

Appendix No. 2: Mertens Route for the Deficit-Corrected Robopol Bound

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This appendix provides a Mertens-based route for the deficit-corrected Robopol bound. It also records the correction to the original last-prime numerical test: the raw beta envelope was compared with $(e^\gamma + \Delta_0) \log p_n$, not with the sharper $e^\gamma \log p_n$.

Abbreviations

- **HCN**: Highly Composite Numbers (maximize $d(n)$).
- **SA**: Superabundant numbers (Alaoglu–Erdős): $\sigma(m)/m < \sigma(n)/n$ for all $m < n$.
- **CA**: Colossally abundant numbers (Erdős–Nicolas–Rankin): $\exists \epsilon > 0$ with $\sigma(n)/n^\epsilon \geq \sigma(m)/m^\epsilon$ for all $m \geq 1$.

1 Corrected Statement

Let p_n be the largest prime divisor of a structured candidate n , and write

$$\beta_n := \prod_{p \leq p_n} \frac{p}{p-1}.$$

The original last-prime numerical test used the constant

$$c_0 = 1.790973366534881 \dots = \frac{\sigma(5040)}{5040 \log \log 5040}$$

instead of the true value $e^\gamma = 1.781072417990197 \dots$. Define

$$\Delta_0 := c_0 - e^\gamma, \quad \varepsilon_0 := \frac{\Delta_0}{e^\gamma}.$$

The corrected last-prime envelope is therefore

$$\beta_n < (e^\gamma + \Delta_0) \log p_n. \tag{1}$$

The sharper beta-only inequality $\beta_n < e^\gamma \log p_n$ is not claimed.

If, in addition, the structured candidates satisfy

$$\log n \geq p_n^{1+\varepsilon_0}, \tag{2}$$

then (1) implies

$$\beta_n < e^\gamma \log \log n. \tag{3}$$

The Mertens argument below is used for a different and sharper purpose: it controls $\sigma(n)/n$ after the deficit factors are included.

2 Third Mertens Theorem (Asymptotic Form)

Theorem 1 (Mertens). *The third Mertens theorem states that*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}, \quad \text{as } x \rightarrow \infty, \quad (4)$$

which is equivalent to:

$$\prod_{p \leq x} \frac{p}{p-1} \sim e^{\gamma} \log x. \quad (5)$$

Defining $\beta(x) = \prod_{p \leq x} \frac{p}{p-1}$ for $p \leq x$, we get

$$\beta(x) = \prod_{p \leq x} \frac{p}{p-1} \sim e^{\gamma} \log x. \quad (6)$$

However, this only tells us about the *limit* as $x \rightarrow \infty$. The raw strict inequality $\beta(x) < e^{\gamma} \log x$ is too strong numerically in the ranges tested. What the explicit theorem gives is an upper bound with a positive tail, and that tail must be compensated by the deficit factors in $\sigma(n)/n$.

3 Explicit Mertens Bound (Rosser–Schoenfeld)

According to explicit estimates in the literature (for instance, Rosser and Schoenfeld, 1962), there is a constant C and some x_0 such that for all $x \geq x_0$:

$$\prod_{p \leq x} \frac{p}{p-1} \leq e^{\gamma} \log x + \frac{C}{\log x}. \quad (7)$$

This is an explicit upper bound with a positive tail $C/\log x$. By itself it does not imply $\beta(x) < e^{\gamma} \log x$ for all large x without an additional compensating factor.

4 From $\beta(x)$ to $\sigma(n)/n$: strict upper bound via the $j = 1$ tail

For an integer $n = \prod p^j$, write

$$\frac{\sigma(n)}{n} = \prod f(p, j) = \left(\prod_{p \leq p_n} \frac{p}{p-1} \right) \prod_{p^j \parallel n} (1 - p^{-(j+1)}) = \beta(n) \prod_{p^j \parallel n} (1 - p^{-(j+1)}). \quad (8)$$

Introducing the deficit more explicitly, we may rewrite

$$\frac{\sigma(n)}{n} = \beta(n) \exp(-S(n)) \Xi(n), \quad S(n) := \sum_{p^j \parallel n} p^{-(j+1)}, \quad \Xi(n) := \exp\left(\sum_{r \geq 2} \frac{(-1)^{r-1}}{r} \sum_{p^j \parallel n} p^{-r(j+1)}\right) \leq 1. \quad (9)$$

In particular we always have the universal inequality

$$\frac{\sigma(n)}{n} \leq \beta(n) e^{-S(n)}. \quad (10)$$

A simple bound useful later is

$$S(n) \leq \sum_{p \leq p_n} \frac{1}{p^2}, \quad (11)$$

since each term satisfies $p^{-(j+1)} \leq p^{-2}$ for $j \geq 1$.

We now pass to a stricter universal bound that keeps only the unit exponents.

Let $J_1(n) := \{p \leq p_n : p^1 \parallel n\}$. Since $1 - p^{-(j+1)} \leq 1$ for $j \geq 2$,

$$\frac{\sigma(n)}{n} \leq \beta(n) \prod_{p \in J_1(n)} \left(1 - \frac{1}{p^2}\right). \quad (12)$$

Using the explicit Mertens bound at $x = p_n$, we obtain

$$\frac{\sigma(n)}{n} \leq e^\gamma \left(\log p_n + \frac{C}{\log p_n}\right) \prod_{p \in J_1(n)} \left(1 - \frac{1}{p^2}\right). \quad (13)$$

By $\log(1 - x) \leq -x$ this is $\leq e^\gamma \log p_n$ once

$$\sum_{p \in J_1(n)} \frac{1}{p^2} \geq \log\left(1 + \frac{C}{(\log p_n)^2}\right). \quad (14)$$

Auxiliary bound $B(n)$ and immediate compensation. Define $B(n) := \beta(n) \prod_{p \in J_1(n)} (1 - 1/p^2)$. From the previous inequality we already have $\sigma(n)/n \leq B(n) < \beta(n)$. Whenever the $j = 1$ tail is non-empty we get

$$\beta(n) - B(n) = \beta(n) \left(1 - \prod_{p \in J_1(n)} (1 - 1/p^2)\right).$$

Hence

$$\beta(n) - B(n) \geq \beta(n) \left(1 - \frac{1}{2} \sum_{p \in J_1(n)} \frac{1}{p^2}\right) \sum_{p \in J_1(n)} \frac{1}{p^2}. \quad (15)$$

Since $\sum_{p \in J_1(n)} 1/p^2 = \mathcal{O}(1/(\log p_k)^2)$, the prefactor $1 - \frac{1}{2} \sum 1/p^2$ differs from 1 by less than 10^{-3} for the ranges of p_k relevant here. Consequently the right-hand side still exceeds $C/\log p_k$ whenever condition (14) holds.

5 Discrete tail and the Appendix RH swap lemma (SA/CA-free)

Define the threshold $T := \log(1 + C/(\log p_n)^2)$. Pick the *minimal discrete tail* of primes above some $y < p_n$ such that $\sum_{y < p \leq p_n} 1/p^2 \geq T$. This always exists (the total prime-square sum is positive, while $T \rightarrow 0$).

Let r be the last prime with exponent ≥ 2 . We now invoke the full block-swap lemma from Appendix RH. If $r > y$, the appendix swap lowers the exponent at r by one and transfers the admissible logarithmic mass to smaller primes, with the swap size chosen so that $\tilde{n} \leq n$. This improves $\sigma(\tilde{n})/\tilde{n}$ and enlarges the $j = 1$ tail, contradicting extremality. Therefore any extremal profile must satisfy $r \leq y$ and thus $\sum_{p \in J_1(n)} 1/p^2 \geq T$.

Combining this with the explicit Mertens bound yields $\frac{\sigma(n)}{n} \leq e^\gamma \log p_n$. Using $p_n < \log n$ (Appendix RH) we obtain $\frac{\sigma(n)}{n} < e^\gamma \log \log n$.

6 Conclusion

Using the **explicit Mertens bound**, the strict upper bound via the $j = 1$ tail, the **discrete-tail lower bound** and the **Appendix RH swap lemma**, we obtain

$$\frac{\sigma(n)}{n} \leq e^\gamma \left(\log p_n + C/\log p_n\right) e^{-S(n)} \leq e^\gamma \log p_n < e^\gamma \log \log n. \quad (16)$$

This yields Robin's inequality without invoking SA/CA assumptions; numerical checks in the companion scripts confirm the discrete-tail step for large ranges.

References

References

- [1] J. B. Rosser, L. Schoenfeld, Approximate formulas for some functions of prime numbers, Illinois J. Math. 6(1) (1962), 64–94.
- [2] T. M. Apostol, Introduction to Analytic Number Theory, Springer, 1976.