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XIV. *The Measurement of Small Differences of Phase.*

By W. E. Sumpner, *D.Sc.**

WHILE investigating recently the behaviour of wattmeters, transformers, and similar apparatus, the writer has found it necessary to measure small phase-differences occurring in the working of alternating-current plant. The methods hitherto available for such measurements are not at all satisfactory when the angles to be determined are small. Phase-meters have been constructed for commercial purposes, but the angular deflexion of the pointer of these instruments is, as a rule, smaller than the phase-difference to be measured, so that when the latter is as small, or smaller, than one degree, such instruments, even if perfectly accurate, are quite unsuited for the purpose. All other known methods necessitate the simultaneous reading of three instruments. The best-known method needs a wattmeter, a voltmeter, and an ammeter. Among wattmeters we may include all instruments of the double current, or double voltage, type, whether dynamometers, current-balances, or electrometers. In all these cases the value of $\cos \theta$, where θ is the angle of phase to be determined, is measured by dividing the wattmeter reading by the product of the readings of the other two instruments. The percentage error made in measuring $\cos \theta$ is thus greater than the corresponding errors made in reading the separate instruments. The method only gives fair results when the angle θ is large, and instruments of suitable range are available. When θ is small the method is hopeless, for since $\cos 1^\circ$ is ·99985, and $\cos 0^\circ{\cdot}5$ is ·99996, it is clear that no such method involving the measurement of three deflexions can be anything like accurate enough for determining values of θ less than one degree.

All other known methods are based upon the three-voltmeter method of measuring power factor, or phase-difference. This method has been much criticised at different times, but its limitations were all fully pointed out in the original paper (see Proc. Roy. Soc. vol. xlix. March 1891) in which Professor Ayrton and the present writer first drew attention to it. It is a method for measuring power, or power factor, which cannot compare in ease or simplicity with the wattmeter method, when a suitable wattmeter is available, and the accuracy of the instrument is not in doubt. But wattmeters and other alternating-current meters, owing to the absence of iron-cored magnetic circuits, are not nearly so sensitive

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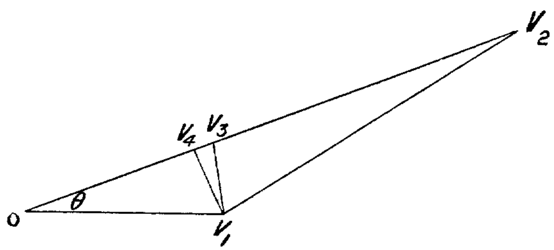
as, and do not possess anything like the range of, the corresponding direct-current instruments. A wattmeter of the right range is not always available, and in such cases the three-voltmeter method, or some modification of it, has often proved a convenient substitute.

The errors arising in practice in measuring phase by the three-voltmeter method are serious for low-power factor circuits, that is for values of θ approaching 90 degrees, but are not so important when high-power factors have to be measured. The perfection of the method in theory, and its limitations in practice, are exactly comparable with the determination of the angles of a triangle from measurements of its three sides. If these sides are measured accurately the angles can be correctly calculated in all cases, but for given percentage errors made in estimating the sides, the resulting error made in calculating the angle will largely depend on the shape of the triangle. Thus, in fig. 1, the

Fig. 1.



angle θ will be determined much less accurately from measurements of the sides of the triangle OV_1V_2, where OV_1 and OV_2 are very different in magnitude, than from equally accurate measurements of the sides of OV_1V_3, in which OV_1 and OV_3 are supposed to be nearly of equal length. The measurement will be most accurate, for given percentage errors in the sides, if the length of the perpendicular V_1V_4 on the line OV_2 can be measured, since then the value of $\sin\theta$ will be known as accurately as the ratio of V_1V_4 to OV_1 can be determined. If the sides of the figure represent voltages in different phases, the measurement of the ratio in question can in this case be determined as accurately as two voltmeter readings can be read and compared.

If we denote by v_1 and v_2 the lengths of the sides of the triangle forming the angle θ, and if the third side of the triangle opposite θ is represented by v, it can easily be shown that

$$\cos\theta = 1 - \tfrac{1}{2}\phi^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $$\phi^2 = \frac{v^2 - d^2}{v_1 v_2}, \text{ and } d = v_1 - v_2.$$

This is true whatever the value of θ, but for small values θ becomes identical with ϕ, and we have in circular measure

$$\theta = \sqrt{\frac{(v-d)(v+d)}{v_1 v_2}} = \frac{v}{\sqrt{v_1 v_2}}\sqrt{1 - \frac{d^2}{v^2}} \quad . \quad . \quad . \quad (2)$$

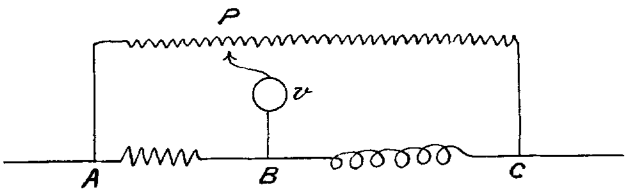
Some years ago the writer tested the phase-difference between the primary and secondary voltages of a small equal-ratio transformer by joining a low-reading hot-wire voltmeter to two terminals, one on each coil, and by connecting the other two terminals with a wire. Some numbers taken from an old note-book and referring to a test with the secondary on open circuit give $v_1 = 72$; $v_2 = 73{\cdot}8$; $v = 1{\cdot}9$. Here $d = 1{\cdot}8$, and the value of θ works out to be 0·0084 radian, or 0·48 degree, for which $\cos\theta$ is 0·99996. But recent tests on the same transformer by a more accurate method have proved that this estimate of θ is no less than 24 times too great, that its real value is only 0·00037 radian, or 0·021 degree, and that consequently the value of $\cos\theta$ only differs from unity by 7 parts in a hundred million.

The explanation of the error made in the earlier test is simply the difficulty of determining the difference d with sufficient accuracy when it is about the same magnitude as v and small compared with v_1 or v_2. If, assuming v_1 correct, v_2 had been read 73·9 instead of 73·8, the angle θ would have worked out to be zero, so that an error of only about one-tenth per cent. in reading one of the voltmeters completely accounts for the difference between the two results. This was not noticed at the time, and as the test was not repeated, the error was not discovered till long afterwards.

If in fig. 1 the voltage OV_2 is adjustable without alteration of phase, the voltmeter measuring the side opposite θ can be made to give a minimum reading by altering OV_2. The minimum reading will be V_1V_4 perpendicular to OV_4, and we can determine $\sin\theta = v/v_1$ as accurately as we can read the voltmeters.

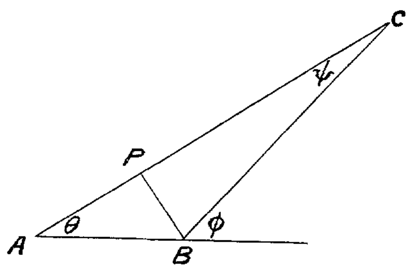
Thus the phase relation of the voltages of two coupled alternators can be determined by a two-voltmeter method in this way, by simply adjusting the excitation of one of them, this one being preferably run on open circuit. But in any case, by shunting the larger of the two voltages forming the angle θ with a non-inductive resistance, it is possible, by tapping this resistance at various points, to get a minimum reading of the voltmeter v, the ratio of which to the unshunted voltage will give the value of $\sin\theta$. This method is particularly suited to determine small angles of phase-difference, since voltmeters having different ranges may be used, and $\sin\theta$ or θ can be measured as accurately as the two readings can be determined. The method is not always applicable, since the conductors absorbing the voltages to be compared may be in series. In such cases the following bridge method can be used. Suppose that in fig. 2 A B represents

Fig. 2.



a non-inductive resistance, and B C an inductive resistance in series with it; the phase-difference ϕ between the current flowing through B C, and the voltage across B C, can be determined by connecting the points A and C to the terminals of a suitable non-inductive resistance A P C, and finding upon it a point P such that when this point is electrically connected with B through a voltmeter the reading v of this instrument is a minimum. The vector figure representing the voltages is shown in fig. 3. In this figure B P

Fig. 3.



represents v and is perpendicular to A C. If v_1 and v_2 are the voltages A B and B C respectively, the phase-angles θ and ψ by which these voltages differ from the voltage A C are such that

$$\sin\theta = \frac{v}{v_1}, \quad \sin\psi = \frac{v}{v_2}, \quad \phi = \theta + \psi.$$

We also have $v_1 v_2 \sin\phi = v v_3$ where v_3 is the voltage A C.

The former equations illustrate a method of determining ϕ by observing v, v_1, and v_2, while the latter equation indicates a method of determining $\sin\phi$ from these measurements together with an observation of v_3. It is to be noted that the conductor APC need not necessarily be noninductive. It should be of the same inductive character all through, and such that it is possible to adjust the point P connected with the voltmeter. The point P will then move along the vector AC (see fig. 3) as the adjustment is varied. Moreover there is no disadvantage in choosing a small noninductive resistance AB such that the voltage v_1 is small compared with v_2, provided always that suitable voltmeters are available for making the requisite measurements. One of the disadvantages of the three-voltmeter method for the measurement of power factor, is that the noninductive resistance AB should absorb about the same voltage as the conductor BC, and hence a greater amount of power than the load under test. If the bridge method just described is used, the load BC may absorb several hundred volts, while v_1 and v may each be less than a volt. Where AB is negligible compared with BC, it of course follows that ψ is small compared with θ, and that θ and ϕ can in general be considered the same. Now in order to determine one of the angles by the above method, all that is really needed is to have two voltages, one like AB in phase with the current, and the other in phase with either BC or AC. One of these voltages should be fixed in magnitude, and the other adjustable. Such voltages can be obtained with all requisite accuracy for most purposes by the aid of current and voltage transformers. The former must have its primary in the main circuit, and its secondary closed through a non-inductive resistance, the voltage on which will be in phase with AB. The latter must be a constant potential transformer with its primary across AC or BC, and its secondary closed through a noninductive resistance, the voltage on which will be in phase with that on the primary. The secondary voltage need not be greater than corresponds with the length AP in the figure. The two secondary circuits must be connected together at one point to determine a common potential represented by the point A in fig. 3. Such an arrangement makes the method adaptable to loads absorbing high potentials, or taking heavy currents. The phase-differences introduced by such transformers, if of good design, can in practice be reduced to about a tenth of a degree, and hence, for values of θ such as are usual, no appreciable error will be introduced into the measurement by them.

The writer has tested this two-voltmeter method in all the various forms above indicated, and found it satisfactory. It is not so simple or so accurate as the wattmeter method when an instrument of the right range is available, and when the value of θ to be determined is large. But its great adaptability to loads of all kinds makes it a useful laboratory method. Under ordinary circumstances, a hot-wire voltmeter reading up to 1 volt can be used for taking the two measurements v and v_1, while the voltage v_2 on the load can be taken by the instrument necessarily used on the circuit. It is then only necessary to find two noninductive resistances, one such as to absorb about one volt when traversed by the load current, and the other suitable for the voltage v_2 or v_3, and having a portion of it (taking about a volt) of such a structure that a voltmeter can be connected to it at various points.

It is, however, when the phase-difference to be determined is small that the method is most useful and most accurate, while the wattmeter method altogether fails. The vector diagram, shown in fig. 3, applies whatever the frequency or wave form of the current, and even when the current is unidirectional but varying in strength. In a test in which a hand-regulated direct-current arc was put in series with a noninductive resistance, it was found, with $v_1=70$ volts and $v_2=40$ volts, that the minimum value of v was less than 0·1 volt. A lower voltage could not be measured with any certainty with the hot-wire voltmeter used. It follows from this test, that the power factor of the direct-current arc cannot differ from unity by more than 5 parts in a million, and the phase-difference ϕ between the voltage and the current works out to be about 0·003 radian. A similar test, made on the small equal-ratio transformer already referred to, with the primary subjected to over 80 volts, gave as minimum voltage v, a value estimated to be only 0·03 volt. This would correspond with a phase-difference of only 0·00037 radian, or 0·021 degree, and a value of cos ϕ differing from unity by only seven parts in a hundred million.

Voltmeters for measuring minute alternate voltages are not procurable. For small phase-angles v will be small compared with either v_1 or v_2. If the apparatus to be tested is such that the application of high voltages to it would either do it injury or alter its working conditions, the method becomes impracticable when the angles to be measured are small.

In order to overcome the difficulty arising from the absence of sensitive alternate-current voltmeters, the writer has adopted the drastic method of commutation. By such means it is possible to make use of sensitive direct-current instruments for the measurement of alternating voltages. Suppose the alternator used for the tests is made to drive a synchronous motor constructed with at least four poles and having on its rotating spindle a commutator with as many metallic sectors as there are current reversals in the time of a revolution. Let half of the sectors (the first, third, &c.) be in metallic connexion with each other, and the other sectors be insulated. A pair of brushes suitably spaced on the commutator will then be in metallic connexion during half the period, and insulated from each other for the other half. But the instant at which the brushes are connected will not be the same as that at which the current reversal takes place, except for special positions of the brushes. If the brushes are adjusted to this position, and the alternating voltage to be measured has its terminals connected to the brushes with a direct-current voltmeter in circuit, this instrument will measure half the mean value of the ordinate of the positive portion of the alternating wave of potential, while the true value of the alternating voltage will be the square root of mean square of this ordinate. The sectors are supposed to be all equal to each other, and the width of the gap between successive sectors should be small compared with the circumferential width of the sectors, though for wave-forms approximately sinuous the gap width can be an appreciable fraction of the sector width without causing any appreciable effect on the readings. Theory shows that if the gap width is a small fraction ϵ of the sector width, the fractional error of the reading is represented by $\pi^2\epsilon^2/8$ for sine currents. It follows that if ϵ is one per cent. the error is only 1/80 per cent., while ϵ has to be 9 per cent. before the error amounts to one per cent. The sensitiveness of the arrangement might have been doubled if the commutator had been constructed so as to reverse the voltage at every half period by connecting alternate sectors to two slip-rings. But further sensitiveness was not required, and hence only the simplest form of commutator was tried.

For each wave-form there is a definite value f for the ratio of the square root of mean square of the ordinates to the arithmetical mean. For sine waves this ratio is

$$f = \frac{\pi}{2\sqrt{2}} = 1{\cdot}11.$$

For such waves the reading of the direct-current voltmeter would have to be multiplied by 2·22 to get the reading of an alternate-current instrument arranged so as to measure the voltage directly. For other wave-forms the multiplier would be $2f$, where f may be called the form factor of the alternating voltage. By using a commutator in this way, the most delicate Thomson reflecting-galvanometer can be utilized for measuring alternating voltages of the same magnitude as the smallest direct-current voltages to which such an instrument responds.

The method would in practice prove very tedious owing to the need for carefully adjusting the brushes before taking a measurement, since the adjustment required will depend on the phase of the voltage under test. But if the commutator be rotated nearly, but not quite, synchronously with the speed corresponding with the frequency, the galvanometer or voltmeter will show "beats," the interval between which lengthens as the speed of synchronism is approached. Under such conditions, the phase of the brush contact will automatically adjust itself whatever the phase of the voltage under test; and the maximum reading of the direct-current instrument will measure half the arithmetical mean of the ordinates of the curve representing the alternating voltage.

Any method of driving in which the speed can be delicately adjusted can be used for running such a commutator. A shunt motor with adjustable resistance either in series or in the shunt circuit would of course do. But for alternating currents of usual frequencies, a particularly convenient piece of apparatus is available in the induction motor. The speed of the rotor of such a motor, when running light, differs from the synchronous speed by a minute amount called the "slip," the amount of which in many cases is less than one-tenth per cent. For currents of 50 cycles per second such a value of the slip would correspond with beats at intervals of 20 seconds, or with a 10 second interval between the maximum positive and the maximum negative readings of the direct-current instrument.

A commutator of very simple construction, as above described, when attached to the spindle of an induction motor, and used with a direct-current instrument, forms in practice a satisfactory means of measuring minute alternate-current voltages. The writer has found it very serviceable in connexion with the measurement of alternating magnetic fields. The reliability of the apparatus was thoroughly tested in various ways, as, for instance, by putting in parallel (1) a standard voltmeter suitable for either direct or alternate voltages, and (2) a direct-current voltmeter in series with the commutator. The ratio of the two readings in volts of the two instruments was found to be exactly 2 on direct-current circuits with the commutator running; but for the alternating voltages produced by a small rotary transformer run from the direct-current side, the ratio was found to be 2·28 when the instruments were shunted to a portion of a non-inductive resistance connected with two slip-rings of the rotary. These ratios were the same whatever instruments were used, and for different resistances put in series with the direct-current instrument, provided due allowance was made for these resistances when interpreting in volts the reading of the latter instrument. But although the ratio of the reading of the alternate-current voltmeter to the maximum reading of the direct-current instrument was always found to be 2·28 when only noninductive resistances were included in the circuit connecting two slip-rings of the rotary, this number was found to vary with the inductiveness of the load current taken from these slip-rings. Thus the parallel arrangement of instrument just described, when shunted to a suitable noninductive resistance traversed by the current passing through an iron-cored choking-coil consisting of one of the coils of an old hedgehog transformer, yielded a ratio of 2·36 for the readings of the two voltmeters when the current was 10·6 amperes, and the frequency 43 cycles per second. A similar experiment on a hand-regulated alternating-current arc for a current of 11 amperes of the same frequency, yielded a ratio of 2·55. In each case about a dozen different observations were taken, and the individual values of the ratio found in each set of tests agreed with each other as accurately as it was possible to read the instruments observed. Since the ratio between the two readings is $2f$ where f is the form factor of the voltage, or the ratio of the square root of mean square of the voltage to its arithmetical mean value, it follows that the form factor of the slip-ring voltage is 1·14, that of the current through the choking-coil is 1·18, and that of the alternating-current arc is 1·275, the theoretical value for sine-currents being 1·11. The wave-forms of the alternating current through an arc, or through a large choking-coil with a strongly magnetized iron core, are quite exceptional. For ordinary currents produced by the rotary, it was found sufficiently accurate to take a constant of 2·3 for direct-current voltmeters used with the commutator.

It now became possible to measure the voltages represented by the lines in figs. 1 and 3, even when some of these voltages were small compared with a millivolt. Moreover, as the angles to be measured are determined from the ratios of voltages, and not from their actual values, it was possible, by measuring each voltage by the commutator method, to eliminate the constant of the instrument, assuming only that the form factor is the same for all the voltages measured.

Thus the inductiveness of a pair of lead plates in slightly acidulated water was tested by the method shown in fig. 3.

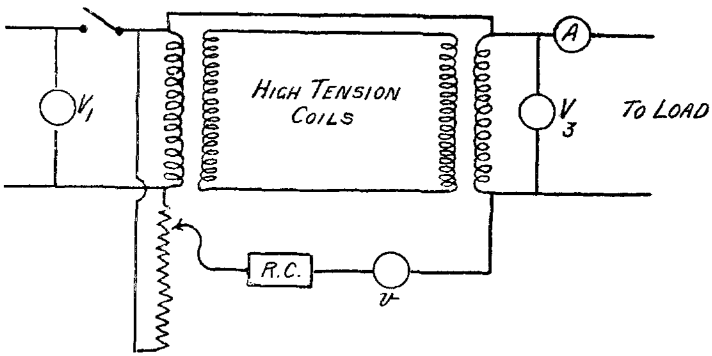
A current of 8·2 amperes was passed through a noninductive resistance AB in series with the lead plates BC. The voltage BC on the plates was 18·7 ; that of AB was 0·6κ, while BP the minimum voltage v was 0·004κ, where κ is the constant previously referred to. It follows that sin θ is 0·0067 and cos θ is 0·999977. A number of similar measurements were made with very satisfactory results.

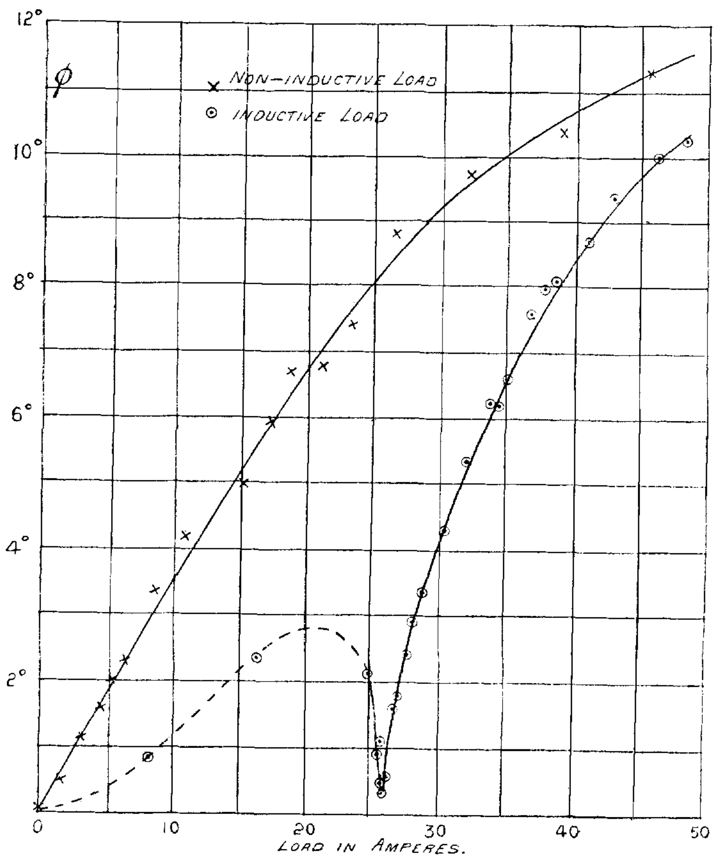
A small current-transformer for oscillograph purposes was tested for phase-difference between primary and secondary currents. This transformer was designed to reduce currents in the ratio of 20 to 1, and for use on a 6000 volt circuit, for a primary current not exceeding 10 amperes, and to supply a secondary current to any closed circuit not absorbing more than 2 volts. Noninductive resistances were put in series with the primary and secondary windings, and were electrically connected so as to form a point represented by O in fig. 1. The former for a current of 8 amperes absorbed 0·66 volt, represented by OV_1. The secondary resistance could be tapped at various points, and the minimum voltage v represented by V_1V_4 produced a reading of 0·50 millivolt. Allowing for the constant 2·3, we have ·00115 volt as the value of v. It follows that sin θ is ·00175, and θ is 0·10 degree for a primary current of 8 amperes. For a primary current of 10 amperes θ was found to be 0·088 degree, and for a current of 7 amperes 0·114 degree. Such values of θ yield values of cos θ differing from unity by about one part in a million, and are such that the wave-forms of primary and secondary currents must correspond with almost absolute accuracy.

Several small instrument transformers as supplied commercially for wattmeters were tested under their normal working conditions, the noninductive resistance put in the secondaries being very small compared with the total secondary resistance, and only absorbing a few millivolts. In one case it was found that the phase-difference between the primary and secondary currents of a current-transformer was 2·32 degrees, and in another as much as 4·2 degrees.

A number of larger transformers for power purposes were tested for phase-differences between primary and secondary voltages. In several cases the phase-difference on open circuit was found to be of the order of a tenth of a degree, and on full load of the order of one degree. On an inductive load the phase-difference found was not so large as a rule as for the same secondary current through a noninductive circuit. The curve shown in fig. 4 represents the result of some tests on two transformers of 3 kilowatt capacity intended to work between voltages of 100 and 1000, and with currents of 100 cycles per second. One transformer was

Fig. 4.


A
V_1
HIGH TENSION COILS
V_3
TO LOAD
R.C.
v


12°
10°
8°
6°
4°
2°
φ
× NON-INDUCTIVE LOAD
⊙ INDUCTIVE LOAD
0
10
20
30
40
50
LOAD IN AMPERES.

used to step up the volts from V_1 to V_2, the other to step down from V_2 to V_3. The arrangement of the circuits is shown in the diagram at the top of fig. 4. A noninductive resistance was put across V_1, and the phase-difference between V_1 and V_3 was tested by the two-voltmeter method already described. The phase-difference found is for the double transformation and is approximately twice that for each transformer. The load for the first curve was composed of lamps, and is noninductive; that for the second curve consisted of a hedgehog transformer, the secondary of which was, for the first test, open-circuited, but afterwards closed through a number of lamps in parallel. The primary took a current of nearly 25 amperes at a power factor of 0·09 when the secondary was open-circuited; but as the lamp-load on the latter was increased the primary current and power factor each rose. The voltages V_1 and V_3 used, instead of being about 100 at 100 cycles, for which the transformers were designed, were about 50 volts at about 50 cycles, so that the magnetic fluxes in the cores were about the same as if the normal voltages and frequencies had been used. The full-load current is 30 amperes for each of the low-voltage coils. The phase-difference between V_1 and V_3 for the two transformers with the second on open circuit, is only 0·109 degree or 0·054 degree per transformer. For a noninductive load it increases regularly with the current. For the full-load current of 30 amperes it is 9·4 degrees, and for 50 amperes it is 11·5 degrees. For the inductive load there was a remarkable, and sharply defined, minimum of 0·36 degree for a current of 25·5 amperes at low-power factor. This minimum was carefully tested. It occurred for a small load on the secondary which could be varied very gradually, the corresponding change in the primary current being only just measurable. For larger loads on the secondary, and therefore for power factors approaching unity as the current increased, the phase-difference curve is seen to approach that corresponding with a noninductive load. The portion of the curve for small inductive currents was not tested owing to the absence of a suitable load, but two special tests were made for lower currents at a power factor of about 0·1, as shown on the dotted portion of the curve, which of course must have the same starting-point for zero current as the curve for noninductive load. In all probability the dotted part of the curve denotes a "leading" condition of current, and the rest of it a "lagging" state of current.

In conclusion the writer desires to express his thanks to Mr. David Owen for much valuable assistance in carrying out several of the measurements above referred to.