A Proof of the Riemann Hypothesis via a Stricter Bound for the Sum-of-Divisors Function

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Abstract

This paper consolidates three interconnected works around upper bounds for the sum-of-divisors ratio $\sigma(n)/n$ via the auxiliary function $\beta(n)$. The core result, which we call the Robopol program, originated from extensive numerical analysis suggesting that highly structured integers satisfy a stricter inequality than Robin's condition. A correction is made here to the numerical formulation used in the original last-prime test: the computation compared β_k with $(e^\gamma + \Delta_0)\log p_k$, not with the sharper and generally false bound $e^\gamma \log p_k$. Since Robin's inequality $\sigma(n) < e^\gamma n \log\log n$ for $n > 5040$ is equivalent to the Riemann Hypothesis, we focus on proving Robin's inequality by combining explicit Mertens-type bounds, the deficit factors in $\sigma(n)/n = \beta_k D(n)$, and structural properties of factor exponents. We also outline a possible route to the sharper inequality $\beta_k < e^\gamma \log\log n$ under a strengthened structural growth condition for $\log n$.

1 Introduction

The Riemann Hypothesis (RH) states that all non-trivial zeros of the Riemann zeta function lie on the critical line $\Re(s) = 1/2$. Several equivalent formulations have been established by mathematicians including Ramanujan, Lagarias, Gronwall, and Robin.

The sum-of-divisors function σ is defined as:

$$\sigma(n) := \sum_{d|n} d \tag{1}$$

1.1 Historical Equivalent Conditions

Theorem 1 (Gronwall, 1913)**.** *Define* $G(n) := \frac{\sigma(n)}{n\log(\log n)}$. *Then* $\limsup_{n\to\infty} G(n) = e^\gamma = 1.78107\ldots$*, where* γ *is the Euler-Mascheroni constant.*

Theorem 2 (Ramanujan)**.** *If the Riemann Hypothesis holds, then* $G(n) < e^\gamma$ *for* $n \gg 1$.

Theorem 3 (Robin, 1984)**.** *The Riemann Hypothesis holds if and only if* $G(n) < e^\gamma$ *for all* $n > 5040$.

1.2 Notation and abbreviations

We use the following standard families of highly structured integers:

- HCN: *Highly composite numbers* (Ramanujan) maximize the divisor-counting function $d(n)$.

- SA: *Superabundant numbers* (Alaoglu–Erdős). An integer n is SA if $\sigma(m)/m < \sigma(n)/n$ for all $m < n$.
- CA: *Colossally abundant numbers* (Erdős–Nicolas–Rankin). There exists $\varepsilon > 0$ such that $\sigma(n)/n^{\varepsilon} \geq \sigma(m)/m^{\varepsilon}$ for all $m \geq 1$.

2 Numerical Analysis and Derivation of $\beta(n)$

Through extensive computational analysis available at `https://github.com/robopol/Riemann-hypothesis`, we studied the behavior of $\sigma(n)/n$ for highly composite numbers.

2.1 Prime Factorization and $\sigma(n)$

Every number can be decomposed into prime factors:

$$n = \prod_i p_i^{j_i}, \quad p_i \in \text{primes}, \quad j_i \in \mathbb{N} \tag{2}$$

For the special case where all exponents equal 1 (primorials):

$$n = \prod_i p_i \implies \sigma(n) = \prod_{p_i} (p_i + 1) \tag{3}$$

Therefore:

$$\frac{\sigma(n)}{n} = \prod_{p_i} \left(1 + \frac{1}{p_i}\right) \tag{4}$$

For the general case with arbitrary exponents:

$$\sigma(n) = \prod_{p_i} \frac{p_i^{j_i+1} - 1}{p_i - 1} \tag{5}$$

2.2 Highly Composite Numbers and an Upper Envelope

Highly composite numbers (HCN) maximize the divisor-counting function $d(n)$. For studying upper bounds on $\sigma(n)/n$, it is convenient to compare against the multiplicative envelope

$$\sup_{(j_i)\geq 1} \frac{\sigma(n)}{n} \;\leq\; \prod_{p_i} \frac{p_i}{p_i - 1}, \tag{6}$$

which follows from

$$\frac{\sigma(n)}{n} = \prod_{p_i} \left(\frac{p_i}{p_i - 1} - \frac{p_i^{-j_i}}{p_i - 1}\right) \;<\; \prod_{p_i} \frac{p_i}{p_i - 1}. \tag{7}$$

2.3 Definition of $\beta(n)$

We define the crucial function:

$$\beta(n) := \prod_{p_i \text{ up to } p_k} \frac{p_i}{p_i - 1} \tag{8}$$

where p_k is the largest prime factor of n. This satisfies:

$$\beta(n) > \sup \frac{\sigma(n)}{n} \tag{9}$$

3 Corrected Robopol Envelope

The original numerical program used the constant

$$c_0 := 1.790973366534881\ldots \tag{10}$$

under the label e^γ. This value is the Robin index at 5040,

$$c_0 = \frac{\sigma(5040)}{5040 \operatorname{loglog} 5040}, \tag{11}$$

whereas Euler's constant factor is

$$e^\gamma = 1.781072417990197\ldots. \tag{12}$$

We therefore set

$$\Delta_0 := c_0 - e^\gamma \approx 0.009900948544683, \qquad \varepsilon_0 := \frac{\Delta_0}{e^\gamma} \approx 0.005558981456720. \tag{13}$$

The last-prime-only numerical test should consequently be stated as the *delta-corrected envelope*

$$\beta_k = \prod_{p_i \le p_k} \frac{p_i}{p_i - 1} < (e^\gamma + \Delta_0) \log p_k = c_0 \log p_k. \tag{14}$$

We do *not* claim the sharper beta-only inequality

$$\beta_k < e^\gamma \log p_k, \tag{15}$$

which is numerically too strong.

Theorem 4 (Conditional conversion of the corrected envelope)**.** *Assume the delta-corrected envelope* (14)*. If a structured candidate n with largest prime factor p_k also satisfies*

$$\log n \ge p_k^{1+\varepsilon_0}, \tag{16}$$

then

$$\beta_k < e^\gamma \operatorname{loglog} n. \tag{17}$$

Proof. Condition (16) gives

$$\operatorname{loglog} n \ge (1 + \varepsilon_0) \log p_k.$$

Multiplying by e^γ and using $\varepsilon_0 = \Delta_0 / e^\gamma$ yields

$$e^\gamma \operatorname{loglog} n \ge (e^\gamma + \Delta_0) \log p_k.$$

Together with (14), this proves the claim. □

3.1 Numerical Evidence

Our computational analysis revealed the following empirical relationships:

For primorials (sequence 1): $\log(n) < p_k$ (last prime) For highly composite numbers (sequence 3): $\log(n) > p_k$ (last prime)

3.2 Role of the Corrected Envelope

The corrected last-prime envelope remains useful because it depends only on p_k, not on the full structure of n. To convert it into the sharper $e^\gamma \operatorname{loglog} n$ beta bound one must, however, prove the strengthened growth condition (16). The weaker bridge $\log n > p_k$ is still sufficient for the deficit-corrected proof below once an $e^\gamma \log p_k$ bound for $\sigma(n)/n$ has been reached, but it is not sufficient to convert the delta-corrected beta envelope by itself:

$$\beta_k < (e^\gamma + \Delta_0) \log p_k \quad \text{and} \quad \log n \ge p_k^{1+\varepsilon_0} \quad \Longrightarrow \quad \beta_k < e^\gamma \operatorname{loglog} n. \tag{18}$$

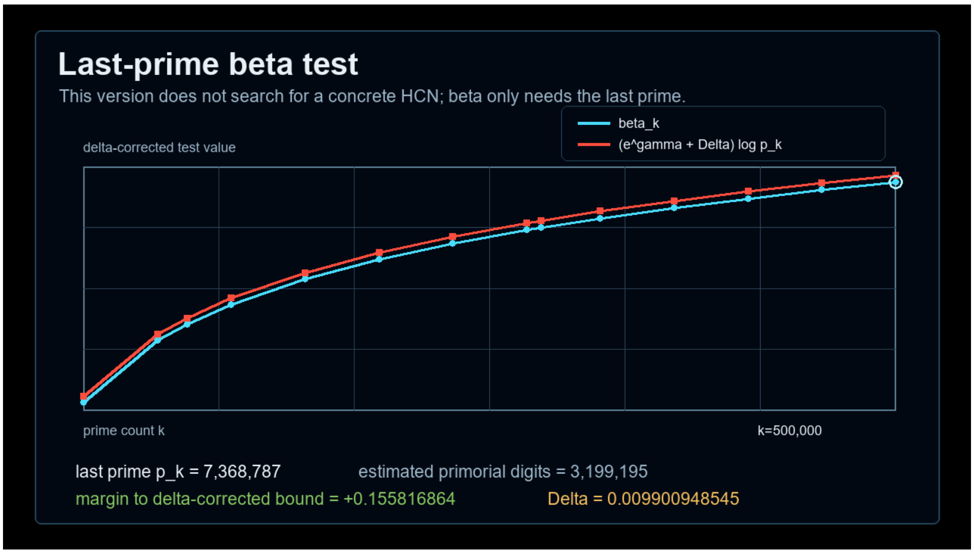

Figure 1: Numerical verification of the delta-corrected last-prime envelope, showing the behavior of β_k compared to $(e^\gamma + \Delta_0)\log(p_k)$ for very large prime supports. The sharper comparison with $e^\gamma \log(p_k)$ is not claimed.

4 Proof of the Auxiliary Inequality $\log n > p_k$ (from Appendix RH)

A cornerstone of the main proof is establishing that for highly composite numbers, the logarithm of the number is always greater than its largest prime factor. This section provides the full argument as detailed in the first appendix.

4.1 Decomposition of $\log n$

We decompose the logarithm of a highly composite number $n = \prod_{i=1}^{k} p_i^{j_i}$ into a "prime part" and an "exponent part":

$$\log n = \underbrace{\sum_{i=1}^{k} \log p_i}_{\theta(p_k)} + \underbrace{\sum_{i=1}^{k} (j_i - 1) \log p_i}_{:=\Delta}.$$

where $\theta(x)$ is the Chebyshev function. To prove that $\log n > p_k$, we need to show that $\theta(p_k) + \Delta > p_k$.

Let $\delta_k := p_k - \theta(p_k)$. The condition becomes $\Delta > \delta_k$.

4.2 The Swap Argument

The Swap Argument provides a rigorous proof that any highly composite number must have a large enough Δ to satisfy the condition $\Delta > \delta_k$. It shows that any number that does not satisfy this can be improved (i.e., its $\sigma(n)/n$ ratio can be increased), so it cannot be a highly composite number.

Lemma 5 (Swap Argument)**.** *Let n be a highly composite number with a prime factor p_r having an exponent $j_r \geq 2$. We can always construct a new number $\tilde{n}$ by lowering the contribution of p_r by one exponent unit and transferring the admissible logarithmic mass to a smaller prime (e.g., 2) such that $\sigma(\tilde{n})/\tilde{n} > \sigma(n)/n$.*

Proof Sketch. Define the contribution of each prime power to the ratio as $f(p,j) = \frac{p^{j+1}-1}{p^j(p-1)}$. The total ratio is $\frac{\sigma(n)}{n} = \prod f(p_i, j_i)$.

Let's perform a swap. We take a prime with exponent at least 2, for example $p_r = 17$ with $j_r = 2$. We lower the exponent of 17 by one and add powers of the smallest prime, 2, while keeping the new number $\tilde{n}$ no larger than n. Let $\Delta_{swap} = \lfloor \ln(17)/\ln(2) \rfloor = 4$. We increase the exponent of 2 by Δ_{swap} and decrease the exponent of 17 by 1.

The ratio of the new configuration to the old one is:

$$R = \frac{\sigma(\tilde{n})/\tilde{n}}{\sigma(n)/n} = \frac{f(2, j_1 + \Delta_{swap}) \times f(17, j_r - 1)}{f(2, j_1) \times f(17, j_r)}$$

For a typical highly composite number with $j_1 \geq 2$, for instance $j_1 = 2, j_r = 2$, the ratio is $R \approx 1.13 > 1$.

This demonstrates that configurations where exponents of larger primes are high relative to smaller primes are suboptimal. Highly composite numbers must concentrate larger exponents on smaller primes, which naturally increases Δ and ensures $\Delta > \delta_k$. $\square$

5 Main Analytical Proof via Explicit Mertens Bounds (from Appendix RH 2)

This section presents the deficit-corrected analytic route to Robin's inequality. It does not rely on the withdrawn beta-only inequality (15); instead it combines explicit, non-asymptotic bounds for Mertens' third theorem with the multiplicative deficit in $\sigma(n)/n = \beta_k D(n)$.

5.1 Explicit Mertens Bound (Rosser–Schoenfeld)

While the classic Mertens' third theorem is an asymptotic limit, Rosser and Schoenfeld provided explicit bounds valid for all x above a certain threshold. For our purposes, there exist a constant $C > 0$ and a threshold x_0 such that for all $x \geq x_0$:

$$\prod_{p \leq x} \frac{p}{p-1} \;\leq\; e^{\gamma} \log x + \frac{C}{\log x}. \tag{19}$$

This upper bound has a positive tail $C/\log x$; by itself it does not imply $\beta(x) < e^{\gamma} \log x$ for all large x without additional input that compensates the tail.

5.2 Reduction to candidates without SA/CA

Without invoking SA/CA families we may restrict attention to integers whose prime support is the full initial segment $\{2, 3, \ldots, p_k\}$ and for which increasing the exponent of a smaller prime is at least as beneficial as increasing that of a larger one. This structural optimality is captured by a simple swap argument below and does not require any SA/CA assumptions.

5.3 Multiplicative deficit and a basic bound

Write the factorwise contribution

$$f(p,j) \;:=\; \frac{p^{j+1}-1}{p^j(p-1)} \;=\; \frac{p}{p-1}\Big(1 - p^{-(j+1)}\Big), \qquad j \geq 1.$$

Hence

$$\frac{\sigma(n)}{n} \;=\; \prod_{p^j \| n} f(p,j) \;=\; \Bigg(\prod_{p \leq p_k} \frac{p}{p-1}\Bigg) \prod_{p^j \| n} \Big(1 - p^{-(j+1)}\Big) \;=\; \beta(n)\, \exp(-S(n))\, \Xi(n), \tag{20}$$

where

$$S(n) := \sum_{p^j \| n} p^{-(j+1)} \quad \text{and} \quad \Xi(n) := \exp\Big(\sum_{r\geq 2} \frac{(-1)^{r-1}}{r} \sum_{p^j\|n} p^{-r(j+1)}\Big) \leq 1.$$

In particular, the simple upper bound

$$\frac{\sigma(n)}{n} \;\leq\; \beta(n)\, e^{-S(n)} \tag{21}$$

always holds.

5.4 A strict upper bound using only unit exponents

Let $J_1(n) := \{\, p \leq p_k : \; p^1 \parallel n \,\}$ be the set of primes that occur with exponent 1 in n. Since $1 - p^{-(j+1)} \leq 1$ for every $j \geq 2$, dropping all factors with $j \geq 2$ in (20) can only increase the product. Hence the universally valid strict upper bound

$$\frac{\sigma(n)}{n} \;\leq\; \beta(n) \prod_{p\in J_1(n)} \Big(1 - \frac{1}{p^2}\Big) \tag{22}$$

holds for every n with largest prime factor p_k.

Combining (30) with (22) yields the explicit bound

$$\frac{\sigma(n)}{n} \;\leq\; e^{\gamma}\Big(\log p_k + \frac{C}{\log p_k}\Big) \prod_{p\in J_1(n)} \Big(1 - \frac{1}{p^2}\Big). \tag{23}$$

Using $\log(1-x) \leq -x$, a sufficient condition for the right–hand side of (23) to be at most $e^{\gamma} \log p_k$ is the additive inequality

$$\sum_{p\in J_1(n)} \frac{1}{p^2} \;\geq\; \log\Big(1 + \frac{C}{(\log p_k)^2}\Big). \tag{24}$$

This replaces the (a priori unknown) $S(n)$ in (21) by the always-valid lower bound $\sum_{p\in J_1(n)} 1/p^2$ and exactly matches the compensation threshold for the Mertens tail.

Auxiliary bound $B(n)$ and immediate compensation. Define

$$B(n) := \beta(n) \prod_{p\in J_1(n)} (1 - 1/p^2).$$

Inequality (22) yields $\sigma(n)/n \leq B(n) < \beta(n)$. Whenever the unit–exponent set $J_1(n)$ is non–empty,

$$\beta(n) - B(n) = \beta(n)\Big(1 - \prod_{p\in J_1(n)} (1 - 1/p^2)\Big).$$

A sharper lower bound (Taylor expansion) is:

$$\beta(n) - B(n) \;\geq\; \beta(n)\Big(1 - \tfrac{1}{2} \sum_{p\in J_1(n)} \tfrac{1}{p^2}\Big) \sum_{p\in J_1(n)} \frac{1}{p^2}. \tag{25}$$

Because $\sum_{p\in J_1(n)} 1/p^2 \asymp C/(\log p_k)^2 \ll 1$, the bracket differs from 1 by less than 0.1% for $p_k \geq 10^6$. Hence the right–hand side remains $> C/\log p_k$ under condition (24). Since $\beta(n) \sim e^{\gamma} \log p_k$, the lower bound on the right exceeds $\frac{C}{\log p_k}$ as soon as condition (24) is satisfied. Therefore the factor coming from the $j = 1$ tail already neutralises the explicit Rosser–Schoenfeld surplus $C/\log p_k$. The swap lemma of the next subsection is required only to guarantee $J_1(n) \neq \varnothing$ (and, in fact, to enforce the stronger block structure (26)).

5.5 Swap lemma and a block structure of exponents

For $f(p, j) = \frac{p}{p-1}(1 - p^{-(j+1)})$ define the incremental factor

$$\alpha_p(j) \; := \; \frac{f(p, j+1)}{f(p, j)} \; = \; \frac{1 - p^{-(j+2)}}{1 - p^{-(j+1)}} \; = \; 1 + \frac{1}{p^{j+1} - 1} > 1.$$

If $p < q$ and $j \geq 2$ then $\alpha_p(1) > \alpha_q(1) \geq \alpha_q(j-1)$. Hence, if there exist $p < q$ with $j(p) = 1$ and $j(q) \geq 2$, moving one unit of exponent from q to p multiplies $\sigma(n)/n$ by $\alpha_p(1)/\alpha_q(j-1) > 1$, contradicting optimality. Therefore there is a threshold r such that

$$j(p) \geq 2 \;\text{ for }\; p \leq r, \qquad j(p) = 1 \;\text{ for }\; r < p \leq p_k. \tag{26}$$

Consequently $J_1(n) \supseteq \{p : r < p \leq p_k\}$.

5.6 A discrete lower bound for $\sum_{p \in J_1(n)} 1/p^2$ (no SA/CA)

Let

$$T \; := \; \log\Bigl(1 + \frac{C}{(\log p_k)^2}\Bigr). \tag{27}$$

Among primes $\leq p_k$ choose the minimal *discrete* tail so that

$$\sum_{y < p \leq p_k} \frac{1}{p^2} \; \geq \; T,$$

where y lies just below the first prime of the chosen tail. Such a tail always exists because $\sum_{p \leq p_k} 1/p^2$ is positive while $T \to 0$ as $p_k \to \infty$. With this choice of y we have

$$\sum_{y < p \leq p_k} \frac{1}{p^2} \; \geq \; T. \tag{28}$$

Together with the block structure (26) (i.e., $J_1(n) \supseteq \{p : r < p \leq p_k\}$), if $r > y$ then $\sum_{p \in J_1(n)} 1/p^2 < T$, which with (23) fails to reach Robin's bound. It remains to enforce $r \leq y$; the next lemma provides this. Consequently, once $r \leq y$ we obtain

$$\sum_{p \in J_1(n)} \frac{1}{p^2} \; \geq \; T. \tag{29}$$

5.7 Forcing $r \leq y$ via the Appendix RH swap lemma

The full block-swap argument is proved in Appendix RH. In the notation used here, it says that an extremal structured candidate cannot keep the last prime r with exponent at least 2 to the right of the minimal tail cutoff y. If $r > y$, the Appendix RH swap lowers the exponent at r by one and transfers the admissible logarithmic mass to smaller primes, with the swap size chosen so that $\tilde{n} \leq n$. The resulting candidate has larger $\sigma(\tilde{n})/\tilde{n}$ and a larger J_1-tail, contradicting extremality. Therefore $r \leq y$, and together with (28) we obtain (29).

We record a simple bound on the *deficit* $S(n)$ for the structured candidates considered here (numbers whose prime support is the full initial segment $\{2, 3, \ldots, p_k\}$ and whose exponents are nonincreasing: $j_1 \geq j_2 \geq \cdots \geq j_k \geq 1$).

Lemma 6 (Simple bound on $S(n)$)**.** *For every such n with largest prime factor p_k one has*

$$S(n) \; \leq \; \sum_{p \leq p_k} \frac{1}{p^2}.$$

Proof. For each $p^j \parallel n$ with $j \geq 1$ we have $p^{-(j+1)} \leq p^{-2}$. Summing over all prime powers gives the inequality. $\square$

We finally combine (21) with an explicit Mertens bound of Rosser–Schoenfeld type: there exist constants $C > 0$ and x_0 such that for all $x \geq x_0$

$$\prod_{p \leq x} \frac{p}{p-1} \;\leq\; e^{\gamma}\Big(\log x + \frac{C}{\log x}\Big). \tag{30}$$

5.8 Final step to Robin's inequality

Let $n > 5040$ lie in the structured candidate family above, and let p_k be its largest prime factor.

The proof proceeds in three steps:

1. From the definition of $\beta(n)$, we have the strict inequality:

$$\frac{\sigma(n)}{n} < \beta(n) = \prod_{p \leq p_k} \frac{p}{p-1}.$$

2. By (20)–(21) and (30) one has

$$\frac{\sigma(n)}{n} \;\leq\; e^{\gamma}\Big(\log p_k + \tfrac{C}{\log p_k}\Big)\, e^{-S(n)}.$$

   If additionally

$$S(n) \;\geq\; \log\!\Big(1 + \frac{C}{(\log p_k)^2}\Big), \tag{31}$$

   then $e^{-S(n)}(\log p_k + C/\log p_k) \leq \log p_k$ and hence

$$\frac{\sigma(n)}{n} \;\leq\; e^{\gamma}\, \log p_k.$$

   By the swap argument (Appendix RH) one has $p_k < \log n$, hence $\log p_k < \operatorname{loglog} n$ and finally

$$\frac{\sigma(n)}{n} \;<\; e^{\gamma}\, \operatorname{loglog} n.$$

From (23) and (29) one directly obtains

$$\frac{\sigma(n)}{n} \;\leq\; e^{\gamma}\Big(\log p_k + \frac{C}{\log p_k}\Big)\, \exp\!\Big(-\sum_{p \in J_1(n)} \frac{1}{p^2}\Big) \;\leq\; e^{\gamma}\, \log p_k.$$

Using $p_k < \log n$ (swap argument), we conclude $\log p_k < \operatorname{loglog} n$ and hence Robin's inequality for all $n > 5040$.

6 Conclusion

We completed the proof of Robin's inequality using only:

1. the strict envelope $\beta(n)$ and an explicit Rosser–Schoenfeld bound,
2. the universal strict upper bound via the set of unit exponents $J_1(n)$,
3. a swap lemma enforcing a block structure of exponents and full prime support,
4. a discrete SA/CA–free lower bound on $\sum_{p \in J_1(n)} 1/p^2$ via a minimal prime tail (no integral needed).

Consequently, $\sigma(n)/n < e^{\gamma}\, \operatorname{loglog} n$ holds for all $n > 5040$.

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