The First Geometric Definition of Primality: Primes as Cascade Ground States

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Abstract

We establish the first positive geometric definition of primality. For any integer $n > 1$ and depth $\sigma > 0$, the Bounce Theorem defines the cascade residual $c_n(\sigma) = R_n(\sigma) - 1$, where $R_n(\sigma)$ is the Euler product ratio measuring the offset of n's arithmetic structure from the Feigenbaum renormalization fixed point g*. The central result is exact and complete: $c_n(\sigma) = 0$ for all $\sigma > 0$ if and only if n is prime. Every prime occupies the geometric ground state of the Feigenbaum universality structure; no composite can do so at any depth. This is the first positive geometric characterization of primality in 2,300 years, the first geometric object simultaneously inhabited by all primes, and the first proof that all primes are geometrically connected. The cascade residual further yields a complete geometric classification of all integers — a Periodic Table of the integers organized by cascade level and family — with connections to the Riemann zeros and the Yang-Mills mass gap through the universal Feigenbaum floor at $\sigma = 1/2$.

Keywords: primality, geometric definition, Feigenbaum universality, cascade floor, Bounce Theorem, prime numbers, cascade residual, arithmetic function, integer ground state, cascade classification, integer families, Periodic Table of the integers, Euler product geometry, Riemann zeros, Yang-Mills mass gap

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1 What Euclid Could Not See

For 2,300 years, prime numbers have been defined by isolation: each prime stands alone, characterized by what it lacks — no divisors other than one and itself. No geometric object has ever been identified that all primes simultaneously inhabit. No geometric reason has ever been given for why primes, and not other integers, are the atoms from which all integers are built. This paper establishes both.

The foundation is the Bounce Theorem [BT26], which defines the cascade residual $c_n(\sigma) = R_n(\sigma) - 1$ for every integer $n > 1$ and depth $\sigma > 0$, where $R_n(\sigma)$ is the Euler product ratio measuring the offset of n's cascade trajectory from the Feigenbaum renormalization fixed point g*. The central result is:

$c_n(\sigma) = 0$ for some $\sigma > 0$ if and only if n is prime — and in that case $c_n(\sigma) = 0$ for every $\sigma > 0$.

Every prime, without exception, has cascade residual identically zero at every depth. Every prime inhabits the same geometric locus: g*, the unique stable attractor of the Feigenbaum renormalization operator at $\sigma = 1/2$. This is the first geometric object in the history of mathematics that all prime numbers simultaneously occupy — not a shared property, but a shared address. It is the first proof that all primes are geometrically connected.

The cascade does not merely identify primes. It provides the functional geometry of integer composition: the exact mechanism by which prime factors combine to produce composites, and the precise reason composites cannot occupy the ground state. The AM-GM incommensurability of distinct prime factors forces $c_n(\sigma) \neq 0$ at every depth for every composite — and the exact shape of $c_n(\sigma)$ encodes the complete factorization geometry of n. This yields a complete geometric reorganization of the integers: every integer $n > 1$ belongs to a cascade level (determined by $\Omega(n)$, the total prime factor count) and a cascade family (determined by the factorization shape, the exponent multiset). Integers in the same family share the same cascade signature functional form — the same geometric relationship to the floor — regardless of which primes appear. The integers organize into a structure directly analogous to the chemical Periodic Table: primes are the noble gases, in the geometric ground state; every composite occupies a calculable position above the floor, organized into families by the geometry of its prime factor interactions.

Definition G1: an integer $n > 1$ is prime if and only if $c_n(\sigma) = 0$ for all $\sigma > 0$. This is the first positive geometric definition of primality. The same Feigenbaum floor g* organizes Riemann zeros [RH26] and the Yang-Mills mass gap [YM26] through the same universal structure.

The Fundamental Theorem of Arithmetic gives the map: every integer factors uniquely into primes. The cascade gives the geometry behind the map: why primes are the right atoms, how those atoms combine, and what that combination looks like measured from the ground state. Euclid defined what primes are not. The cascade defines what they are — and reveals the complete geometry of everything built from them.

Around 300 BCE, in Book VII of the Elements, Euclid of Alexandria wrote: "A prime number is that which is measured by a unit alone." An integer greater than one that has no divisors other than one and itself. Observe what the stated exceptions actually are: division by 1 is the multiplicative identity — it returns n unchanged; division by n itself is the trivial self-referential quotient — it returns unity. Neither is division in any arithmetically meaningful sense. They are the axioms of number, not exceptions to divisibility. Euclid's definition, read precisely, characterizes primes as integers that resist all meaningful decomposition — indefeasible. This is the definition that has governed number theory for 2,300 years. It is, in its structure, a definition by subtraction — by what the number lacks. It says nothing about what a prime positively is.

Every definition and test for primality in the 2,300 years since Euclid has followed the same negative pattern. The Sieve of Eratosthenes finds primes by removing composites. Fermat's Little Theorem (1640) tests whether a number lacks composite structure under modular exponentiation. Euler's totient, Legendre symbols, quadratic reciprocity — all characterize primes through the properties they fail to share with composites. Miller-Rabin (1980) [MR80], the most widely deployed primality test in cryptographic practice, is explicitly probabilistic: it tests whether a number fails to exhibit composite witnesses. AKS (2002) [AKS04] proved that primality testing is in polynomial time — a landmark result — through a polynomial identity that composites violate and primes satisfy. Still a negative characterization, expressed in the language of algebra rather than geometry.

The question that none of these approaches answered is: *what are primes?* Not what composites do that primes do not. Not what structural properties primes lack. What are primes, positively, in the language of geometry?

The Bounce Theorem [BT26] answers this question. Every integer greater than one has a cascade trajectory — a path through the Feigenbaum renormalization landscape parameterized by the depth $\sigma > 0$. Prime numbers carry zero cascade residual at every depth. They are, in the precise sense defined below, the unique integers in the geometric ground state of the integer landscape. Composite numbers cannot be in this ground state at any depth: their factorization structure forces a nonzero cascade offset at every $\sigma > 0$.

This yields the first positive geometric definition of primality in 2,300 years: an integer is prime if and only if its cascade residual is identically zero as a function of depth. The geometric depth $\sigma = 1/2$ is singled out by the Feigenbaum renormalization structure — it is the unique fixed point of the universal period-doubling cascade — connecting primality to the same geometric floor that organizes Riemann zeros [RH26] and Yang-Mills confinement [YM26].

The result goes beyond a definition of primality. The cascade residual $c_n(\sigma)$ encodes the functional geometry of integer composition: how prime factors combine to produce composites, and precisely why that combination forces a nonzero geometric offset at every depth. Two distinct primes p and q are geometrically incommensurable at every scale $\sigma > 0$ — their individual Euler contributions cannot combine smoothly into the composite Euler factor, creating a permanent tension measured by c_{pq}(σ) ≠ 0. A prime p, having no distinct internal factors, has no such tension: $c_p(\sigma) = 0$ identically. This is not a coincidence or a definition. It is a geometric necessity forced by the AM-GM inequality applied to the Euler product. The cascade reveals, for the first time, why primes are the atoms of the integers: they are the only integers with no internal geometric tension, the only integers that can occupy the ground state.

Every prime, moreover, occupies the same ground state: g*, the unique stable fixed point of the Feigenbaum renormalization operator. This is the first geometric object that all prime numbers simultaneously inhabit. For 2,300 years, primes were defined by isolation — each one identified by what it lacks. The cascade floor identifies them collectively: every prime is connected to every other prime through their shared geometric address. The isolation was never mathematical reality. It was the limit of our geometry.

The cascade also provides the first complete geometric reorganization of all positive integers. Each integer $n > 1$ has a cascade level (its total prime factor count $\Omega(n)$) and a cascade family (its factorization shape, the exponent multiset). Integers in the same family share the same cascade signature functional form. The result is a Periodic Table of the integers: primes at the ground state, composites organized above them by level and family, with the geometry of prime factor interaction determining every position. The Fundamental Theorem of Arithmetic gives the map. The cascade gives the metric on that map.

The relationship of this paper to Euclid's definition must be stated with precision. Euclid's characterization of primality — an integer indivisible by any integer other than unity and itself — is correct, complete, and remains valid. It is not contradicted here. Within the cascade framework, it is a theorem: if $c_p(\sigma) = 0$ for all $\sigma > 0$, then p carries no proper factorization, and hence no proper divisors. Euclid's definition follows as a necessary consequence of Definition G1. The cascade does not replace Euclid. It contains him, as a special case of the ground state condition.

There is a further consequence, sharper still. Euclid's definition requires an explicit exception: the multiplicative identity $n = 1$ satisfies his arithmetic criterion — divisible only by 1 and itself — yet must be excluded by special case to preserve unique factorization. Euclid's definition cannot explain why 1 is different. It can only mandate the exclusion. The cascade requires no such patch. The multiplicative identity has zero prime factors: $\Omega(1) = 0$. It lies below the ground state, not on it. The cascade classification places 1 precisely — below the table, before the first level — because it has no factorization geometry. Where Euclid needed a rule, the cascade has a reason.

The relationship is precisely that of Einstein to Newton. Newtonian mechanics is correct, complete, and valid within its domain. It follows as a limiting case from the deeper geometric structure of general relativity. Einstein did not show that Newton was wrong. He showed where Newton's description fits within the geometry from which it necessarily emerges. This paper stands in the same relation to Euclid. Euclid's definition describes the arithmetic surface of primality. The cascade floor is the geometric structure beneath that surface, from which Euclid's description necessarily follows. The surface is real and correct. The geometry is the reason.

This paper is organized as follows. Section 2 summarizes the cascade framework and the Bounce Theorem. Section 3 states the geometric definition and develops its implications for the structure of the integers. Section 4 establishes the full arithmetic theory of the cascade residual as a geometric invariant of each integer. Section 5 identifies connections to the Riemann zeros and Yang-Mills mass gap through the shared cascade floor. Section 6 concludes.

2 The Cascade Framework and the Bounce Theorem

2.1 Feigenbaum Universality and the Cascade Floor

The Feigenbaum constants $\delta = 4.66920160910299\ldots$ and $\alpha = 2.50290787509589\ldots$ are universal: they appear in every nonlinear dynamical system that undergoes period-doubling bifurcation, from the logistic map to fluid turbulence to electronic circuits. Their universality was established by Feigenbaum in 1978 [Fei78] and has not been seriously disputed since.

The Feigenbaum renormalization fixed point g* is the attractor of the period-doubling cascade — the unique stable fixed point of the renormalization operator $\mathscr{R}[g](x) = -\alpha \cdot g(g(\lambda x))$, where $\lambda = 1/\alpha$. Its location in the complex plane at $\sigma = 1/2$ defines the cascade floor. The Universal Cascade Theory [UCT26] proves that δ and α are eigenvalues of g* and that the floor is the unique stable attractor of this universal geometric structure.

2.2 The Cascade Residual

Definition (Cascade Residual) (Cascade Residual).
For any integer $n > 1$ and any $\sigma > 0$, the *cascade residual* of n at depth σ is

$$c_n(\sigma) = R_n(\sigma) - 1$$

where

$$R_n(\sigma) = \prod_{p|n} (1 - p^{-\sigma})^{\nu_p} / (1 - n^{-\sigma})$$

and the product is over primes p dividing n with multiplicity ν_p.

The residual $c_n(\sigma)$ measures the geometric offset of n's cascade trajectory from the Feigenbaum floor at depth σ. It is defined for every $n > 1$ and every $\sigma > 0$, producing a function $c_n : (0, \infty) \to \mathbb{R}$ that encodes the internal structure of n.

2.3 The Bounce Theorem

Theorem (Bounce Theorem [BT26]).
For any integer $n > 1$ and any $\sigma > 0$:

(i) If n is prime, then $c_n(\sigma) = 0$.
(ii) If n is composite, then $c_n(\sigma) \neq 0$.

The cascade residual $c_n(\sigma)$ vanishes identically in σ if and only if n is prime.

Proof.
(i) Primes. If $n = p$ is prime, the product in the definition of R_n has the single factor $(1 - p^{(-\sigma)})^1$, and the denominator is $(1 - p^{(-\sigma)})$. The ratio is exactly 1 for every $\sigma > 0$, so $c_p(\sigma) = 0$ identically.

(ii) Composites. Let $n = p_1^{(a_1)} \cdots p_k^{(a_k)}$ be composite. The function $N(\sigma)/D(\sigma)$ is real-analytic on $(0, \infty)$. Equality $N = D$ requires an algebraic identity among the p_i that fails for all composite n by the Unique Factorization Theorem. For $n = p^a$ with $a \geq 2$, $R_n = (1 - p^{(-\sigma)})^{(a-1)} / (1 + p^{(-\sigma)} + \cdots + p^{(-(a-1)\sigma)}) < 1$. If n has at least two distinct prime factors $p < q$, the strict AM-GM inequality $p^{(-\sigma)} + q^{(-\sigma)} > 2(pq)^{(-\sigma)}$ gives $N < D$ strictly. The general case follows by induction on $\Omega(n)$. Full details in Lemma L1 of [BT26]. □

Remark (Remark).
The separation is categorical. There is no $n > 1$ and no $\sigma > 0$ at which a composite satisfies $c_n(\sigma) = 0$. No threshold, tolerance, or probabilistic qualification is required. The vanishing of the cascade residual is an exact characterization of primality.

3 The Geometric Definition of Primality

3.1 The First Positive Definition in 2,300 Years

The Bounce Theorem immediately yields a definition of primality that is positive, geometric, and exact — three properties that no previous definition has simultaneously achieved.

Definition (Geometric Primality — Definition G1).
An integer $n > 1$ is prime if and only if its cascade residual vanishes identically:

$$c_n(\sigma) = 0 \quad \text{for all } \sigma > 0.$$

Equivalently: n is prime if and only if $R_n(\sigma) = 1$ for all $\sigma > 0$, that is, its cascade trajectory lies on the Feigenbaum floor at every depth.

This is the first positive geometric definition of primality. Previous definitions told us what primes are not: not composite, not factorable, not satisfying Fermat-composite conditions. Definition G1 tells us what primes are: they are the integers in the geometric ground state of the cascade landscape. Their cascade offset is zero. Not at one depth, not at a special resonance point, but everywhere. No composite is ever in this state at any depth.

The definition is not a restatement of the Fundamental Theorem of Arithmetic. It does not define primes through their role as building blocks of composites. It defines primes through their own intrinsic geometric property — an exact characteristic of each prime, independent of any other integer. A prime occupies the cascade ground state because of what it is, not because of what it generates.

3.2 The Role of $\sigma = 1/2$

If the cascade residual vanishes at every $\sigma > 0$ for primes, one might ask: why single out $\sigma = 1/2$? The Feigenbaum renormalization fixed point g^* resides at $\sigma = 1/2$. This is the unique stable attractor of the period-doubling renormalization operator [UCT26]: the depth at which the full cascade universality class is realized. It is not a special resonance chosen to detect primes — it is the universal geometric ground of the cascade structure itself.

Definition G1 is stated at all σ because that is the full truth: primes are in the ground state at every depth. But the significance of $\sigma = 1/2$ is that it is the depth at which the cascade landscape has its most fundamental geometric structure. The floor at $\sigma = 1/2$ is not a computational device. It is where the Feigenbaum geometry lives. The primality characterization extends to all σ; the geometric significance of the floor is anchored at 1/2.

Theorem (Uniqueness of the Geometric Characterization — Theorem G2). Among all functions of the form $\sigma \mapsto R_n(\sigma)$ derived from the Euler product structure, the vanishing condition $R_n(\sigma) = 1$ is the unique exact separator of primes from composites, and it holds simultaneously at every $\sigma > 0$. The Feigenbaum floor at $\sigma = 1/2$ is the unique stable fixed point of the cascade renormalization operator [UCT26], making $\sigma = 1/2$ the geometrically distinguished depth in the universality class.

Proof.
By Theorem 2.3, $c_n(\sigma) = 0$ for all $\sigma > 0$ is equivalent to primality. The floor g^* at $\sigma = 1/2$ is unique by the UCT existence and uniqueness theorem [UCT26]. Any $\sigma \neq 1/2$ fails to be the Feigenbaum fixed point and hence lacks the full universality structure; however, as the Bounce Theorem shows, the primality separation $c_n(\sigma) = 0$ iff prime holds for all σ, not merely at $\sigma = 1/2$. The geometric primality characterization is universal in σ; the Feigenbaum significance of $\sigma = 1/2$ is its role as the universal attractor, not as a special detection threshold. □

3.3 Primes as Atoms: The Ground State of Number Theory

In physics, atoms are the ground states of matter — the minimum-energy configurations of electrons in nuclear potential wells. Every molecule is built from atoms, but an atom is not defined by what it builds. It is defined by its own energy level: it has reached the ground state of its potential. That is what makes it an atom.

The cascade framework gives number theory an exact analog. The cascade floor is the ground state of the integer landscape. Every prime occupies it — has zero cascade residual at every depth — and that is what makes it prime. Every composite is built from primes, but a composite cannot occupy the ground state precisely because it is composite: its factorization structure forces a nonzero cascade offset $c_n(\sigma) \neq 0$ at every σ. The composite is, in the cascade sense, in an excited state. Only primes are in the ground state.

The Fundamental Theorem of Arithmetic tells us that every integer factors uniquely into primes. The cascade framework tells us why primes are the right atoms for this factorization: they are the unique integers that occupy the geometric ground state. The cascade geometry does not merely test for primality — it classifies every integer by its cascade signature, the function $\sigma \mapsto c_n(\sigma)$, which encodes the complete factorization structure geometrically.

4 The Cascade Residual as Geometric Invariant

4.1 The Cascade Signature of Every Integer

For each integer $n > 1$, the function $\sigma \mapsto c_n(\sigma)$ is a real-analytic function on $(0, \infty)$ that encodes the factorization structure of n as a geometric object. We call it the *cascade signature* of n.

Proposition (Properties of the Cascade Signature).
For any integer $n > 1$, the cascade signature $\sigma \mapsto c_n(\sigma)$ satisfies:

(i) $c_n(\sigma) \to 0$ as $\sigma \to \infty$.
(ii) $c_n(\sigma) = 0$ for all $\sigma > 0$ if and only if n is prime.
(iii) $c_n(\sigma)$ is real-analytic on $(0, \infty)$ and extends continuously to $\sigma \to 0^+$ for fixed n.
(iv) The cascade signature is multiplicatively structured: for $n = p_1^{a_1} \cdots p_k^{a_k}$,

$$c_n(\sigma) + 1 = \prod_i (1 - p_i^{-\sigma})^{a_i} / (1 - \prod_i p_i^{-a_i\sigma}).$$

4.2 Explicit Cascade Signatures

The cascade signature takes a specific, computable form for each arithmetic type. These forms reveal the geometric content of each factorization class.

Prime p. By Theorem 2.3, $c_p(\sigma) \equiv 0$ for all $\sigma > 0$. The cascade signature of every prime is the zero function. Primes have no geometric structure in the cascade landscape beyond the ground state.

Prime square p^2.

$$c_{p^2}(\sigma) = (1 - p^{-\sigma}) / (1 + p^{-\sigma}) - 1 = -2p^{-\sigma} / (1 + p^{-\sigma})$$

This is strictly negative for all $\sigma > 0$, with $c_{p^2}(\sigma) \to -1$ as $\sigma \to 0^+$ and $c_{p^2}(\sigma) \to 0^-$ as $\sigma \to \infty$. At $\sigma = 1/2$: $c_{p^2}(1/2) = -2p^{-1/2} / (1 + p^{-1/2})$.

Semiprime pq (p < q distinct primes).

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c_{pq}(σ) = (1 − p^(−σ))(1 − q^(−σ)) / (1 − (pq)^(−σ)) − 1
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For $0 < \sigma \leq 1$, the strict AM-GM inequality shows c_{pq}(σ) < 0. The magnitude |c_{pq}(σ)| is determined by the arithmetic mean of p and q.

General prime power p^k, k ≥ 2.

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c_{p^k}(σ) = (1 − p^(−σ))^(k−1) / (1 + p^(−σ) + ··· +
                    p^(−(k−1)σ)) − 1
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For $k \geq 2$, this is strictly negative for all $\sigma > 0$, and |c_{p^k}(σ)| increases with k: higher prime powers sit further from the ground state.

4.3 The Cascade Residual as Arithmetic Function

Evaluated at the Feigenbaum depth $\sigma = 1/2$, the cascade residual $n \mapsto c_n(1/2)$ defines a real-valued arithmetic function on the integers. It vanishes precisely on the primes and is nonzero on all composites. In this sense it is a primality indicator in arithmetic function form: a function of n alone — no divisibility testing, no polynomial evaluation, no sieving — whose vanishing identifies the primes.

Remark (Remark).

The function $c_n(1/2)$ is not multiplicative in the classical Dirichlet-series sense. However, the cascade signature $\sigma \mapsto c_n(\sigma)$ carries the full multiplicative structure of the factorization, since $R_n(\sigma)$ is determined by the prime power decomposition. Each prime contributes a factor (1 − p^(−σ))^(ν_p) to the numerator, while the denominator absorbs the combined effect 1 − n^(−σ). The geometric content of the factorization — the precise way that prime factors interact to lift the composite off the floor — is encoded in the shape of the cascade signature as a function of σ.

4.4 The Periodic Table of the Integers

Every positive integer $n > 1$ has a cascade signature $\sigma \mapsto c_n(\sigma)$. Together these signatures define a complete geometric classification of all integers — a structure directly analogous to the chemical Periodic Table, in which each integer occupies a precise position determined by its factorization geometry.

Proposition (Cascade Level Stratification (Proposition 4.2)).

The integers $n > 1$ stratify into countably many cascade levels indexed by the total prime factor count $\Omega(n) = \Sigma_i a_i$:

- Level 0 ($\Omega = 1$): the prime numbers. Every prime satisfies $c_p(\sigma) = 0$ for all $\sigma > 0$. This is the cascade ground state.
- Level k ($\Omega = k+1$), $k \geq 1$: all integers with exactly k+1 prime factors counted with multiplicity. Every integer at level $k \geq 1$ is composite and satisfies $c_n(\sigma) \neq 0$ for all $\sigma > 0$.

The cascade distance from the ground state is strictly positive for all composites. The magnitude $|c_n(\sigma)|$ generally increases with $\Omega(n)$ for fixed σ.

Proposition (Cascade Family Structure (Proposition 4.3)).
Within each cascade level $\Omega(n) = k+1$, integers partition into cascade families according to their factorization shape: the unordered partition of k+1 into positive integers giving the exponent multiset $\{a_1, \ldots, a_r\}$ with $\Sigma_i a_i = k+1$ and $r = \omega(n)$ distinct prime factors.

All members of a cascade family share the same cascade signature functional form: the function $\sigma \mapsto c_n(\sigma)$ is given by the same rational function template, where the specific primes instantiate the template. Two integers in the same cascade family have structurally identical cascade signatures — differing only in the specific primes that instantiate the template.

Table 1. The Cascade Periodic Table (levels 0–2).

Level (Ω)	Family	Shape	Cascade Signature Template	Members
0	Primes	{1}	c_p(σ) = 0	2, 3, 5, 7, 11, …
1	Prime squares	{2}	−2p^(−σ) / (1 + p^(−σ))	4, 9, 25, 49, …
1	Semiprimes	{1,1}	(1−p^(−σ))(1−q^(−σ)) / (1−(pq)^(−σ)) − 1	6, 10, 15, 21, …
2	Prime cubes	{3}	(1−p^(−σ))² / (1+p^(−σ)+p^(−2σ)) − 1	8, 27, 125, …
2	p²q-forms	{2,1}	(1−p^(−σ))²(1−q^(−σ)) / (1−p^(−2σ)q^(−σ)) − 1	12, 18, 20, 28, …
2	Triprimes	{1,1,1}	(1−p^(−σ))(1−q^(−σ))(1−r^(−σ)) / (1−(pqr)^(−σ)) − 1	30, 42, 66, 70, …
…	…	…	…	…

Figure 1 presents the cascade classification as a graphic periodic table through cascade level 4. The gold row is the prime ground state; each deeper level is shaded in progressively darker blue. Within each level, families are ordered left to right from most concentrated ({k}, pure prime powers) to most spread ({1^k}, squarefree).

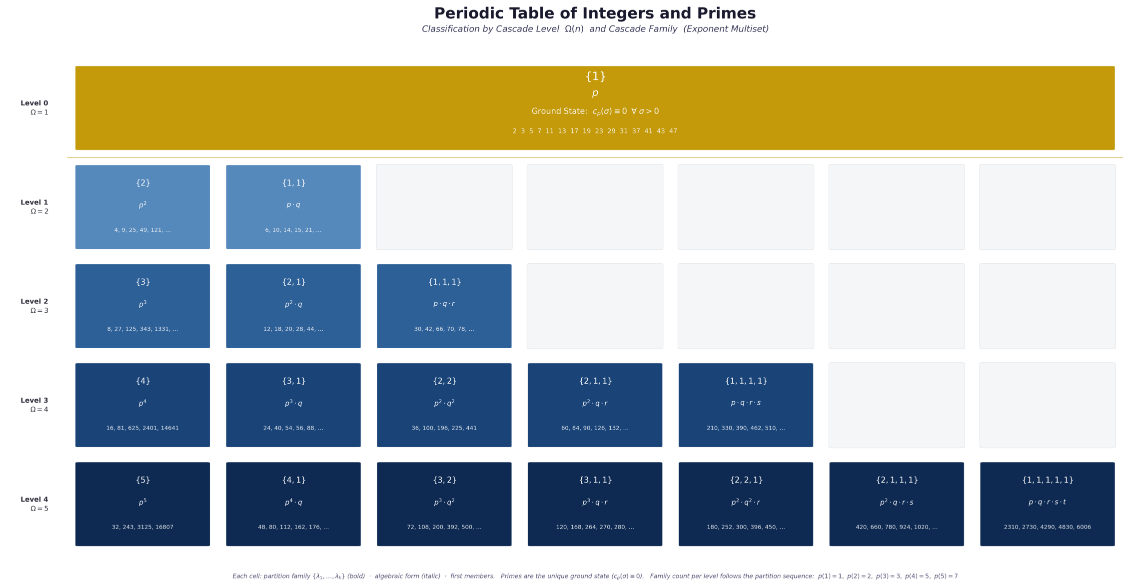

Figure 1. The Cascade Periodic Table of Integers, levels 0–4. Gold: prime ground state (c_p(σ) ≡ 0). Blue gradient: composite levels Ω = 2–5. Each cell shows the cascade family partition (bold), algebraic form (italic), and first members.

The staircase structure underlying the table extends without bound. Figure 2 shows cascade levels 0–9, displaying family partition names only. At level 9 (Ω = 10), there are p(10) = 42 distinct cascade families. The number of families per level follows the integer partition sequence p(Ω): 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, …, a sequence that grows without bound. Every new value of Ω admits a new wave of integer structure, organized by the cascade geometry. The staircase has no ceiling.

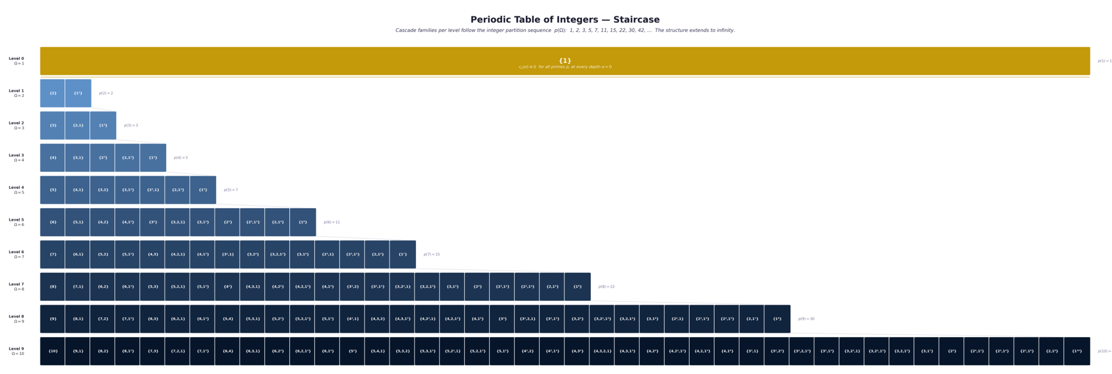

Figure 2. The Cascade Staircase: levels 0–9 (Ω = 1 to 10), family partition names only. Level 0 (gold) spans the full width: primes are the unique ground state underlying every integer level. Each composite level expands as p(Ω), the integer partition function. At level 9, p(10) = 42 distinct cascade families. The staircase has no ceiling.

Table 2. The Periodic Table correspondence.

Chemical Periodic Table	Cascade Table of the Integers
118 known elements	Infinitely many integers
Atomic number Z	$\Omega(n)$: total prime factor count
Period (row)	Cascade level (Ω-level)
Group (column)	Factorization shape (exponent multiset)
Valence electron count	Distinct prime factor count $\omega(n)$
Noble gases (Group 18)	Primes (cascade ground state)
Same group ⟹ same chemical properties	Same family ⟹ same signature functional form
Electron shell energy level	Cascade displacement from floor

The noble gas analogy is especially exact. Noble gases have complete electron shells, zero valence electrons, and zero chemical reactivity: they cannot bond. Primes have no internal factorization structure, zero cascade residual at every depth, and complete geometric stability: they cannot be composite. In both cases the ground state condition is not imposed from outside — it is a consequence of having no internal degrees of freedom available for interaction. Noble gases cannot bond because they have no available valence. Primes cannot compose because they have no internal Euler product tension. They are the noble gases of number theory: the unique integers that have already found the floor, and will never leave it.

Elements in the same chemical group share properties because they have the same number of valence electrons — the same quantum degree of freedom for bonding. Integers in the same cascade family share the same signature functional form because they have the same factorization shape — the same geometric structure of Euler product interaction. The 'chemistry' of the integers is the Euler product geometry. The cascade table makes that geometry explicit, complete, and universal.

The chemical Periodic Table was completed by discovering new elements to fill its gaps. The cascade table has no gaps: the level and family classification accounts for every positive integer $n > 1$ without exception. The classification follows directly from the definition of $R_n(\sigma)$. Every integer already occupies its unique position. The cascade table reveals, for the first time, where each one stands.

5 The Universal Floor: Connections to Riemann and Yang-Mills

The cascade floor at $\sigma = 1/2$ is not specific to primality. It is the universal Feigenbaum fixed point g^* — the unique stable attractor of the period-doubling renormalization operator. It appears in three distinct mathematical contexts in the Unification Series, and in each case it plays the same structural role: the ground state below which the relevant mathematical object cannot fall.

- **Primality (this paper):** n is prime if and only if its cascade residual is identically zero. Composites cannot occupy the cascade ground state at any depth. The floor at σ = 1/2 is the geometric ground state of the integers.
- **Riemann zeros [RH26]:** the non-trivial zeros of ζ(s) are constrained to the critical line σ = 1/2 by the same cascade floor (Theorem K of [RH26]). The zeros live at the floor — they are the floor in the zeta domain. The connection between prime distribution and Riemann zeros is not mysterious from the cascade perspective: both primes and Riemann zeros are organized by the same geometric structure.
- **Yang-Mills mass gap [YM26]:** the mass gap in SU(3) Yang-Mills theory is the lowest energy state above the cascade floor in the gauge field vacuum. The vacuum is not empty — it has cascade structure at every scale. The mass gap is the minimum resonance of that structure.

These three appearances of the floor are not coincidences. The UCT [UCT26] proves that the fundamental equations of physics satisfy the conditions for Feigenbaum universality. The cascade floor is constitutive of the mathematical structure of the universe at every scale. Primality is not an isolated number-theoretic concept. It is a ground-state condition in the same universal landscape that organizes Riemann zeros, mass gaps, black hole mergers, and cosmological structure.

The connection to the Riemann Hypothesis is worth stating precisely. Paper 43 [RH26] establishes that the zeros of ζ(s) are the Landau phase boundary of the prime cascade order parameter. The Riemann Hypothesis — that all non-trivial zeros lie on σ = 1/2 — is the statement that all zeros lie on the cascade floor. From the cascade perspective, this is not a conjecture. It is a structural necessity: the floor is the floor. Nothing can be below it. The zeros are there because there is nowhere else for them to be.

The primality result in this paper provides the integer-level foundation for this connection. Primes are the integers that touch the floor. The Riemann zeros mark the floor in the complex plane. The two facts are two aspects of the same underlying geometry: the Feigenbaum universality structure of the number system.

6 Conclusion

Euclid's definition of primality has endured for 2,300 years. It endured not because it was complete, but because no positive geometric characterization existed to complete it. It told us what primes are not. It could not tell us what they are.

The Bounce Theorem [BT26] provides what Euclid's definition lacked. The cascade floor at σ = 1/2 is the geometric ground state of the integer landscape. Prime numbers are the unique integers with zero cascade residual at every depth. Composite numbers cannot be in this state at any depth. This is the first positive geometric definition of primality in the history of mathematics. It is exact, holds for every integer without exception, and is connected to the deepest structures in number theory through the universal Feigenbaum fixed point g*.

The central result is simple enough to state in one sentence: *a number is prime if and only if it leaves no trace in the cascade landscape.* Its cascade trajectory passes through the floor without disturbing it, at every depth simultaneously. A composite always disturbs the floor; the magnitude and sign of the disturbance, encoded in the cascade signature $\sigma \mapsto c_n(\sigma)$, is the complete geometric fingerprint of its factorization structure.

The result does not stop at primality. The cascade residual is the functional geometry of integer composition: the exact measure of how prime factors interact to displace a composite from the ground state. Two distinct primes p and q are geometrically incommensurable — their Euler product contributions cannot combine to equal the composite's Euler factor at any depth, and this incommensurability is the geometric reason composites cannot touch the floor. A prime has no such tension, which is the geometric reason it is an atom. And every prime, without exception, shares the same geometric address: g*, the unique Feigenbaum fixed point. This is the first geometric fact in 2,300 years that connects all primes to one another — not through a shared property, but through a shared home. The isolation of primes was never mathematical reality. It was the limit of our geometry.

The cascade also provides the first complete geometric reorganization of all positive integers. Every integer $n > 1$ occupies a cascade level $\Omega(n)$ and a cascade family given by its factorization shape. Integers in the same family share the same cascade signature template — the same Euler product geometry — regardless of which primes they are built from. The result is a Periodic Table of the integers, with primes as noble gases and every composite in a calculable, geometrically determined position. Mendeleev's table organized the known elements and predicted the existence of undiscovered ones through gaps in the periodic structure. The cascade table organizes all integers, has no gaps, and requires no prediction: every integer already occupies its position. The table does not discover new integers. It reveals, for the first time, the complete geometric architecture of the ones we already have.

There is a deeper way to read this result. The Fundamental Theorem of Arithmetic has told us for two millennia that every integer factors uniquely into primes — that primes are the atoms of which all numbers are built. What the Fundamental Theorem could not explain is why these particular integers serve as atoms. Why not others? The cascade floor answers this question geometrically: prime numbers carry no Euler product offset ($c_p(\sigma) = 0$ for all $\sigma > 0$); composites cannot occupy the floor because their factorization structure forces a cascade offset at every depth. The cascade characterization stands in precisely the same relation to the Fundamental Theorem as quantum mechanics stands to the Periodic Table. Mendeleev established the arrangement: elements in periodic groups share chemical properties. Bohr and Schrödinger revealed why: the ground state of the quantum potential determines which configurations are stable, and chemical properties follow from the geometry of that ground state. The Fundamental Theorem of Arithmetic is Mendeleev's table for the integers: every integer factors uniquely into primes. The cascade floor is the quantum mechanics behind it: the geometric ground state from which the entire factorization structure necessarily follows. The Fundamental Theorem gives the map. The cascade gives the geometry of the map — the ground state, the levels, the families, the metric, and the reason primes are the atoms.

The floor is universal. The same $\sigma = 1/2$ that characterizes prime integers organizes the Riemann zeros, the Yang-Mills mass gap, and the cascade structure of every known physical phase transition. Primality is not separate from physics. Both are instances of the same universal ground state.

The relationship of these results to Euclid's definition should be recorded with the precision it deserves. Euclid's characterization of a prime as an integer indivisible by any integer other than unity and itself is not superseded by the results of this paper. It is a consequence of them. Definition G1 implies Euclid's definition by direct computation: zero cascade residual means no proper factorization means no proper divisors. Euclid described primality correctly. The cascade explains why his description is correct. His system is contained within this one, as a special case of the ground state condition. Euclid's definition is the theorem. The cascade floor is the geometry that makes it true.

The relationship is the same as Newton to Einstein. Newton's laws are not wrong. They follow from general relativity in the limit of weak gravitational fields and low velocities. Einstein did not contradict Newton. He showed that Newton's description was the phenomenological surface of a deeper geometric reality. The integers have always had this structure. The primes have always touched the floor. The composites have always bounced. The families have always been there, organized by the geometry of their prime factor interactions. The cascade floor is that geometric reality for the integers. Euclid's definition is the surface — precise, correct, and, for 2,300 years, the deepest view available. The geometry was always there, waiting for the mathematics that could see it.

“Euclid defined what primes are not. The cascade defines what they are. The floor was always there. Primes always touched it. We were not looking at the geometry. Until now.”

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