ estimate of Section 13. The key point is to avoid differentiating an unspecified Taylor remainder. Instead, one differentiates the exact analytic expression for $R_z(y)$ at fixed y.

Recall

$$R_z(y) = F_z\left(u_z + \frac{y}{\sqrt{a_z}}\right) - F_z(u_z) + \frac{1}{2}y^2.$$

Set

$$w_z(y) = u_z + \frac{y}{\sqrt{a_z}}.$$

Since

$$\partial_z F_z(u) = 2\log u, \qquad F_z'(u_z) = 0,$$

differentiating at fixed y gives the exact identity

$$\boxed{\partial_z R_z(y) = 2\log\left(\frac{w_z(y)}{u_z}\right) + F_z'(w_z(y))\left(u_z' - \frac{y a_z'}{2a_z^{3/2}}\right).}$$

Equivalently,

$$\partial_z R_z(y) = 2\log\left(1 + \frac{y}{u_z\sqrt{a_z}}\right) + F_z'(w_z(y))\left(u_z' - \frac{y a_z'}{2a_z^{3/2}}\right).$$

This identity is the basis for the differentiated estimates below.

Lemma 14.1 (Higher derivative scales of u_z and a_z)**.** *For $1 \le j \le 4$,*

$$u_z^{(j)} = O_j(z^{-j}), \qquad a_z^{(j)} = O_j(a_z z^{-j}).$$

Consequently,

$$\partial_z^j a_z^{-1/2} = O_j(a_z^{-1/2} z^{-j}).$$

Proof. The bounds for $u_z^{(j)}$ were established in Lemma 8.3. We prove the estimate for a_z. Write

$$a_z = zG(u_z) + 9, \qquad G(u) = 4u^{-1} + 2u^{-2}.$$

For every fixed $m \ge 0$,

$$G^{(m)}(u_z) = O_m((\log z)^{-1-m}).$$

By Faà di Bruno's formula,

$$\frac{d^r}{dz^r} G(u_z)$$

is a finite sum of products

$$G^{(m)}(u_z)\prod_{\nu=1}^{m} u_z^{(\ell_\nu)}, \qquad \ell_1+\cdots+\ell_m=r.$$

Using $u_z^{(\ell)}=O_\ell(z^{-\ell})$, each such product is

$$O_r((\log z)^{-1}z^{-r}).$$

Now differentiate $zG(u_z)$. For $r\geq 1$,

$$\frac{d^r}{dz^r}\big(zG(u_z)\big)=z\frac{d^r}{dz^r}G(u_z)+r\frac{d^{r-1}}{dz^{r-1}}G(u_z),$$

since z has no derivatives of order 2 or higher. Hence

$$a_z^{(r)}=O_r\left(\frac{1}{\log z}z^{1-r}\right)=O_r(a_z z^{-r}),$$

because $a_z \asymp z/\log z$. Finally, the estimate

$$\partial_z^j a_z^{-1/2}=O_j(a_z^{-1/2}z^{-j})$$

follows from Faà di Bruno applied to $x\mapsto x^{-1/2}$, using the already established bounds for $a_z^{(r)}$. □

Lemma 14.2 (Uniform differentiated Taylor coefficient bounds). *For fixed $k\geq 3$ and $0\leq j\leq 4$, define*

$$\alpha_k(z)=\frac{F_z^{(k)}(u_z)}{k!\,a_z^{k/2}}.$$

Then

$$\alpha_k^{(j)}(z)=O_{k,j}\left(a_z^{1-k/2}z^{-j}\right).$$

Proof. The case $j=0$ is Lemma 8.5. For $j\geq 1$, put

$$Q_k(z)=F_z^{(k)}(u_z).$$

Faà di Bruno's formula for $F_z^{(k)}(u_z)$ gives a finite sum of terms of the form

$$\partial_z^\mu F_z^{(k+m)}(u_z)\prod_{\nu=1}^{m} u_z^{(\ell_\nu)}, \qquad \mu+\ell_1+\cdots+\ell_m=j.$$

Because $F_z(u)=2z\log u+\frac{9}{2}u-\pi e^{2u}$ is affine in z, one has $\mu\in\{0,1\}$. If $\mu=0$, then

$$F_z^{(k+m)}(u_z)=O_{k,m}(a_z),$$

while the product of $u_z^{(\ell_\nu)}$'s contributes $O(z^{-j})$. If $\mu=1$, then

$$\partial_z F_z^{(k+m)}(u_z)=O_{k,m}(u_z^{-k-m})=O_{k,m}(a_z/z),$$

and the remaining product contributes $O(z^{-(j-1)})$. In both cases,

$$Q_k^{(j)}(z)=O_{k,j}(a_z z^{-j}).$$

Now apply Leibniz' rule to

$$\alpha_k(z)=\frac{1}{k!}Q_k(z)a_z^{-k/2}.$$

By Lemma 14.1,

$$\partial_z^r a_z^{-k/2} = O_{k,r}(a_z^{-k/2} z^{-r}),$$

and hence

$$\alpha_k^{(j)}(z) = O_{k,j}\left(a_z^{1-k/2} z^{-j}\right).$$

$\square$

Lemma 14.3 (Parity and coefficient bookkeeping)**.** *Fix $0 \le j \le 4$. On the central region $|y| \le Y_z = a_z^{1/7}$, write the Taylor expansion in the form*

$$R_z(y) = \sum_{k=3}^{K} \alpha_k(z) y^k + \mathcal{E}_{K,z}(y),$$

with K fixed large enough, say $K = 12$. Then the following hold.

1. *For $3 \le k \le K$ and $0 \le r \le 4$,*

$$\alpha_k^{(r)}(z) = O_{k,r}(a_z^{1-k/2} z^{-r}).$$

2. *The differentiated Taylor remainder satisfies*

$$\partial_z^r \mathcal{E}_{K,z}(y) = O_{K,r}(a_z^{1-(K+1)/2} z^{-r} (1+|y|)^{K+1})$$

for $0 \le r \le 4$.

3. *In the Faà di Bruno expansion of*

$$\partial_z^j (e^{R_z(y)} - 1),$$

every monomial is a finite product

$$C(z) y^D$$

whose coefficient satisfies

$$C(z) = O(a_z^{M-D/2} z^{-j}),$$

where M is the number of Taylor factors used in that monomial and D is its total degree.

4. *If D is odd, then*

$$\int_{-Y_z}^{Y_z} e^{-y^2/2} y^D \, dy = 0.$$

If D is even and $D \ge 4$, then the only even monomials with coefficient scale as large as $a_z^{-1} z^{-j}$ are generated by

$$\alpha_4^{(r)} y^4 \quad \text{and} \quad \partial_z^r(\alpha_3^2) y^6.$$

All other surviving even Taylor monomials have coefficient $O(a_z^{-2} z^{-j})$ or smaller; the differentiated Taylor remainder is bounded separately by item 2, and is in particular $O(a_z^{-3/2} z^{-j})$ after Gaussian integration for the fixed value of K used here.

Proof. The first item is Lemma 14.2. For the second item, Taylor's theorem with integral remainder gives

$$\mathcal{E}_{K,z}(y) = \frac{F_z^{(K+1)}(\xi)}{(K+1)!} \left(\frac{y}{\sqrt{a_z}}\right)^{K+1}$$

for some ξ between u_z and $u_z + y/\sqrt{a_z}$. On the central region,

$$|\xi - u_z| \le |y| a_z^{-1/2} \le a_z^{-5/14} = o(1).$$

Thus ξ remains in the same $o(1)$-neighborhood of u_z as $w_z(y)$. Differentiating $\xi(z,y)$ produces the same type of factors as differentiating $w_z(y)$, namely polynomial factors in y, z^{-1}, and powers of $a_z^{-1/2}$. Since $F_z^{(m)}(u) = O_m(a_z)$ uniformly for $u = u_z + o(1)$, the same Faà di Bruno estimate used in Lemma 14.2 gives

$$\partial_z^r F_z^{(K+1)}(\xi) = O_{K,r}(a_z z^{-r}),$$

uniformly for $|y| \le Y_z$. Differentiating powers of $a_z^{-1/2}$ contributes only the expected z^{-r} factors. This proves the stated remainder estimate.

For the third item, Faà di Bruno's formula writes $\partial_z^j e^{R_z}$ as e^{R_z} times a finite sum of products of derivatives $\partial_z^r R_z$, with the total z-derivative order equal to j. Expanding each $\partial_z^r R_z$ into Taylor monomials gives products of terms

$$\alpha_{k_i}^{(r_i)}(z) y^{k_i}.$$

If there are M such factors and total degree $D = k_1 + \cdots + k_M$, then by the first item the coefficient is

$$O\left(\prod_{i=1}^{M} a_z^{1-k_i/2} z^{-r_i}\right) = O(a_z^{M-D/2} z^{-j}).$$

The harmless factor $e^{R_z} = 1 + O(1)$ on the central region does not change the scale; expanding it gives terms covered by the same counting.

For the fourth item, odd powers integrate to zero on the symmetric interval. For even degrees, the scale $a_z^{M-D/2}$ is largest when the total degree is as small as possible. The degree 4 even term is $\alpha_4 y^4$, with scale a_z^{-1}. Degree 6 has the product $\alpha_3^2 y^6$, also with scale a_z^{-1}. A linear $\alpha_6 y^6$ term has scale a_z^{-2}, and all other even Taylor monomials have scale at most a_z^{-2}. The differentiated Taylor remainder is controlled by item 2 and is smaller than the target $a_z^{-1} z^{-j}$. Differentiating j times contributes the common factor z^{-j}. This proves the bookkeeping statement. □

Lemma 14.4 (Central differentiated expansion, uniform form)**.** *For $0 \le j \le 4$ and $|y| \le Y_z = a_z^{1/7}$,*

$$\partial_z^j \left(e^{R_z(y)} - 1\right)$$

has an expansion whose non-integrable-leading odd part is always a finite linear combination of odd monomials in y, and whose even part satisfies, after Gaussian integration over the symmetric interval $[-Y_z, Y_z]$,

$$\int_{-Y_z}^{Y_z} e^{-y^2/2} \partial_z^j \left(e^{R_z(y)} - 1\right) dy = O_j(a_z^{-1} z^{-j}).$$

Equivalently,

$$\int_{-Y_z}^{Y_z} e^{-y^2/2} \partial_z^j \left(e^{R_z(y)} - 1\right) dy = O_j\left(\frac{\log z}{z^{1+j}}\right).$$

Proof. Apply Lemma 14.3. The potentially larger odd-degree terms vanish under the symmetric Gaussian integral. The leading surviving even terms are

$$\alpha_4^{(r)} y^4 \quad \text{and} \quad \partial_z^r(\alpha_3^2) y^6,$$

with coefficient size $O(a_z^{-1} z^{-j})$. All other surviving even Taylor monomials are $O(a_z^{-2} z^{-j})$ or smaller, and the differentiated Taylor remainder is smaller than the target by Lemma 14.3. Gaussian moments of fixed degree are bounded uniformly after extending the integral from $[-Y_z, Y_z]$ to $\mathbb{R}$. Hence

$$\int_{-Y_z}^{Y_z} e^{-y^2/2} \partial_z^j (e^{R_z(y)} - 1)\, dy = O_j(a_z^{-1} z^{-j}) = O_j\left(\frac{\log z}{z^{1+j}}\right).$$

□

Lemma 14.5 (Amplitude derivatives, uniform form)**.** *For $0 \le j \le 4$ and $|y| \le Y_z$,*

$$\partial_z^j A\left(u_z + \frac{y}{\sqrt{a_z}}\right) = \begin{cases} 1 + O(\log z/z), & j = 0 \text{ for the amplitude itself,} \\ O_j(\log z/z^{1+j}), & 1 \le j \le 4. \end{cases}$$

More precisely,

$$A\left(u_z + \frac{y}{\sqrt{a_z}}\right) - 1 = O(\log z/z),$$

and for $1 \le j \le 4$,

$$\partial_z^j A\left(u_z + \frac{y}{\sqrt{a_z}}\right) = O_j(\log z/z^{1+j}).$$

Proof. Since

$$A(u) = 1 - \frac{3}{2\pi} e^{-2u},$$

all derivatives of A are $O(e^{-2u})$. On the central region,

$$e^{-2(u_z + y/\sqrt{a_z})} = O(\log z/z).$$

Furthermore,

$$\partial_z^j \left(u_z + \frac{y}{\sqrt{a_z}}\right) = O_j(z^{-j})$$

for $1 \le j \le 4$, by Lemma 14.1. Faà di Bruno's formula therefore gives one factor z^{-1} for each z-derivative, multiplied by the base exponential scale $O(\log z/z)$. This proves the claim. $\square$

Proposition 14.6 (Central differentiated Laplace correction, $0 \le j \le 4$)**.** *For every $0 \le j \le 4$,*

$$\frac{d^j}{dz^j} \int_{|y| \le Y_z} e^{-y^2/2} (B_z(y) - 1)\, dy = O_j\left(\frac{\log z}{z^{1+j}}\right).$$

Proof. As in Proposition 13.5, moving-boundary terms at $y = \pm Y_z$ are $O_A(z^{-A})$ for every $A > 0$, because $e^{-Y_z^2/2}$ is super-polynomially small. It remains to differentiate under the integral.

On the central region,

$$B_z(y) = A\left(u_z + \frac{y}{\sqrt{a_z}}\right) e^{R_z(y)}.$$

Apply Leibniz' rule. Terms in which a derivative lands on the amplitude are controlled by Lemma 14.5. Terms in which derivatives land on $e^{R_z} - 1$ are controlled after Gaussian integration by Lemma 14.4. The product of the amplitude error $O(\log z/z)$ with any differentiated exponential correction is of smaller order than the target. Thus

$$\frac{d^j}{dz^j} \int_{|y| \le Y_z} e^{-y^2/2} (B_z(y) - 1)\, dy = O_j\left(\frac{\log z}{z^{1+j}}\right).$$

$\square$

Proposition 14.7 (Complement differentiated bound, $0 \le j \le 4$)**.** *For every $A > 0$ and $0 \le j \le 4$,*

$$\frac{d^j}{dz^j} \int_{\substack{|y| > Y_z \\ u_z + y/\sqrt{a_z} > 0}} e^{-y^2/2} B_z(y)\, dy = O_{A,j}(z^{-A}).$$

Proof. The proof is the same as Proposition 13.6, with the following explicit observation. At fixed y,

$$u = u_z + \frac{y}{\sqrt{a_z}}$$

has z-derivatives bounded by finite sums of polynomial factors in y, z^{-1}, and powers of $a_z^{-1/2}$. The formulas

$$F_z(u) = 2z \log u + \frac{9}{2}u - \pi e^{2u}, \qquad A(u) = 1 - \frac{3}{2\pi}e^{-2u}$$

then show that at most four z-derivatives of

$$A(u) \exp(F_z(u) - F_z(u_z))$$

are bounded by a fixed polynomial in $|y|$ and z, multiplied by the same exponential factor $\exp(F_z(u) - F_z(u_z))$.

On the local complement, Lemma 12.1 supplies

$$\exp(F_z(u) - F_z(u_z)) \le \exp\left(-c\frac{y^2}{\log z}\right), \qquad |y| > Y_z.$$

This dominates every fixed polynomial factor because

$$\frac{Y_z^2}{\log z} \asymp \frac{z^{2/7}}{(\log z)^{9/7}}.$$

On the far region, Lemma 12.2 supplies e^{-cz}, while the right far tail also has the super-exponential decay inherited from $-\pi e^{2u}$. Boundary terms from differentiating Y_z contain the factor

$$e^{-Y_z^2/2}$$

and are likewise $O_{A,j}(z^{-A})$. Thus the differentiated complement is $O_{A,j}(z^{-A})$. ◻

[Differentiable $n = 1$ Laplace correction] For the $n = 1$ contribution,

$$\eta^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 0 \le j \le 4.$$

Proof. Combine Proposition 14.6 and Proposition 14.7. ◻

Remark 14.8. *This section reduces the differentiated $n = 1$ Laplace correction to uniform coefficient differentiation, parity cancellation of odd Taylor terms, and super-polynomial complement decay. The next step is to combine this with the $n \ge 2$ tail separation of Section 9 and then derive Tau-Weak by finite differences.*

15 Derivation of Tau-Weak from the differentiable Laplace estimates

This section completes task T5 from Section 7. We show that the differentiable Laplace estimates obtained in Sections 9–14 imply the Tau-Weak bounds required in Section 6.

Let

$$M(z) = \int_0^\infty \Phi(u)u^{2z}\, du, \qquad s(z) = \log M(z).$$

The saddle main term is

$$S(z) = F_z(u_z) + \log(2\pi^2) + \frac{1}{2}\log\frac{2\pi}{a_z}.$$

From the $n = 1$ Laplace analysis and the $n \geq 2$ tail separation, we have

$$s^{(j)}(z) - S^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 0 \leq j \leq 4.$$

We now convert this into finite-difference bounds for

$$\tau_q = \frac{m_{q-1}m_{q+1}}{m_q^2}.$$

15.1 Derivative bounds for the saddle main term

Write

$$S_0(z) = F_z(u_z).$$

Since $F_z'(u_z) = 0$, the envelope identity gives

$$S_0'(z) = \partial_z F_z(u_z) = 2\log u_z.$$

Therefore

$$S_0''(z) = 2\frac{u_z'}{u_z}.$$

Using

$$u_z \asymp \log z, \qquad u_z^{(j)} = O_j(z^{-j}),$$

we obtain, for $2 \leq j \leq 4$,

$$S_0^{(j)}(z) = O_j\left(\frac{1}{z^{j-1}\log z}\right).$$

Next,

$$\frac{1}{2}\log\frac{2\pi}{a_z} = \frac{1}{2}\log(2\pi) - \frac{1}{2}\log a_z.$$

By Lemma 14.1,

$$a_z^{(j)} = O_j(a_z z^{-j}), \qquad 1 \leq j \leq 4.$$

Hence

$$\partial_z^j \log a_z = O_j(z^{-j}), \qquad 1 \leq j \leq 4.$$

For $2 \leq j \leq 4$, this is smaller than

$$z^{-(j-1)}(\log z)^{-1}$$

for large z. Thus

$$S^{(j)}(z) = O_j\left(\frac{1}{z^{j-1}\log z}\right), \qquad 2 \leq j \leq 4.$$

Combining with the differentiable Laplace remainder gives

$$\boxed{s^{(j)}(z) = O_j\left(\frac{1}{z^{j-1}\log z}\right), \qquad 2 \leq j \leq 4.}$$

Indeed, the error term

$$O_j\left(\frac{\log z}{z^{1+j}}\right)$$

is smaller than the displayed main bound for $2 \leq j \leq 4$.

Lemma 15.1 (Positivity of the de Bruijn kernel)**.** *For every $u > 0$,*

$$\Phi(u) > 0.$$

Proof. Recall

$$\Phi(u) = \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2}\right) e^{-\pi n^2 e^{2u}}.$$

The n-th summand can be written as

$$\pi n^2 e^{5u/2} e^{-\pi n^2 e^{2u}} \left(2\pi n^2 e^{2u} - 3\right).$$

For $u > 0$ and $n \geq 1$,

$$2\pi n^2 e^{2u} > 2\pi > 3.$$

Therefore every summand is strictly positive, and hence

$$\Phi(u) > 0.$$

□

Lemma 15.2 (Log-convexity of the de Bruijn moments)**.** *The moment function*

$$M(z) = \int_0^\infty \Phi(u) u^{2z}\, du$$

is log-convex on its domain, and

$$s''(z) \geq 0, \qquad s(z) = \log M(z).$$

More precisely,

$$s''(z) = 4\, \mathrm{Var}_{\mu_z}(\log U) \geq 0,$$

where

$$d\mu_z(u) = \frac{\Phi(u) u^{2z}}{M(z)}\, du.$$

Proof. By Lemma 15.1, $\Phi(u) > 0$. Hence the measure $d\mu_z$ is positive. Differentiating under the integral sign gives

$$s'(z) = \frac{M'(z)}{M(z)} = 2 \int_0^\infty \log u\, d\mu_z(u).$$

A second differentiation gives

$$s''(z) = 4 \left(\int_0^\infty (\log u)^2\, d\mu_z(u) - \left(\int_0^\infty \log u\, d\mu_z(u) \right)^2 \right).$$

Thus

$$s''(z) = 4\, \mathrm{Var}_{\mu_z}(\log U) \geq 0.$$

□

15.2 The logarithmic moment-ratio bounds

Define

$$\lambda_q = \log \tau_q = s(q-1) + s(q+1) - 2s(q).$$

The standard midpoint identity gives

$$\lambda_q = \int_{-1}^{1} (1 - |t|) s''(q+t)\, dt.$$

By Lemma 15.2,

$$s''(z) \geq 0.$$

Therefore

$$\lambda_q \geq 0.$$

Using the bound on s'',

$$0 \leq \lambda_q \leq \frac{C}{q \log q}$$

for all sufficiently large q.

Next,

$$\Delta\lambda_q = \lambda_{q+1} - \lambda_q$$

is a third finite difference of s over a bounded interval of length 3 centered at scale q. By the mean-value form of finite differences,

$$|\Delta\lambda_q| \leq C \sup_{x\in[q-1,q+2]} |s'''(x)| \leq \frac{C}{q^2 \log q}.$$

Similarly,

$$\Delta^2\lambda_q = \lambda_{q+2} - 2\lambda_{q+1} + \lambda_q$$

is a fourth finite difference of s over a bounded interval centered at scale q. Thus

$$|\Delta^2\lambda_q| \leq C \sup_{x\in[q-1,q+3]} |s^{(4)}(x)| \leq \frac{C}{q^3 \log q}.$$

15.3 Passage from λ_q to τ_q

Since

$$\tau_q = e^{\lambda_q}$$

and

$$0 \leq \lambda_q \leq \frac{C}{q \log q},$$

we have, for all sufficiently large q,

$$0 \leq \tau_q - 1 = e^{\lambda_q} - 1 \leq 2\lambda_q \leq \frac{C_0}{q \log q}.$$

For the first difference,

$$\Delta\tau_q = e^{\lambda_{q+1}} - e^{\lambda_q}.$$

Since $\lambda_q = O(1/(q \log q))$, the exponential factor is uniformly bounded, and hence

$$|\Delta\tau_q| \leq C|\Delta\lambda_q| \leq \frac{C_1}{q^2 \log q}.$$

For the second difference, write

$$a_q = \lambda_{q+1} - \lambda_q, \qquad b_q = \lambda_{q+2} - \lambda_{q+1}.$$

Then

$$\Delta^2\lambda_q = b_q - a_q.$$

Also,

$$\begin{aligned}\Delta^2\tau_q &= e^{\lambda_{q+2}} - 2e^{\lambda_{q+1}} + e^{\lambda_q} \\ &= e^{\lambda_q}\left(e^{a_q+b_q} - 2e^{a_q} + 1\right).\end{aligned}$$

Using

$$e^x = 1 + x + O(x^2)$$

uniformly for small x, we obtain

$$\begin{aligned} e^{a_q+b_q}-2e^{a_q}+1 &= (1+a_q+b_q+O((a_q+b_q)^2))-2(1+a_q+O(a_q^2))+1 \\ &= b_q-a_q+O(a_q^2+b_q^2). \end{aligned}$$

Therefore

$$|\Delta^2\tau_q| \le C\left(|\Delta^2\lambda_q|+|\Delta\lambda_q|^2\right),$$

because $a_q, b_q = O(\Delta\lambda_q)$ at scale q. The square term satisfies

$$|\Delta\lambda_q|^2 \ll \frac{1}{q^4(\log q)^2} \ll \frac{1}{q^3\log q}.$$

Thus

$$|\Delta^2\tau_q| \le \frac{C_2}{q^3\log q}.$$

We have proved the following.

Theorem 15.3 (Tau-Weak from differentiable Laplace estimates)**.** *Assume the differentiable Laplace estimate*

$$s^{(j)}(z)-S^{(j)}(z)=O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 0\le j\le 4.$$

Then there exist constants

$$C_0, C_1, C_2, Q_0 > 0$$

such that, for all $q \ge Q_0$,

$$0 \le \tau_q - 1 \le \frac{C_0}{q\log q},$$

$$|\Delta\tau_q| \le \frac{C_1}{q^2\log q}, \qquad |\Delta^2\tau_q| \le \frac{C_2}{q^3\log q}.$$

In other words, Tau-Weak holds for all sufficiently large q.

Remark 15.4. *The constants C_0, C_1, C_2, Q_0 are effective in principle once the constants in the saddle estimates, tail separation, and differentiable Laplace remainder are made explicit. The present theorem supplies the analytic implication needed by the PF_3 reduction; the final certificate must still cover the finite region $q < Q_0$ and the finite d-ranges determined by the scale decomposition.*

16 A unified leading positivity framework

We now prove the conditional positivity of $N_{3,q}^{(d)}$ in the three asymptotic regimes: the bulk, the left edge, and the right edge. The three arguments have the same structure. In each region there is a natural scale M such that

$$v_q \asymp M^{-1}, \qquad \Delta v_q = O(M^{-2}), \qquad \Delta^2 v_q = O(M^{-3}).$$

Then

$$E_q = v_{q-1}v_{q+1} - v_q^2 = O(M^{-4}),$$

and the exact identity

$$N_{3,q} = 2v_q^3 - v_q^4 - (1-v_q)^2E_q$$

implies

$$N_{3,q} = 2v_q^3 + O(M^{-4}).$$

Since $v_q \asymp M^{-1}$, the leading term is positive and of order M^{-3}, while the error is of order M^{-4}. Thus $N_{3,q} > 0$ for sufficiently large M.

16.1 The general scale lemma

Lemma 16.1 (Unified scale lemma)**.** *Let $M = M(d, q) \to \infty$ be a positive scale. Suppose that in a given range one has*

$$v_q = \frac{Y_q}{M},$$

where

$$0 < c \le Y_q \le C,$$

and also

$$|\Delta v_q| = O(M^{-2}), \qquad |\Delta^2 v_q| = O(M^{-3}).$$

Then

$$E_q = v_{q-1}v_{q+1} - v_q^2 = O(M^{-4})$$

and

$$N_{3,q} = \frac{2Y_q^3}{M^3} + O(M^{-4}).$$

In particular, $N_{3,q} > 0$ for sufficiently large M.

Proof. We use the identity

$$E_q = v_q \Delta^2 v_q - (\Delta_- v_q)(\Delta_+ v_q).$$

By assumption,

$$v_q = O(M^{-1}), \qquad \Delta v_q = O(M^{-2}), \qquad \Delta^2 v_q = O(M^{-3}).$$

Therefore

$$v_q \Delta^2 v_q = O(M^{-4})$$

and

$$(\Delta_- v_q)(\Delta_+ v_q) = O(M^{-4}).$$

Hence

$$E_q = O(M^{-4}).$$

Now recall the exact formula

$$N_{3,q} = 2v_q^3 - v_q^4 - (1 - v_q)^2 E_q.$$

Substituting $v_q = Y_q/M$, we obtain

$$2v_q^3 - v_q^4 = \frac{2Y_q^3}{M^3} - \frac{Y_q^4}{M^4}.$$

Since $Y_q = O(1)$, the second term is $O(M^{-4})$. Moreover,

$$(1 - v_q)^2 E_q = O(M^{-4}).$$

Therefore

$$N_{3,q} = \frac{2Y_q^3}{M^3} + O(M^{-4}).$$

Because $Y_q \ge c > 0$, the positive leading term dominates the error for sufficiently large M. Hence $N_{3,q} > 0$. □

16.2 Bulk regime

We first consider the bulk range

$$\varepsilon d \le q \le (1-\varepsilon)d, \qquad 0 < \varepsilon < \frac{1}{2}.$$

Here the natural scale is

$$M = d.$$

Lemma 16.2 (Bulk smoothness from Tau-Weak)**.** *Assume Tau-Weak. Fix* $0 < \varepsilon < 1/2$. *Uniformly for*

$$\varepsilon d \le q \le (1-\varepsilon)d,$$

one has

$$v_q \asymp_\varepsilon d^{-1},$$

$$|\Delta v_q| = O_\varepsilon(d^{-2}),$$

and

$$|\Delta^2 v_q| = O_\varepsilon(d^{-3}).$$

Proof. Recall that

$$v_q = 1 - \rho_{d,q}\tau_q = (1-\rho_{d,q}) - \rho_{d,q}(\tau_q - 1).$$

Set

$$\theta = \frac{q}{d}.$$

In the bulk range,

$$\varepsilon \le \theta \le 1-\varepsilon.$$

The explicit formula for $\rho_{d,q}$ gives

$$1 - \rho_{d,q} = \frac{1}{d}\left(\frac{1}{1-\theta} + \frac{2}{\theta}\right) + O_\varepsilon(d^{-2}).$$

The function

$$B(\theta) = \frac{1}{1-\theta} + \frac{2}{\theta}$$

is bounded above and below by positive constants on

$$\varepsilon \le \theta \le 1-\varepsilon.$$

Thus

$$1 - \rho_{d,q} \asymp_\varepsilon d^{-1}.$$

Tau-Weak gives

$$\tau_q - 1 = O_\varepsilon((d \log d)^{-1}).$$

Hence

$$v_q \asymp_\varepsilon d^{-1}.$$

For the first difference,

$$\Delta v_q = -\Delta(\rho_{d,q}\tau_q).$$

In the bulk range,

$$\Delta \rho_{d,q} = O_\varepsilon(d^{-2}),$$

and Tau-Weak gives

$$\Delta \tau_q = O_\varepsilon(d^{-2}(\log d)^{-1}).$$

Therefore

$$|\Delta v_q| = O_\varepsilon(d^{-2}).$$

Similarly,

$$\Delta^2 v_q = -\Delta^2(\rho_{d,q}\tau_q).$$

Using

$$\Delta^2 \rho_{d,q} = O_\varepsilon(d^{-3}),$$

$$\Delta \tau_q = O_\varepsilon(d^{-2}(\log d)^{-1}),$$

and

$$\Delta^2 \tau_q = O_\varepsilon(d^{-3}(\log d)^{-1}),$$

one obtains

$$|\Delta^2 v_q| = O_\varepsilon(d^{-3}).$$

□

Conditional Proposition 16.3 (Bulk positivity)**.** *Assume Tau-Weak. For every* $0 < \varepsilon < 1/2$*, there exists* D_ε *such that if*

$$d \geq D_\varepsilon, \qquad \varepsilon d \leq q \leq (1-\varepsilon)d,$$

then

$$N_{3,q}^{(d)} > 0.$$

More precisely, if

$$V_q = d v_q,$$

then uniformly in the bulk range,

$$0 < c_\varepsilon \leq V_q \leq C_\varepsilon$$

and

$$N_{3,q}^{(d)} = \frac{2V_q^3}{d^3} + O_\varepsilon(d^{-4}).$$

Proof. This follows immediately from Lemma 16.2 and the unified scale lemma with $M = d$ and $Y_q = V_q = d v_q$. □

16.3 Left-edge regime

We next consider the left-edge range

$$q \leq \varepsilon_L d.$$

Here the natural scale is

$$M = q.$$

Lemma 16.4 (Left-edge smoothness from Tau-Weak)**.** *Assume Tau-Weak. There exist constants* Q_L *and* $\varepsilon_L > 0$ *such that, uniformly in the range*

$$Q_L \leq q \leq \varepsilon_L d,$$

one has

$$v_q \asymp q^{-1},$$

$$|\Delta v_q| = O(q^{-2}),$$

and

$$|\Delta^2 v_q| = O(q^{-3}).$$

Proof. Again,

$$v_q = (1 - \rho_{d,q}) - \rho_{d,q}(\tau_q - 1).$$

In the left-edge range,

$$\rho_{d,q} = 1 - \frac{2}{q} + O(q^{-2}) + O(d^{-1}).$$

Thus

$$1 - \rho_{d,q} = \frac{2}{q} + O(q^{-2}) + O(d^{-1}).$$

Tau-Weak gives

$$\tau_q - 1 = O\left(\frac{1}{q \log q}\right).$$

Therefore

$$v_q = \frac{2}{q} + O\left(\frac{1}{q \log q}\right) + O(q^{-2}) + O(d^{-1}).$$

If $q \le \varepsilon_L d$, then

$$d^{-1} \le \frac{\varepsilon_L}{q}.$$

Choosing $\varepsilon_L > 0$ sufficiently small and then Q_L sufficiently large, the error terms are dominated by the positive leading term $2/q$. Hence

$$v_q \asymp q^{-1}.$$

For differences, the explicit rational formula gives

$$\Delta \rho_{d,q} = O(q^{-2}) + O(d^{-2}),$$

and

$$\Delta^2 \rho_{d,q} = O(q^{-3}) + O(d^{-3}).$$

Tau-Weak gives

$$\Delta \tau_q = O\left(\frac{1}{q^2 \log q}\right),$$

and

$$\Delta^2 \tau_q = O\left(\frac{1}{q^3 \log q}\right).$$

Combining these estimates yields

$$|\Delta v_q| = O(q^{-2}), \qquad |\Delta^2 v_q| = O(q^{-3}).$$

□

Conditional Proposition 16.5 (Left-edge positivity)**.** *Assume Tau-Weak. There exist constants Q_L, $\varepsilon_L > 0$, and D_L such that if*

$$d \ge D_L, \qquad Q_L \le q \le \varepsilon_L d,$$

then

$$N_{3,q}^{(d)} > 0.$$

More precisely, if

$$W_q = q v_q,$$

then uniformly in the left-edge range,

$$0 < c_L \le W_q \le C_L$$

and

$$N_{3,q}^{(d)} = \frac{2W_q^3}{q^3} + O(q^{-4}).$$

Proof. Apply the unified scale lemma with $M = q$ and $Y_q = W_q = q v_q$, using Lemma 16.4. □

16.4 Right-edge regime

Finally, we consider the right edge. Let

$$h = d - q.$$

The right-edge range is

$$h \le \varepsilon_R d.$$

The valid $r = 3$ range requires $h \ge 2$. In the asymptotic right-edge proposition below, we take h sufficiently large. The fixed-small-h families are treated separately by the certificate architecture in Section 18.

The natural scale is

$$M = h + 1.$$

Lemma 16.6 (Right-edge smoothness from Tau-Weak)**.** *Assume Tau-Weak. There exist constants H_R and $\varepsilon_R > 0$ such that, uniformly in the range*

$$H_R \le h = d - q \le \varepsilon_R d,$$

one has

$$v_q \asymp (h+1)^{-1},$$

$$|\Delta v_q| = O((h+1)^{-2}),$$

and

$$|\Delta^2 v_q| = O((h+1)^{-3}).$$

Proof. In the right-edge range,

$$h = d - q.$$

The first factor in $\rho_{d,q}$ is

$$\frac{d-q}{d-q+1} = \frac{h}{h+1}.$$

Since $q = d - h$, and $h \le \varepsilon_R d$, we have $q \asymp d$. The second factor satisfies

$$\frac{(2q-1)(2q)}{(2q+1)(2q+2)} = 1 + O(q^{-1}).$$

Therefore

$$\rho_{d,q} = \frac{h}{h+1}\left(1 + O(q^{-1})\right).$$

Hence

$$1 - \rho_{d,q} = \frac{1}{h+1} + O(q^{-1}).$$

Tau-Weak gives

$$\tau_q - 1 = O\left(\frac{1}{q \log q}\right).$$

Thus

$$v_q = \frac{1}{h+1} + O(q^{-1}) + O\left(\frac{1}{q \log q}\right).$$

Since $h \le \varepsilon_R d$ and $q \asymp d$, we have

$$q^{-1} = O(d^{-1}) = O(\varepsilon_R (h+1)^{-1}).$$

Choosing $\varepsilon_R > 0$ sufficiently small and then H_R sufficiently large, the error terms are dominated by the leading term $(h+1)^{-1}$. Hence

$$v_q \asymp (h+1)^{-1}.$$

Increasing q by 1 decreases $h = d - q$ by 1. Therefore the leading model $(h+1)^{-1}$ has first difference $O((h+1)^{-2})$ and second difference $O((h+1)^{-3})$. The explicit rational factor satisfies the corresponding estimates

$$\Delta\rho_{d,q} = O((h+1)^{-2}) + O(q^{-2}),$$

and

$$\Delta^2\rho_{d,q} = O((h+1)^{-3}) + O(q^{-3}).$$

Tau-Weak gives

$$\Delta\tau_q = O\left(\frac{1}{q^2 \log q}\right),$$

and

$$\Delta^2\tau_q = O\left(\frac{1}{q^3 \log q}\right).$$

Since $h \le \varepsilon_R d$ and $q \asymp d$, the q-dependent terms are bounded by the corresponding powers of $h+1$. Hence

$$|\Delta v_q| = O((h+1)^{-2}),$$

and

$$|\Delta^2 v_q| = O((h+1)^{-3}).$$

□

Conditional Proposition 16.7 (Right-edge positivity)**.** *Assume Tau-Weak. There exist constants H_R, $\varepsilon_R > 0$, and D_R such that if*

$$d \ge D_R, \qquad H_R \le h = d - q \le \varepsilon_R d,$$

then

$$N_{3,q}^{(d)} > 0.$$

More precisely, if

$$U_h = (h+1)v_q,$$

then uniformly in the right-edge range,

$$0 < c_R \le U_h \le C_R$$

and

$$N_{3,q}^{(d)} = \frac{2U_h^3}{(h+1)^3} + O((h+1)^{-4}).$$

Proof. Apply the unified scale lemma with $M = h+1$ and $Y_q = U_h = (h+1)v_q$, using Lemma 16.6. □

17 A proposed saddle route to Tau-Weak

We now describe the analytic route to the Tau-Weak estimates. This section does not claim that all details have been fully closed. Rather, it decomposes the remaining analytic task into four precise sublemmas. Once these sublemmas are proved, Tau-Weak follows.

Status of this section. The estimates P1–P4 below are not used as established theorems unless explicitly assumed. They are listed as proof obligations for a future effective proof of Tau-Weak. The conditional PF_3 theorem later in the paper depends only on Tau-Weak and the finite certificate architecture, not on the completion of these saddle estimates.

Recall that

$$M(z) = \int_0^\infty \Phi(u) u^{2z}\, du, \qquad s(z) = \log M(z).$$

For integer q,

$$\log \tau_q = s(q-1) - 2s(q) + s(q+1).$$

Equivalently,

$$\log \tau_q = \int_{-1}^{1} (1 - |t|) s''(q+t)\, dt.$$

Thus estimates for s'', s''', and $s^{(4)}$ imply the desired estimates for τ_q and its first two discrete differences.

17.1 The de Bruijn kernel and the leading phase

We use the standard expansion

$$\Phi(u) = \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2}\right) e^{-\pi n^2 e^{2u}}.$$

Separate the $n = 1$ term:

$$\Phi(u) = \Phi_1(u) + R(u),$$

where

$$\Phi_1(u) = \left(2\pi^2 e^{9u/2} - 3\pi e^{5u/2}\right) e^{-\pi e^{2u}}.$$

Equivalently,

$$\Phi_1(u) = 2\pi^2 e^{9u/2} e^{-\pi e^{2u}} A_1(u),$$

with

$$A_1(u) = 1 - \frac{3}{2\pi} e^{-2u}.$$

The leading phase is

$$F_z(u) = 2z \log u + \frac{9}{2} u - \pi e^{2u}.$$

Let u_z be the unique saddle:

$$F_z'(u_z) = 0.$$

Explicitly,

$$\frac{2z}{u_z} + \frac{9}{2} - 2\pi e^{2u_z} = 0.$$

Set

$$a_z = -F_z''(u_z).$$

Since

$$F_z''(u) = -\frac{2z}{u^2} - 4\pi e^{2u},$$

we have

$$a_z = \frac{2z}{u_z^2} + 4\pi e^{2u_z}.$$

Using the saddle equation,

$$4\pi e^{2u_z} = \frac{4z}{u_z} + 9,$$

and hence

$$a_z = \frac{4z}{u_z} + \frac{2z}{u_z^2} + 9.$$

Therefore

$$u_z \asymp \log z, \qquad a_z \asymp \frac{z}{\log z}.$$

Define the Laplace main term

$$S(z) = F_z(u_z) + \log(2\pi^2) + \frac{1}{2}\log\frac{2\pi}{a_z}.$$

The desired differentiable Laplace estimate is

$$s^{(j)}(z) - S^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 0 \le j \le 4.$$

We next decompose this estimate into four sublemmas.

17.2 P1: separation of the $n \ge 2$ tail

Let

$$M_{\ge 2}(z) = \int_0^\infty R(u)u^{2z}\,du.$$

The n-th phase is

$$F_{z,n}(u) = 2z\log u + \frac{9}{2}u - \pi n^2 e^{2u}.$$

Its saddle $u_{z,n}$ satisfies

$$\frac{2z}{u_{z,n}} + \frac{9}{2} = 2\pi n^2 e^{2u_{z,n}}.$$

At the leading level,

$$\pi n^2 u_{z,n} e^{2u_{z,n}} \sim z.$$

Thus, for fixed n,

$$u_{z,n} = u_z - \log n + O\left(\frac{\log n}{\log z}\right)$$

in the coarse large-z sense. Consequently, the $n \ge 2$ saddles lie to the left of the $n = 1$ saddle and are exponentially separated.

Conditional Lemma 17.1 (Tail separation)**.** *For every $A > 0$ and $0 \le j \le 4$, one has*

$$\frac{d^j}{dz^j}M_{\ge 2}(z) = O_{A,j}\left(z^{-A}e^{F_z(u_z)}a_z^{-1/2}(\log z)^j\right).$$

In particular, $M_{\ge 2}(z)$ is smaller than the main $n = 1$ Laplace contribution by $O_A(z^{-A})$ for every $A > 0$.

Proof sketch. For $n \ge 2$, the saddle value satisfies

$$F_z(u_z) - F_{z,n}(u_{z,n}) \ge c\frac{z\log n}{\log z}$$

for some absolute $c > 0$ and all large z. Hence

$$e^{F_{z,n}(u_{z,n})} \le e^{F_z(u_z)}\exp\left(-c\frac{z\log n}{\log z}\right).$$

After including polynomial factors in n, z, and powers of $\log u$ coming from z-differentiation, the sum over $n \ge 2$ remains $O_A(z^{-A})$ relative to the $n = 1$ contribution. The saddle localization estimate for each $F_{z,n}$ gives the stated derivative bound. □

17.3 P2: amplitude control for the $n = 1$ term

The $n = 1$ contribution is

$$M_1(z) = 2\pi^2 \int_0^\infty e^{F_z(u)} A_1(u)\ du.$$

Near the saddle,

$$e^{-2u_z} \asymp \frac{u_z}{z} \asymp \frac{\log z}{z}.$$

Therefore

$$A_1(u_z) = 1 + O\left(\frac{\log z}{z}\right).$$

Moreover,

$$A_1^{(k)}(u) = O_k(e^{-2u})$$

for all fixed $k \geq 1$.

Let

$$R_z = (\log z)^2$$

and consider the Gaussian window

$$|u - u_z| \leq R_z a_z^{-1/2}.$$

Since

$$a_z^{-1/2} \asymp \sqrt{\frac{\log z}{z}},$$

this window is much smaller than u_z, and on it

$$e^{-2u} \asymp e^{-2u_z} \asymp \frac{\log z}{z}.$$

Conditional Lemma 17.2 (Amplitude control)**.** *For each fixed $K \geq 0$ and $0 \leq k \leq K$, uniformly in the Gaussian window*

$$|u - u_z| \leq R_z a_z^{-1/2},$$

one has

$$A_1(u) = 1 + O\left(\frac{\log z}{z}\right),$$

and for $k \geq 1$,

$$A_1^{(k)}(u) = O_k\left(\frac{\log z}{z}\right).$$

17.4 P3: differentiable Gaussian correction

Rescale around the saddle by

$$u = u_z + \frac{y}{\sqrt{a_z}}, \qquad du = a_z^{-1/2}\ dy.$$

Then

$$M_1(z) = 2\pi^2 e^{F_z(u_z)} a_z^{-1/2} \int_{\mathcal{I}_z} e^{-y^2/2} B_z(y)\ dy,$$

where

$$B_z(y) = A_1\left(u_z + \frac{y}{\sqrt{a_z}}\right) \exp\left(F_z\left(u_z + \frac{y}{\sqrt{a_z}}\right) - F_z(u_z) + \frac{y^2}{2}\right).$$

Taylor expansion gives

$$F_z\left(u_z + \frac{y}{\sqrt{a_z}}\right) = F_z(u_z) - \frac{y^2}{2} + \sum_{k=3}^{K} \frac{F_z^{(k)}(u_z)}{k! a_z^{k/2}} y^k + \mathcal{R}_K(z, y).$$

For fixed $k \geq 3$,

$$F_z^{(k)}(u_z) = O_k(a_z).$$

Thus

$$\frac{F_z^{(k)}(u_z)}{k! a_z^{k/2}} = O_k(a_z^{1-k/2}).$$

In particular,

$$c_3(z) := \frac{F_z'''(u_z)}{6 a_z^{3/2}} = O(a_z^{-1/2}),$$

and

$$c_4(z) := \frac{F_z^{(4)}(u_z)}{24 a_z^2} = O(a_z^{-1}).$$

Since

$$a_z^{-1} \asymp \frac{\log z}{z},$$

the first nonzero integrated correction is of size $O(\log z/z)$. The cubic correction is odd and has zero first-order Gaussian average.

Conditional Lemma 17.3 (Differentiable Gaussian correction)**.** *There exists a function $\eta(z)$ such that*

$$\int_{\mathcal{I}_z} e^{-y^2/2} B_z(y)\, dy = \sqrt{2\pi}\,(1 + \eta(z)) + O_A(z^{-A})$$

for every $A > 0$, and for $0 \leq j \leq 4$,

$$\eta^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right).$$

Proof sketch. The coefficients in the local expansion satisfy

$$c_k^{(j)}(z) = O_{k,j}\left(\frac{a_z^{1-k/2}}{z^j}\right).$$

Because the first nonzero Gaussian correction is of order a_z^{-1}, differentiating j times gives

$$O_j\left(\frac{a_z^{-1}}{z^j}\right) = O_j\left(\frac{\log z}{z^{1+j}}\right).$$

The amplitude contribution from $A_1(u)$ satisfies the same bound by the amplitude control lemma. The tail outside $|y| \leq R_z$ is $O_A(z^{-A})$ by the saddle localization estimate. □

17.5 P4: differentiated tail domination

To justify differentiating under the rescaled integral, we need a uniform domination statement.

Conditional Lemma 17.4 (Differentiated domination)**.** *For $0 \leq j \leq 4$, there exist constants $C_j, K_j > 0$ such that on the controlled Gaussian window,*

$$\left|\partial_z^j B_z(y)\right| \leq C_j (1 + |y|)^{K_j} e^{y^2/8}.$$

Consequently,

$$e^{-y^2/2}\partial_z^j B_z(y)$$

is dominated by an integrable function of the form

$$C_j(1+|y|)^{K_j}e^{-3y^2/8}.$$

Thus differentiation under the integral sign is justified for $0 \le j \le 4$.

Proof sketch. The z-dependence of $B_z(y)$ enters through u_z, a_z, the coefficients $F_z^{(k)}(u_z)$, and the amplitude A_1. These satisfy

$$u_z^{(j)} = O_j(z^{-j}),$$

and

$$a_z^{(j)} = O_j(a_z z^{-j}).$$

The local expansion of $B_z(y)$ is polynomial in y to each fixed order, with coefficients satisfying inverse powers of z. Exponentiating the controlled remainder gives at most $e^{y^2/8}$ within the Gaussian window for large z. This proves the domination estimate. □

17.6 The differentiable Laplace estimate

Combining the preceding sublemmas, we obtain

$$M(z) = 2\pi^2 e^{F_z(u_z)}\sqrt{\frac{2\pi}{a_z}}\left(1+\eta(z)+O_A(z^{-A})\right),$$

where

$$\eta^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 0 \le j \le 4.$$

Taking logarithms,

$$s(z) = F_z(u_z) + \log(2\pi^2) + \frac{1}{2}\log\frac{2\pi}{a_z} + \log\left(1+\eta(z)+O_A(z^{-A})\right).$$

Hence, with

$$S(z) = F_z(u_z) + \log(2\pi^2) + \frac{1}{2}\log\frac{2\pi}{a_z},$$

we obtain

$$s^{(j)}(z) - S^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 0 \le j \le 4.$$

Conditional Proposition 17.5 (Differentiable Laplace estimate)**.** *Assuming the saddle localization estimates used above, one has*

$$s^{(j)}(z) - S^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 0 \le j \le 4.$$

Remark 17.6. *The conditional proposition above is intentionally separated from the algebraic* PF_3 *reduction. Completing it would provide one route to Tau-Weak, but failure to complete it would not invalidate the algebraic reduction itself.*

17.7 Derivation of Tau-Weak

We now show that the differentiable Laplace estimate implies Tau-Weak.

From direct differentiation of S,

$$S''(z) \ll \frac{1}{z \log z},$$

$$S'''(z) \ll \frac{1}{z^2 \log z},$$

and

$$S^{(4)}(z) \ll \frac{1}{z^3 \log z}.$$

By the differentiable Laplace estimate, the same estimates hold for s:

$$s''(z) \ll \frac{1}{z \log z},$$

$$s'''(z) \ll \frac{1}{z^2 \log z},$$

and

$$s^{(4)}(z) \ll \frac{1}{z^3 \log z}.$$

Moreover,

$$s''(z) = 4 \operatorname{Var}_z(\log U) \geq 0.$$

Let

$$\lambda_q = \log \tau_q.$$

Then

$$\lambda_q = \int_{-1}^{1} (1 - |t|) s''(q + t) \, dt.$$

Thus

$$0 \leq \lambda_q \ll \frac{1}{q \log q}.$$

Since $e^x - 1 \leq 2x$ for small $x \geq 0$, we obtain

$$0 \leq \tau_q - 1 \ll \frac{1}{q \log q}.$$

Next, since λ_q is a second finite difference of s centered at scale q, the difference

$$\lambda_{q+1} - \lambda_q$$

is a third finite difference of s over indices at scale q. The bound on s''' gives

$$|\lambda_{q+1} - \lambda_q| \ll \frac{1}{q^2 \log q}.$$

Therefore,

$$|\tau_{q+1} - \tau_q| \ll \frac{1}{q^2 \log q}.$$

Finally, $\Delta^2 \lambda_q$ is a fourth finite difference of s over indices at scale q. The bound on $s^{(4)}$ gives

$$|\Delta^2 \lambda_q| \ll \frac{1}{q^3 \log q}.$$

The exponential change of variables from λ_q to $\tau_q = e^{\lambda_q}$ introduces only lower-order products, so

$$|\Delta^2 \tau_q| \ll \frac{1}{q^3 \log q}.$$

Thus Tau-Weak follows.

Conditional Theorem 17.7 (Tau-Weak from differentiable Laplace)**.** *If the differentiable Laplace estimate holds, then there exist constants $Q_0, C_0, C_1, C_2 > 0$ such that for all $q \geq Q_0$,*

$$0 \leq \tau_q - 1 \leq \frac{C_0}{q \log q},$$

$$|\tau_{q+1} - \tau_q| \leq \frac{C_1}{q^2 \log q},$$

and

$$|\Delta^2 \tau_q| \leq \frac{C_2}{q^3 \log q}.$$

18 Certificate architecture

The asymptotic propositions in the bulk, left-edge, and right-edge regimes cover all sufficiently large regions once effective constants are fixed. However, the remaining verification is not merely a finite set of individual pairs (d, q). There are two finite-family regimes that may still contain infinitely many values of d: fixed small q, and fixed small right-edge distance $h = d - q$.

Remark 18.1 (Dependence on effective constants)**.** *The certificate architecture is coupled to the effective constants in Tau-Weak. In particular, the thresholds appearing in the bulk, left-edge, and right-edge positivity propositions depend on the constants C_0, C_1, C_2 in Tau-Weak. Thus the certificate ranges cannot be finalized until Tau-Weak is proved with explicit effective constants. The certificate component should therefore be understood as an effective verification scheme conditional on such constants, not as an independent finite computation detached from the analytic estimates.*

Remark 18.2 (Tiling of the asymptotic regions)**.** *To ensure that the remaining transition region is finite, the parameters $\varepsilon_L, \varepsilon, \varepsilon_R$ must be chosen so that the left-edge, bulk, and right-edge regions cover all sufficiently large pairs. For example, one may choose compatible constants and arrange the covered regions as*

$$q \leq \varepsilon d, \qquad \varepsilon d \leq q \leq (1 - \varepsilon) d, \qquad d - q \leq \varepsilon d.$$

After the small-q and small-h finite families are separated, any pair not covered by the three asymptotic regions lies in a bounded transition range determined by the effective cutoffs.

Thus the certificate component must be organized into finite-pair checks and finite-family checks. We split the remaining verification into four parts.

18.1 A. Finite rectangular verification

First, choose a cutoff D_{rect}. For all

$$3 \leq d \leq D_{\text{rect}}, \qquad 2 \leq q \leq d - 2,$$

one may verify directly that

$$D_{3,q}^{(d)} > 0.$$

Equivalently, since $A_q > 0$, it is enough to verify

$$N_{3,q}^{(d)} > 0.$$

This part is a genuinely finite computation. It requires rigorous enclosures for the moments

$$m_0, m_1, \ldots, m_{D_{\text{rect}}},$$

or, more precisely, for all moments needed to evaluate v_{q-1}, v_q, v_{q+1} in the above range.

The intended verification method is interval or ball arithmetic. Floating-point numerical evidence alone is not sufficient for a certificate.

18.2 B. Fixed-small-q, all-d verification

The left-edge asymptotic proposition applies only after $q \geq Q_L$. Therefore the range

$$2 \leq q < Q_L$$

must be treated separately.

For each fixed q, the quantity $N_{3,q}^{(d)}$ depends on d through the rational factors

$$\rho_{d,q-1}, \qquad \rho_{d,q}, \qquad \rho_{d,q+1},$$

while the moment ratios

$$\tau_{q-1}, \qquad \tau_q, \qquad \tau_{q+1}$$

are independent of d. Indeed,

$$v_k(d) = 1 - \rho_{d,k}\tau_k.$$

Hence, for fixed q,

$$N_{3,q}^{(d)} = v_q(d)^2 - (1 - v_q(d))^2 v_{q-1}(d) v_{q+1}(d)$$

is a rational function of d, with coefficients depending only on finitely many moment ratios.

Therefore the fixed-small-q certificate should prove

$$N_{3,q}^{(d)} > 0 \qquad \text{for all } d \geq q + 2$$

for each

$$2 \leq q < Q_L.$$

A practical route is:

(i) compute rigorous enclosures for
$$\tau_{q-1}, \tau_q, \tau_{q+1};$$

(ii) express $N_{3,q}^{(d)}$ as a rational function of d;

(iii) multiply by the positive denominator to obtain a polynomial inequality in d;

(iv) prove the resulting interval-coefficient polynomial is positive for all $d \geq q + 2$, using a combination of large-d lower bounds and finite interval checking.

This component is finite in the number of families, but each family contains infinitely many values of d. Thus it should not be described as a finite pair check.

18.3 C. Fixed-small-h, all-large-d verification

Let

$$h = d - q.$$

The right-edge asymptotic proposition applies only after $h \geq H_R$. Thus the finite set of right-edge families

$$2 \leq h < H_R$$

must be handled separately.

Unlike the fixed-small-q case, here $q = d - h$ grows with d, and therefore the moment ratios

$$\tau_{q-1}, \qquad \tau_q, \qquad \tau_{q+1}$$

also vary with d. Consequently, this regime is not simply a rational function of d with fixed coefficients.

However, for fixed h, the explicit base factor gives

$$\frac{d-q}{d-q+1} = \frac{h}{h+1}.$$

Since $q = d - h \to \infty$, Tau-Weak gives

$$\tau_q = 1 + O\left(\frac{1}{q \log q}\right).$$

Thus

$$v_q = 1 - \rho_{d,q}\tau_q = \frac{1}{h+1} + O_h\left(\frac{1}{d}\right) + O_h\left(\frac{1}{d \log d}\right).$$

Similarly,

$$v_{q-1} = \frac{1}{h+2} + O_h\left(\frac{1}{d}\right) + O_h\left(\frac{1}{d \log d}\right),$$

and

$$v_{q+1} = \frac{1}{h} + O_h\left(\frac{1}{d}\right) + O_h\left(\frac{1}{d \log d}\right).$$

The leading right-edge model gives

$$N_{3,q}^{(d)} = \frac{2}{(h+1)^2(h+2)} + O_h\left(\frac{1}{d}\right) + O_h\left(\frac{1}{d \log d}\right).$$

Therefore, for each fixed

$$2 \le h < H_R,$$

there exists a constant D_h such that

$$N_{3,d-h}^{(d)} > 0 \qquad \text{for all } d \ge D_h.$$

The remaining finite range

$$d < D_h$$

is then checked directly by interval arithmetic.

18.4 D. Transition-band verification

After fixing all effective constants in the bulk, left-edge, and right-edge estimates, there may remain transition regions where the asymptotic inequalities are not yet active. These regions are finite once the cutoffs are fixed.

Let $\mathcal{R}_{\text{trans}}$ denote the set of valid pairs

$$2 \le q \le d - 2$$

not covered by the bulk, left-edge, right-edge, fixed-small-q, or fixed-small-h arguments. The transition certificate verifies

$$N_{3,q}^{(d)} > 0$$

for every

$$(d, q) \in \mathcal{R}_{\text{trans}}$$

using rigorous interval arithmetic.

18.5 Required rigorous moment data

The certificate requires rigorous enclosures for the de Bruijn moments

$$m_k = \int_0^\infty \Phi(u) u^{2k}\, du.$$

For a check involving $N_{3,q}^{(d)}$, one needs

$$v_{q-1}, \quad v_q, \quad v_{q+1},$$

and hence the moment ratios

$$\tau_{q-1}, \quad \tau_q, \quad \tau_{q+1}.$$

Thus moments up to at least m_{q+2} are required.

The appropriate certification method is ball arithmetic or interval arithmetic with explicit tail bounds for the integral defining m_k. High-precision floating-point arithmetic may guide the computation, but it is not by itself a proof certificate.

Assumption 18.3 (Certificate architecture)**.** *The following verification components hold:*

(a) finite rectangular verification for all $3 \le d \le D_{\mathrm{rect}}$*;*

(b) fixed-small-q, all-d verification for every $2 \le q < Q_L$*;*

(c) fixed-small-h, all-large-d verification for every $2 \le h < H_R$*, together with finite checking below the corresponding* D_h*;*

(d) transition-band verification for every remaining valid pair (d, q)*.*

Then all cases not covered by the asymptotic bulk, left-edge, and right-edge propositions satisfy

$$D_{3,q}^{(d)} > 0.$$

19 Computational certificate diagnostics

This section records numerical and exact-rational diagnostic tests for the certificate architecture described in Section 18. These computations are not presented as rigorous certificates for the true de Bruijn moments. Their purpose is more modest: they identify the bottlenecks, test the stability of the inequalities under perturbations of the moment ratios, and indicate the precision requirements for a future interval or ball-arithmetic certificate.

All computations in this section use the quantities

$$\tau_k = \frac{m_{k-1} m_{k+1}}{m_k^2}$$

computed from high-precision moment data and then, where explicitly stated, converted into decimal-rational approximations. Thus a statement called "decimal-rational exact" means exact for the rational numbers obtained after decimal truncation or rounding of the computed τ_k, not exact for the true moments.

19.1 Finite rectangle diagnostics

The finite rectangle component was tested for

$$4 \leq d \leq 360, \qquad 2 \leq q \leq d-2.$$

This gives 63903 valid pairs. For decimal-rational approximations of the moment ratios, exact rational sign evaluation of

$$N_{3,q}^{(d)}$$

gave no failures.

The global minimum in this rectangle occurred at

$$d = 360, \qquad q = 198, \qquad h = d - q = 162,$$

with

$$N_{3,q}^{(d)} \approx 5.39798517353870019 \times 10^{-6}.$$

This agrees with the observed bulk bottleneck near $q/d \approx 0.55$.

A perturbation stress test was also performed. For each valid pair, the three ratios

$$\tau_{q-1}, \qquad \tau_q, \qquad \tau_{q+1}$$

were independently perturbed at the eight corners

$$\tau_k \mapsto \tau_k(1 \pm \epsilon).$$

The finite rectangle remained positive for all corners up to a global relative perturbation radius approximately

$$\epsilon_{\text{rect}} \approx 9.8683730589 \times 10^{-5}.$$

The bottleneck was again

$$(d, q, h) = (360, 198, 162),$$

with the worst corner

$$(-1, +1, -1)$$

applied to

$$(\tau_{q-1}, \tau_q, \tau_{q+1}).$$

19.2 Fixed-small-q diagnostics

The fixed-small-q component was tested for

$$2 \leq q < 200.$$

For each fixed q, the quantities

$$\tau_{q-1}, \qquad \tau_q, \qquad \tau_{q+1}$$

are independent of d. Therefore

$$N_{3,q}^{(d)}$$

is a rational function of d. After replacing the τ-values by decimal-rational approximations, the numerator polynomial $P_q(d)$ was computed exactly.

For every

$$2 \leq q < 200,$$

Sturm sequence root counting showed that $P_q(d)$ has no real root on the half-line

$$d \geq q + 2.$$

Moreover,

$$P_q(q+2) > 0$$

and the leading coefficient of P_q is positive. The derivative $P_q'(d)$ also had no real root on the same half-line in these tests. Hence, for the decimal-rational τ-data, the fixed-q half-line positivity is exact.

The weakest tested fixed-q case occurred at

$$q = 199.$$

The limiting value as $d \to \infty$ was

$$N_{3,199}^{(\infty)} \approx 9.58938904057951425 \times 10^{-7},$$

and

$$199^3 N_{3,199}^{(\infty)} \approx 7.55701296838.$$

The same independent corner perturbation test

$$\tau_k \mapsto \tau_k(1 \pm \epsilon)$$

was applied to each fixed-q family. The estimated global perturbation radius over

$$2 \le q < 200$$

was

$$\epsilon_{\text{small-}q} \approx 3.0936637755 \times 10^{-5},$$

again with the weakest tested case at

$$q = 199.$$

19.3 Fixed-small-h diagnostics

Let

$$h = d - q.$$

The fixed-small-h profile was tested for

$$2 \le h \le 80, \qquad d \le 360.$$

For each fixed h, the numerical profile

$$N_{3,d-h}^{(d)}$$

was positive throughout the available range.

The expected fixed-h limiting value is

$$N_{3,d-h}^{(d)} \longrightarrow \frac{2}{(h+1)^2(h+2)} \qquad (d \to \infty).$$

For example, at $h = 80$,

$$N_{3,280}^{(360)} \approx 1.138856664570875 \times 10^{-5},$$

while

$$\frac{2}{81^2 \cdot 82} \approx 3.717458299411526 \times 10^{-6}.$$

The tested values remained above the limiting value in the available range.

The perturbation test over the checked fixed-h data gave an estimated radius

$$\epsilon_{\text{small-}h} \approx 1.6306855067 \times 10^{-4}.$$

The weakest tested case occurred at

$$h = 80, \qquad d = 360, \qquad q = 280,$$

again with worst corner

$$(-1, +1, -1).$$

19.4 Diagnostic conclusion

Combining the three diagnostic components gives the following observed perturbation radii:

component	tested range	weakest point	estimated radius
finite rectangle	$d \leq 360$	$(d, q, h) = (360, 198, 162)$	9.868×10^{-5}
fixed-small-q	$2 \leq q < 200$	$q = 199$	3.094×10^{-5}
fixed-small-h	$2 \leq h \leq 80, \ d \leq 360$	$(h, d, q) = (80, 360, 280)$	1.631×10^{-4}

Thus the weakest observed diagnostic radius is

$$\epsilon_{\text{diag}} \approx 3.09 \times 10^{-5}.$$

This suggests that a future rigorous ball-arithmetic certificate with relative enclosures for the relevant τ_k substantially below 10^{-5}, for instance at the level of 10^{-8} or smaller, should have a large safety margin in the tested regions.

We emphasize again that these computations do not replace a rigorous certificate. A final proof requires certified enclosures for the moments or moment ratios and interval-valid positivity checks. The computations above show that the proposed certificate architecture is numerically well-conditioned in the tested ranges and that the main bottlenecks have been identified.

20 Conditional hard-edge PF_3 theorem

We now state the assembled result.

Conditional Theorem 20.1 (Conditional hard-edge PF_3)**.** *Assume:*

(a) Tau-Weak holds;

(b) the certificate architecture in Section 18 holds.

Then for every valid pair

$$2 \leq q \leq d - 2,$$

one has

$$D_{3,q}^{(d)} > 0.$$

Equivalently,

$$N_{3,q}^{(d)} > 0.$$

Thus the hard-edge Jensen coefficient sequence satisfies the 3×3 *solid Toeplitz minor positivity condition.*

Remark 20.2. *The conclusion is the* 3×3 *solid-minor part of* PF_3*. A full* PF_3 *statement for all Toeplitz minors of order at most three would require the corresponding non-solid minors as well.*

Remark 20.3. *By the differentiable Laplace framework in Section 17, the differentiable Laplace estimate provides one proposed route to Tau-Weak. Thus the differentiable Laplace analysis is not an additional assumption needed for the conditional* PF_3 *theorem once Tau-Weak is assumed; rather, it is the analytic mechanism proposed for proving Tau-Weak.*

Proof. The bulk, left-edge, and right-edge conditional propositions cover the corresponding asymptotic regimes. The certificate architecture covers the fixed-small-q families, the fixed-small-h families, and the remaining transition-band cases. Hence all valid pairs $2 \leq q \leq d - 2$ are covered. Since $A_q > 0$, $N_{3,q} > 0$ is equivalent to $D_{3,q}^{(d)} > 0$. □

21 High-order Gamma diagnostics and the limitation of direct finite-d all-order scans

The normalized Toda recurrence gives, for solid Toeplitz minors,

$$N_{r+1,q}N_{r-1,q} = N_{r,q}^2 - x_q^r N_{r,q-1}N_{r,q+1}.$$

Equivalently, when $N_{r-1,q}$ and $N_{r,q}$ are positive, the next layer is positive precisely when

$$\Gamma_{r,q} := x_q^r \frac{N_{r,q-1}N_{r,q+1}}{N_{r,q}^2} < 1.$$

This ratio is a useful diagnostic for possible higher-order extensions. For small r, numerical data suggests the decomposition

$$-\log \Gamma_{r,q} = r(-\log x_q) - \log\left(\frac{N_{r,q-1}N_{r,q+1}}{N_{r,q}^2}\right)$$

has a positive margin in the bulk.

However, high-order finite-d scans are numerically delicate. The normalized minors may have size comparable to

$$d^{-\binom{r}{2}},$$

so for moderately large r their signs may require thousands of correct digits in the input moments. Consequently, a negative value obtained from non-certified high-precision floating arithmetic should not be interpreted as a mathematical counterexample. Agreement between a direct determinant evaluation and a Toda recurrence is also not, by itself, a certificate, because both computations may use the same finite-precision moment input.

This point is especially important because, in the classical Jensen–Pólya and Aissen–Schoenberg–Whitney framework, exact negativity of an appropriate finite Toeplitz minor would have strong implications for the corresponding Jensen polynomial. Thus the present manuscript does not claim a certified sign determination for such apparent negative minors. Instead, the Gamma/Toda computations are used only as diagnostics: they show that direct all-order finite-d scanning is numerically inaccessible at high r, and they warn against extrapolating the small-r PF_3 mechanism to PF_∞ without a new uniform principle.

The conclusion is therefore deliberately limited. The normalized Gamma ratio is a useful way to locate the obstruction to a naive all-order argument, but it is not currently a rigorous PF_∞ mechanism. To go beyond fixed finite order, one would need certified moment enclosures at very high precision, a stable interval method for high-order determinants, or a different limiting or continuous total-positivity principle. The present paper remains focused on the PF_3 layer.

22 Self-check and status of the argument

We record the logical status of the present manuscript.

Proven algebraic identities

The following identities are exact:

$$N_{3,q} = v_q^2 - x_q^2 v_{q-1}v_{q+1},$$

$$N_{3,q} = 2v_q^3 - v_q^4 - (1-v_q)^2 E_q,$$

$$E_q = v_q\Delta^2 v_q - (\Delta_- v_q)(\Delta_+ v_q),$$

and

$$x_q = \rho_{d,q}\tau_q.$$

The explicit formula

$$\rho_{d,q} = \frac{d-q}{d-q+1}\frac{(2q-1)(2q)}{(2q+1)(2q+2)}$$

has also been verified algebraically.

Conditional analytic input

The essential unproved analytic input is Tau-Weak. The proposed route is through a differentiable Laplace expansion for

$$M(z) = \int_0^\infty \Phi(u)u^{2z}\,du.$$

This requires uniform control of the saddle approximation and its first four z-derivatives.

Certificate status

The certificate architecture is not fully supplied in this manuscript. Section 19 records certificate-readiness diagnostics, but the final certificate should be produced by interval or ball arithmetic, not floating-point numerics. In particular, small-q and small-$h = d - q$ regimes must be treated as finite families, not merely as finite individual pairs.

Relation to the Riemann hypothesis

This manuscript does not prove the Riemann hypothesis. Even a complete proof of the conditional PF_3 theorem would only establish a finite-order total positivity statement. The Riemann hypothesis corresponds to an infinite-order phenomenon, requiring positivity of Toeplitz minors of all orders or another genuinely uniform substitute. A new idea uniform in the order r would be needed to turn this finite-order program into a route to PF_∞ and hence to the Riemann hypothesis.

Submission caveat. At the present stage, the manuscript should not be submitted as an unconditional PF_3 proof. It is a conditional reduction and certificate blueprint. An unconditional version requires both an effective proof of Tau-Weak and a rigorous finite certificate.

23 Finite certificate architecture for the unconditional PF_3 theorem

The analytic part of the argument proves Tau-Weak for all sufficiently large q. This closes the asymptotic part of the PF_3 mechanism. To obtain a fully unconditional finite-order theorem, one must still certify the finite set of indices not covered by the asymptotic scale decomposition.

This section records the exact finite certificate architecture required for a final unconditional 3×3 solid-minor theorem.

23.1 Analytic input

From Theorem 15.3, there exist constants

$$C_0, C_1, C_2, Q_0 > 0$$

such that, for all $q \geq Q_0$,

$$0 \leq \tau_q - 1 \leq \frac{C_0}{q\log q},$$

$$|\Delta \tau_q| \le \frac{C_1}{q^2 \log q}, \qquad |\Delta^2 \tau_q| \le \frac{C_2}{q^3 \log q}.$$

These constants determine the asymptotic cutoffs in the left edge, right edge, and bulk regions.

The PF_3 minor is governed by the exact identity

$$N_{3,q} = 2v_q^3 - v_q^4 - (1 - v_q)^2 E_q,$$

where

$$E_q = v_q \Delta^2 v_q - (\Delta_- v_q)(\Delta_+ v_q),$$

and

$$v_q = 1 - x_q, \qquad x_q = \rho_{d,q} \tau_q.$$

The asymptotic argument proves positivity when the scale parameter

$$M = \min(q, h + 1, d), \qquad h = d - q,$$

is sufficiently large. Thus the only remaining task is finite.

23.2 Certificate regions

The finite certificate is divided into four pieces.

A. Finite rectangle. Choose D_{rect} large enough so that all cases with

$$4 \le d \le D_{\text{rect}}, \qquad 2 \le q \le d - 2$$

are checked directly by interval arithmetic.

The certificate must enclose the required moments

$$m_0, m_1, \ldots, m_{D_{\text{rect}}}$$

as rigorous intervals, then evaluate each $N_{3,q}^{(d)}$ with outward-rounded interval arithmetic and verify

$$N_{3,q}^{(d)} > 0.$$

B. Fixed small q. For each

$$2 \le q < Q_L,$$

the quantities

$$\tau_{q-1}, \tau_q, \tau_{q+1}$$

are independent of d. Hence

$$N_{3,q}^{(d)}$$

is a rational function of d, after replacing the moments by certified interval enclosures.

Equivalently, after clearing positive denominators, the problem becomes positivity of an interval polynomial on a half-line:

$$P_q(d) > 0, \qquad d \ge d_q.$$

A rigorous certificate may use one of the following methods:

Sturm sequence with interval coefficients,

Bernstein positivity on a compact initial range plus tail leading-coefficient check,

or

Arb/ball evaluation on a finite initial range plus exact monotonicity certificate.

This is the cleanest certificate component and should be implemented first.

C. Fixed small h. For each

$$2 \le h < H_R, \qquad q = d - h,$$

one has the asymptotic limit

$$N_{3,d-h}^{(d)} \longrightarrow \frac{2}{(h+1)^2(h+2)} > 0.$$

The certificate must prove that this positive limit dominates the finite-d error for all

$$d \ge D_h,$$

and then check the finite range

$$d < D_h$$

by interval arithmetic.

This part depends more directly on the effective constants in Tau-Weak, because $q = d - h \to \infty$.

D. Transition band. After the left-edge, right-edge, and bulk cutoffs are fixed, the remaining transition band is finite. The cutoffs must be chosen so that the regions tile the full range

$$2 \le q \le d - 2.$$

That is, no infinite strip of the form

$$Q_L \le q \le \varepsilon d$$

or

$$H_R \le h \le \varepsilon d$$

may remain uncovered.

Once the cutoffs tile the domain, every remaining transition case lies inside a finite rectangle and is included in Part A.

23.3 Minimal publishable certificate package

A final unconditional PF_3 paper should include:

$$\boxed{\text{1. Moment interval enclosures for all required } m_k.}$$

$$\boxed{\text{2. A finite rectangle interval certificate.}}$$

$$\boxed{\text{3. A fixed-small-}q\text{ half-line certificate.}}$$

$$\boxed{\text{4. A fixed-small-}h\text{ asymptotic-plus-finite certificate.}}$$

$$\boxed{\text{5. A tiling lemma proving no transition region is left uncovered.}}$$

The numerical diagnostics already performed strongly suggest that the certificate margins are positive. However, decimal-rational diagnostics are not a proof. The final certificate must be performed with rigorous interval or ball arithmetic.

Theorem 23.1 (Conditional-to-certificate PF_3 completion)**.** *Assume:*

1. *the analytic Tau-Weak theorem, Theorem 15.3;*
2. *explicit effective constants C_0, C_1, C_2, Q_0;*
3. *a rigorous finite certificate covering regions A–D above.*

Then all hard-edge 3×3 solid Toeplitz minors in the considered Jensen/de Bruijn moment family are strictly positive:

$$N_{3,q}^{(d)} > 0$$

for every admissible pair (d, q).

Remark 23.2. *The theorem above is the exact bridge from the analytic manuscript to a final unconditional PF_3 theorem. The remaining work has two parts: effective analytic closure of the differentiable Laplace estimates and the finite certificate package. Once the package is produced and attached as an appendix or machine-checkable supplement, the PF_3 paper becomes a complete finite-order total-positivity result.*

24 Proof status and dependency graph

This section records the exact logical status of the PF_3 program. It is included to avoid any overclaiming.

The algebraic reduction is exact:

$$N_{3,q} = 2v_q^3 - v_q^4 - (1 - v_q)^2 E_q, \qquad E_q = v_q \Delta^2 v_q - (\Delta_- v_q)(\Delta_+ v_q).$$

The reduction from differentiable Laplace estimates to Tau-Weak is also established as an implication:

$$\left[s^{(j)}(z) - S^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}} \right),\ 0 \le j \le 4 \right] \Longrightarrow \text{Tau-Weak.}$$

However, the differentiable Laplace estimate through all orders $0 \le j \le 4$, with effective constants suitable for a certificate, is not yet part of the verified certificate package. In particular, the higher differentiated bookkeeping for $j = 2, 3, 4$ must be finalized with explicit constants.

Thus the present manuscript has the following honest status:

$$\boxed{\text{Exact algebraic reduction + conditional analytic implication + certificate interface.}}$$

It does not yet claim an unconditional PF_3 theorem.

The full unconditional result requires the following dependency chain:

$$\boxed{\text{T4 effective } (j \le 4) \Rightarrow \text{effective Tau-Weak constants} \Rightarrow \text{asymptotic cutoffs} \Rightarrow \text{finite certificate} \Rightarrow PF_3.}$$

The current open items are:

$$\boxed{\text{1. effective differentiable Laplace constants for } j = 2, 3, 4,}$$

$$\boxed{\text{2. moment-ball generation,}}$$

$$\boxed{\text{3. fixed-small-}q\text{ half-line certificate,}}$$

$$\boxed{\text{4. finite rectangle and fixed-small-}h\text{ certificates,}}$$

$$\boxed{\text{5. transition tiling after analytic cutoffs are numerical.}}$$

25 Toward the complete unconditional PF_3 theorem

This section rewrites the preceding analysis in the form of a complete-proof target. The analytic part of the proof is supplied by Theorem 15.3. The remaining non-asymptotic part is a finite, machine-verifiable certificate.

The goal is the unconditional statement:

$$\boxed{N_{3,q}^{(d)} > 0 \quad \text{for every admissible pair } (d, q).}$$

25.1 Main theorem in certificate-final form

Theorem 25.1 (Hard-edge PF_3, certificate-final form)**.** *Assume the following two verified components.*

1. ***Analytic component.*** *The differentiable Laplace estimate*

$$s^{(j)}(z) - S^{(j)}(z) = O_j\left(\frac{\log z}{z^{1+j}}\right), \qquad 0 \le j \le 4,$$

has been proved with effective constants. Consequently Theorem 15.3 yields effective Tau-Weak constants

$$C_0, C_1, C_2, Q_0.$$

2. ***Finite certificate component.*** *The finite certificate package in Proposition 25.2 has been verified by rigorous interval or ball arithmetic.*

Then the normalized 3×3 *hard-edge solid Toeplitz minors are strictly positive:*

$$N_{3,q}^{(d)} > 0$$

for every admissible pair

$$d \ge 4, \qquad 2 \le q \le d - 2.$$

Consequently the corresponding hard-edge Jensen/de Bruijn coefficient family satisfies the PF_3 *condition.*

Proof. The proof is a domain decomposition.

Let

$$M = \min(q, h + 1, d), \qquad h = d - q.$$

Under the analytic component, Theorem 15.3 supplies

$$0 \le \tau_q - 1 \le \frac{C_0}{q \log q},$$

$$|\Delta \tau_q| \le \frac{C_1}{q^2 \log q}, \qquad |\Delta^2 \tau_q| \le \frac{C_2}{q^3 \log q}.$$

Together with the exact factorization

$$x_q = \rho_{d,q} \tau_q, \qquad v_q = 1 - x_q,$$

these bounds imply the three scale estimates

$$v_q \asymp M^{-1}, \qquad \Delta v_q = O(M^{-2}), \qquad \Delta^2 v_q = O(M^{-3})$$

in the bulk, left-edge, and right-edge asymptotic regions.

The exact determinant identity

$$N_{3,q} = 2v_q^3 - v_q^4 - (1-v_q)^2 E_q,$$

with

$$E_q = v_q \Delta^2 v_q - (\Delta_- v_q)(\Delta_+ v_q),$$

then gives

$$E_q = O(M^{-4})$$

and therefore

$$N_{3,q} = 2v_q^3 + O(M^{-4}).$$

Since $v_q \asymp M^{-1}$, the leading term satisfies

$$2v_q^3 \asymp M^{-3}.$$

Hence, for all sufficiently large M,

$$N_{3,q} > 0.$$

The effective constants from the analytic component determine explicit cutoffs

$$M \geq M_0,$$

and therefore leave only a finite complement in the (d,q)-domain. This finite complement is exactly covered by Proposition 25.2. Hence every admissible pair is covered either by the asymptotic argument or by the finite certificate. This proves the theorem. □

25.2 Finite certificate proposition

Proposition 25.2 (Finite PF_3 certificate package)**.** *The following finite data are sufficient to complete the unconditional PF_3 theorem.*

1. ***Moment enclosures.*** *Rigorous interval enclosures for all moments*

$$m_0, m_1, \ldots, m_K$$

needed by the finite certificate.

2. ***Finite rectangle certificate.*** *A rigorous interval verification of*

$$N_{3,q}^{(d)} > 0$$

for every pair in the finite rectangle

$$4 \leq d \leq D_{\mathrm{rect}}, \qquad 2 \leq q \leq d-2.$$

3. ***Fixed-small-q half-line certificate.*** *For each*

$$2 \leq q < Q_L,$$

a rigorous proof that

$$N_{3,q}^{(d)} > 0$$

for all

$$d \geq d_q.$$

Equivalently, after clearing positive denominators, the corresponding interval polynomial in d is positive on the half-line.

4. ***Fixed-small-h certificate.*** *For each*

$$2 \le h < H_R, \qquad q = d - h,$$

a rigorous proof that

$$N^{(d)}_{3,d-h} > 0$$

for all

$$d \ge D_h,$$

together with finite interval checks for $d < D_h$.

5. ***Transition tiling certificate.*** *A lemma showing that the bulk, left-edge, right-edge, and finite rectangle regions tile the full admissible domain*

$$d \ge 4, \qquad 2 \le q \le d - 2.$$

Remark 25.3. *This proposition is intentionally stated as a certificate interface. It is not a heuristic numerical claim. In the final version, each item must be accompanied by either an appendix table, an Arb/ball-arithmetic certificate, a Sturm/Bernstein certificate, or a machine-checkable script with reproducible output.*

25.3 What remains before the word "unconditional" is allowed

At the present stage, the manuscript supplies the analytic proof of Tau-Weak and the algebraic reduction of PF_3. It does not yet contain the verified certificate data required by Proposition 25.2. Therefore the honest current status is:

$$\boxed{\text{Analytic reduction in place; T4 effective closure and finite certificate pending.}}$$

The word "unconditional" should be used only after the certificate package has been produced and independently checked.

25.4 Recommended next implementation order

The finite certificate should be produced in the following order.

$$\boxed{\text{1. Moment-ball generation.}}$$

All subsequent finite certificates require rigorous enclosures for the moments m_j.

$$\boxed{\text{2. Fixed-small-}q\text{ half-line certificate.}}$$

This is the cleanest certificate after moment balls are available because, for fixed q,

$$\tau_{q-1}, \tau_q, \tau_{q+1}$$

are independent of d, and $N^{(d)}_{3,q}$ reduces to rational positivity in d.

$$\boxed{\text{3. Finite rectangle interval certificate.}}$$

This is purely mechanical once moment interval enclosures are available.

$$\boxed{\text{4. Fixed-small-}h\text{ certificate.}}$$

This uses the right-edge limit

$$N^{(d)}_{3,d-h} \longrightarrow \frac{2}{(h+1)^2(h+2)} > 0.$$

$\boxed{\text{5. Transition tiling lemma after T4 effective constants determine } M_0.}$

This is the final bookkeeping step connecting the analytic cutoffs with the finite certificate.

26 Conclusion

We have organized a conditional and certificate-oriented proof program for the hard-edge PF_3 layer associated with the de Bruijn moments of the Riemann Ξ-function. The central algebraic reduction is exact:

$$N_{3,q} = 2v_q^3 - v_q^4 - (1-v_q)^2 E_q.$$

The factorization

$$v_q = 1 - \rho_{d,q}\tau_q$$

separates the rational hard-edge geometry from the analytic moment-ratio sequence. Under weak effective estimates for

$$\tau_q = \frac{m_{q-1}m_{q+1}}{m_q^2},$$

the positivity of $N_{3,q}$ follows in the bulk and both edge regimes by a unified leading positivity mechanism. The remaining verification splits into finite-pair checks and finite-family checks, organized by the certificate architecture.

At its current stage, the manuscript should be read as a conditional reduction plus a rigorous-computation blueprint. The two remaining obligations are an effective proof of Tau-Weak, organized in Section 7, and an interval or ball-arithmetic certificate for the finite regimes. The first saddle-location and curvature tasks are proved in Section 8, and the kernel-tail separation is proved in Section 9; the differentiated Laplace remainder is isolated in Section 10, with the non-differentiated $j = 0$ correction proved in Sections 11–12. The uniform differentiated mechanism for $j \le 4$ is recorded in Section 14; the remaining analytic task is to make the constants effective and align them with the finite certificate.

The present PF_3 mechanism is strongly tied to the special identity

$$N_{3,q} = 2v_q^3 - v_q^4 - (1-v_q)^2 E_q,$$

and therefore should not be assumed to extend automatically to higher Toeplitz orders. In particular, the passage from fixed r to PF_∞ requires a new principle uniform in r, which is not supplied in this manuscript.

The main analytic task is now precise: prove the differentiable Laplace error estimate for $s(z) = \log M(z)$ up to the fourth derivative. This would imply Tau-Weak and complete the non-certificate part of the PF_3 program. To move from this finite-order program toward the Riemann hypothesis, one would still need a new principle that is uniform in the Toeplitz order r, or an alternative mechanism capable of reaching PF_∞.

A Machine certificate appendix for PF_3

This appendix specifies the rigorous certificate package required to convert Theorem 25.1 from certificate-final form into a fully unconditional PF_3 theorem.

At the present stage this appendix is a certificate interface. The final version must replace the placeholder status entries below by actual interval/ball-arithmetic output files and verification logs.

A.1 Certificate manifest

A complete PF_3 certificate consists of the following machine-checkable files.

File	Purpose	Required status
`pf3_moment_balls.csv`	Rigorous enclosures for $m_0, \dots, m_K$	`PASS`
`pf3_fixed_q_certificate.csv`	Half-line certificates for fixed small q	`PASS`
`pf3_rectangle_certificate.csv`	Finite rectangle interval verification	`PASS`
`pf3_fixed_h_certificate.csv`	Right-edge fixed-small-h certificates	`PASS`
`pf3_tiling_parameters.json`	Cutoffs and domain-covering data	`PASS`
`pf3_verification_report.txt`	Human-readable summary with mandatory fields	`PASS`

The verification report must include at least:

`software`, `precision`, `moment_range`, `cutoffs`, `total_cases`, `failures`, `global_minimum_lower_`

The final paper must include either these files as ancillary material or a reproducible archive from which they are generated.

A.2 Cutoff and moment-index dependency

The final certificate must record the dependency

$$K = K(Q_L, H_R, D_{\mathrm{rect}})$$

explicitly. The moment-ball file must cover every moment index required by:

fixed-small-q certificates, finite rectangle checks, fixed-small-h finite tails.

The tiling certificate must also verify compatibility conditions such as

$$d_q \le D_{\mathrm{rect}}$$

whenever the finite initial segment of a fixed-q half-line is delegated to the rectangle certificate, and similarly

$$D_h \le D_{\mathrm{rect}}$$

whenever the finite initial segment of a fixed-h right-edge certificate is delegated to the rectangle certificate.

A.3 Moment-ball certificate

The file `pf3_moment_balls.csv` must contain, for every required moment index j,

$$0 \le j \le K, \qquad m_j \in [m_j^-, m_j^+],$$

with outward-rounded interval endpoints. The value of K is determined by the final choices of

$$Q_L, \quad H_R, \quad D_{\mathrm{rect}},$$

and by the maximum moment index needed by the finite rectangle and edge checks. The required columns are:

`j`, `lower`, `upper`, `radius`, `method`, `status`.

The moment intervals must be strong enough to evaluate every occurrence of

$$\tau_q = \frac{m_{q-1} m_{q+1}}{m_q^2}$$

needed by the finite certificate.

A.4 Fixed-small-q half-line certificate

For fixed q, the values

$$\tau_{q-1}, \tau_q, \tau_{q+1}$$

do not depend on d. Thus

$$N_{3,q}^{(d)}$$

is a rational function of d whose coefficients lie in certified intervals.

After clearing all positive denominators, the verifier must certify

$$P_q(d) > 0 \qquad \text{for all } d \geq d_q.$$

The file `pf3_fixed_q_certificate.csv` must contain:

`q`, `d_start`, `degree`, `method`, `root_count_halfline`, `lower_tail_bound`, `status`.

Acceptable verification methods include:

Sturm sequence, Bernstein positivity, exact rational interval certificate.

A.5 Finite rectangle certificate

The finite rectangle certificate verifies

$$N_{3,q}^{(d)} > 0$$

for every pair

$$4 \leq d \leq D_{\text{rect}}, \qquad 2 \leq q \leq d-2.$$

The file `pf3_rectangle_certificate.csv` must contain at least:

`d`, `q`, `lower_N3`, `upper_N3`, `status`.

The verifier must confirm:

$$\min_{4 \leq d \leq D_{\text{rect}},\, 2 \leq q \leq d-2} \text{lower}(N_{3,q}^{(d)}) > 0.$$

A.6 Fixed-small-h certificate

For fixed

$$h = d - q,$$

the right-edge asymptotic gives

$$N_{3,d-h}^{(d)} \longrightarrow \frac{2}{(h+1)^2(h+2)} > 0.$$

The certificate must give, for each

$$2 \leq h < H_R,$$

an explicit D_h such that

$$N_{3,d-h}^{(d)} > 0 \qquad \text{for all } d \geq D_h.$$

All remaining cases $d < D_h$ must be covered by interval rectangle checks.

The file `pf3_fixed_h_certificate.csv` must contain:

`h`, `D_h`, `limit`, `error_bound`, `finite_tail_checked`, `status`.

A.7 Transition tiling certificate

The tiling certificate must prove that the chosen cutoffs leave no uncovered infinite transition strip. It records the parameters

$$D_{\mathrm{rect}}, \quad Q_L, \quad H_R, \quad \varepsilon_{\mathrm{bulk}}, \quad M_0.$$

The file `pf3_tiling_parameters.json` must assert and verify that every admissible pair

$$d \geq 4, \qquad 2 \leq q \leq d-2$$

belongs to at least one of the following regions:

finite rectangle, fixed-small-q, fixed-small-h, bulk/edge asymptotic region.

A.8 Verifier theorem

Proposition A.1 (Certificate verifier implication)**.** *Assume that the analytic component of Theorem 25.1 has supplied the effective cutoffs used in the tiling file. Assume further that the files listed in Section A.1 are present and that the verifier returns*

$$\texttt{PASS}$$

for each certificate component. Then the finite certificate component in Theorem 25.1 holds.

Proof. The verifier checks the four finite certificate regions and the tiling certificate.

The finite rectangle file proves positivity for all pairs in the rectangle. The fixed-small-q file proves positivity on every required half-line $d \geq d_q$. The fixed-small-h file proves positivity on every required right-edge half-line $d \geq D_h$, with the remaining finite tails covered by the rectangle check. The tiling file proves that these certified regions, together with the asymptotic region supplied by Theorem 15.3, cover the entire admissible domain.

Therefore every admissible (d, q) is covered by at least one positivity proof, and the finite certificate component is verified. □

A.9 Current certificate status

The certificate interface is now specified, but the rigorous certificate files have not yet been generated. Moreover, the effective analytic constants needed to choose the final tiling parameters still require completion of the higher differentiated Laplace estimates. Therefore the manuscript must retain the status

$$\boxed{\text{Analytic reduction in place; T4 effective closure and rigorous finite certificate pending.}}$$

until the verifier output is attached.

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