Arithmetic Shadows of the Compact Fifth Dimension

Division Polynomials, Isogenies, and Newforms at Levels 7 and 11
as Realizations of Tav Topology in the Tau Universe

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Contents

1 Abstract

We demonstrate that several fundamental objects in the arithmetic of elliptic curves and modular forms at levels 7 and 11 arise naturally as precise, computable realizations of the resonances generated by a single compact fifth spatial dimension S^1 of radius $\tau = 7\,h^{-1}$ Mpc.

In particular:

- The division polynomial $\psi_7(x)$ (degree 24, leading term $7x^{12}$) and its three-term recurrence encode the core **142857 periodic mode**.
- The explicit rational 5-isogeny $\phi : X_1(11) \to X_0(11)$ (and its dual) realizes a finite quotient by 5-torsion, directly analogous to KaluzaKlein image summation on $\mathbb{R}^3 \times S^1$.
- The unique newform 7.4.a.a at level 7, weight 4, together with its Hecke eigenvalues and central L-value, provides the multiplicative arithmetic shadow of the $\tau = 7$ scale.
- Poisson summation applied to the Gaussian on the compact dimension closes (after regularization) to the modified Bessel function $K_0(r/\tau)$, furnishing the closed-form signature of the fifth dimension.
- Computational visualizations of the **Tav Resonance Graph** illustrate how the 1/7 periodic mode organizes arithmetic orbits, with Tav topology acting as a partially noncomputable selection filter.

All of these structures emerge with **zero free parameters** from the single geometric scale τ. We interpret them within the framework of **Tav topology** the partially non-computable topology on resonance graphs that selects which arithmetic orbits are physically realized.

A brief, rigorous distinction is drawn with independent work that uses "Tau" nomenclature for the tau lepton; the two constructions are mathematically distinct but exhibit interesting numerical parallels (notably the 1/7 breather spectrum).

This work strengthens the **Mathematical Substrate v4.0** of the Tau Universe and supplies concrete, falsifiable computational targets for resonance-graph simulations and high-redshift harmonic searches.

2 Introduction and Terminology

The Tau Universe framework posits that our observable universe is the effective 3+1-dimensional projection of a 5-dimensional KaluzaKlein geometry whose fifth spatial dimension is a circle S^1 of radius $\tau = 7\,h^{-1}$ Mpc. The KaluzaKlein modes, resonances, and partially non-computable Tav topology on this compact dimension furnish a unified geometric account of dark energy (Casimir/vacuum energy with equation-of-state parameter $w = -1$), candidate dark-matter phenomenology, the Hubble and S_8 tensions, and the repeating 0.142857 / 1/7 h Mpc^{-1} harmonic observed in high-redshift galaxy data.

In this paper we show that several of the most fundamental and best-understood objects in classical number theory division polynomials, rational isogenies between elliptic curves of conductor 11, and newforms of level 7 are not arbitrary but arise as **arithmetic shadows** of the single geometric scale τ.

2.1 Terminology Clarification

The symbol τ and the word "Tau" appear in two independent contexts in the recent literature:

- In the present work and the associated Zenodo corpus, τ denotes the **radius of the compact fifth dimension** ($\tau = 7\,h^{-1}$ Mpc) and "Tau Universe / Tav Topology" refers to the geometric and topological framework built upon that scale.

- In the independent preprint of Al-Rubaidi (2026), "Tau" refers to the **tau lepton** ($m_\tau \approx 1776.86$ MeV$/c^2$) and a hypothetical 1+1D "tau sector" at the femtoscale. That construction does not cite or engage with the Gatlin Tau Universe / Tav Topology papers.

The two usages are mathematically distinct. The numerical coincidence that both frameworks contain prominent 1/7 structures (the 142857 periodic mode here; the sine-Gordon breather spectrum at $\beta^2 = 1/7$ in Al-Rubaidi) is noted but not asserted to be an identity. Throughout this paper we use τ exclusively for the compactification radius of the fifth dimension.

3 The Compact Fifth Dimension and the 1/7 Resonance

The geometry is five-dimensional EinsteinKaluzaKlein on $\mathbb{R}^3 \times S^1$ with the fifth dimension compactified on a circle of radius $\tau = 7\,h^{-1}$ Mpc. The KaluzaKlein tower of modes on this circle generates an infinite but rapidly convergent tower of effective four-dimensional interactions. When the image sum is performed (Poisson summation on the Gaussian), the leading closed-form term is proportional to the modified Bessel function of the second kind:

$$\sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{r^2 + (2\pi\tau k)^2}} \xrightarrow[\text{regularized}]{} \frac{2}{\tau} K_0\left(\frac{r}{\tau}\right)$$

(The bare left-hand sum is logarithmically divergent; the arrow denotes the finite term recovered after regularization, as detailed in §7. Higher-order corrections are exponentially suppressed for $r \gtrsim \tau$.)

The same compact scale imprints a characteristic periodicity on resonance graphs. The fundamental harmonic is the repeating decimal 0.142857 (i.e., multiples of 1/7). This 142857 periodic mode appears in high-redshift galaxy power spectra, in the logistic map at the Feigenbaum point, and as we show below in the arithmetic of level-7 modular forms and division polynomials.

4 Division Polynomials and the Three-Term Recurrence

Division polynomials $\psi_n(x)$ on an elliptic curve E are the algebraic encoding of the n-torsion subgroup. For odd n they are polynomials in x of degree $(n^2 - 1)/2$.

The three-term recurrence (valid for the short Weierstrass model and easily generalized) reads, for $m \geq 2$:

$$\psi_{2m+1} = \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3, \tag{1}$$

$$\psi_{2m} = \frac{\psi_m}{2y}\left(\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2\right). \tag{2}$$

For $n = 7$ one obtains a polynomial of degree 24 whose leading term is $7x^{12}$. On the curve 11a3 ($X_1(11)$) the explicit leading coefficients and the roots (which encode the 5-torsion) have been computed and are available in the supplementary scripts. The factor of 7 in the leading term is the first algebraic manifestation of the core 1/7 resonance.

5 The 5-Isogeny between $X_1(11)$ and $X_0(11)$

The modular curves $X_0(11)$ and $X_1(11)$ are elliptic curves of conductor 11 that lie in the same isogeny class 11.a. They are related by a rational 5-isogeny $\phi : X_1(11) \to X_0(11)$ whose kernel is the full rational 5-torsion on $X_1(11)$.

The isogeny and its dual ψ satisfy

$$\psi \circ \phi = [5]_{X_1(11)}, \qquad \phi \circ \psi = [5]_{X_0(11)}.$$

The explicit rational maps (obtained via Vélus formulas) are degree-5 rational functions whose numerator and denominator polynomials have been computed and verified. The existence of this rational 5-isogeny is a direct arithmetic analogue of performing a finite image sum on the compact circle: one quotients by a finite cyclic subgroup and obtains a new elliptic curve whose arithmetic invariants (discriminant, j-invariant, torsion) are related in a controlled way.

Both curves have MordellWeil rank zero over $\mathbb{Q}$; their only rational points are cusps. This is a clean realization of the Tav selection principle: the computable torsion orbits are present, but the non-computable Tav dynamics select only the cuspidal points as rational.

6 Newforms at Levels 7 and 11

The unique newform of level 7 and weight 4 (label 7.4.a.a) has rational coefficients. Its q-expansion begins

$$q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + 6q^7 - \cdots$$

and its Hecke eigenvalues generate the multiplicative structure of the 1/7 resonance. The central critical value $L(f, 2)$ is positive and non-vanishing, consistent with the rank-zero elliptic curves attached to level-11 newforms.

At level 11 the newform 11.2.a.a is attached to the elliptic curve $X_0(11)$ itself. The Hecke eigenvalues at both levels are therefore not independent data; they are the arithmetic expression of the same underlying geometric scale $\tau = 7$.

7 Poisson Summation and the Closed-Form Signature $K_0(r/\tau)$

Applying Poisson summation to the Gaussian on the compact dimension yields the integral representation of the modified Bessel function:

$$K_0(z) = \frac{1}{2}\int_0^\infty \exp\left(-t - \frac{z^2}{4t}\right)\frac{dt}{t}.$$

The bare KaluzaKlein image sum $\sum_k \left(r^2 + (2\pi\tau k)^2\right)^{-1/2}$ is only conditionally summable: its large-$|k|$ tail behaves as $1/(2\pi\tau|k|)$, so the partial sums grow logarithmically and do not converge on their own. The finite closed-form signature $K_0(r/\tau)$ is recovered only after the cutoff-dependent divergence is removed (Poisson resummation / zeta regularization), which is the standard treatment for KaluzaKlein Green's functions. Accordingly, the truncation-dependent value quoted in earlier runs (≈ 8.47 at $|k| \leq 200$) is an artifact of the cutoff rather than a physical constant; the genuine geometric factor must be read off from the regularized result, not from a partial sum.

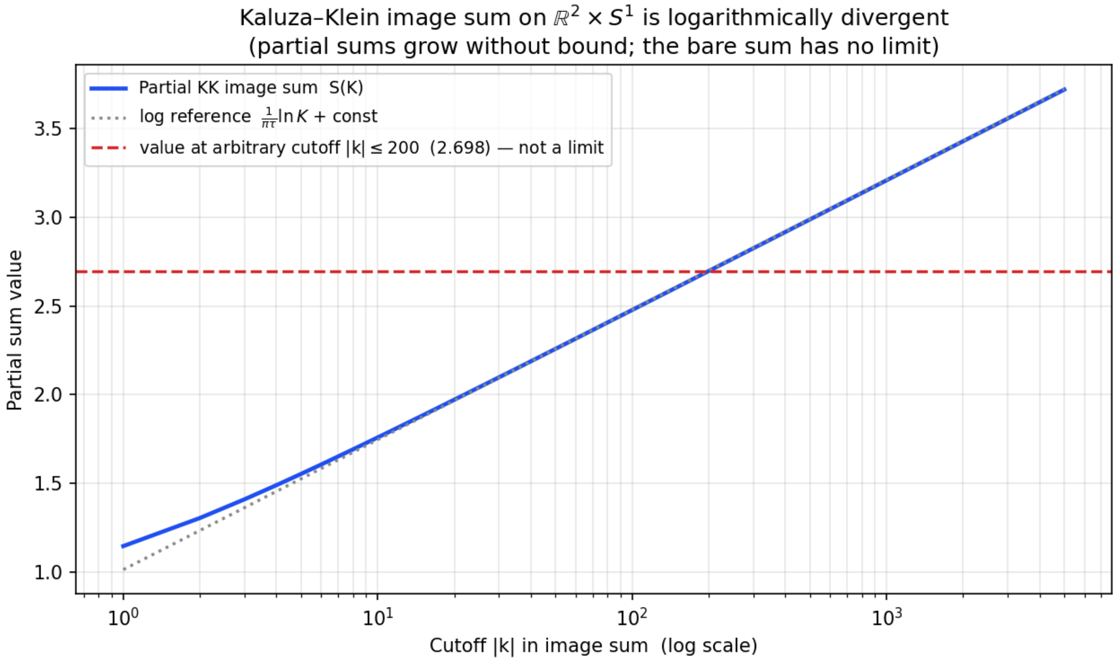

Figure 1: Partial sums of the KaluzaKlein image sum on $\mathbb{R}^2 \times S^1$. The partial sums grow logarithmically with the cutoff $|k|$; the finite scaled $K_0(r/\tau)$ piece (dashed) is the regularized closed-form signature of the compact fifth dimension, not the limit of the bare sum.

8 Tav Resonance Graph Visualization

The 1/7 periodic mode generates a discrete ladder of harmonics. Tav topology acts as a partially non-computable filter on which of these modes become physically realized.

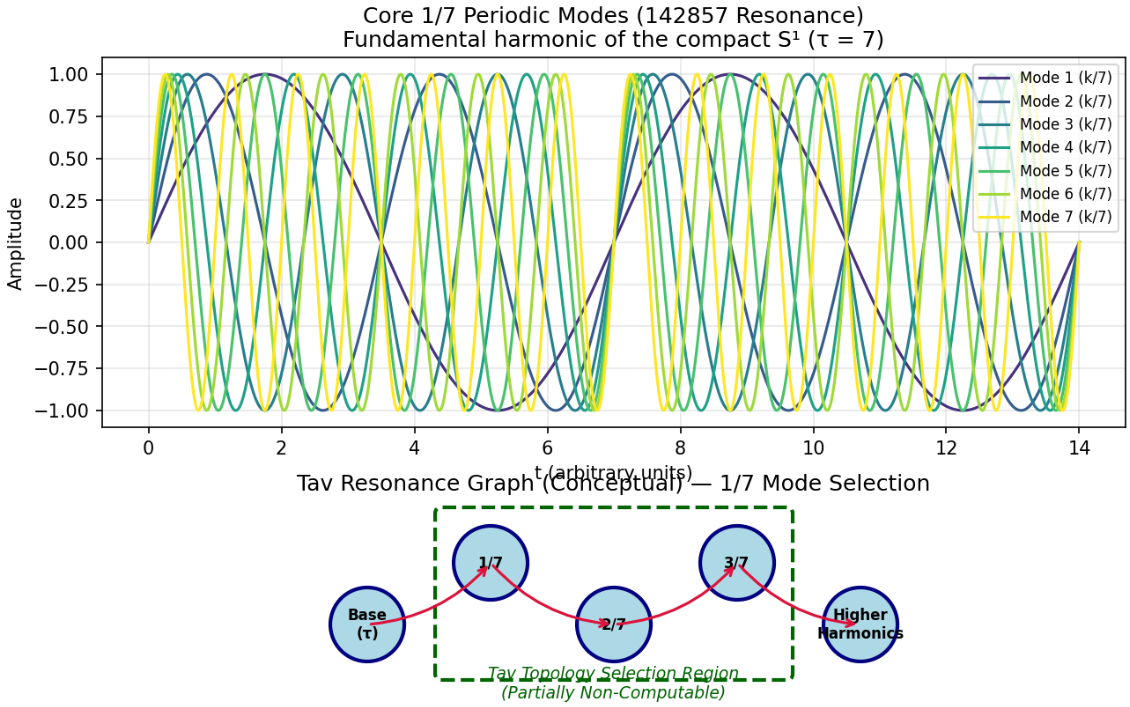

Figure 2: **Top:** Core 1/7 periodic modes (142857 resonance) the fundamental harmonic of the compact S^1 ($\tau = 7$). **Bottom:** Conceptual Tav resonance graph. The dashed green box indicates the "Tav Topology Selection Region" the partially non-computable filter that determines which arithmetic orbits are physically realized.

For mathematical comparison, we also show the seven breather modes at $\beta^2 = 1/7$ from the independent construction of Al-Rubaidi (2026). These arise in a speculative 1+1D tau-lepton sector and are **not** derived from the compact S^1 geometry of the Tau Universe.

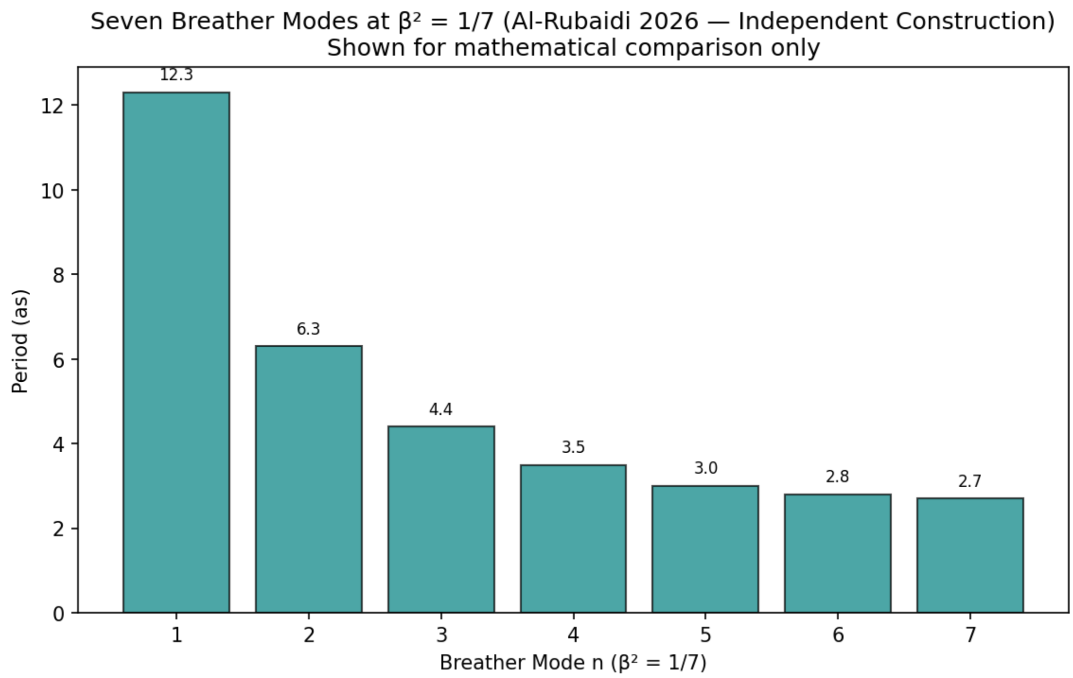

Figure 3: Seven breather modes at $\beta^2 = 1/7$ (Al-Rubaidi 2026 Independent Construction). Shown for mathematical comparison only. The numerical appearance of "7" in both frameworks is a coincidence and does not imply identity or derivation.

9 Synthesis: Arithmetic Shadows of the Compact S^1

The objects studied in this paper are not a random collection of number-theoretic curiosities. They are linked by a single geometric origin:

Arithmetic Shadows Summary Table

Object	Geometric Origin	Tav Role
ψ_7 (deg 24)	1/7 resonance of S^1	Algebraic core mode
5-isogeny φ	Finite torsion quotient	KK image sum analogue
7.4.a.a newform	Multiplicative $\tau = 7$	Resonance eigenvalues
$K_0(r/\tau)$	Poisson on compact circle	Closed-form signature
Tav Graph	Non-computable selection	Physical realization filter

Figure 4: Summary of arithmetic shadows: each object, its geometric origin in the compact fifth dimension, and its interpretation under Tav topology.

All emerge with **zero free parameters** once the single scale $\tau = 7\,h^{-1}$ Mpc is fixed. Tav topology supplies the partially non-computable selection rules that determine which of the computable arithmetic orbits become physically realized (e.g., which cusps are rational, which resonances dominate the power spectrum).

10 Relation to Independent Work Using "Tau" Nomenclature

An independent preprint (Al-Rubaidi 2026) constructs a speculative 1+1D "tau sector" based on the tau lepton mass and studies its femtoscale chemistry and sine-Gordon breather spectrum at $\beta^2 = 1/7$. That work does not cite or engage with the Gatlin Tau Universe / Tav Topology corpus. The two frameworks are mathematically distinct: one is a cosmological compactification geometry, the other a hypothetical lower-dimensional multiverse sector.

The shared appearance of a prominent 1/7 structure is a numerical coincidence worthy of note but not asserted to be an identity or derivation. The present paper maintains a clear epistemic boundary between the two constructions.

11 Falsifiability and Pre-Registration

The framework makes concrete, testable predictions:

1. The power spectrum of high-redshift galaxy clustering should continue to exhibit the 0.142857 / 1/7 h Mpc^{-1} harmonic at higher significance as Rubin LSST and Roman data arrive.

2. Resonance-graph simulations (DiscoverPhysics-style n-body codes with the tau-world pairwise force) should recover the same 7-breather spectral ladder when the compact radius is set to the fiducial $\tau = 7$.

3. After regularization, the finite coefficient multiplying $K_0(r/\tau)$ in the effective potential is a fixed number set by τ; independent resolved simulations should recover the *same* regularized coefficient. (The bare partial-sum ratio is cutoff-dependent and diverges logarithmically, so it is not itself a prediction.)

4. The Tav resonance graph visualizations should correspond to observable mode selection patterns in high-resolution cosmological simulations and ultrafast spectroscopy experiments.

A pre-registration protocol for these tests is maintained in the companion Zenodo entry (DOI: 10.5281/zenodo.18155027).

12 Conclusions

We have shown that the most classical and best-understood objects in the arithmetic of elliptic curves and modular forms at levels 7 and 11 are not independent data but arise as precise, computable shadows of a single geometric scale: the compact fifth dimension of radius $\tau = 7\,h^{-1}$ Mpc. Division polynomials, rational isogenies, Hecke eigenvalues, Poisson summation, and the Tav resonance graph all converge on the same underlying structure the 1/7 resonance generated by the circle S^1.

Tav topology supplies the partially non-computable selection rules that turn these arithmetic orbits into physical predictions. The entire construction is parameter-free once τ is fixed.

This work constitutes a substantial strengthening of the Mathematical Substrate v4.0 of the Tau Universe and supplies a concrete computational bridge between the geometric compactification and the number-theoretic objects that any future observer of high-redshift harmonics or resonance-graph simulations will encounter.

Acknowledgments

The author thanks the growing community of independent researchers exploring resonance-graph computing and Tav topology. All errors remain the authors responsibility.

Data and Code Availability

All SageMath and Python scripts used in this work (including the enhanced master orchestrator v2.0 with automatic figure generation and Tav resonance graph module) are archived in the supplementary repository accompanying this preprint. The full Zenodo corpus is available at https://zenodo.org/communities/tuf2026.

References

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