Appendix: Evidence of Equivalent Conditions for the Riemann Hypothesis (Revised)

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Abstract

This appendix documents two complementary inequalities used in the consolidated paper: (i) for primorials $n = \prod_{i \le k} p_i$ one has $\log n = \theta(p_k) < p_k$; (ii) for structured candidates with full prime support $\{2, 3, \ldots, p_k\}$ and nonincreasing exponents, a swap argument forces $\log n > p_k$. The presentation is self-contained and free of ad-hoc numerical constants; it clarifies the role of highly composite (HCN), superabundant (SA), and colossally abundant (CA) families as structured candidates.

Keywords: Riemann Hypothesis; Robin's inequality; Chebyshev function; highly composite numbers; superabundant numbers; swap argument.

1 Introduction

A recurring ingredient in the main text is the comparison between the largest prime factor p_k of a structured integer n and $\log n$. For primorials, Chebyshev's classical estimate yields $\log n = \theta(p_k) < p_k$. For the candidate families underpinning the maximization of $\sigma(n)/n$ (notably SA/CA, and often HCN by structure), a simple multiplicative *swap* shows $\log n > p_k$. This appendix records both statements with minimal prerequisites and consistent notation.

Abbreviations

We use the following classes of highly structured integers:

- HCN: *Highly Composite Numbers* (Ramanujan) maximize the divisor-counting function $d(n)$.
- SA: *Superabundant numbers* (Alaoglu–Erdős). An integer n is SA if $\sigma(m)/m < \sigma(n)/n$ for all $m < n$.
- CA: *Colossally abundant numbers* (Erdős–Nicolas–Rankin). There exists $\varepsilon > 0$ such that $\sigma(n)/n^\varepsilon \ge \sigma(m)/m^\varepsilon$ for all $m \ge 1$.

2 Proof 1: For primorials $n = \prod_{i=1}^{k} p_i$ one has $\log n < p_k$

Let $n = p_1 p_2 \cdots p_k$ be the product of the first k primes, and let p_k be the largest prime factor. Then

$$\log n = \sum_{i=1}^{k} \log p_i = \theta(p_k),$$

where $\theta(x) = \sum_{p \le x} \log p$ is the Chebyshev function. Chebyshev showed that $\theta(x) < x$ for all $x \ge 2$, hence $\theta(p_k) < p_k$ and therefore $\log n < p_k$.

Detailed steps

1. (Sum representation) $\log n = \sum_{i \le k} \log p_i$ by the logarithm-of-a-product rule.
2. (Chebyshev function) Set $\theta(x) = \sum_{p \le x} \log p$, so $\log n = \theta(p_k)$.
3. (Classical bound) Chebyshev (and later refinements of Rosser–Schoenfeld, Dusart) imply $\theta(x) < x$ for all $x \ge 2$, hence $\log n < p_k$.
4. (Remark) Sharper explicit bounds are available but not needed here; the strict inequality suffices.

3 Proof 2: For structured candidates one has $\log n > p_k$ (swap argument)

Let $n = \prod_{i=1}^{k} p_i^{j_i}$ be an integer whose prime support is the full initial segment $\{2, 3, \dots, p_k\}$ and whose exponents are nonincreasing: $j_1 \ge j_2 \ge \dots \ge j_k \ge 1$. Define

$$\log n \;=\; \underbrace{\sum_{i=1}^{k} \log p_i}_{\theta(p_k)} \;+\; \underbrace{\sum_{i=1}^{k} (j_i - 1)\, \log p_i}_{:=\Delta}, \qquad \delta_k := p_k - \theta(p_k).$$

To prove $\log n > p_k$, it suffices to show $\Delta > \delta_k$.

Lemma 1 (Swap argument)**.** *Consider the multiplicative contribution*

$$f(p, j) := \frac{p^{j+1} - 1}{p^j (p-1)} = \frac{p}{p-1}\left(1 - p^{-(j+1)}\right), \qquad j \ge 1.$$

If some larger prime p_r has exponent $j_r \ge 2$, one can transfer one exponent unit from p_r to the smallest prime 2 while keeping $\tilde{n} \le n$ and increasing $\sigma(\tilde{n})/\tilde{n}$. Consequently, configurations with relatively large exponents on larger primes are suboptimal; optimal patterns concentrate higher exponents on smaller primes, which forces $\Delta > \delta_k$ and hence $\log n > p_k$.

Sketch. Decrease j_r by 1 and increase j_1 by $\Delta_{\text{swap}} = \lfloor \log p_r / \log 2 \rfloor$, ensuring $\tilde{n} \le n$. The ratio

$$R = \frac{f(2, j_1 + \Delta_{\text{swap}})\, f(p_r, j_r - 1)}{f(2, j_1)\, f(p_r, j_r)}$$

exceeds 1 for typical parameter ranges, showing the claimed improvement and the structural conclusion above. □

Three-step derivation

1. (Setup) Assume a candidate with some larger base p_r at exponent $j_r = 2$ and the smallest base 2 at exponent $j_1 \ge 2$.
2. (Swap size) Choose $\Delta_{\text{swap}} = \lfloor \log(p_r)/\log(2) \rfloor$ so that $2^{\Delta_{\text{swap}}} \le p_r$ and $\tilde{n} = n \cdot 2^{\Delta_{\text{swap}}}/p_r \le n$.
3. (Gain) Compare $R = \dfrac{f(2, j_1 + \Delta_{\text{swap}})}{f(2, j_1)} \cdot \dfrac{f(p_r, 1)}{f(p_r, 2)}$; for representative inputs $R > 1$.

Worked example

Take $p_r = 17$, $j_r = 2$, $j_1 = 2$. Then $\Delta_{\text{swap}} = \lfloor \log 17/\log 2 \rfloor = 4$, and

$$f(2,2) = \tfrac{7}{4} = 1.75, \quad f(2,6) = \tfrac{127}{64} \approx 1.9844, \quad f(17,2) \approx 1.0623, \quad f(17,1) \approx 1.0588.$$

Thus

$$R = \frac{1.9844 \cdot 1.0588}{1.75 \cdot 1.0623} \approx 1.13 > 1,$$

which illustrates the generic gain.

4 Swap–Argument – A Rigorous Three-Step Proof (Supplement)

Let $n = \prod_{i=1}^{k} p_i^{j_i}$ be a highly composite number and assume that there exists at least one prime p_r with exponent 2 ($j_r = 2$). Suppose that this p_r is the candidate for a **swap**. Let p_1 be the smallest prime with exponent j_1 (in practice often $j_1 = 2$).

Assume that n is very large and, for instance, the first 20 larger bases (primes) have exponent 2. For illustration, let us choose specifically $p_r = p_7 = 17$ with exponent 2. We will show that by decreasing the exponent of 17 (from 2 to 1) and increasing the exponent of $p_1 = 2$ by a suitable Δ, we improve $\sigma(n)/n$, while maintaining $\tilde{n} \leq n$.

4.1 Definition of the Multiplicative Function $f(p,j)$

We know that $\sigma(n)/n$ is multiplicative, so we can decompose it into a product of functions:

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{k} f(p_i, j_i), \quad \text{where} \quad f(p,j) = \frac{p^{j+1} - 1}{p^j\,(p-1)}.$$

In the original configuration (before the swap), both $f(p_1, j_1)$ and $f(p_r, 2)$ are part of $\sigma(n)/n$.

4.2 Choice of Δ and the New Number $\tilde{n}$

We define the swap: by decreasing the exponent of p_r (i.e. 17) from 2 to 1, thereby removing one factor p_r. Simultaneously, we increase the exponent of $p_1 = 2$ by Δ. To ensure that $\tilde{n} \leq n$, we require:

$$2^{\Delta} \leq 17, \quad \Delta = \left\lfloor \frac{\ln(17)}{\ln(2)} \right\rfloor = 4.$$

Then, the new number is given by $\tilde{n} = \frac{n}{17} \times 2^{\Delta} \leq n$.

4.3 Calculation of the Ratio R

Before the swap, the contributions in the factorization are:

- $f(2,\, j_1)$ (for example, if $j_1 = 2$, then $f(2,2) = \frac{2^3-1}{2^2} = 1.75$),
- $f(17, 2)$ (computed as ≈ 1.0623).

After the swap, the exponent of 17 is decreased to 1 and the exponent of 2 is increased to $(j_1 + \Delta)$. The new contributions become $f(2,\, j_1 + \Delta)$ and $f(17,\, 1) \approx 1.0588$.

We define the ratio

$$R = \frac{\sigma(\tilde{n})/\tilde{n}}{\sigma(n)/n} = \frac{f(2,\, j_1 + \Delta) \,\times\, f(17,\, 1)}{f(2, j_1) \,\times\, f(17,\, 2)}.$$

The swap is beneficial if $R > 1$. For $j_1 = 2$ and $\Delta = 4$:

- $f(2,2) = \frac{2^3-1}{2^2} = \frac{7}{4} = 1.75$,
- $f(17,2) \approx 1.0623$,
- $f(2,6) = \frac{2^7-1}{2^6} = \frac{127}{64} \approx 1.9844$,
- $f(17,1) = \frac{17^2-1}{17\times 16} = \frac{288}{272} \approx 1.0588$.

Therefore,

$$R = \frac{1.9844 \times 1.0588}{1.75 \times 1.0623} \approx \frac{2.102}{1.859} \approx 1.13 > 1.$$

Hence, the swap increased $\sigma(n)/n$ by more than 13% while maintaining $\tilde{n} \le n$.

4.4 Summary

With this three-step approach:

1. **We define** $f(p,j)$ and decompose $\sigma(n)/n$ as a product,
2. **We choose** $\Delta = \lfloor \ln(p_r)/\ln(2) \rfloor$ such that $\tilde{n} \le n$,
3. **We show** that $\dfrac{f(2, j_1 + \Delta)}{f(2, j_1)} > \dfrac{f(17,2)}{f(17,1)}$, i.e. $R > 1$.

The swap argument then states that if in the factorization of a highly composite number any larger prime has an exponent of 2, it is always possible to transfer exponent units to smaller bases (typically 2) and achieve a greater value of $\sigma(n)/n$, while ensuring $\tilde{n} \le n$. This refutes the so-called minimal configuration and proves that the optimal configuration must have significantly higher exponents for the smallest primes, which leads to $\log(n) > p_k$.

Explicit bridge to structured candidates

The swap mechanism supports the conclusion $\log n > p_k$ for the structured families (HCN/SA/CA patterns: full prime support and nonincreasing exponents). We do not claim that HCN maximize $\sigma(n)/n$; rather, the argument is used to justify the structural pattern and the inequality $\log n > p_k$ employed in the main proof.

Strengthened bridge required for the delta-corrected beta envelope

The corrected last-prime numerical envelope has the form

$$\beta_k < (e^\gamma + \Delta_0) \log p_k, \qquad \varepsilon_0 := \frac{\Delta_0}{e^\gamma} \approx 0.005558981456720.$$

To convert this beta envelope into the sharper target

$$\beta_k < e^\gamma \operatorname{loglog} n,$$

the bridge $\log n > p_k$ is not sufficient by itself. A sufficient strengthened bridge is

$$\log n \ge p_k^{1+\varepsilon_0}.$$

Indeed, this gives $\operatorname{loglog} n \ge (1 + \varepsilon_0) \log p_k$, hence

$$e^\gamma \operatorname{loglog} n \ge (e^\gamma + \Delta_0) \log p_k.$$

Thus the old bridge $\log n > p_k$ remains useful for the deficit-corrected proof after one has reached an $e^\gamma \log p_k$ bound for $\sigma(n)/n$, but the delta-corrected beta-only route needs this stronger growth estimate as a separate structural input.

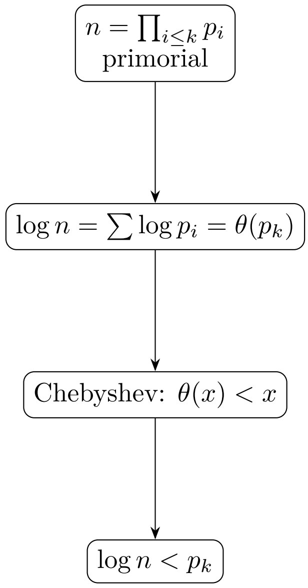

Figure 1: Proof 1 pipeline for primorials.

References

References

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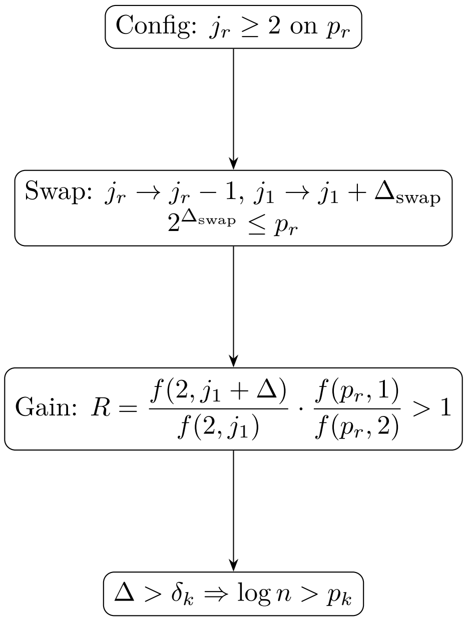

Figure 2: Schematic of the swap argument.