# TRANSPOSED POISSON STRUCTURE ON THE WITT-TYPE ALGEBRA $\mathcal{W}(a, -1)$: DERIVATIONS, AUTOMORPHISMS, AND ROTA–BAXTER OPERATORS

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ABSTRACT. In this paper, we provide a comprehensive study of the structural properties of the transposed Poisson algebra $\mathcal{W}(a, -1)$. We classify several types of linear maps, including derivations, local derivations, quasi-derivations, and $\delta$-derivations, showing that non-trivial $\delta$-derivations exist only for $\delta = 1$ and $\delta = \frac{1}{2}$. Furthermore, we describe the groups of automorphisms, local automorphisms, 2-local automorphisms, and quasi-automorphisms.

We also investigate Rota–Baxter operators of weight 1 on $\mathcal{W}(a, -1)$. Specifically, we classify operators that are homogeneous with respect to both the standard $\mathbf{Z}$-grading and a $\mathbf{Z}_2$-grading, establishing a rigidity result for the latter case. Finally, we classify all $\mathbf{W}$-compatible Novikov–Poisson structures, demonstrating that the associative product on the Witt algebra is universally compatible with its known Novikov structures.

## CONTENTS



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## 1. Transposed Poisson algebras

1.1. **Introduction.** Poisson algebras emerged from the Poisson brackets of Hamiltonian mechanics. These structures are widely studied in symplectic geometry, the theory of integrable systems, and classical and quantum mechanics [9, 21, 23, 25, 29]. Extensions of this concept, particularly in the context of hydrodynamic-type brackets and Hamiltonian operators, led to the discovery of Novikov–Poisson algebras [27]. These algebras are connected to Poisson brackets of hydrodynamic type and the classification of Hamiltonian operators [14], and they describe specific aspects of fluid dynamics and integrable systems.

The structure of a transposed Poisson algebra emerged from the study of Novikov–Poisson algebras. In particular, the commutator of a Novikov product, $[x,y] = x \circ y - y \circ x$, defines a Lie bracket. While it is natural to expect this bracket, together with a commutative associative product, to form a standard Poisson algebra, these operations satisfy the transposed Leibniz identity [5]. Furthermore, the Novikov product can be modified by an associative term: for any fixed element $\nu$ in the algebra, the new operation defined by

$$x \circ' y := x \circ y - \nu \cdot x \cdot y$$

provides a new Novikov–Poisson structure $(\mathfrak{L}, \circ', \cdot)$ on the original algebra [27].

This algebraic structure appears naturally in several other settings. For instance, given a derivation $D$ on a commutative associative algebra, a transposed Poisson bracket can be defined by:

$$[x,y] := x \cdot D(y) - y \cdot D(x).$$

Furthermore, C. Bai, R. Bai, and L. Guo [8] showed that transposed Poisson algebras arise as classical limits of Novikov deformations. In this framework, the Lie bracket is obtained as:

$$[x,y] := \frac{x \cdot_h y - y \cdot_h x}{h} \pmod{h}$$

where $x \cdot_h y \equiv x \cdot y \pmod{h}$ and $h$ is a parameter of the base field of characteristic 0.

These algebras also have natural generalizations and connections to other algebraic theories. One active research direction, introduced in [1], involves adding a parameter to the definitions of Poisson and transposed Poisson algebras. This results in $\delta$-Poisson and $\delta$-transposed Poisson algebras, where the identities linking the operations are scaled by $\delta$. Recent studies, such as the classification of transposed $\delta$-Poisson structures on null-filiform algebras [10] and the work on the $\delta$-first Whitehead Lemma for Jordan algebras [30], demonstrate how deformation and cohomological techniques apply in this broader context. Additionally, B. Sartayev [24] has investigated varieties of generalized transposed Poisson algebras, establishing their basic identities and comparing them with classical varieties. The landscape of non-associative structures, including results relevant to Poisson-type algebras, has been comprehensively surveyed by I. Kaygorodov [15]. Finally, connections to other areas, such as skew braces and Rota–Baxter operators on semi-direct products [7], suggest links between transposed Poisson algebras and the theory of solutions to the Yang–Baxter equation, opening new directions for future research.

In the present paper, we study the algebra $\mathcal{W}(a,b)$, which is the semi-direct product of the Witt algebra $\mathbf{W}$ and the tensor density module $I(a,b)$. We focus on the case $b=-1$. This choice is motivated by the work of Ferreira, Kaygorodov, and Lopatkin [11], who proved that all $\frac{1}{2}$-derivations of this Lie algebra are trivial when $b \neq -1$. According to [16], this implies that non-trivial transposed Poisson structures cannot exist in that case.

In fact, finding non-trivial $\frac{1}{2}$-derivations has become a common and effective way to identify such structures. For instance, recent studies have computed all $\frac{1}{2}$-derivations for deformative Schrödinger–Witt algebras, as well as non-finitely graded Witt and Heisenberg–Witt algebras, thereby classifying all their transposed Poisson structures [18] (see more cases in [2, 19, 28]).

Beyond these specific classifications, it is worth noting that generalized structures are just as significant as classical ones. For example, generalized derivations, introduced by G. Leger and E. Luks in [22], were recently used by R. Andruszkiewicz, T. Brzeziński, and K. Radziszewski in [3]. They established a one-to-one correspondence between Lie affgebras and pairs $(\mathfrak{g}, D)$, where $\mathfrak{g}$ is a Lie algebra and $D$ is a generalized derivation (supplemented by an element of $\mathfrak{g}$). This result underscores the deep connections between generalized derivations, non-associative structures, and the development of transposed Poisson algebras.

Parallel to the study of generalized operators, the concept of "local" structures has also gained prominence. This idea originally migrated into algebra from functional analysis. In some finite-dimensional algebras, local structures do not coincide with global ones. A notable example is the Cayley algebra $\mathcal{C}$ over a field $k$ with norm $n$. In [4], S. Ayupov, A. Elduque, and K. Kudaybergenov established the following result:

- The local derivations of $\mathcal{C}$ form the Lie algebra $\{d \in \mathfrak{so}_n(\mathbf{C}) \mid d(1) = 0\}$;
- The 2-local derivations coincide with the local derivations.

1.2. **Structure of the paper.** The paper is organized as follows. **Section 2** focuses on linear maps related to the Lie structure. We classify derivations, local derivations, and quasi-derivations. A key result here is that non-trivial $\delta$-derivations only exist for $\delta = 1$ and $\delta = \frac{1}{2}$. **Section 3** describes the symmetries of the algebra. We classify automorphisms and their generalizations, such as local and 2-local automorphisms. **Section 4** deals with Rota–Baxter operators. We classify operators that are homogeneous with respect to the **Z**-grading and prove a rigidity result for the $\mathbf{Z}_2$-grading. Finally, **Section 5** examines **W**-compatible Novikov–Poisson structures. We show that the associative product on the Witt algebra is compatible with all known Novikov structures on **W**

1.3. **Main results.** In this work, we classify both classical structures, such as derivations and automorphisms, and their various generalizations, including quasi-automorphisms and quasi-derivations. Our results show the structural rigidity of $\mathcal{W}(a,-1)$ by establishing that many "local" versions of these maps actually coincide with the global ones. Specifically, we show that all 2-local structures are identical to the standard ones. Furthermore, we find that while the **Z**-grading allows for several types of Rota–Baxter operators, the $\mathbf{Z}_2$-grading is so restrictive that it allows only the trivial operator.

1.4. **Notation.** In this paper, all algebras and linear spans are considered over the field of complex numbers **C**, and all linear maps are assumed to be **C**-linear. For the tensor density module $I(a,-1)$, we set $I_a = 0$ if the parameter $a$ is not an integer. Similarly, for any map $f$ with an integer domain, we define $f(a) = 0$ whenever $a \notin \mathbf{Z}$.

## 2. Derivations of $\mathcal{W}(a,-1)$

Let us define the primary objects of this study. Let $\mathfrak{L}$ be a linear space equipped with two bilinear operations:

$$\cdot,\ [\ ,\ ]\colon \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L}.$$

**Definition 1.** *The triple $(\mathfrak{L}, \cdot, [\ ,\ ])$ is called a* ***transposed Poisson algebra*** *if $(\mathfrak{L}, \cdot)$ is a commutative and associative algebra, $(\mathfrak{L}, [\ ,\ ])$ is a Lie algebra, and the transposed Leibniz identity holds:*

$$2z \cdot [x,y] = [z \cdot x, y] + [x, z \cdot y].$$

Let $\mathbf{W} = \langle L_n \mid n \in \mathbf{Z} \rangle$ be the Witt algebra and $I(a,-1) = \langle I_m \mid m \in \mathbf{Z} \rangle$ be the tensor density module. On the semi-direct product $\mathcal{W}(a,-1) = \mathbf{W} \ltimes I(a,-1)$ we define the following relations on the generators $L_n$ and $I_m$:

$$\begin{array}{lll} L_n \cdot L_m := L_{n+m} & I_n \cdot I_m := 0 & L_n \cdot I_m := I_{n+m}, \\ [L_n, L_m] := (n-m)L_{n+m} & [I_n, I_m] := 0 & [L_n, I_m] := -(m+a-n)I_{n+m}, \end{array}$$

for all integers $n$ and $m$. These relations define a **transposed Poisson algebra structure on** $\mathcal{W}(a,-1)$.

2.1. **Derivations.** X. Tang described the derivations of the Lie structure $(\mathcal{W}(a,-1), [\ ,\ ])$ in [26]. Every such derivation is a sum of an inner derivation and a projector onto the tensor density module:

$$\mathfrak{Der}_{\mathrm{Lie}}\big(\mathcal{W}(a,-1)\big) = \mathrm{Inn}\big(\mathcal{W}(a,-1)\big) \oplus \mathbf{C}\, D_1,$$

where $D_1(L_m) = 0,\ D_1(I_m) = I_m$. Derivations of the transposed Poisson structure must be compatible with both products. Let $D$ be an arbitrary derivation of the Lie structure:

$$D = \mathrm{ad}_\xi + \lambda D_1,$$

where $\xi = \sum\limits_j \alpha_j L_j + \beta_j I_j$. We obtain the following classification theorem:

**Theorem 2.** *The operator* $D$ *is a derivation of the transposed Poisson algebra* $\mathcal{W}(a,-1)$ *if and only if* $\xi = \alpha L_0 + \beta I_{-a}$ *and* $\lambda = 0$ *when* $a \in \mathbf{Z}$*. In other words,*

$$\mathfrak{Der}\big(\mathcal{W}(a,-1)\big) = \mathbf{C}\,\mathrm{ad}_{L_0} + \mathbf{C}\,\mathrm{ad}_{I_{-a}} + \mathbf{C}\,\delta_{a\notin \mathbf{Z}} D_1.$$

**Remark 3.** *In operator terms, these derivations are:*

$$\begin{pmatrix} \mathrm{diag}\big(-n \mid n \in \mathbf{Z}\big) & 0 \\ 0 & \mathrm{diag}\big(-(m+a) \mid m \in \mathbf{Z}\big) \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \big(i\delta_{i,i+a}(i,j)\big)_{i,j} & 0 \end{pmatrix}, \delta_{a\notin\mathbf{Z}} \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

*Proof.* The derivation satisfies the following condition:

$$[\xi, x] \cdot y + \lambda D_1(x) \cdot y + x \cdot [\xi, y] + x \cdot \lambda D_1(y) = [\xi, x \cdot y].$$

Since $D$ is a linear operator, we prove this for the generators of our algebra.
<u>Case 1</u>: $x = L_n$ and $y = L_m$.

$$\sum_j \big((j-n)\alpha_j + (j-m)\alpha_j\big) L_{n+m+j} + \sum_j \big((j+a-n)\beta_j + (j+a-m)\beta_j\big) I_{n+m+j}$$
$$= \sum_j (j-n-m)\alpha_j L_{n+m+j} + (j+a-n-m)\beta_j I_{n+m+j}.$$

Comparing the coefficients, we obtain:

$$j\alpha_j = 0,$$
$$(j+a)\beta_j = 0.$$

This implies:

$$\alpha_j = \alpha\delta_{0,j}, \quad \alpha \in \mathbf{C}, \tag{1}$$
$$\beta_j = \beta\delta_{-a,j}, \quad \beta \in \mathbf{C}. \tag{2}$$

<u>Case 2</u>: $x = L_n$ and $y = I_m$.

$$n\alpha I_{n+m} + (m+a)\alpha I_{n+m} + \lambda I_{n+m} = \alpha(n+m+a) I_{n+m}.$$

After computation, we get:

$$\alpha\big(m+a+n-(m+n+a)\big) I_{n+m} + \lambda I_{n+m} = 0.$$

This identity implies that $\lambda = 0$ if $a \in \mathbf{Z}$. Finally, we obtain:

$$D(\_) = [\alpha L_0 + \beta I_{-a}, \_] + \lambda\delta_{a\notin\mathbf{Z}} D_1(\_).$$

□

**Corollary 4.** *Consider the transposed Poisson algebra structure on the Witt algebra induced from* $\mathcal{W}(a,-1)$*. The derivations are described as:*

$$\mathfrak{Der}(\mathbf{W}) = \mathbf{C}\,\mathrm{ad}_{L_0},$$

*or in operator form:*

$$\mathfrak{Der}(\mathbf{W}) = \big\langle \mathrm{diag}\big(-n \mid n \in \mathbf{Z}\big)\big\rangle.$$

2.2. $\frac{1}{2}$**-derivations.** The class of $\frac{1}{2}$-derivations is important for transposed Poisson algebras. The transposed Leibniz rule implies that the map $y \mapsto x \cdot y$ is a $\frac{1}{2}$-derivation. This is analogous to how the map $y \mapsto [x, y]$ is a derivation for a Poisson algebra.

**Definition 5.** *A linear map* $D$ *is called a* $\frac{1}{2}$***-derivation*** *if it satisfies the following condition:*

$$D(x \star y) = \frac{1}{2}\big(D(x) \star y + x \star D(y)\big),$$

*for all elements* $x, y$ *and all products* $\star$ *in the algebra.*

In [11], the authors established that any element $D \in \frac{1}{2}\text{-}\mathfrak{Der}_{\mathrm{Lie}}\big(\mathcal{W}(a,-1)\big)$ acts on the generators as follows:

$$D(L_n) = \sum_j \alpha_j L_{j+n} + \beta_j I_{j+n}, \quad D(I_m) = \sum_j \alpha_j I_{j+m},$$

where the sequences $\alpha_*, \beta_* \subset \mathbf{C}$ have finite support.

**Theorem 6.** *Every $\frac{1}{2}$-derivation $D \in \frac{1}{2}$-$\mathfrak{Der}(\mathcal{W}(a,-1))$ acts on the generators according to:*

$$D(L_n) = \sum_j \alpha_j L_{j+n} + \beta_j I_{j+n}, \quad D(I_m) = \sum_j \alpha_j I_{j+m},$$

*for sequences $\alpha_*, \beta_* \subset \mathbf{C}$ with finite support.*

**Remark 7.** *In operator terms, the $\frac{1}{2}$-derivations of $\mathcal{W}(a,-1)$ are generated by:*

$$A_{t\in\mathbf{Z}} := \begin{pmatrix} \mathcal{J}^t & 0 \\ 0 & \mathcal{J}^t \end{pmatrix}, \quad B_{t\in\mathbf{Z}} := \begin{pmatrix} 0 & 0 \\ \mathcal{J}^t & 0 \end{pmatrix}.$$

*Proof.* We verify the condition for the generators $L_n$ and $I_m$.
Case 1: $x = L_n$ and $y = L_m$.

$$\sum_j \alpha_j L_{j+n+m} + \beta_j I_{j+n+m} = \frac{1}{2}\Big(\sum_j \alpha_j L_{j+n+m} + \beta_j I_{j+n+m} + \sum_j \alpha_j L_{j+m+n} + \beta_j I_{j+m+n}\Big).$$

The condition holds automatically, so this case imposes no additional restrictions.
Case 2: $x = L_n$ and $y = I_m$. Here we have

$$D(L_n \cdot I_m) = \sum_j \alpha_j I_{j+n+m},$$
$$D(L_n) \cdot I_m + L_n \cdot D(I_m) = \sum_j 2\alpha_j I_{j+n+m}.$$

It follows that $D(L_n \cdot I_m) = \frac{1}{2}\big(D(L_n) \cdot I_m + L_n \cdot D(I_m)\big)$. This case also imposes no restrictions on $\alpha_j$ and $\beta_j$. □

**Corollary 8.** *Consider the transposed Poisson algebra structure on the Witt algebra induced from $\mathcal{W}(a,-1)$. The space $\frac{1}{2}$-$\mathfrak{Der}(\mathbf{W})$ is generated by the following operators:*

$$A_{t\in\mathbf{Z}} := \mathcal{J}^t.$$

2.3. **Local derivations.**

**Definition 9.** *A linear map $D$ is a **local derivation** if for every element $x$ in an algebra, there exists a derivation $\Delta_x$ such that:*

$$D(x) = \Delta_x(x).$$

**Theorem 10.** *The local derivations of the transposed Poisson algebra $\mathcal{W}(a,-1)$ are spanned by the following operators:*

$$\begin{pmatrix} \operatorname{diag}(\alpha_L(n)n \mid n \in \mathbf{Z}) & 0 \\ 0 & \operatorname{diag}(\alpha_I(m)(m+a) \mid m \in \mathbf{Z}) \end{pmatrix},$$
$$\begin{pmatrix} 0 & \big(i\beta(i)\delta_{i,j+a}\big)_{i,j} \\ 0 & 0 \end{pmatrix},$$
$$\delta_{a\notin\mathbf{Z}} \begin{pmatrix} 0 & 0 \\ 0 & \operatorname{diag}(\lambda(m) \mid m \in \mathbf{Z}) \end{pmatrix},$$

*where $\alpha_L, \alpha_I, \beta, \lambda\colon \mathbf{Z} \longrightarrow \mathbf{C}$ are arbitrary functions.*

*Proof.* Every local derivation is represented as:

$$D(\_) = \alpha(\_)[L_0, \_] + \beta(\_)[I_{-a}, \_] + \lambda(\_)\delta_{a\notin\mathbf{Z}} D_1(\_),$$

where $\alpha, \beta, \lambda\colon \mathcal{W}(a,-1) \longrightarrow \mathbf{C}$ are arbitrary maps. Since $D$ is linear, it is sufficient to define its action on the generators:

$$\begin{aligned} D(L_n) &= \alpha(L_n)[L_0, L_n] + \beta(L_n)[I_{-a}, L_n] \\ &= -\alpha(L_n)nL_n + \beta(L_n)nI_{n-a}, \\ D(I_m) &= \alpha(I_m)[L_0, I_m] + \lambda(I_m)\delta_{a\notin\mathbf{Z}} D_1(I_m) \\ &= -\alpha(I_m)(m+a)I_m + \lambda(I_m)\delta_{a\notin\mathbf{Z}} I_m. \end{aligned}$$

We set the following functions: $\alpha_L(n) := \alpha(L_n)$, $\alpha_I(m) := \alpha(I_m)$, $\beta(n) := \beta(L_n)$, and $\lambda(m) := \lambda(I_m)$. This completes the proof. □

**Corollary 11.** *Consider the transposed Poisson algebra structure on the Witt algebra* $\mathbf{W}$ *induced from* $\mathcal{W}(a,-1)$. *The local derivations of* $\mathbf{W}$ *have the following form:*

$$\mathfrak{loc}\,\mathfrak{Der}(\mathbf{W}) = \big\langle \operatorname{diag}\big(\alpha(n)n \mid n \in \mathbf{Z}\big)\big\rangle,$$

*where* $\alpha\colon \mathbf{Z} \longrightarrow \mathbf{C}$ *is an arbitrary function.*

### 2.4. **2-local derivations.**

**Definition 12.** *A (not necessarily linear) map* $D$ *is called a* ***2-local derivation*** *if for any elements* $x$ *and* $y$ *in an algebra, there exists a derivation* $\Delta_{x,y}$ *such that:*

$$D(x) = \Delta_{x,y}(x),$$
$$D(y) = \Delta_{x,y}(y).$$

**Theorem 13.** *Every 2-local derivation of* $\mathcal{W}(a,-1)$ *is a derivation. In other words,*

$$2\text{-}\mathfrak{loc}\,\mathfrak{Der}\big(\mathcal{W}(a,-1)\big) = \mathfrak{Der}\big(\mathcal{W}(a,-1)\big).$$

*Proof.* We represent a 2-local derivation as follows:

$$D(x) = \alpha(x,y)[L_0,x] + \beta(x,y)[I_{-a},x] + \lambda(x,y)\delta_{a\notin\mathbf{Z}}D_1(x),$$

where $\alpha, \beta, \lambda$ are scalar functions. For any $y$ and $y'$, we have:

$$D(x) = \alpha(x,y)[L_0,x] + \beta(x,y)[I_{-a},x] + \lambda(x,y)\delta_{a\notin\mathbf{Z}}D_1(x),$$
$$D(x) = \alpha(x,y')[L_0,x] + \beta(x,y')[I_{-a},x] + \lambda(x,y')\delta_{a\notin\mathbf{Z}}D_1(x).$$

Subtracting these equations gives:

$$0 = (\alpha(x,y) - \alpha(x,y'))[L_0,x] + (\beta(x,y) - \beta(x,y'))[I_{-a},x] + (\lambda(x,y) - \lambda(x,y'))\delta_{a\notin\mathbf{Z}}D_1(x).$$

This identity shows that $\alpha(x,y)$ is independent of $y$. Due to the symmetry between $x$ and $y$, we conclude that $\alpha$, $\beta$, and $\lambda$ are constants. Thus, $D$ is a derivation. □

**Corollary 14.** *Consider the transposed Poisson algebra structure on the Witt algebra* $\mathbf{W}$ *induced from* $\mathcal{W}(a,-1)$. *The 2-local derivations of* $\mathbf{W}$ *are derivations:*

$$2\text{-}\mathfrak{loc}\,\mathfrak{Der}(\mathbf{W}) = \mathfrak{Der}(\mathbf{W}).$$

### 2.5. **Quasi-derivations.**

**Definition 15.** *A pair of linear maps* $(D,F)$ *is called a* ***quasi-derivation*** *if it satisfies the following condition:*

$$D(x \star y) = F(x) \star y + x \star F(y),$$

*for all elements* $x, y$ *and all products* $\star$ *in the algebra.*

In [17], I. Kaygorodov, A. Khudoyberdiyev, and Z. Shermatova established that the space of quasi-derivations of a Lie algebra is the sum of the space of derivations and the space of $\frac{1}{2}$-derivations:

$$\mathfrak{QDer}_{\text{Lie}}\big(\mathcal{W}(a,-1)\big) = \mathfrak{Der}_{\text{Lie}}\big(\mathcal{W}(a,-1)\big) \oplus \frac{1}{2}\text{-}\mathfrak{Der}_{\text{Lie}}\big(\mathcal{W}(a,-1)\big).$$

**Theorem 16.** *Any quasi-derivation* $(D,F)$ *of the transposed Poisson algebra* $\mathcal{W}(a,-1)$ *is defined by its action on the generators:*

$$F(L_n) = \sum_j \alpha_j L_{n+j} + \sum_j \beta_j I_{n+j} - \gamma\, n L_n - \tau n I_{n-a},$$
$$F(I_m) = \sum_j \alpha_j I_{m+j} - (m+a)\,\gamma I_m + \lambda\delta_{a\notin\mathbf{Z}} I_m,$$

*and*

$$D(L_n) = \sum_j 2\alpha_j L_{n+j} + 2\beta_j I_{n+j} - \gamma\, n L_n - \tau n I_{n-a},$$
$$D(I_m) = \sum_j 2\alpha_j I_{m+j} - (m+a)\,\gamma I_m + \lambda\delta_{a\notin\mathbf{Z}} I_m,$$

*where* $\tau, \gamma \in \mathbf{C}$ *and the sequences* $\alpha_j, \beta_j \subset \mathbf{C}$ *have finite support.*

**Remark 17.** *In operator terms, the quasi-derivations* $(D,F)$ *on* $\mathcal{W}(a,-1)$ *are generated by:*

$$A_{t\in\mathbf{Z}} := \begin{pmatrix} \partial^t & 0 \\ 0 & \partial^t \end{pmatrix}, \quad \Gamma := \begin{pmatrix} \mathrm{diag}(n \mid n\in\mathbf{Z}) & 0 \\ 0 & \mathrm{diag}(m+a \mid m\in\mathbf{Z}) \end{pmatrix}, \quad B_{t\in\mathbf{Z}} := \begin{pmatrix} 0 & 0 \\ \partial^t & 0 \end{pmatrix},$$

$$T := \begin{pmatrix} 0 & 0 \\ (i\delta_{i,j+a})_{i,j} & 0 \end{pmatrix}, \quad \Lambda := \delta_{a\notin\mathbf{Z}} \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

*These operators satisfy the following relations:*

$$F = \sum_t (C_t A_t + C'_t B_t) - C''\Gamma - C'''T + \widetilde{C}\Lambda,$$
$$D = 2\sum_t (C_t A_t + C'_t B_t) - C''\Gamma - C'''T + \widetilde{C}\Lambda,$$

*for arbitrary complex coefficients.*

*Proof.* The expressions for $F$ follow directly from Theorems 2 and 6 and the results in [17]. The action of $D$ is obtained by direct computation using the relations:

$$D(L_n) = F(L_n)\cdot L_0 + L_n \cdot F(L_0),$$
$$D(I_m) = F(I_m)\cdot L_0 + I_m \cdot F(L_0).$$

Calculating these gives:

$$D(L_n) = \sum_j 2\alpha_j L_{n+j} + 2\beta_j I_{n+j} - \gamma n L_n - \tau n I_{n-a},$$
$$D(I_m) = \sum_j 2\alpha_j I_{m+j} - (m+a)\gamma I_m + \lambda\delta_{a\notin\mathbf{Z}} I_m.$$

This describes the action of $D$ on the generators. □

**Corollary 18.** *Consider the transposed Poisson algebra structure on the Witt algebra* $\mathbf{W}$ *induced from* $\mathcal{W}(a,-1)$. *Any quasi-derivation* $(D,F)$ *of* $\mathbf{W}$ *is described by:*

$$F(L_n) = \sum_j \alpha_j L_{n+j} - \gamma n L_n,$$
$$D(L_n) = \sum_j 2\alpha_j L_{n+j} - \gamma n L_n,$$

*where* $\gamma \in \mathbf{C}$ *and the sequence* $(\alpha_j)_j \subset \mathbf{C}$ *has finite support.*

2.6. **$\delta$-derivations.**

**Definition 19.** *A linear map* $D$ *is called a* ***$\delta$-derivation*** *if it satisfies the following condition:*

$$D(x \star y) = \delta\left(D(x)\star y + x \star D(y)\right),$$

*where* $\delta \in \mathbf{C}$. *This condition holds for all elements* $x, y$ *and all products* $\star$ *in the algebra. Note that in terms of Definition 15, this means* $F = \delta D$.

**Proposition 20.** *For the transposed Poisson algebra* $\mathcal{W}(a,-1)$ *and the induced structure on the Witt algebra, non-trivial* $\delta$*-derivations exist only for* $\delta = 1$ *or* $\delta = \frac{1}{2}$.

*Proof.* We solve the equation $F = \delta D$. First, we consider the generator $L_n$:

$$F(L_n) = \sum_j \alpha_j L_{n+j} + \sum_j \beta_j I_{n+j} - \gamma n L_n - \tau n I_{n-a},$$
$$\delta D(L_n) = \sum_j 2\delta\alpha_j L_{n+j} + 2\delta\beta_j I_{n+j} - \delta\gamma n L_n - \delta\tau n I_{n-a}.$$

Comparing the coordinates, we obtain the following system:

$$\begin{aligned}\alpha_j(1-2\delta)&=0, \quad j\neq 0,\\ \alpha_0(1-2\delta)-\gamma n(1-\delta)&=0,\\ \beta_j(1-2\delta)&=0, \quad j\neq -a,\\ \beta_{-a}(1-2\delta)-\tau n(1-\delta)&=0.\end{aligned}$$

If $\delta=\frac{1}{2}$, then the coefficients $\alpha_j$ and $\beta_j$ are arbitrary, while $\tau=0$ and $\gamma=0$. If $\delta=1$, then $\alpha_j=0$ and $\beta_j=0$ for all j, while $\tau$ and $\gamma$ are arbitrary complex numbers. If $\delta\notin\{\frac{1}{2},1\}$, then $\alpha_j=0$ for $j\neq 0$ and $\beta_j=0$ for $j\neq -a$. The remaining equations are:

$$\begin{aligned}\alpha_0(1-2\delta)-\gamma n(1-\delta)&=0,\\ \beta_{-a}(1-2\delta)-\tau n(1-\delta)&=0.\end{aligned}$$

For $n=0$, we find $\alpha_0=0$ and $\beta_{-a}=0$. Consequently, $\tau=0$ and $\gamma=0$. Thus, there are no non-trivial $\delta$-derivations for $\delta\notin\{\frac{1}{2},1\}$.

Next, we check the generator $I_m$:

$$\begin{aligned}F(I_m)&=\sum_j \alpha_j I_{m+j}-(m+a)\gamma I_m+\lambda\delta_{a\notin\mathbf{Z}}I_m,\\ \delta D(I_m)&=\sum_j 2\delta\alpha_j I_{m+j}-\delta(m+a)\gamma I_m+\delta\lambda\delta_{a\notin\mathbf{Z}}I_m.\end{aligned}$$

We test the cases $\delta=1$ and $\delta=\frac{1}{2}$.
If $\delta=1$, the equation becomes:

$$-(m+a)\gamma I_m+\lambda\delta_{a\notin\mathbf{Z}}I_m=-(m+a)\gamma I_m+\lambda\delta_{a\notin\mathbf{Z}}I_m.$$

This identity holds for any $\gamma$ and $\lambda$.
If $\delta=\frac{1}{2}$, the equation becomes:

$$\sum_j \alpha_j I_{m+j}+\lambda\delta_{a\notin\mathbf{Z}}I_m=\sum_j \alpha_j I_{m+j}+\frac{1}{2}\lambda\delta_{a\notin\mathbf{Z}}I_m.$$

This implies $\lambda=0$. □

## 3. AUTOMORPHISMS OF $\mathcal{W}(a,-1)$

S. Gao, C. Jiang, and Y. Pei described the automorphisms of the Lie algebra structure on $\mathcal{W}(a,-1)$ in [12]. They established the following isomorphism:

$$\mathrm{Aut}_{\mathrm{Lie}}\big(\mathcal{W}(a,-1)\big)\cong\begin{cases}\mathbf{C}^\infty\rtimes(\mathbf{C}^*\times\mathbf{C}^*) & a\notin\frac{1}{2}\mathbf{Z},\\ \mathbf{C}^\infty\rtimes(\mathbf{Z}_2\ltimes(\mathbf{C}^*\times\mathbf{C}^*)) & \text{otherwise.}\end{cases}$$

Here, $\mathbf{C}^\infty$ denotes the set of all sequences of complex numbers with finite support. The action on the generators is determined by the following subgroup structures.

*The Normal Abelian Subgroup* $\mathfrak{I}$. The subgroup $\mathfrak{I}:=\langle\exp(k\,\mathrm{ad}\,I_i)\mid i\in\mathbf{Z},k\in\mathbf{C}\rangle$ of $\mathrm{Aut}\big(\mathcal{W}(a,-1)\big)$ acts on the generators as follows:

$$\begin{aligned}&\prod_j\exp(k_j\mathrm{ad}\,I_j)L_n=L_n+\sum_j^t k_j(j+a-n)I_{j+n},\\ &\prod_j\exp(k_j\mathrm{ad}\,I_j)I_m=I_m,\end{aligned}$$

where $(k_*)\subset\mathbf{C}$ is a sequence with finite support.

*Subgroups of Scale and Reflection.* When $a \notin \frac{1}{2}\mathbf{Z}$, the automorphism group is determined by $\mathfrak{a}_1 := \{\overline{\sigma}_{\alpha,\mu} \mid \alpha, \mu \in \mathbf{C}^*\} \cong \mathbf{C}^* \times \mathbf{C}^*$:

$$\overline{\sigma}_{\alpha,\mu}(L_n) = \alpha^n L_n,$$
$$\overline{\sigma}_{\alpha,\mu}(I_m) = \alpha^m \mu I_m.$$

For the remaining cases, the structure involves $\mathfrak{a}_2 \cong \mathbf{Z}_2$ and $\mathfrak{b}_2 \cong \mathbf{C}^* \times \mathbf{C}^*$. Their actions are defined as follows:

(1) **Action of** $\mathfrak{a}_2 = \{\overline{\sigma}_\varepsilon \mid \varepsilon = \pm 1\}$:

$$\overline{\sigma}_\varepsilon(L_n) = \begin{cases} \varepsilon L_{\varepsilon n} & a \neq 0 \\ \varepsilon L_{\varepsilon n} + \lambda_\varepsilon n I_{\varepsilon n} & a = 0 \end{cases}; \qquad \overline{\sigma}_\varepsilon(I_m) = I_{\varepsilon(m+a)-a}.$$

(2) **Action of** $\mathfrak{b}_2 = \{\overline{\sigma}_{\alpha,\mu} \mid \alpha, \mu \in \mathbf{C}^*\}$:

$$\overline{\sigma}_{\alpha,\mu}(L_n) = \begin{cases} \alpha^n L_n & a \neq 0 \\ \alpha^n L_n + \alpha^n \lambda_1 n I_n & a = 0 \end{cases}; \qquad \overline{\sigma}_{\alpha,\mu}(I_m) = \alpha^m \mu I_m.$$

Using this notation, we express the automorphisms of the Lie structure as follows:

$$\mathrm{Aut}_{\mathrm{Lie}}\big(\mathcal{W}(a,-1)\big) \cong \begin{cases} \mathfrak{J} \rtimes \mathfrak{a}_1 & a \notin \frac{1}{2}\mathbf{Z}, \\ \mathfrak{J} \rtimes (\mathfrak{a}_2 \ltimes \mathfrak{b}_2) & \text{otherwise.} \end{cases}$$

**Theorem 21.** *The group of automorphisms of the transposed Poisson algebra $\mathcal{W}(a,-1)$ is generated by:*

$$\begin{pmatrix} \mathrm{diag}\big(\alpha^n \mid n \in \mathbf{Z}\big) & 0 \\ 0 & \mathrm{diag}\big(\mu\alpha^n \mid n \in \mathbf{Z}\big) \end{pmatrix},$$

*where $\alpha, \mu \in \mathbf{C}^*$. In particular,*

$$\mathrm{Aut}\big(\mathcal{W}(a,-1)\big) \cong \mathbf{C}^* \times \mathbf{C}^*.$$

*Proof.* To prove the theorem, we check which Lie automorphisms preserve the associative product $x \cdot y$. Since $\mathcal{W}(a,-1)$ is generated by $L_n$ and $I_m$, it is sufficient to verify the conditions $f(L_n \cdot L_m) = f(L_n) \cdot f(L_m)$ and $f(L_n \cdot I_m) = f(L_n) \cdot f(I_m)$.

Case 1: $a \notin \frac{1}{2}\mathbf{Z}$.

We begin by fixing the generators of the Lie algebra automorphisms: $\tau \in \mathfrak{J}$ and $\sigma_{\alpha,\mu} \in \mathfrak{a}_1$. Define the map $f(x) := \tau(\sigma_{\alpha,\mu}(x))$.

First, consider the action on $L_n$ and $L_m$:

$$f(L_{n+m}) = f(L_n \cdot L_m) = f(L_n) \cdot f(L_m).$$

Expanding both sides:

$$f(L_{n+m}) = \alpha^{n+m}\left(L_{n+m} + \sum_i k_i(i+a-n-m)I_{i+n+m}\right),$$
$$f(L_n)\cdot f(L_m) = \alpha^{n+m}\left(L_n + \sum_i k_i(i+a-n)I_{i+n}\right)\cdot\left(L_m + \sum_i k_i(i+a-m)I_{i+m}\right)$$
$$= \alpha^{n+m}\left(L_{n+m} + \sum_i k_i(2i+2a-n-m)I_{i+n+m}\right).$$

Comparing the coefficients at $I_{i+n+m}$, we obtain the condition:

$$k_i(i+a) = 0.$$

This implies $k_i = c\delta_{i,-a}$ for some $c \in \mathbf{C}$. Thus, $\tau(L_n) = L_n - cnI_{n-a}$ and $\tau(I_m) = I_m$.

Next, we apply the condition to the product $L_n \cdot I_m$:

$$f(I_{n+m}) = f(L_n \cdot I_m) = f(L_n) \cdot f(I_m),$$
$$\alpha^{n+m}\mu I_{n+m} = \alpha^n(L_n - cnI_{n-a}) \cdot (\alpha^m \mu I_m)$$
$$= \alpha^{n+m}\mu I_{n+m} - \alpha^{n+m}\mu cnI_{n-a+m}.$$

This forces $c = 0$, yielding $f(L_n) = \alpha^n L_n$ and $f(I_m) = \alpha^m \mu I_m$.

Case 2: $a \in \frac{1}{2}\mathbf{Z}$ and $a \notin \mathbf{Z}$.
Let $f(x) := \tau(\sigma_\varepsilon(\sigma_{\alpha,\mu}(x)))$ where $\sigma_\varepsilon \in \mathfrak{a}_2$ and $\sigma_{\alpha,\mu} \in \mathfrak{b}_2$. Repeating the calculation for $L_n$ and $L_m$:

$$f(L_{n+m}) = \varepsilon\alpha^{n+m}\left(L_{\varepsilon(n+m)} + \sum_i k_i(i + a - \varepsilon(n+m))I_{i+\varepsilon(n+m)}\right),$$

$$f(L_n)\cdot f(L_m) = \varepsilon^2\alpha^{n+m}\left(L_{\varepsilon(n+m)} + \sum_i k_i\big[2i + 2a - \varepsilon(n+m)\big]I_{i+\varepsilon(n+m)}\right).$$

Comparing the coefficients of $L_{\varepsilon(n+m)}$ gives $\varepsilon = \varepsilon^2$, hence $\varepsilon = 1$. The coefficients of $I_{i+\varepsilon(n+m)}$ again yield $k_i(i + a) = 0$. The rest of the argument follows Case 1.

Case 3: $a \in \mathbf{Z}$.
It is known from [12] that $\mathcal{W}(x, b) \cong \mathcal{W}(x + n, b)$ for all integers $n$, so we reduce this to $a = 0$. Let $f(x) := \tau(\sigma_\varepsilon(\sigma_{\alpha,\mu}(x)))$. Following [12], we have:

$$\begin{aligned} f(L_\xi) &= \tau\left(\alpha^\xi\big[\varepsilon L_{\varepsilon\xi} + (\lambda_\varepsilon + \lambda_1)\xi I_{\varepsilon\xi}\big]\right) \\ &= \alpha^\xi\left(\varepsilon\Big[L_{\varepsilon\xi} + \sum_j k_j(j - \varepsilon\xi)I_{j+\varepsilon\xi}\Big] + (\lambda_\varepsilon + \lambda_1)\xi I_{\varepsilon\xi}\right). \end{aligned}$$

To check the preservation of the associative product, we expand both sides of equation $f(L_{n+m}) = f(L_n)\cdot f(L_m)$. Comparing the coefficients of $L_{\varepsilon(n+m)}$, we immediately obtain $\varepsilon = \varepsilon^2$, which forces $\varepsilon = 1$. Substituting $\varepsilon = 1$ back into the expressions, we get:

$$\begin{aligned} f(L_{n+m}) = \alpha^{n+m}\Bigg(&(L_{n+m} + \sum_{j\neq 0} k_j(j - (n+m))I_{j+n+m} \\ &+ \big[(\lambda_1 + \lambda_1) - k_0\big](n+m)I_{n+m}\Bigg), \end{aligned}$$

$$\begin{aligned} f(L_n)\cdot f(L_m) = \alpha^n &\left(L_n + \sum_j k_j(j-n)I_{j+n} + 2\lambda_1 n I_n\right) \\ \cdot\, \alpha^m &\left(L_m + \sum_j k_j(j-m)I_{j+m} + 2\lambda_1 m I_m\right). \end{aligned}$$

Using the property $L_p \cdot I_q = I_{p+q}$ and $I_p \cdot I_q = 0$, the product $f(L_n)\cdot f(L_m)$ simplifies to:

$$\begin{aligned} f(L_n)\cdot f(L_m) = \alpha^{n+m}\Bigg(&L_{n+m} + \sum_{j\neq 0} k_j(j-n)I_{j+n+m} + \sum_{j\neq 0} k_j(j-m)I_{j+n+m} \\ &+ \big[k_0(0-n) + k_0(0-m) + 2\lambda_1 n + 2\lambda_1 m\big]I_{n+m}\Bigg). \end{aligned}$$

Now, we compare the coefficients for each basis element $I_{j+n+m}$:

- For $j \neq 0$:
$$k_j(j - n - m) = k_j(j-n) + k_j(j-m) = k_j(2j - n - m).$$
This simplifies to $k_j j = 0$, which implies $k_j = 0$ for all $j \neq 0$.
- For $j = 0$ (the coefficient of $I_{n+m}$):
$$(2\lambda_1 - k_0)(n+m) = (2\lambda_1 - 2k_0)(n+m).$$
This yields $k_0(n+m) = 0$, so $k_0 = 0$.

Since $k_j = \lambda_j/\varepsilon_j$ (where $\varepsilon_j$ are non-zero constants from the Lie structure), the vanishing of all $k_j$ implies that all $\lambda_j$ are also zero. Thus, the only automorphisms that preserve the associative product are the scaling maps $\sigma_{\alpha,\mu}$, where $\alpha, \mu \in \mathbf{C}^*$. ☐

### 3.1. **Local automorphisms.**

**Definition 22.** *A linear map* $A$ *is called a* ***local automorphism*** *if for each element* $x$ *there exists an automorphism* $\mathcal{A}_x$ *such that:*

$$A(x)=\mathcal{A}_x(x).$$

**Theorem 23.** *Local automorphisms of the transposed Poisson algebra* $\mathcal{W}(a,-1)$ *have the following form:*

$$\mathfrak{loc}\,\mathfrak{Aut}\big(\mathcal{W}(a,-1)\big)=\begin{pmatrix}\operatorname{diag}\big(\alpha_L^n(n)\mid n\in\mathbf{Z}\big) & 0\\ 0 & \operatorname{diag}\big(\mu(m)\alpha_I^m(m)\mid m\in\mathbf{Z}\big)\end{pmatrix},$$

*where* $\alpha_L,\alpha_I,\mu\colon\mathbf{Z}\longrightarrow\mathbf{C}^*$ *are arbitrary functions.*

*Proof.* The proof follows the same logic as the proof for local derivations. Let $A\in\mathfrak{loc}\,\mathfrak{Aut}(\mathcal{W}(a,-1))$. By definition, for any $x$, the action of $A$ is given by $A(x)=\big(\operatorname{diag}(\alpha^n(x)\mid n\in\mathbf{Z})\oplus\operatorname{diag}(\mu(x)\alpha^n(x)\mid n\in\mathbf{Z})\big)x$. Applying this to the basis generators, we have:

$$\begin{aligned}A(L_n)&=\alpha^n(L_n)L_n,\\ A(I_m)&=\mu(I_m)\alpha^m(I_m)I_m.\end{aligned}$$

By setting $\alpha_L(n):=\alpha(L_n)$, $\alpha_I(m):=\alpha(I_m)$, and $\mu(m):=\mu(I_m)$, we obtain the required diagonal form. □

**Corollary 24.** *Local automorphisms of the transposed Poisson Witt algebra* $\mathbf{W}$ *are given by the operators* $\big\langle\operatorname{diag}\big(\alpha^n(n)\mid n\in\mathbf{Z}\big)\big\rangle$ *for an arbitrary function* $\alpha\colon\mathbf{Z}\longrightarrow\mathbf{C}^*$.

### 3.2. **2-local automorphisms.**

**Definition 25.** *A (not necessarily linear) map* $A$ *is called a* ***2-local automorphism*** *if for every pair of elements* $x,y$ *there exists an automorphism* $\mathcal{A}_{x,y}$ *such that:*

$$\begin{aligned}A(x)&=\mathcal{A}_{x,y}(x),\\ A(y)&=\mathcal{A}_{x,y}(y).\end{aligned}$$

**Theorem 26.** *Every 2-local automorphism of the transposed Poisson algebra* $\mathcal{W}(a,-1)$ *is an automorphism. In other words, any* $A\in 2\text{-}\mathfrak{loc}\,\mathfrak{Aut}\,(\mathcal{W}(a,-1))$ *has the form:*

$$\begin{pmatrix}\operatorname{diag}(\alpha^n\mid n\in\mathbf{Z}) & 0\\ 0 & \operatorname{diag}(\mu\alpha^n\mid n\in\mathbf{Z})\end{pmatrix},$$

*for some constants* $\alpha,\mu\in\mathbf{C}^*$.

*Proof.* Let $A$ be a 2-local automorphism. For any pairs $(x,y)$ and $(x,y')$, there exist automorphisms $\mathcal{A}_{x,y}$ and $\mathcal{A}_{x,y'}$ such that:

$$\begin{aligned}A(x)&=\mathcal{A}_{x,y}(x)=(\operatorname{diag}(\alpha^n(x,y)\mid n\in\mathbf{Z})\oplus\mu(x,y)\operatorname{diag}(\alpha^n(x,y)\mid n\in\mathbf{Z}))\,x,\\ A(x)&=\mathcal{A}_{x,y'}(x)=\big(\operatorname{diag}(\alpha^n(x,y')\mid n\in\mathbf{Z})\oplus\mu(x,y')\operatorname{diag}(\alpha^n(x,y')\mid n\in\mathbf{Z})\big)\,x.\end{aligned}$$

Thus, the difference must be zero:

$$0=\operatorname{diag}\big(\alpha^n(x,y)-\alpha^n(x,y')\mid n\in\mathbf{Z}\big)\oplus\operatorname{diag}\big(\mu(x,y)\alpha^n(x,y)-\mu(x,y')\alpha^n(x,y')\mid n\in\mathbf{Z}\big)x.$$

Setting $x=L_n$ and $y'=L_0$, we get:

$$\begin{aligned}\alpha^n(L_n,y)-\alpha^n(L_n,L_0)&=0,\\ \alpha(L_n,y)&=\zeta\alpha(L_n,L_0),\end{aligned}$$

where $\zeta$ is an $n$-th root of unity compatible with the action on $L_0$. Since the left-hand side must be independent of $y$, the function $\alpha$ is constant across the section $L_n\times\mathcal{W}(a,-1)$. By the symmetry of the arguments $x$ and $y$, it follows that $\alpha(x,y)\equiv\alpha$ is a global constant in $\mathbf{C}^*$.

Now consider $x=I_m$ and $y'=L_0$:

$$\begin{aligned}\mu(I_m,y)\alpha^m(I_m,y)&=\mu(I_m,L_0)\alpha^m(I_m,L_0),\\ \mu(I_m,y)&=\mu(I_m,L_0)\frac{\alpha^m(I_m,L_0)}{\alpha^m(I_m,y)}.\end{aligned}$$

Since $\alpha$ is already proven to be constant, we have $\mu(I_m,y)=\mu(I_m,L_0)=\mu$. Thus, $\alpha(x,y)\equiv\alpha$ and $\mu(x,y)\equiv\mu$ are constants, and $A$ is a global automorphism. □

**Corollary 27.** *Any 2-local automorphism of the transposed Poisson algebra* $\mathbf{W}$ *is an automorphism.*

### 3.3. **Quasi-automorphisms.**

**Definition 28.** *A pair of linear maps* $(A, F)$ *is a* ***quasi-automorphism*** *if it satisfies the following condition:*

$$A(x \star y) = F(x) \star F(y),$$

*for all elements* $x, y$ *and all products* $\star$ *in the algebra.*

Before proceeding to the main theorem, we state a useful lemma.

**Lemma 29.** *The only idempotents in the associative structure of the transposed Poisson algebra* $\mathcal{W}(a, -1)$ *are* $0$ *and* $L_0$.

*Proof.* This follows directly from the fact that the associative structure of $\mathcal{W}(a, -1)$ is an algebra of extended Laurent polynomials, where the only solutions to $p^2 = p$ are $0$ and $1$ (corresponding to $L_0$). □

**Theorem 30.** *Every quasi-automorphism* $(A, F)$ *of* $\mathcal{W}(a, -1)$ *is an automorphism, where* $A = F$. *Specifically:*

$$A = F = \begin{pmatrix} \mathrm{diag}(\alpha^n \mid n \in \mathbf{Z}) & 0 \\ 0 & \mathrm{diag}(\mu\alpha^n \mid n \in \mathbf{Z}) \end{pmatrix}, \quad \alpha, \mu \in \mathbf{C}^*.$$

*Proof.* Applying the definition to $L_0$:

$$A(L_0) = A(L_0 \cdot L_0) = F(L_0^{\cdot N} \cdot L_0^{\cdot M}) = F(L_0)^{\cdot N} \cdot F(L_0)^{\cdot M} = F(L_0) \cdot F(L_0).$$

By Lemma 29, $F(L_0) = L_0$ (excluding 0 since $F$ and $A$ are invertible).

Assume the map $F$ has the block matrix form $F = \begin{pmatrix} X & V \\ Y & W \end{pmatrix}$. Using the relations $L_{m+t} = L_m \cdot L_t = \frac{1}{m+t}[L_m, L_t]$, we compare the associative and Lie actions:

$$F(L_m) \cdot F(L_t) = \sum_k \sum_i x_i^m x_{k-i}^t L_k + \sum_k \sum_i (y_i^m x_{k-i}^t + x_i^m y_{k-i}^t) I_k,$$

$$F(L_{m+t}) \cdot L_0 = \sum_i x_i^{m+t} L_i + \sum_i y_i^{m+t} I_i,$$

$$\frac{1}{m+t}[F(L_{m+t}), L_0] = \frac{1}{m+t}\left(\sum_i i x_i^{m+t} L_i + \sum_i (i+a) y_i^{m+t} I_i\right).$$

Comparing the last two expressions coordinate-wise, we obtain:

$$x_i^{m+t} = \frac{i}{m+t} x_i^{m+t} \implies (m+t-i) x_i^{m+t} = 0, \tag{X-eq}$$

$$y_i^{m+t} = \frac{i+a}{m+t} y_i^{m+t} \implies (m+t-(i+a)) y_i^{m+t} = 0. \tag{Y-eq}$$

The equation (X-eq) implies that $X$ is a diagonal matrix ($x_i^n = 0$ for $i \neq n$). Substituting this into the associative product condition gives $x_m^m x_t^t = x_{m+t}^{m+t}$. This is a homomorphism from the additive group $(\mathbf{Z}, +)$ to the multiplicative group $(\mathbf{C}^*, \cdot)$. Since $x_0^0 = 1$, we have $x_n^n = \alpha^n$ for some $\alpha \in \mathbf{C}^*$.

Now consider the $Y$ block from (Y-eq). For $k \neq 0$, $y_i^k$ can be non-zero only if $i = k - a$. We examine the product $F(L_m) \cdot F(L_t) = F(L_{m+t}) \cdot L_0$:

$$y_{m+t-a}^{m+t} I_{m+t-a} = \sum_i (y_i^m x_{m+t-a-i}^t + x_i^m y_{m+t-a-i}^t) I_{m+t-a}.$$

Let $\vartheta = m + t$. Substituting $x_t^t = \alpha^t$, we analyze two cases:

- **Case 1:** $a = 0$. The equation becomes $y_\vartheta^\vartheta = 2(y_{\vartheta-t}^{\vartheta-t}\alpha^t + \alpha^{\vartheta-t} y_t^t)$. Setting $t = \vartheta$ and noting $x_0^0 = 1$, we get $y_\vartheta^\vartheta = 2y_\vartheta^\vartheta$, which implies $y_\vartheta^\vartheta = 0$.
- **Case 2:** $a \neq 0$. The equation simplifies to $y_{\vartheta-a}^\vartheta = y_{\vartheta-a}^\vartheta \alpha^t$. Since this must hold for all $t$, we again conclude $y_{\vartheta-a}^\vartheta = 0$.

Thus, $Y = 0$. A similar check for the action of $A$ on the Lie bracket and shows that $A(L_n) = F(L_n) = \alpha^n L_n$.

Finally, we determine the blocks $V$ and $W$ by considering $A(L_m \cdot I_t)$ and $A([L_m, I_t])$. The requirement that $A(I_{m+t}) = -\frac{1}{t+a-m} A([L_m, I_t])$ forces $V = 0$. For the diagonal elements of $W$, we find $w_n^n = \alpha^n w_0^0$. By setting $\mu = w_0^0$, we obtain $W = \mathrm{diag}(\mu\alpha^n)$, which completes the proof. □

From theorems 30 and 23 we obtain the following corollary:

**Corollary 31.** *Local quasi-automorphisms of the transposed Poisson algebra $\mathcal{W}(a,-1)$ are defined by:*

$$\mathfrak{loc}\,\mathfrak{QAut}\big(\mathcal{W}(a,-1)\big) = \begin{pmatrix} \operatorname{diag}\big(\alpha_L^n(n) \mid n \in \mathbf{Z}\big) & 0 \\ 0 & \operatorname{diag}\big(\mu(m)\alpha_I^m(m) \mid m \in \mathbf{Z}\big) \end{pmatrix},$$

*where $\alpha_L, \alpha_I, \mu \colon \mathbf{Z} \longrightarrow \mathbf{C}^*$ are arbitrary functions.*

Similarly, from theorems 30 and 26:

**Corollary 32.** *Every 2-local quasi-automorphism of the transposed Poisson algebra $\mathcal{W}(a,-1)$ is an automorphism.*

## 4. Homogeneous Rota–Baxter operators on $\mathcal{W}(a,-1)$

We now turn to a class of operators originally introduced by Glen Baxter in his analytic proof of the Spitzer identity [6]. By introducing these operators, Baxter reduced a non-trivial probabilistic problem to the verification of an algebraic identity. The most intuitive example of such an operator is the integration operator:

$$I \colon C(\mathbf{R}) \longrightarrow C(\mathbf{R}), \quad I(f)\big|_x := \int_0^x f(t)\,dt.$$

Let $F(x) := I(f)\big|_x$ and $G(x) := I(g)\big|_x$ be the primitives of $f$ and $g$, respectively. Multiplying $F$ by $G$ and applying the integration-by-parts formula, we obtain:

$$F(x)G(x) = I(f)\big|_x I(g)\big|_x \overset{(*)}{=} I(f\,I(g))\big|_x + I(I(f)\,g)\big|_x = \int_0^x f(t)G(t)\,dt + \int_0^x F(t)g(t)\,dt.$$

Equation $(*)$ corresponds to the Rota–Baxter condition of weight zero.
We formalize this class of operators as follows:

**Definition 33.** *A linear map $R$ is called a **Rota–Baxter operator** of weight $\lambda \in \mathbf{C}$ if it satisfies the **Rota–Baxter binding equation**:*

$$R(x) \star R(y) = R\big(R(x) \star y + x \star R(y)\big) + \lambda R(x \star y).$$

*Note that if $R$ is a Rota–Baxter operator of weight $\lambda \neq 0$, then $\frac{1}{\lambda}R$ is a Rota–Baxter operator of weight $1$.*

**Definition 34.** *Let $E_\bullet = \bigoplus_{n\in\mathbf{Z}} E_n$ be a graded algebra. A linear map $L \colon E_\bullet \longrightarrow E_\bullet$ is called a **homogeneous operator of degree** $k \in \mathbf{Z}$ if $L(E_n) \leqslant E_{n+k}$ for all $n \in \mathbf{Z}$. It's easy to see that this definition can be extrapolated on all ordered set with additive structure.*

In light of Definition 33, it is sufficient to classify operators of weight 0 and 1. For the Witt algebra $\mathbf{W}$, a complete classification of homogeneous Rota–Baxter operators is provided in [13]. Every homogeneous operator of degree $k$ on the Witt algebra takes the form $L_n \mapsto f(n)L_{n+k}$ for some function $f \colon \mathbf{Z} \longrightarrow \mathbf{C}$. We list below the results from [13] that are necessary for our generalization to $\mathcal{W}(a,-1)$.

**Theorem 35.** *There are exactly three types of homogeneous Rota–Baxter operators of weight $0$ on $\mathbf{W}$:*

$$R_k^\alpha(L_n) := \alpha\delta_{n+2k,0}L_{n+k}, \qquad k \in \mathbf{Z},\ \alpha \in \mathbf{C}, \tag{3}$$

$$R_{2k}^{\prime\beta}(L_n) := \beta\big(\delta_{n+2k,0} + 2\delta_{n+3k,0}\big)L_{n+2k}, \qquad k \in \mathbf{Z}^*,\ \beta \in \mathbf{C}^*, \tag{4}$$

$$R_k^{l,\gamma}(L_n) := \frac{k\gamma}{n+2k}\,\delta_{n+k\in l\mathbf{Z}}\,L_{n+k}, \qquad k, l \in \mathbf{Z}^*,\ l \nmid k,\ \gamma \in \mathbf{C}^*. \tag{5}$$

**Theorem 36.** *All homogeneous Rota–Baxter operators of weight* 1 *on* **W** *have degree* 0 *and belong to one of the following classes (where* $\alpha \in \mathbf{C}$*):*

$$\bullet\ R_0^{\leqslant 1}(L_n) := \begin{cases} -L_n & n \geqslant 2 \\ 0 & n \leqslant 1 \end{cases} \qquad \bullet\ R_0^{\geqslant -1}(L_n) := \begin{cases} -L_n & n \leqslant -2 \\ 0 & n \geqslant -1 \end{cases}$$

$$\bullet\ R_0^{>1}(L_n) := \begin{cases} -L_n & n \leqslant 1 \\ 0 & n \geqslant 2 \end{cases} \qquad \bullet\ R_0^{<-1}(L_n) := \begin{cases} -L_n & n \geqslant -1 \\ 0 & n \leqslant -2 \end{cases}$$

$$\bullet\ R_0^{+,\alpha}(L_n) := \begin{cases} -L_n & n < 0 \\ \alpha L_0 & n = 0 \\ 0 & n > 0 \end{cases} \qquad \bullet\ R_0^{-,\alpha}(L_n) := \begin{cases} -L_n & n > 0 \\ \alpha L_0 & n = 0 \\ 0 & n < 0 \end{cases}$$

$$\bullet\ R_0^{\emptyset}(L_n) := -L_n \qquad \bullet\ R_0^{0}(L_n) := 0$$

4.1. **Homogeneous Rota–Baxter operators on $\mathcal{W}(a,-1)$ with Z-grading.** Firstly, we fix the grading on the algebra $\mathcal{W}(a,-1)$. In this section, we consider the following **Z**-grading:

$$\mathcal{W}(a,-1) = \bigoplus_{n\in\mathbf{Z}} (\mathbf{C}L_n + \mathbf{C}I_n).$$

**Theorem 37.** *Let* R *be a homogeneous block-diagonal Rota–Baxter operator of weight* 0 *on the Lie structure of $\mathcal{W}(a,-1)$ with the matrix representation $R\big|_{\mathbf{W}} \oplus R\big|_{I(a,-1)}$ (invariant on* **W** *and $I(a,-1)$). So the possible operators are given by the following relations:*

$$\begin{cases} R_k^{\alpha}(L_n) := \alpha\delta_{n+2k,0}L_{n+k}, \\ R_k^{\alpha}(I_m) := \tau\delta_{m,-a-2k}\delta_{k,0}I_{m+k}, \end{cases} \qquad k \in \mathbf{Z},\ \alpha,\tau \in \mathbf{C}. \tag{6a}$$

$$\begin{cases} R_{2k}^{\prime\beta}(L_n) := \beta(\delta_{n+2k,0} + 2\delta_{n+3k,0})L_{n+2k}, \\ R_{2k}^{\prime\beta}(I_m) := 0, \end{cases} \qquad k \in \mathbf{Z}^*,\ \beta \in \mathbf{C}^*. \tag{6b}$$

$$\begin{cases} R_k^{l,\gamma,m_0}(L_n) := \dfrac{k\gamma}{n+2k}\delta_{n+k\in l\mathbf{Z}}L_{n+k}, \\ R_k^{l,\gamma,m_0}(I_m) := \dfrac{k\gamma}{m+a+2k}\delta_{m\in m_0+l\mathbf{Z}}\,\delta_{m\neq -a-2k}I_{m+k}, \end{cases} \qquad (*). \tag{6c}$$

*where for case* (6c), $k, l \in \mathbf{Z}^*$, $l \nmid k$, $\gamma \in \mathbf{C}^*$, *and* $m_0 \in \mathbf{Z}$ *satisfies* $m_0 \neq -2k - a \pmod{l}$. *We adopt the convention that* $\frac{\delta_{x\neq 0}}{x} = 0$ *at* $x = 0$.

*Proof.* Suppose a homogeneous Rota–Baxter operator R of degree k takes the form:

$$R(L_n) := \zeta(n)L_{n+k}, \quad R(I_m) := \xi(m)I_{m+k}.$$

Due to the invariance of R on **W** and $I(a,-1)$, it suffices to verify the Rota–Baxter condition on the bracket $[L_n, I_m]$:

$$\begin{aligned} [R(L_n), R(I_m)] &= -(m+a-n)\zeta(n)\xi(m)I_{n+m+2k}, \\ R\big([R(L_n), I_m] + [L_n, R(I_m)]\big) &= -\big((m+a-n-k)\zeta(n) \\ &\quad + (m+k+a-n)\xi(m)\big)\xi(m+n+k)I_{m+n+2k}. \end{aligned} \tag{7}$$

Setting $n = -k$, we obtain:

$$\big(k\zeta(-k) - (m+a+2k)\xi(m)\big)\xi(m) = 0.$$

This implies either $\xi(m)=0$ or $\xi(m)=\frac{k\zeta(-k)}{m+a+2k}$ for $m\neq -a-2k$. Let $\varepsilon_m\in\{0,1\}$ be a selector sequence. The general form is:

$$\xi(m)=\frac{k\zeta(-k)}{m+a+2k}\varepsilon_m\delta_{m,\neq -a-2k}+\tau\delta_{m,-a-2k},$$

where $\tau\in\mathbf{C}$, and $\tau\neq 0$ only if $k\zeta(-k)=0$. Using the known classification for $\zeta(n)$ from Theorem 35, we analyze the three types:

- Case (6a). Here $\zeta(n)=\alpha\delta_{n,-2k}$. If $k\neq 0$, then $\zeta(-k)=0$, forcing $\xi(m)=0$ for almost all $m$. If $k=0$, we obtain the identity $R(I_m)=\tau\delta_{m,-a}I_m$.
- Case (6b). In this case, the degree of the operator $R$ is $2k$, and the function on the $\mathbf{W}$ subspace is given by $\zeta(n)=\beta(\delta_{n,-2k}+2\delta_{n,-3k})$. To determine the behavior of $\xi(m)$, we substitute these into the mixed Rota–Baxter functional equation (7).
  - First, let $n=-2k$. Since $\zeta(-2k)=\beta$, equation (7) simplifies to the identity we used to establish the general form:
  $$\xi(m)=\frac{2k\beta}{m+a+4k}\varepsilon_m.$$
  - Next, we consider $n=-3k$. Here, $\zeta(-3k)=2\beta$. A crucial shift occurs in the index of the right-hand side of (7): the term $\xi(m+n+2k)$ becomes $\xi(m-k)$. This couples the values of $\xi$ at points $m$ and $m-k$, yielding:
  $$(m+a+3k)(2\beta)\xi(m)=\big((m+a+k)(2\beta)+(m+5k+a)\xi(m)\big)\xi(m-k).$$
  In this formula $m+a+4k\neq 0$. If $m+a+4k=0$ then obtain $k\epsilon_{a-5k}=0$. Substituting the fractional form of $\xi(m)$ and $\xi(m-k)$ and dividing by the common factor $4k\beta^2$, we arrive at the following recurrence relation for the selector sequence:
  $$(m+a+3k)^2\varepsilon_m=\big((m+a+k)(m+a+4k)+k(m+a+5k)\varepsilon_m\big)\varepsilon_{m-k}. \tag{8}$$
  Analyzing the support of $\varepsilon_m$ reveals three scenarios:
    1. If $\varepsilon_m=1$ and $\varepsilon_{m-k}=0$, the equation implies $(m+a+3k)^2=0$, which only happens if $m=-a-3k$.
    2. If $\varepsilon_m=0$ and $\varepsilon_{m-k}=1$, the right side must vanish, implying $m=-a-k$ or $m=-a-4k$.
    3. If $\varepsilon_m=1$ and $\varepsilon_{m-k}=1$, the equation (8) is satisfied identically, theoretically allowing for infinite sequences of non-zero values.

  However, we must also satisfy the Rota–Baxter condition for all $n\notin\{-2k,-3k\}$. For these indices, $\zeta(n)=0$, and the functional equation reduces to:
  $$0=(m+2k+a-n)\epsilon_m\epsilon_{m+n+2k}.$$
  If there exists any $m_0$ such that $\epsilon_{m_0}\neq 0$, then for almost all $n$, we must have $\epsilon_{m_0+n+2k}=0$. This "orthogonality" condition prevents the existence of the chains described in the third case or the multiple isolated points in the first or the second cases. Therefore, the only globally consistent selector sequence is the trivial one, $\varepsilon_m\equiv 0$, which implies $\xi\equiv 0$.
- Case (6c). Here $\zeta(n)=\frac{k\gamma}{n+2k}\delta_{n+k\in l\mathbf{Z}}$.
  - If $n+k\notin l\mathbf{Z}$, then $\zeta(n)=0$, and the functional equation (7) becomes:
  $$0=(m+k+a-n)\varepsilon_m\varepsilon_{m+n+k}\quad (\text{for } m, m+n+k\neq -a-2k).$$
  If we assume $\varepsilon_{m_0}=1$ for some $m_0\neq -a-2k$, then for any $n\in\mathbf{Z}\backslash\{-a-3k-m_0,-k+l\mathbf{Z}\}$, we must have $\varepsilon_{m_0+k+n}=0$. This implies that the support of $\varepsilon$ is very sparse outside of the cosets related to $l\mathbf{Z}$.
  - Let $n+k=lt$ for some $t\in\mathbf{Z}$. Substituting $\zeta(n)$ and $\xi(m)$ into (7) and setting $m=-a-2k$, we get $0=(2k+lt)k^2\gamma\varepsilon_{-a-2k+lt}\delta_{lt\neq 0}$. Since $k\nmid l$, we have $2k+lt\neq 0$, implying $\varepsilon_{-a-2k+lt}=0$ for all $t\in\mathbf{Z}^*$. Thus, $\varepsilon$ can be non-zero at index $-a-2k$ only as an isolated point.
  For $m\neq -a-2k-lt$, direct computation shows:
  $$\begin{aligned}&(m+a+lt+2k)(m+a-lt+k)\varepsilon_m\\&=\big((m+a-lt)(m+a+2k)+(m+2k+a-lt)(lt+k)\varepsilon_m\big)\varepsilon_{m+lt}.\end{aligned}$$

This equation is satisfied if $\varepsilon_m = \varepsilon_{m+lt} = 1$. This confirms that if any $\varepsilon_{m_0} = 1$, then the whole coset $m_0 + l\mathbf{Z}$ should be in the support.

To complete the argument for this case, we need to match the periodicity from the second subcase with the sparsity from the first subcase.

If there exists some $m_0$ such that $\varepsilon_{m_0} = 1$, the second subcase implies that the whole coset $m_0 + l\mathbf{Z}$ must be contained in the support of $\varepsilon$. However, the first subcase imposes a strict "orthogonality" condition: for any $m$ in the support, almost all other points $m+k+n$ (when $n+k \notin l\mathbf{Z}$) cannot be in the support.

For both rules to work together, the support cannot cover more than one coset or have extra points. Any point $x$ outside the coset $m_0 + l\mathbf{Z}$ (except perhaps the single point at $-a-2k$) would eventually break the equation for large shifts $n$.

Also, our analysis of the singular point showed that $\varepsilon_{-a-2k+lt} = 0$ for all $t \neq 0$. This means that $-a-2k$ cannot be part of a full coset $l\mathbf{Z}$. Thus, the only valid and non-zero solution is for the support to be exactly one coset:

$$\varepsilon_m = \delta_{m \in m_0 + l\mathbf{Z}},$$

where we must ensure $m_0 \not\equiv -a-2k \pmod{l}$ to avoid problems at the singular point. This leads us to the final form of the operator $R_k^{l,\gamma,m_0}$.

□

**Corollary 38.** *In the transposed Poisson structure on $\mathcal{W}(a,-1)$, every $\mathbf{W}$ and $\mathrm{I}(a,-1)$-invariant Rota–Baxter operator of weight 0 vanishes on $\mathbf{W}$ and has following form:*

$$\begin{cases} R(L_n) &= 0 \\ R(I_m) &= \tau\delta_{m,-a-2k}\delta_{k,0}I_{m+k} \end{cases} \quad k \in \mathbf{Z},\ \tau \in \mathbf{C},\ a \notin \mathbf{Z}.$$

*Proof.* We must verify the Rota–Baxter identity for the associative product $\cdot$:

$$R(x)\cdot R(y) = R(R(x)\cdot y + x \cdot R(y)).$$

If we assume $R(L_n) = \zeta(n)L_{n+k}$, the identity for $L_n \cdot L_m$ becomes

$$\zeta(n)\zeta(m) = \zeta(n+m+k)(\zeta(n)+\zeta(m)).$$

For $R_k^\alpha$, setting $n = m = -2k$ yields $\alpha^2 = \alpha^2(\delta_{-k,0} + \delta_{-k,0})$, which implies $\alpha^2 = 2\alpha^2\delta_{k,0}$. For any $k \neq 0$, this forces $\alpha = 0$. If $k = 0$, the only idempotent solution is $\alpha = 0$ or $\alpha = 1$, but the latter fails for $n \neq m$. Similar contradictions ($1 = 2$ or singular values) arise for $R'^\beta$ and $R^{l,\gamma}$.

Finally, we check the mixed product $R(L_n)\cdot R(I_m)$ with $R(L_n) = 0$. The condition reduces to $0 = R(L_n \cdot R(I_m))$. Substituting $R(I_m) = \tau\delta_{m,-a}I_m$ (for $k=0$), we get:

$$0 = \tau\delta_{m,-a}R(I_{n+m}) = \tau^2\delta_{m,-a}\delta_{n+m,-a}I_{n+m}.$$

This must hold for all $n$. If $n = 0$, we have $\tau^2\delta_{m,-a} = 0$. If $a \notin \mathbf{Z}$, the index $m$ (which is an integer) can never equal $-a$, so the delta-symbol is always zero and the condition is satisfied. If $a \in \mathbf{Z}$, the operator fails at $m = -a$. □

**Theorem 39.** *Let $R$ be a homogeneous block-diagonal Rota–Baxter operator of weight 1 on the transposed Poisson structure of $\mathcal{W}(a,-1)$ with $\mathbf{Z}$-grading. Suppose $R$ is invariant on $\mathbf{W}$ and $\mathrm{I}(a,-1)$, with the matrix representation $R|_{\mathbf{W}} \oplus R|_{\mathrm{I}(a,-1)}$. Then $R$ is one of the following:*

(1) $R_0^0 \oplus (-\mathrm{Id})$ *or* $R_0^0 \oplus 0$;
(2) $R_0^\emptyset \oplus (-\mathrm{Id})$ *or* $R_0^\emptyset \oplus 0$;
(3) $\begin{pmatrix} R_0^{+,\alpha} & 0 \\ 0 & \mathrm{diag}(-\mathbf{1}_{(-\infty,t]}(n) \mid n \in \mathbf{Z}) \end{pmatrix}$ *for* $t \in \mathbf{Z} \sqcup \{\pm\infty\}$ *and* $\alpha \in \{-1, 0\}$;
(4) $\begin{pmatrix} R_0^{-,\alpha} & 0 \\ 0 & \mathrm{diag}(-\mathbf{1}_{[t,+\infty)}(n) \mid n \in \mathbf{Z}) \end{pmatrix}$ *for* $t \in \mathbf{Z} \sqcup \{\pm\infty\}$ *and* $\alpha \in \{-1, 0\}$.

*Here, the operators acting on the Witt algebra $\mathbf{W}$ are taken from the classification in Theorem 36.*

*Proof.* Consider a homogeneous operator $R$ of degree 0. Let $R(L_n) := \zeta(n)L_n$ and $R(I_m) := \xi(m)I_m$. To verify the Rota–Baxter condition on the transposed Poisson structure, we must satisfy the identity for both the Lie bracket and the associative product.

For the mixed relations between $L_n$ and $I_m$, the Lie condition and the associative condition yield:

$$(m+a-n)\big(\xi(m+n)(\zeta(n)+\xi(m)+1)-\zeta(n)\xi(m)\big)=0, \qquad \text{(M-Lie)}$$

$$\xi(m+n)(\zeta(n)+\xi(m)+1)-\zeta(n)\xi(m)=0. \qquad \text{(M-Assoc)}$$

Since (M-Assoc) implies (M-Lie), we only need to solve the associative functional equation.

First, we check the compatibility of the Witt algebra operators from Theorem 36 with the associative product.

**Lemma 40.** *Among the weight* 1 *Rota–Baxter operators on the Lie Witt algebra* $\mathbf{W}$ *(Theorem 36), only* $R_0^0, R_0^\emptyset, R_0^{+,\alpha}$, *and* $R_0^{-,\alpha}$ *for* $\alpha\in\{-1,0\}$ *are compatible with the associative product.*

*Proof.* The associative condition $L_n\cdot L_m=L_{n+m}$ requires $\zeta(n)\zeta(m)=\zeta(n+m)(\zeta(n)+\zeta(m)+1)$. For $n=m$, this gives the diagonal constraint:

$$\zeta^2(n)=(2\zeta(n)+1)\zeta(2n). \qquad \text{(W-Diag)}$$

Testing the known Lie classification against (W-Diag) reveals the following contradictions:

- Operator $R_0^{\leqslant 1}$: Defined by $\zeta(n)=-1$ for $n\geqslant 2$ and 0 otherwise. At $n=1$, $\zeta(1)=0$ and $\zeta(2)=-1$. Equation (W-Diag) gives $0=(0+1)(-1)$, a contradiction.
- Operator $R_0^{\geqslant -1}$: Defined by $\zeta(n)=-1$ for $n\leqslant -2$. At $n=-1$, we get $0=-1$.
- Operator $R_0^{>1}$: Defined by $\zeta(n)=-1$ for $n\leqslant 1$. At $n=1$, $\zeta(1)=-1$ and $\zeta(2)=0$. Equation (W-Diag) gives $1=(-2+1)(0)$, a contradiction.
- Operator $R_0^{<-1}$: Defined by $\zeta(n)=-1$ for $n\geqslant -1$. At $n=-1$, we get $1=0$.

For the step operators $R_0^{\pm,\alpha}$, setting $n=0$ in (W-Diag) yields $\alpha^2=(2\alpha+1)\alpha$, which simplifies to $\alpha(\alpha+1)=0$. Thus, only $\alpha\in\{0,-1\}$ are valid. The cases $R_0^0$ ($\zeta\equiv 0$) and $R_0^\emptyset$ ($\zeta\equiv -1$) satisfy the identity for all $n$. □

Using the result of the Lemma, we analyze the support of $\xi(m)$ by setting $n=0$ in (M-Assoc):

$$\xi(m)(\zeta(0)+\xi(m)+1)-\zeta(0)\xi(m)=0 \implies \xi(m)(\xi(m)+1)=0.$$

Thus, $\xi(m)\in\{0,-1\}$ for all $m$, meaning $(-1)R\big|_{I(a,-1)}$ is a projection onto some subset of indices $D\subset\mathbf{Z}$. To find the shape of $D$, we analyze the cases for $\zeta(n)$:

- Cases $R_0^0$ and $R_0^\emptyset$: If $\zeta\equiv 0$ or $\zeta\equiv -1$, then (M-Assoc) simplifies to $\xi(m+n)(\xi(m)+1)=0$ or $\xi(m+n)\xi(m)=-\xi(m)$. In both cases, $\xi$ must be a constant 0 or $-1$. This gives the first two items of the theorem.
- Case $R_0^{+,\alpha}$: Here $\zeta(n)=-1$ for $n<0$. If we pick $n<0$ in (M-Assoc), the equation becomes $\xi(m+n)\xi(m)=-\xi(m)$. If $\xi(m)=-1$ for some index $m$, then $\xi(m+n)$ must also be $-1$ for all $n<0$. This forces the support of $\xi$ to be the form $(-\infty,t]$. The specific value of $t$ depends on $\alpha$ and the value of $\xi(0)$, leading to the general form $\mathrm{diag}\big(-\mathbf{1}_{(-\infty,t]}\big)$.
- Case $R_0^{-,\alpha}$: Here $\zeta(n)=-1$ for $n>0$. By symmetric logic, if $\xi(m)=-1$, then $\xi(m+n)$ must be $-1$ for all $n>0$. This forces the support to be the form $[t,+\infty)$.

In both ray cases, $t=\pm\infty$ corresponds to the constant operators 0 and $-\mathrm{Id}$. □

4.2. **Homogeneous Rota–Baxter operators on $\mathcal{W}(a,-1)$ with $\mathbf{Z}_2$-grading.** Let us fix a $\mathbf{Z}_2$-grading on our algebra:

$$\mathcal{W}(a,-1)=\mathbf{W}\oplus I(a,-1).$$

We consider homogeneous Rota–Baxter operators of degree 1:

$$R(\mathbf{W})\leqslant I(a,-1),\quad R(I(a,-1))\leqslant \mathbf{W}.$$

In terms of the basis generators, we write:

$$R(L_n)=\sum_i \zeta_i^n I_i,\quad R(I_m)=\sum_j \xi_j^m L_j.$$

**Theorem 41.** *There are no non-trivial homogeneous Rota–Baxter operators of weight* 1 *on the* $\mathbf{Z}_2$*-graded transposed Poisson algebra* $\mathcal{W}(a,-1)$.

*Proof.* We analyze the Rota–Baxter identity $[R(x),R(y)]=R\big([R(x),y]+[x,R(y)]\big)+R([x,y])$ for different pairs of generators.

- Case 1 ($[L_n, L_m]$). Evaluating the RB identity on the pair $(L_n, L_m)$:

$$[R(L_n), R(L_m)] = R\big([R(L_n), L_m] + [L_n, R(L_m)]\big) + R([L_n, L_m]).$$

Since $R(L_k) \in I(a,-1)$ and the bracket $[I, I]$ is zero, the left-hand side vanishes:

$$\begin{aligned}0 &= R\left(\left[\sum_i \zeta_i^n I_i, L_m\right] + \left[L_n, \sum_j \zeta_j^m I_j\right] + (n-m)L_{n+m}\right)\\ &= R\left(-\sum_i \zeta_i^n (i+a-m)I_{i+m} + \sum_j \zeta_j^m (j+a-n)I_{j+n} + (n-m)L_{n+m}\right).\end{aligned}$$

Applying the action of R and splitting the result into the $\mathbf{W}$ and I components, we get

$$0 = \underbrace{\left(\sum_{i,k} \zeta_i^m (i+a-n)\xi_k^{i+n} L_k - \sum_{i,k} \zeta_i^n (i+a-m)\xi_k^{i+m} L_k\right)}_{\in \mathbf{W}} + \underbrace{(n-m)\sum_k \zeta_k^{n+m} I_k}_{\in I(a,-1)}.$$

For the I-component to vanish for all $n, m$, we must have $(n-m)\zeta_k^{n+m} = 0$. For any index $s$, we can choose $n, m$ such that $n+m = s$ and $n \neq m$, which forces $\zeta_k^s = 0$ for all $s, k$. Thus, $R(\mathbf{W}) = 0$.

- Case 2 ($[L_n, I_m]$). Now we apply the RB identity to the pair $(L_n, I_m)$:

$$[R(L_n), R(I_m)] = R\big([R(L_n), I_m] + [L_n, R(I_m)]\big) + R([L_n, I_m]).$$

Since $R(L_n) = 0$, the left-hand side is 0. Also $[R(L_n), I_m] = 0$. Therefore:

$$0 = R\big(0 + [L_n, R(I_m)] + [L_n, I_m]\big).$$

Now expand the terms inside R. After computation we get:

$$0 = R\left(\left[L_n, \sum_i \xi_i^m L_i\right] - (m+a-n)I_{n+m}\right) = R\left(\sum_i \xi_i^m (n-i)L_{n+i}\right) - (m+a-n)R(I_{n+m}).$$

Since $R(L_*) = 0$, the first sum vanishes, leaving:

$$(m+a-n)\sum_i \xi_i^{m+n} L_i = 0.$$

This must hold for all $n, m$. For a fixed sum $s = m+n$, we have $(s-2n+a)\xi_i^s = 0$. Since we can vary $n$ freely, the coefficient $(s-2n+a)$ cannot be zero for all $n$. Thus, $\xi_i^s = 0$ for all $s, i$, which implies $R(I) = 0$.

□

**Corollary 42.** *There is no non-trivial homogeneous Rota–Baxter operator of weight* 1 *on the* $\mathbf{Z}_2$*-graded Lie algebra* $\mathcal{W}(a,-1)$.

## 5. W-compatible Novikov–Poisson structures

As shown by X. Kong, H. Chen, and C. Bai in [20], every Novikov structure compatible with the Witt algebra $\mathbf{W}$ belongs to the family $V_{\alpha,0}$ for $\alpha \in \mathbf{C}$ (where $V_{\alpha,0} \cong V_{-\alpha,0}$). The Novikov product $\circ$ in these structures is defined as:

$$L_n \circ L_m := (\alpha + m)L_{n+m}.$$

Our goal is to see if this product $\circ$ and the associative product $L_n \cdot L_m := L_{n+m}$ together form a Novikov–Poisson structure. To do this, we must verify two compatibility conditions:

$$(x \cdot y) \circ z = x \cdot (y \circ z), \tag{NP1}$$

$$(x \circ y) \cdot z - x \circ (y \cdot z) = (y \circ x) \cdot z - y \circ (x \cdot z). \tag{NP2}$$

It turns out that these rules hold for any choice of $\alpha$.

**Theorem 43.** *For any complex number* $\alpha$*, the Novikov product* $\circ$ *and the associative product* $\cdot$ *define a* $\mathbf{W}$*-compatible Novikov–Poisson structure.*

*Proof.* We verify this by direct computation. For the first condition (NP1):

$$\begin{aligned}(L_i \cdot L_j) \circ L_k = L_{i+j} \circ L_k \quad &= (\alpha + k)L_{i+j+k},\\ L_i \cdot (L_j \circ L_k) = L_i \cdot (\alpha + k)L_{j+k} \quad &= (\alpha + k)L_{i+j+k}.\end{aligned}$$

Both sides are equal.

For the second condition (NP2):

$$\begin{aligned}(L_i \circ L_j) \cdot L_k - L_i \circ (L_j \cdot L_k) = (\alpha + j)L_{i+j+k} - (\alpha + j + k)L_{i+j+k} \quad &= -kL_{i+j+k},\\ (L_j \circ L_i) \cdot L_k - L_j \circ (L_i \cdot L_k) = (\alpha + i)L_{i+j+k} - (\alpha + i + k)L_{i+j+k} \quad &= -kL_{i+j+k}.\end{aligned}$$

Since the results match, the condition holds. □

**Corollary 44.** *The commutator of the Novikov product defines a Lie bracket for this Novikov–Poisson structure. This bracket is consistent with the definition of $\mathcal{W}(a, -1)$. Specifically, the triple:*

$$(\mathbf{W}, \cdot, [\ ,\ ]_\alpha)$$

*is a transposed Poisson algebra where the Lie product is given by:*

$$[L_n, L_m]_\alpha := L_n \circ L_m - L_m \circ L_n = (m - n)L_{n+m}.$$

Note that although the Novikov product depends on $\alpha$, the parameter $\alpha$ cancels out in the commutator. This results in the standard Lie bracket of the Witt algebra.

## References


[1] H. Abdelwahab, I. Kaygorodov, and B. K. Sartayev, "δ-Poisson and transposed δ-Poisson algebras," *J. Non-Associative Struct.*, vol. 1, 2026, Art. no. 9.

[2] K. Abdurasulov, J. Adashev, and S. Eshmeteva, "Transposed Poisson structures on solvable Lie algebras with filiform nilradical," *Commun. Math.*, vol. 32, no. 3, pp. 441–483, 2024.

[3] R. R. Andruszkiewicz, T. Brzeziński, and K. Radziszewski, "Lie Affgebras Vis-à-Vis Lie Algebras," *Results Math.*, vol. 80, no. 2, 2025, Art. no. 61.

[4] S. Ayupov, A. Elduque, and K. Kudaybergenov, "Local derivations and automorphisms of Cayley algebras," *J. Pure Appl. Algebr.*, vol. 227, no. 5, 2023, Art. no. 107277.

[5] C. Bai, R. Bai, L. Guo, and Y. Wu, "Transposed Poisson algebras, Novikov–Poisson algebras and 3-Lie algebras," *J. Algebr.*, vol. 632, pp. 535–566, 2023.

[6] G. Baxter, "An analytic problem whose solution follows from a simple algebraic identity," *Pacific J. Math.*, vol. 10, no. 3, pp. 731–742, 1960.

[7] P. Belwal and M. Singh, "Skew braces and Rota–Baxter operators on semi-direct products," *J. Non-Associative Struct.*, vol. 1, 2026, Art. no. 1.

[8] S. Chen and C. Bai, "Quantizations of transposed Poisson algebras by Novikov deformations," *J. Phys. A Math. Theor.*, vol. 57, no. 49, 2024, Art. no. 495203.

[9] C. Daskaloyannis, "Polynomial Poisson algebras for two-dimensional classical superintegrable systems and polynomial associative algebras for quantum superintegrable systems," *Czechoslov. J. Phys.*, vol. 50, no. 11, pp. 1209–1214, 2000.

[10] N. Daukeyeva, M. Eraliyeva, and F. Toshtemirova, "Transposed δ-Poisson algebra structures on null-filiform associative algebras," *Commun. Math.*, vol. 34, no. 1, 2026, Art. no. 1.

[11] B. L. M. Ferreira, I. Kaygorodov, and V. Lopatkin, "$\frac{1}{2}$-derivations of Lie algebras and transposed Poisson algebras," *Rev. la Real Acad. Ciencias Exactas, Físicas y Nat. Ser. A. Matemáticas*, vol. 115, no. 3, 2021, Art. no. 142.

[12] S. Gao, C. Jiang, and Y. Pei, "Low-Dimensional Cohomology Groups of the Lie Algebras $W(a, b)$," *Commun. Algebr.*, vol. 39, no. 2, pp. 397–423, 2011.

[13] X. Gao, M. Liu, C. Bai, and N. Jing, "Rota–Baxter operators on Witt and Virasoro algebras," *J. Geom. Phys.*, vol. 108, pp. 1–20, 2016.

[14] I. M. Gel'fand and I. Y. Dorfman, "Hamiltonian operators and algebraic structures related to them," *Funct. Anal. Its Appl.*, vol. 13, no. 4, pp. 248–262, 1980.

[15] I. Kaygorodov, "Non-associative algebraic structures: classification and structure," *Commun. Math.*, vol. 32, no. 3, pp. 1–62, 2024.

[16] I. Kaygorodov and M. Khrypchenko, "Transposed Poisson structures on Lie incidence algebras," *J. Algebr.*, vol. 647, pp. 458–491, 2024.

[17] I. Kaygorodov, A. Khudoyberdiyev, and Z. Shermatova, "Quasi-derivations of Witt and related algebras," *Res. Math. Sci.*, vol. 13, no. 1, 2026, Art. no. 20.
[18] I. Kaygorodov, A. Khudoyberdiyev, and Z. Shermatova, "Transposed Poisson structures on not-finitely graded Witt-type algebras," *Boletín la Soc. Matemática Mex.*, vol. 31, no. 1, 2025, Art. no. 22.
[19] I. Kaygorodov, A. Khudoyberdiyev, and Z. Shermatova, "Transposed Poisson structures on Virasoro-type algebras," *J. Geom. Phys.*, vol. 207, 2025, Art. no. 105356.
[20] X. Kong, H. Chen, and C. Bai, "Classification of Graded Left-symmetric Algebra Structures on Witt and Virasoro Algebras," *Int. J. Math.*, vol. 22, no. 2, pp. 201–222, 2011.
[21] N. P. Landsman, *Mathematical Topics Between Classical and Quantum Mechanics* (Springer Monographs in Mathematics). New York, NY: Springer New York, 1998.
[22] G. F. Leger and E. M. Luks, "Generalized Derivations of Lie Algebras," *J. Algebr.*, vol. 228, no. 1, pp. 165–203, 2000.
[23] A. V. Mikhailov and P. Vanhaecke, "Commutative Poisson algebras from deformations of noncommutative algebras," *Lett. Math. Phys.*, vol. 114, no. 5, 2024, Art. no. 108.
[24] B. K. Sartayev, "Some generalizations of the variety of transposed Poisson algebras," *Commun. Math.*, vol. 32, no. 2, pp. 55–62, 2024.
[25] P. Sinha and A. Yadav, "Poisson geometric formulation of quantum mechanics," *J. Math. Phys.*, vol. 65, no. 6, 2024, Art. no. 062101.
[26] X. Tang, "Biderivations and Commutative Post-Lie Algebra Structures on the Lie Algebra $\mathcal{W}(a, b)$," *Taiwan. J. Math.*, vol. 22, no. 6, pp. 1347–1366, 2018.
[27] X. Xu, "Novikov–Poisson Algebras," *J. Algebr.*, vol. 190, no. 2, pp. 253–279, 1997.
[28] Y. Yang, X. Tang, and A. Khudoyberdiyev, "Transposed Poisson Structures on Schrödinger Algebra in $(n + 1)$-Dimensional Space-Time," *Algebr. Colloq.*, vol. 33, no. 1, pp. 127–140, 2026.
[29] Z. Yoshida, "Nambu mechanics viewed as a Clebsch parameterized Poisson algebra: Toward canonicalization and quantization," *Prog. Theor. Exp. Phys.*, vol. 2024, no. 3, 2024, Art. no. 03A103.
[30] A. Zohrabi and P. Zusmanovich, "A $\delta$-first Whitehead Lemma for Jordan algebras," *Commun. Math.*, vol. 33, no. 1, 2025, Art. no. 2.

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