A Dynamic Graph Fluid System

Toward a Graph-Theoretic Analog of Fluid Flow,
Self-Assembly, and Terminal Circulation

Edward Lendward Smith

Independent Mathematician

June 2, 2026

Abstract

We introduce the Dynamic Graph Fluid System (dgfs) — a mathematical framework in which vertices behave as particles that bond, break, and rebond dynamically, flowing through a self-assembling graph topology under an intrinsic driving force. Particles bond with their lowest-resistance neighbors subject to a uniform valence constraint and a type-affinity function, carving out graph structure as they flow. Persistent bonds among same-type particles give rise to coherent subgraphs called *eigenstructures*, characterised by a dominant leading eigenvalue. All particles ultimately drain into a terminal basin — the *lake* — whose boundary is dynamically expanding and whose interior supports perpetual, potentially stratified circulation with no escape.

We prove that when the lake stabilises to a complete graph K_m on m same-type particles with valence $v \geq m-1$, and under the steady-accumulation condition that each particle enters L via exactly one boundary-crossing bond, the inflow Φ and internal circulation ΔC satisfy the exact relation

$$\Phi = \frac{2m}{(m-1)(m-2)}\,\Delta C.$$

We further prove a *Necessary Growth Condition*: the circulation balance implies that $m(t)$ must be strictly increasing whenever $\Phi(t) > 0$ — a static lake under nonzero inflow is inconsistent with the balance equation. The framework is entirely intrinsic, requiring no background geometry, and applies universally to any system whose components interact and accumulate according to least-resistance bonding principles.

Keywords: graph theory, fluid dynamics, dynamic networks, self-assembly, porous medium equation, flow networks, eigenstructures, terminal basin, circulation balance.

MSC 2020: 05C21, 76S05, 37N10, 05C82.

Contents

1 Motivation

Classical flow networks treat graphs as fixed scaffolds through which flow moves. The graph is given in advance; the flow is assigned to its edges. This framework inverts that relationship. Here the graph is not given — it is carved out by the flow itself. Particles are the primitive objects. Edges are what happen between particles when they meet and bond. The topology of the system is a consequence of the dynamics, not a precondition for them.

The motivating physical analogy is a river system. Water does not follow a pre-drawn map of channels — it carves those channels by flowing. Tributaries form, merge, and recombine. Particles bond transiently, travel together, then break and rebond with new neighbours. Not all particles are of the same type: oil flowing within water bonds preferentially to other oil particles, maintaining internal coherence — an eigenstructure — even while sharing the same pathway as the surrounding water. Everything ultimately drains to a terminal basin, a lake, where flow accumulates and circulates without escape, potentially in stratified layers of different particle types.

Critically, the framework is not specific to water or to any physical fluid. It applies to any system whose components interact, bond, and accumulate according to least-resistance principles — information flowing through a network, resources moving through a supply chain, influence spreading through social connections, or dependencies resolving in a computational graph. The mathematics is the same in all cases.

2 The Particle Space

Definition 2.1 (Particle Space)**.** Let $P = \{p_1, p_2, \ldots, p_n\}$ be a finite collection of particles. The Dynamic Graph Fluid System at time $t \geq 0$ is the pair $G(t) = (V(t), E(t))$ where:

- $V(t) = P$ for all t. The vertex set is fixed. Particles are never created or destroyed.
- $E(t) \subseteq P \times P$ is the edge set at time t, encoding which particles are currently bonded. $E(t)$ evolves continuously.

The graph $G(t)$ is a time-indexed family of graphs on a fixed vertex set. All structural complexity lives in the evolution of $E(t)$.

3 Valence and Availability

Assumption 3.1 (Uniform Valence)**.** Every particle $p_i \in P$ has the same valence $v \in \mathbb{N}$, $v \geq 1$ — the maximum number of simultaneous bonds it may hold.

Definition 3.2 (Degree and Availability)**.** The *degree* of p_i at time t is $\deg(p_i, t) = |\{p_j : (p_i, p_j) \in E(t)\}|$. A particle is *available* if $\deg(p_i, t) < v$ and *saturated* if $\deg(p_i, t) = v$.

Remark 3.3 (Valence–Lake Compatibility). The lake axioms (Section 9) require that incoming particles can bond to boundary particles in $L(t)$ and that internal circulation can be maintained. Both conditions impose a lower bound on valence. Specifically, for Theorem 10.3 to apply — where L stabilises to the complete graph K_m — each particle in L must hold $m - 1$ bonds simultaneously, so the system must satisfy $v \geq m - 1$. When $v < m - 1$ the lake cannot reach full connectivity and the circulation formula of Theorem 10.3 does not apply; the generalisation to valence-constrained lakes is addressed in Open Question OQ2. Throughout the paper, whenever Theorem 10.3 is invoked, the condition $v \geq m - 1$ is assumed.

4 Particle Types and Affinity

Definition 4.1 (Particle Type and Affinity)**.** Assign to each particle p_i a type $\sigma(p_i) \in \Sigma$, where Σ is a finite set of types. The *affinity function* $\alpha : \Sigma \times \Sigma \to [0, 1]$ satisfies:

- $\alpha(\sigma, \sigma) = 0$ for all σ: same-type bonding carries no penalty.
- $\alpha(\sigma, \sigma') > 0$ for $\sigma \neq \sigma'$: cross-type bonding carries a positive penalty.

Remark 4.2 (Bonding Cost Scale). The bonding cost $c(p_i, p_j, t) = \varrho(p_j, t) + \alpha(\sigma(p_i), \sigma(p_j))$ adds an occupation ratio $\in [0, 1]$ to an affinity penalty $\in [0, 1]$. The two terms are on the same scale by construction: the occupation ratio measures fractional saturation, and the affinity penalty is normalised to $[0, 1]$ by definition. When $\alpha = 1$ the cross-type penalty equals the maximum possible occupation cost, making a fully free cross-type partner no cheaper than a half-saturated same-type partner. The relative weight between terms is a modelling parameter; multiplicative alternatives are discussed in Open Question OQ5.

5 The Intrinsic Driving Force

Definition 5.1 (Occupation Ratio and Driving Force)**.** The *occupation ratio* of p_i at time t is

$$\varrho(p_i, t) = \frac{\deg(p_i, t)}{v} \in [0, 1].$$

The *intrinsic driving force* is $F(p_i, t) = 1 - \varrho(p_i, t)$, ranging from 1 (completely free) to 0 (fully saturated). F is entirely intrinsic: no background geometry or external physics is required.

6 The Bonding Rule

Definition 6.1 (Bonding Cost)**.** The *bonding cost* between available particles p_i and p_j at time t is

$$c(p_i, p_j, t) = \varrho(p_j, t) + \alpha(\sigma(p_i), \sigma(p_j)) \, .$$

Definition 6.2 (Bonding Rule — Mutual Least-Cost Selection)**.** A bond (p_i, p_j) forms at time t if and only if:

1. Both p_i and p_j are available,
2. $p_j = \arg\min_{p_k \in \mathcal{N}(p_i, t)} c(p_i, p_k, t)$, and
3. $p_i = \arg\min_{p_k \in \mathcal{N}(p_j, t)} c(p_j, p_k, t)$.

Bonding requires mutual least-cost selection. A bond forms only where both parties prefer each other simultaneously.

Property 6.3 (Conflict-Free Bonding)**.** The mutual least-cost selection principle eliminates bonding conflicts entirely. If p_i prefers p_j but p_j prefers p_k, no bond forms between p_i and p_j. No tiebreaker rule is required for consistency.

Property 6.4 (Tie-Breaking Independence)**.** When multiple partners present equal cost simultaneously, the choice among them is arbitrary. The large-scale behaviour of the system — eigenstructure formation, lake growth, circulation balance, and the validity of all theorems — is independent of how ties are resolved. Tie-breaking affects only particle labelling, not structural outcomes.

7 Continuous-Time Evolution

Definition 7.1 (Active Neighbourhood)**.** The *active neighbourhood* of p_i at time t is $\mathcal{N}(p_i, t) = \{p_j \in P : p_j \neq p_i,\ p_j \text{ available at } t\}$.

Definition 7.2 (Entry Points and Active Set)**.** Let $\mathcal{E} \subseteq P$ be the *entry points* — particles beginning with $F(p_i, 0) = 1$, fully free. Let $A(t) \subseteq P$ be the active set at time t. Particles may enter $A(t)$ at any time $t \geq 0$, modelling continuous arrival of new flow across the watershed.

Definition 7.3 (Continuous-Time Evolution Rule)**.** $G(t)$ evolves by the following process running simultaneously and continuously:

Bond formation. (p_i, p_j) forms when mutual least-cost selection is satisfied.

Bond breaking. Bond (p_i, p_j) breaks when $\exists\, p_k \in \mathcal{N}(p_i, t)$ with $c(p_i, p_k, t) < c(p_i, p_j, t)$, or symmetrically for p_j. The particle immediately seeks a new bond.

Simultaneity. Every particle evaluates both conditions independently and continuously. No particle waits for any other.

Property 7.4 (Bonding Stability Under Continuous Evolution)**.** Under the mutual least-cost bonding rule, oscillatory bond-switching — in which two particles p_i and p_j indefinitely alternate between bonded and unbonded states — cannot occur when all bonding costs are strictly ordered.

To see this: suppose p_i breaks from p_j to bond p_k because $c(p_i, p_k, t) < c(p_i, p_j, t)$. After this bond forms, p_k's occupation ratio increases, raising $c(p_i, p_k, t)$. For p_i to return to p_j, p_j's occupation ratio must have decreased, which requires p_j to have broken a bond. Each bond break increases the breaking particle's drive F, so the system is monotonically pushed toward saturation at the local level. In the presence of ties, tie-breaking may produce labelling differences but not structural oscillation, by Property 2.

Note: a full global stability proof for arbitrary network configurations is an open problem and is listed as Open Question OQ6.

Property 7.5 (Asymmetric Boundary Conditions)**.** The dgfs is open at its source (particles may enter $A(t)$ at any time) and closed at its terminus (no particle ever leaves $L(t)$). Flow is permanently directed: source to lake, never the reverse.

8 Eigenstructures

Not all bonds are equally transient. When same-type particles maintain bonds long enough, the bonded subgraph develops internal coherence — a stable mode characterising the cluster during its lifetime. The affinity function α makes this natural: same-type bonding is cheap, so same-type clusters resist dissolution against the disaggregating pressure of flow.

Definition 8.1 (Eigenstructure)**.** A connected subgraph $H(t) \subseteq G(t)$ is an *eigenstructure* if:

1. Its bond configuration persists for duration $\tau \geq \tau_{\min}$, and
2. All particles share the same type: $\sigma(p_i) = \sigma(p_j)$ for all $p_i, p_j \in H(t)$.

Definition 8.2 (Coherence)**.** The *coherence* of eigenstructure H is the spectral gap $\Delta(H) = \lambda_1(A_H) - \lambda_2(A_H)$, where A_H is the adjacency matrix of H with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots$

- $\Delta(H) \gg 0$: strongly coherent — one dominant mode, ice-like.
- $\Delta(H) \approx 0$: weakly coherent — beginning to dissolve.
- $\Delta(H) = 0$: dissolved — particles rejoin free flow.

9 The Terminal Basin (The Lake)

Definition 9.1 (The Lake)**.** Let $L(t) \subseteq P$ denote the *lake* at time t. Axiom 1 is contingent on the availability condition: a bond between an outside particle and a lake boundary particle can only form if the boundary particle has an available valence slot ($\deg(p_j, t) < v$). When all boundary particles are saturated, Axiom 1 is vacuously satisfied and the lake boundary is temporarily closed to inflow — consistent with the upstream stagnation described in Theorem 10.5.

Axiom 9.2 (Total Inflow)**.** If $p_i \notin L(t)$ bonds with $p_j \in L(t)$, then $p_i \in L(t+\varepsilon)$. Particles enter but never leave.

Axiom 9.3 (Internal Circulation)**.** The subgraph induced by $L(t)$ contains directed cycles. Particles rebond perpetually inside. There is no drain.

Axiom 9.4 (Dynamic Boundary)**.** $L(t) \subseteq L(t+\varepsilon)$ for all $\varepsilon > 0$. The lake only grows.

9.1 Stratified Circulation

Because particles carry types and eigenstructures may survive entry into the lake, the internal circulation need not be uniform. Different type clusters may maintain coherence within the lake — circulating as distinct strata while remaining permanently enclosed.

10 The Circulation Condition and Main Results

Define the boundary $\partial L(t)$ as particles in $L(t)$ sharing at least one bond with a particle outside:

$$\partial L(t) = \left\{\, p_i \in L(t) : \exists\, p_j \notin L(t),\ (p_i, p_j) \in E(t) \right\}.$$

Definition 10.1 (Boundary Inflow and Internal Circulation)**.** The *total inflow* is

$$\Phi(t) = \left|\{(p_i, p_j) \in E(t) : p_i \notin L(t),\ p_j \in \partial L(t)\}\right|.$$

The *total internal circulation* $C(t)$ is the cyclomatic number (number of independent cycles) of the subgraph induced by $L(t)$ on $G(t)$.

Definition 10.2 (Steady Accumulation)**.** A dgfs is in *steady accumulation* on an interval $[t_0, t_1]$ if:

1. **Single-bond entry.** Each particle enters $L(t)$ via exactly one boundary-crossing bond. That is, at the moment p_i joins L, exactly one edge (p_i, p_j) with $p_j \in \partial L$ exists. Equivalently, $\Phi(t)$ counts one inflow bond per newly admitted particle.
2. **Valence sufficiency.** $v \geq m - 1$, so that all particles in L can hold the $m - 1$ bonds required by K_m.
3. **Type homogeneity.** All particles in L share the same type ($\alpha = 0$ within L), so K_m is the natural terminal configuration.

Condition 1 is the substantive dynamical assumption; conditions 2 and 3 are structural prerequisites for K_m formation. Generalisations are addressed in Open Questions OQ1 and OQ2.

10.1 The Circulation Theorem

Theorem 10.3 (Circulation Balance)**.** *For a* dgfs *in steady accumulation (Definition 10.2) where $L(t)$ has stabilised to a complete graph K_m on m same-type particles:*

$$\Phi = \frac{2m}{(m-1)(m-2)}\,\Delta C,$$

where $\Delta C = C(t+\varepsilon) - C(t)$ is the growth in internal circulation.

Proof. **Step 1: Count Φ.** Under the steady-accumulation condition (Definition 10.2, condition 1), each of the m particles in K_m entered L via exactly one boundary-crossing bond. Therefore $\Phi = m$. (This is the content of the single-bond entry condition: it rules out the case where a particle holds multiple boundary bonds at the moment of entry, which would give $\Phi > m$ for the same m particles.)

Step 2: Count ΔC. The cyclomatic number of a connected graph is $|E| - |V| + 1$. For K_m: $|V| = m$, $|E| = \binom{m}{2} = \frac{m(m-1)}{2}$. Therefore:

$$\Delta C = \frac{m(m-1)}{2} - m + 1 = \frac{m^2 - 3m + 2}{2} = \frac{(m-1)(m-2)}{2}.$$

Step 3: Compute the ratio.

$$\frac{\Phi}{\Delta C} = \frac{m}{\frac{(m-1)(m-2)}{2}} = \frac{2m}{(m-1)(m-2)}.$$

Hence $\Phi = \dfrac{2m}{(m-1)(m-2)}\,\Delta C.$ □ □

Remark 10.4. The theorem holds under the steady-accumulation conditions of Definition 10.2. For mixed-type lakes ($\alpha \neq 0$ within L), valence-constrained systems ($v < m - 1$), or configurations where particles enter L via multiple simultaneous bonds, the formula requires adjustment. These generalisations are addressed in Open Questions OQ1 and OQ2.

10.2 Verification

m	Φ	ΔC	$\frac{2m}{(m-1)(m-2)}\Delta C$	Checks
3	3	1	$3 \times 1 = 3$	✓
4	4	3	$\frac{4}{3} \times 3 = 4$	✓
5	5	6	$\frac{5}{6} \times 6 = 5$	✓
6	6	10	$\frac{3}{5} \times 10 = 6$	✓

10.3 Necessary Growth Condition

Theorem 10.5 (Necessary Growth Condition)**.** *For a* dgfs *in steady accumulation, the circulation balance of Theorem 10.3 implies that* $m(t)$ *must be strictly increasing whenever* $\Phi(t) > 0$*. Equivalently: a static lake is inconsistent with the circulation balance under nonzero inflow.*

Proof. Suppose $m(t)$ is constant over some interval $[t_0, t_0 + \delta]$. Then no new particles enter L, so $\Delta C(t) = 0$ on that interval. But $\Phi(t) > 0$ by assumption. The circulation balance gives

$$\Phi = \frac{2m}{(m-1)(m-2)} \cdot 0 = 0,$$

contradicting $\Phi(t) > 0$. Therefore $m(t)$ must be strictly increasing. □ □

Remark 10.6 (Interpretation). Theorem 10.5 is a consequence of the circulation balance equation: it shows that the equation itself is inconsistent with a static lake under nonzero inflow. The physical interpretation — that inflow bonds accumulate without generating new internal cycles, causing particles to form upstream eigenstructures that never drain — is consistent with the axioms and with the stress-test results of Section 11, but constitutes an interpretation of the mathematical constraint rather than a separate derived result.

11 Concrete Example and Validation

11.1 Six-Particle System, $v = 2$

$P = \{p_1, \dots, p_6\}$, $\Sigma = \{A, B\}$, types A: $\{p_1, p_2, p_3\}$, types B: $\{p_4, p_5, p_6\}$, $\alpha(A, B) = 0.8$, $\alpha(A, A) = \alpha(B, B) = 0$, lake $= \{p_6\}$. Particles enter at $t = 0, 1, \dots, 5$.

The type A particles form a triangle K_3 and saturate at $v = 2$. p_4 and p_5 cannot bond across the affinity barrier and wait. p_6 enters and bonds p_4, p_5. All type B particles saturate via p_6.

Result: $v = 2$ is insufficient for internal lake circulation. Each particle uses both bond slots just to connect to the lake, leaving no capacity for internal cycles. Full K_m formation for $m = 3$ requires $v \geq 2$ (Remark 3.3); for larger lakes the valence requirement increases accordingly.

11.2 Six-Particle System, $v = 3$

Same setup, $v = 3$. The type A particles form K_3 but are not saturated — each retains one open slot. Type B particles $\{p_4, p_5, p_6\}$ also form K_3 inside the lake. With remaining open slots, cross-type bonds (p_1, p_4), (p_2, p_5), (p_3, p_6) form. All particles saturate.

Circulation check: $\Phi = 3$ (three cross-type inflow bonds), $\Delta C = 1$ (one independent cycle in K_3), $\frac{2\times3}{2\times1} \times 1 = 3 = \Phi$. ✓

11.3 Stress Tests

Test	**Result**
Preference loops under mutual minimisation	No loops arise. System always reaches a stable bonding state. ✓
Frozen lake under nonzero inflow	Upstream deadlock confirmed. Particles form permanent eigenstructures that never drain. Confirms Theorem 10.5. ✓
Circulation formula for $m = 4, 5$	Formula verified exactly for all tested values. ✓
Tie-breaking independence	Large-scale structure invariant under all tie-breaking strategies. Only particle labels change, not structural outcomes. ✓

12 The Continuous Limit

Does the dgfs recover classical fluid equations as $|P| \to \infty$? Under a natural velocity ansatz — that particles flow down the density gradient — the answer is yes: the dgfs reduces to the porous medium equation, the physically correct limit for resistance-dominated, low-Reynolds-number flow. The derivation of this ansatz from the microscopic bonding dynamics is an open problem (Open Question OQ4).

12.1 Scaling Parameters

Introduce:

- $h > 0$: spatial scale — typical distance between neighbouring particles,
- $\tau > 0$: time scale — typical bond duration,
- $N = |P|$: number of particles.

The continuous limit is $h \to 0$, $\tau \to 0$, $N \to \infty$ simultaneously with total volume and total flow conserved.

12.2 Continuous Fields from Discrete Quantities

In this limit, the discrete occupation ratio becomes a continuous density field $\varrho(p_i, t) \to \rho(x, t) \in [0, 1]$, and the intrinsic driving force becomes $F \to 1 - \rho(x, t)$. Since particles flow toward lower bonding cost, and bonding cost increases with the neighbour's occupation ratio, it is natural to posit that the continuous velocity field points down the density gradient:

$$u(x,t) = -\nabla\rho(x,t).$$

This is the *velocity ansatz.* It is the natural continuous analogue of least-cost bonding — particles move toward lower density — but its derivation from the discrete bonding rule via a formal hydrodynamic limit remains to be established (Open Question OQ4).

12.3 Conservation Law

The dgfs conserves particles: vertices are never created or destroyed. In the continuous limit this becomes the continuity equation:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\, u) = 0.$$

12.4 The Continuous Limit Equation

Proposition 12.1 (Continuous Limit under Velocity Ansatz)**.** *In the limit $h, \tau \to 0$, $N \to \infty$, with a single particle type ($\alpha = 0$) and under the velocity ansatz $u = -\nabla\rho$, the* dgfs *reduces to:*

$$\frac{\partial\rho}{\partial t} = \tfrac{1}{2}\,\nabla^2(\rho^2).$$

This is the porous medium equation with exponent $m = 2$.

Proof. Substituting $u = -\nabla\rho$ into the continuity equation:

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\, u) = \nabla\cdot(\rho\,\nabla\rho) = \rho\,\nabla^2\rho + |\nabla\rho|^2 = \tfrac{1}{2}\,\nabla^2(\rho^2),$$

using the identity $\nabla^2(\rho^2) = 2\rho\,\nabla^2\rho + 2|\nabla\rho|^2$. □ □

Remark 12.2. This is a conditional result: given the velocity ansatz, the porous medium equation follows by direct computation. The proposition is stated separately from the theorems to reflect that the ansatz has not been derived from the microscopic bonding rule. A full derivation would follow the structure of a hydrodynamic limit argument (analogous to the derivation of Euler equations from Boltzmann) and is listed as Open Question OQ4.

12.5 Physical Interpretation

The porous medium equation governs slow, resistance-dominated flow: groundwater filtration, gas seeping through porous rock, heat radiation in plasmas, and biological population spreading. It is the correct macroscopic limit for a system in which particles find paths of least resistance and carry no inertia.

Real water in a river exhibits two regimes:

- *Molecular scale* — slow, viscous, resistance-dominated. Porous medium behaviour. This is what the dgfs models.

- *River scale* — fast, inertial, potentially turbulent. Full Navier-Stokes with inertial term $(u \cdot \nabla)u$.

The dgfs does not recover full Navier-Stokes because particles carry no momentum. The porous medium equation is therefore not a limitation but the correct and honest continuous limit of a purely resistance-driven system.

12.6 The Lake in the Continuous Limit

In the continuous limit, the lake $L(t)$ becomes a compact absorbing region $\Omega(t) \subset \mathbb{R}^n$ with boundary $\partial\Omega(t)$. Under the velocity ansatz, the discrete circulation balance corresponds to the classical Green's theorem identity:

$$\oint_{\partial\Omega} u \cdot d\mathbf{s} = \iint_{\Omega} (\nabla \times u)\, dA.$$

Inflow across the boundary equals the curl integrated over the interior. The discrete result is the finite-particle analogue of this continuous identity, confirming consistency between the two regimes under the velocity ansatz.

13 Related Work

Flow networks. Classical flow network theory, originating with Ford and Fulkerson [1], assigns flows to a fixed graph subject to capacity constraints. The graph is given in advance. The dgfs inverts this: the graph is carved out by the flow itself. The topology is a consequence of the dynamics, not a precondition.

Adaptive network models. Work on Hagen-Poiseuille adaptive networks [3] studies graphs whose edge conductivities evolve to minimise dissipated energy, producing steady-state tree geometries connecting sources to sinks. These models produce static terminal structures. The dgfs differs in two respects: the graph forms from discrete bonding particles rather than continuous conductivity adaptation, and the terminal basin supports perpetual internal circulation rather than settling to a static tree.

Self-assembling graphs. Mathematical models of self-assembly study how small objects autonomously associate into larger structures. These models typically target static equilibrium configurations. The dgfs introduces flow as the primary mechanism — the assembly is driven by a continuous directed force, not by attraction to a fixed target structure.

Discrete Green's theorem and Helmholtz-Hodge decomposition on graphs. Knill [2] and others have established graph-theoretic versions of Green-Stokes identities and the Helmholtz-Hodge decomposition, defining curl, divergence, and gradient operators in a purely discrete setting. The dgfs circulation condition connects to this framework: Theorem 10.3 is a discrete circulation identity whose continuous limit (under the velocity ansatz) recovers the classical Green's theorem form. The dgfs provides a dynamic context in which this identity arises naturally from particle interactions rather than being imposed as a structural constraint.

Active flow networks. Recent work [4] connects graph theory and self-organisation principles to study how topology controls dynamics in actively driven flow networks. The dgfs shares the self-organisation philosophy but operates at the particle level, where the graph itself is the emergent object rather than the given scaffold.

Porous medium equations. The porous medium equation $\partial_t \rho = \nabla^2(\rho^m)$ is classical [5, 6], arising in groundwater hydrology, plasma physics, and mathematical biology. Its appearance as the continuous limit of the dgfs (under the velocity ansatz) establishes a connection between discrete graph-particle dynamics and a well-studied continuum PDE, providing both a validation of the framework and a bridge to the existing analytical theory of porous medium flows.

The combination of self-assembling topology, intrinsic driving force, eigenstructure formation, dynamic terminal circulation, and an explicit continuous limit derivation does not appear as a unified framework in any prior work known to the author.

14 Discussion

14.1 What the Framework Models

The dgfs is not a model of water specifically. It is a mathematical language for any system whose components interact, bond, and accumulate according to least-resistance principles. The physical analogy of a river system provides intuition, but the framework applies wherever the following conditions hold:

- Components can form and break connections dynamically.
- Connection formation follows a least-resistance or least-cost principle.
- Components have finite bonding capacity.
- There exists a terminal accumulation region from which nothing escapes.

Candidate applications include information propagation through communication networks, resource flow through supply chains, influence spreading through social graphs, dependency resolution in computational systems, and biological signalling cascades. In each case the particles are the entities, bonds are the interactions, types encode compatibility, and the lake is the terminal state — the database, the warehouse, the decision point, the activated receptor.

14.2 The Significance of Necessary Growth

Theorem 10.5 establishes a precise mathematical constraint: the circulation balance of Theorem 10.3 is inconsistent with a static lake under nonzero inflow. This is a non-obvious consequence of the circulation formula itself, not an independent assumption about the system.

In classical flow network theory, a sink node simply absorbs whatever arrives. In the dgfs, the sink has internal structure — it circulates, it stratifies, it has a boundary governed by a quantitative balance equation. The requirement for growth emerges from that equation. The real-world interpretation — databases that cannot expand, institutions that cannot absorb new members, ecosystems that cannot expand habitat — follows from the mathematics, but should be understood as interpretation rather than derived consequence.

14.3 Eigenstructures and Stratification

The emergence of eigenstructures from the affinity function is one of the framework's more striking features. Without engineering coherent structures into the system, they arise naturally whenever same-type particles travel together long enough. The spectral gap $\Delta(H)$ provides a continuous, computable measure of structural persistence requiring no additional parameters.

The stratified circulation inside the lake — different type clusters maintaining coherence while permanently enclosed — is a direct consequence of the same mechanism. Oil does not mix with water not because we forbid it but because the bonding cost makes cross-type bonding expensive. The separation is energetic, not imposed.

14.4 Limitations and Future Directions

The current framework has four honest limitations.

First, the continuous limit derivation assumes the velocity ansatz $u = -\nabla\rho$ directly (Proposition 12.1). A rigorous derivation would establish this velocity-density relationship from the microscopic bonding dynamics through a formal hydrodynamic limit argument, analogous to the derivation of Euler equations from the Boltzmann equation.

Second, Theorem 10.3 assumes the lake stabilises to K_m under the steady-accumulation conditions of Definition 10.2. This holds for same-type, fully connected lakes with $v \geq m - 1$, but requires generalisation for mixed-type or valence-constrained systems. The correct circulation formula in those cases remains open.

Third, the framework is currently purely deterministic. Real fluid systems exhibit thermal fluctuations and stochastic bonding. Introducing a probabilistic bonding rule — where cost determines probability rather than certainty — would produce a richer stochastic version of the dgfs with connections to statistical mechanics.

Fourth, Property 7.4 establishes bonding stability under strict cost ordering but does not prove global convergence for arbitrary network configurations. A complete stability proof remains open.

15 Open Questions

OQ1. Under what conditions on v, Σ, α, and the initial configuration does the system guarantee all particles reach $L(t)$ in finite time? Theorem 10.5 establishes that a growing lake is necessary for sustained flow, but sufficient conditions for complete drainage remain open.

OQ2. For mixed-type lakes or valence-constrained systems where L does not stabilise to K_m, what is the correct generalisation of the circulation balance $\Phi = \frac{2m}{(m-1)(m-2)}\,\Delta C$?

OQ3. Can eigenstructures survive entry into the lake indefinitely, or does perpetual internal circulation always dissolve them? What determines the dissolution timescale as a function of $\Delta(H)$ at the moment of entry?

OQ4. Proposition 12.1 establishes the continuous limit under the velocity ansatz $u = -\nabla\rho$. A rigorous hydrodynamic limit derivation — establishing this velocity-density relation from the microscopic bonding dynamics — remains to be carried out.

OQ5. What is the correct stochastic generalisation of the dgfs in which bonding cost determines bonding probability rather than certainty? Does this recover known results in statistical mechanics or non-equilibrium thermodynamics? How does the additive bonding cost structure compare to a multiplicative alternative?

OQ6. Property 7.4 establishes bonding stability under strict cost ordering. What are the conditions for global convergence of the bond-switching dynamics in arbitrary network configurations, including cases with persistent ties?

Acknowledgments

The author used AI assistance for writing and exposition. All mathematical ideas, constructions, proofs, and definitions are the author's own.

References

[1] L. R. Ford Jr. and D. R. Fulkerson, Maximal flow through a network, *Canadian Journal of Mathematics* **8** (1956), 399–404. DOI: 10.4153/CJM-1956-045-5.

[2] O. Knill, The theorems of Green–Stokes, Gauss–Bonnet and Poincaré–Hopf in graph theory, arXiv preprint arXiv:1201.6049 (2012).

[3] H. Ronellenfitsch and E. Katifori, Global optimization, local adaptation, and the role of growth in distribution networks, *Physical Review Letters* **117** (2016), 138301. DOI: 10.1103/PhysRevLett.117.138301.

[4] F. G. Woodhouse, A. Forrow, J. B. Fawcett, and J. Dunkel, Stochastic cycle selection in active flow networks, *Proceedings of the National Academy of Sciences* **113** (2016), 8200–8205. DOI: 10.1073/pnas.1603351113.

[5] D. G. Aronson, The porous medium equation, in *Nonlinear Diffusion Problems* (A. Fasano and M. Primicerio, eds.), Lecture Notes in Mathematics 1224, Springer-Verlag, New York, 1986, pp. 1–46.

[6] J. L. Vázquez, *The Porous Medium Equation: Mathematical Theory*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2007.