V.—KANT'S PROOF OF THE PROPOSITION, "MATHEMATICAL JUDGMENTS ARE ONE AND ALL SYNTHETICAL".

By Bruce McEwen.

It needs no very profound analysis of the *Critique of Pure Reason* to demonstrate that the Æsthetic, as it stands in the forefront of Kant's system, fulfils a distinctly double purpose. Leaving aside the doctrine of Time and regarding only the doctrine of Space, we find that it presents itself to us under two quite different aspects, according as we look at it from one or the other of two points of view towards which the author himself repeatedly directs our attention. Considered alone and by itself, the Æsthetic has a purely dogmatical, assertoric or didactical value: it is a theory as to the nature of Space and of Spatial existences. But when we pass beyond this first crude conception of the Æsthetic, and begin to trace out the various lines of connexion, which lead to every part of Kant's speculations, when we examine the internal arrangement of the matter of the *Critique* and the relations of interdependence and mutual defence, which Kant strives to establish between all portions of his system, there is gradually unfolded before our eyes the higher purpose of the Æsthetic, as a first apology for the whole. However much we may prize the content of the doctrine of Space for its own sake, still to the true Kantian it will always seem relatively more important to regard the Æsthetic as giving us, in Kant's own words, "one part of the answer necessary for the solution of the great general problem of the Transcendental Philosophy, namely the problem: How are synthetical propositions *a priori* possible?" The vindication of those forms of knowledge, which are at once synthetic and *a priori*, is the object of the new Criticism from beginning to end, and with his doctrine of Space and the corresponding real science, Pure Mathematics, Kant introduces the thin end of the wedge destined to cleave asunder the sceptical denial of the reality of all knowledge of this type. "When Hume felt the truly philosophic impulse to cast his glance over the

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whole field of pure (synthetic) knowledge *a priori*, he somewhat recklessly excluded a whole province of such knowledge, —perhaps indeed the most important province,—namely Pure Mathematics." Ask Kant for an instance of this real knowledge, and he is found too cautious to mention the name of Metaphysics: he "points to a Euclid" and says, Here! "Pure Mathematics, in spite of the attacks of the hardiest sceptic, stands, like a Colossus, for the proof of the reality of a knowledge enlarged and augmented through Pure Reason alone."

Now from the latter of the points of view, thus briefly indicated, Kant's object is to prove two leading characteristics of Mathematics; first, that it is synthetic knowledge; second, that it is knowledge *a priori;* and for the proper understanding of the opening chapters of the *Critique* it is absolutely necessary to keep the two parts of this proof distinct. Neither of them depends in the very least degree upon the other; either might be invalid, while the strength of the other remained unweakened. The terms 'synthetic' and '*a priori*' are far from being indispensable to one another; the often repeated use of them in conjunction, to describe different kinds of knowledge, implies in each case two distinct investigations of the nature of the knowledge so described. In the present essay we shall examine only the process, by which Kant sought to prove the synthetic nature of Mathematics, and leave the other investigation strictly out of account. Kant's final view on the point, which concerns us here, is summed up in his brief statement: "Mathematical judgments are *one and all* synthetical"; we shall see that Kant reached this result only after a long struggle, in the course of which all varieties of mathematical judgments came directly under his consideration: we shall try to dispose of those objections to Kant's doctrine, which deny the perfection of his induction, and seek to overthrow it by producing particular instances from mathematics of analytic judgments supposed to have been overlooked by Kant. Questions as to the *a priori* nature of mathematics, or the validity of the distinction between analytic and synthetic judgments generally, will be treated as irrelevant: our procedure is primarily exegetical, and, if at any point it be necessary to defend Kant's doctrine, our defence will be confined entirely within the limits already assigned to our investigation.

To the historian of philosophy nothing is more obvious than that Kant was merely following the example of a long and illustrious line of predecessors when he entered upon the realm of Philosophy from the side of mathematical

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science. Indeed so commonplace is this observation, that in consequence the historian is usually led to neglect the vast difference which distinguishes Kant's first appearance from that of the many other "mathematical philosophers". The followers of Descartes, especially on the continent of Europe, had pinned their faith in mathematical matters to the analytical method introduced into geometry by their great master. The comparatively modern science of Algebra was applied to the problems of Space and Motion, and in the hands of such colossal geniuses as the Bernouillis, Leibniz and Euler problems, which had long been the despair of the Euclideans, now found a simple and accurate solution. For the time there appeared to be no limit to the powers of the new mathematical analysis. In Geometry, in Dynamics, even in the mixed sciences, Astronomy and Physics, it absorbed, transformed and reproduced all previous results with such startling rapidity that the credit of discovery has often been wholly filched away from the original author by those who merely borrowed his thoughts.[1] But alongside of this rapidly advancing Cartesian system the old conservative methods, which had come down from the days of Pythagoras and Plato, always retained many illustrious intellects in their service. The English school of Scientists, represented by Boyle, Hooke, Newton, and the other founders of the Royal Society, continued to give ample proof of the vitality of pure geometry and experimental investigation in science. Still, as we have said, their discoveries were no sooner proclaimed than appropriated by the analytical method, all signs of the source from which they came being thereby obliterated.

Accordingly it must have been with feelings of surprise that Kant's contemporaries beheld one who was born into and educated under the new analytical school throwing off the traditions of his time, and returning to the methods they were wont to consider superseded. From the outset Kant attached himself unreservedly to the "Natural Philosophy" of Newton, adopting at the same time the peculiarly Newtonian preference for pure geometrical methods. Kant, it must never be forgotten, was a mathematician of no mean order; even if we disregard his professedly mathematical writings, we have sufficient indication of this fact in the many references to Mathematics in his philosophical works,

[1] A most striking instance of this tendency may be noticed in the case of the invention of the Differential Calculus originally discovered by Newton, but afterwards almost universally ascribed to Leibniz, who gave it the convenient algebraical form in which it now appears.

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and in his private notes (*Löse Blätter*), which are full of mathematical calculations upon knotty points, that occupied his attention from moment to moment. And yet there is not a single instance in all Kant's writings of his using the Cartesian method in geometry. He repeatedly praises it as being "rich in discovery" (*erfindungsreich*), he quotes its formulæ with a facility which betokens long acquaintance with them, yet in his own investigations he altogether disdains its aid.

And Kant was not content to limit his mathematical speculations to the pure geometrical method without giving good reasons for his preference. In Dynamics Newton had often gone considerably out of his way to preserve the purity of his procedure; his *Principia* explain the motion of a body, moving under certain forces, by actually following out its path on paper from point to point before the eyes of the reader. Thus by strictly geometrical methods the orbits of the moving masses are depicted *in concreto*, and when the process is finished the nature of these orbits is intuitively deducible from the nature of their construction. In direct contrast with this somewhat slow and laborious process the Leibnitian method simply writes down a symbolical expression for the forces acting on a mass, and by a single swift stroke (Integration) another is deduced, which is the perfect algebraic expression for the curve described. So far, the analysis appears to have been concerned with mere signs in the abstract, but a serious difficulty now presents itself, when we try to interpret our results. We work with symbols but these have no signification, except indeed as interpreted, and for this interpretation the analytical method must ultimately call in the aid of pure geometry, for every equation used depends absolutely for its meaning upon the propositions of the Euclideans. If we use an example quoted by Kant himself we might suppose the formula, $ax=y^2$, to be the result of an algebraic analysis of the data in some problem or other: now we know that this represents a certain curve described, but it is only by a knowledge of the pure geometrical properties of Conic Sections that we are enabled to identify it with a parabola.

It is a distinct merit of Kant's, though, I believe, he has never received the full credit of it, that he clearly perceived and as boldly proclaimed the incompleteness of the Cartesian method at the very time when it was generally considered to be an instrument of mathematical research far more powerful than the pure geometry, which it really involves, and from which it was originally derived. And it was this

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clear insight into what is fundamental in Mathematics that enabled Kant, as every student knows, to confine the question of the nature of all geometry with the most systematic consistency within the limits of the Euclidean method. As early as 1763, in his treatise on the *Principles of Natural Theology*, Kant takes his stand definitely within these limits and announces the doctrine that "the definitions of Mathematics are entirely synthetic, while those of Philosophy are analytic". "The conceptions, which I expound in Mathematics, are not given before the definition, but have their origin always from it. A 'cone' may mean what it will, but, within the realm of Mathematics, it is formed by the arbitrary representation of the revolution of a right-angled triangle about one of its sides. Here and in all other cases the exposition finds its origin through Synthesis." The definitions of Philosophy on the other hand, Kant at present considers to be analytic in general, yet instances may be produced, he says, "from which we might draw the conclusion that the expositions of philosophers may sometimes be synthetic and those of the mathematician analytic". "Leibniz," he explains, "formed the conception of a simple substance, which had only confused representations, and called it a slumbering Monad. Here he has not expounded (*erklärt*) this monad, but merely formed a conception of it (*erdacht*), for the conception was not given to him, but was first created by him." Kant then is quite willing to believe that philosophic definitions may be of two kinds but he sturdily denies that the same is true of Mathematics. "The expositions of the mathematicians," he says, "have often been analytic in their nature, I admit it, but yet there has always been an error involved. For example, Wolf considered geometrical similarity from a philosophic point of view, so as to bring this particular case under the general conception of similarity." But Euclid's definition of similarity runs: 'Rectilinear figures are said to be *similar*, when they are equi-angular and have the sides about the equal angles proportional with a corresponding definition, if need be, for curvilinear figures, so that "for Geometry nothing depends on the general definition of similarity".

Here then we have Kant's first detailed account of the nature of mathematical definitions, and this account he never saw fit to alter, save perhaps by multiplying and improving his illustrations of its truth. Throughout the remainder of his writings it is always laid down that the mathematician in defining proceeds "by constructing his conceptions, and not by analysing them". When this

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favourite phrase is contrasted with Kant's more general statement, that Mathematical judgments are always synthetical, it has again and again been pointed out that this latter expresses more than was really necessary for Kant's purpose. It is admitted that the three moments of all mathematical procedure consist, as Kant has said, of Definition, Axiom and Demonstration, yet the synthetic nature of Mathematics generally is sufficiently proved when it is shown that the Definitions are synthetic. But with any such partial result Kant is not satisfied: he extends his inquiries at once to every part of mathematical processes. Here, unfortunately, Kant's opinions were slow to crystallise into their final form, and the result is that there is always an appearance of wavering about his statements. At times he seems to admit analytical principles into mathematical method, again to deny them, sometimes to give expression to both views almost in the same sentence.

The difficulty lies in the fact that, in Algebra and Geometry alike, the mathematician speedily passes away from an actual representation of real things to a consideration of letters, figures, or words, which, for the time being, are treated as mere signs. Take for example an ordinary mathematical syllogism—

$$x = y$$
$$\text{and } w = v$$
$$\therefore x + w = y + v \text{ and } x + w > y.$$

The letters, x, y, w, v, here used, may be mere algebraic symbols, or on the other hand in pure geometry they may stand for lines, or plane angles, and so on, but in any case certain fundamental propositions (*Grundsätze*) are involved in manipulating them, namely two well-known Axioms of Euclid: If equals be added to equals the wholes are equal; and: The whole is greater than its part. Now these "axioms" are quite obviously of a different class from such axioms as: Two straight lines cannot enclose a space; which is Kant's best example of a purely synthetic and intuitive proposition. This latter axiom is applicable only in geometry, while the two former seem to extend to Algebra, and even to the non-mathematical procedure of human Reason. Are such axioms then synthetic? Are they not rather analytic? It is possible, says Kant in one passage, "to prove them in strict philosophic fashion out of mere conceptions"; it can never be "the proper business of Mathematics to prove that the whole is greater than the part". And yet we may quote a passage from the Transcendental Æsthetic which runs: "Every geometrical axiom is derived from intuition

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a priori, and with apodeictic certainty and never from general conceptions". At first sight there appears to be considerable confusion here, and many of Kant's opponents have promptly taken advantage of it to deny his theory, that mathematical judgments are one and all synthetical. Still the fact remains that in Kant's earlier view a synthetic nature is claimed only for the Definitions of Mathematics: the definitions are synthetic, but the remainder of mathematical procedure Kant still considers to be in great part analytic.

"In Arithmetic," he says, "we treat not the things themselves but their signs which indicate the magnitude of the things and their relations as greater and less, etc. Afterwards we work with these signs according to easy and sure rules by transposition, addition or subtraction, and different sorts of operations, so that the things indicated by the signs are in the meantime completely dropped out of consideration, until finally in the conclusion the meaning of the symbolic result is deciphered." By Arithmetic Kant means "both the general doctrine of indeterminate magnitudes," or Algebra, and the "particular doctrine of numbers, where the relation of the magnitudes to unity is strictly determined," Arithmetic in the ordinary narrow sense. The view expressed in the foregoing quotation is that, so long as we work with signs, we proceed *in abstracto* and analytically without being assured that any real object corresponds to our signs *in concreto*. But in Mathematics "the meaning of the signs must be sure" at every point, and therefore a synthetic and intuitive process must precede our using signs; in fact when we say, Let x or let the figure 3 represent the objects, this process is always subject to the condition that it be "possible for these signs actually to represent the things" and that "one should clearly understand what meaning he assigns to them". Otherwise the symbolical result we reach can never be properly deciphered, indeed may be perfectly absurd. Every schoolboy knows how clearly he must lay down the meaning of his symbols before he begins to use them, for he has already experienced, if at least he is conscientious, many an instance of the complete bewilderment which ensues, when after a long exercise in Arithmetic he obtains an answer in figures, say 19, but knows not whether it is 19 men, 19 shillings, 19 eggs, or 19 of anything else real or imaginary. Suppose an answer so obtained to be numerically correct, then from his earlier point of view (in 1763) Kant would say that the analytical process of sign-manipulation was correct, the synthetic and intuitive adoption of signs instead of things equally cor-

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rect, and their "*relations as greater or less*," *etc.*, properly understood, but the corresponding synthetic and intuitive retranslation of the signs into the concrete was defective. In Pure Geometry, on the other hand, we certainly do work with signs "by transposition, addition or subtraction, and other operations," but we retain throughout a concrete sense representation of our process from point to point in the figure which we construct, we are continually passing from mere signs to their meaning in the figure, and therefore in this case the same difficulty does not appear. Mathematics, Kant says in fact, always presupposes synthetic processes, yet its method may be largely analytical.

Now this double view of the nature of mathematics in general continues to pervade Kant's work at least down to the publication of the first edition of the *Critique*. In the *Dissertation* of 1770, where "the distinction of Sense and Intellect had come to Kant's aid," his doctrine receives a new verbal setting, but in its essence it remains unchanged. The intuitive and synthetic procedure of Mathematics is represented as characteristic of the use of the faculty of sense, intellect proceeds by analysis. "Of the objects of the intellect," says Kant, "men are given no intuition, but only a symbolical cognition, and the use of the intellect is possible to us only by means of universal concepts *in abstracto* and not by singular instances *in concreto*." This use of the intellect through universal concepts *in abstracto* is the exact equivalent of the "Metaphysics," which Kant so clearly distinguished from geometry in his *Monadologia Physica* of 1756 and later in many detached passages. When the Geometrician sets out to prove that space is infinitely divisible he constructs a figure, in which he says: Draw an "infinite" number of straight lines from a fixed point. Using this symbolical figure *in concreto*, he sees intuitively that the division of Space must go on without end. The philosopher, on the other hand, argues that he can, in thought, remove all composition from a material body, which nevertheless must still consist of real substance, and therefore he concludes that, when all composition is removed nothing but simple substances are left. Here we have neither figures constructed, nor signs introduced, but from mere consideration of universals *in abstracto* the philosopher can draw his conclusion: All composite substances consist of simple parts. Such, in short, is the essence of Kant's earlier treatment of this 'Antinomy,' but now he can give a new colouring to his older outline. Philosophic procedure is concerned with the objects of intellect, mathematics with those of sense. But

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just as Kant formerly felt unable to characterise all the procedure of Mathematics as synthetic and intuitive, so now he cannot yet affirm that it is entirely concerned with objects of the sense-faculty. "Pure Mathematics," he says, "considers Space in geometry, and Time in pure Mechanics. To these we must add a certain conception, namely the conception of Number, dealt with by Arithmetic. This conception, in itself indeed, is of an intellectual nature, but nevertheless its realisation *in concreto* requires the aid of the notions, Space and Time (in adding together quantities in succession and combining them)." And so it comes about that, while Kant can announce his result: "Therefore pure Mathematics expounds the form of all our sense-knowledge and is the organon of every kind of intuitive and distinct cognition"; he is still forced to except Arithmetical procedure, a most important exception from a general treatment of Mathematics. To be sure, in "Arithmetic" we have "*cognitio symbolica*" if we only use that phrase vaguely enough, and we might therefore suppose it was something akin to the symbolical knowledge we have of objects of the intellect in Philosophy.

Now after 1770, if Kant intended to make his treatment of "Arithmetic" consistent, two courses were open to him. He might boldly deny that it was in any sense a part of Mathematics, and no doubt the consequences of this denial would, in Kant's time, have been easily accepted by many philosophers. He might affirm that the principles of Arithmetic were purely analytic, that it was concerned only with objects of the intellect and therefore proceeded not by intuition but only by analysing universal concepts, in fact, by purely symbolic cognition. But by proclaiming such a result Kant would at once have ranged himself on the side of the philosophic defenders of the Cartesian 'mathematical analysis,' for the central assumption of their theory of mathematics lies in some distinction between Algebraical and Geometrical method, such as that which would have been directly implied by Kant's adopting the course we have suggested. Or if Kant hesitated to regard "Arithmetic" as a branch of Philosophy and wished rather to bring it into strict accord with pure Mathematics (*Mathesis Pura*), which he formerly spoke of as limited to Geometry and Mechanics, it was necessary for him to prove that the procedure of "Arithmetic" was really non-philosophical. Now Kant chose this latter alternative, and accordingly, in 1781, we can easily trace the presence of a distinct improvement in his doctrine.

In the paragraph of the *Critique* on the Axioms of Intuition

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he introduces for the first time a statement to the effect that "self-evident propositions as to the relations of numbers are certainly synthetical," and produces as proof an examination of his much-misunderstood instance of Arithmetical procedure, the addition of seven and five to make twelve. "The proposition, that $7 + 5 = 12$, is not analytical." "For neither in the representation of seven, nor of five, nor of the composition of the two numbers, do I cogitate the number twelve." If we borrow from the later defence of this thesis in the *Prolegomena*, we find that Kant considers it absolutely necessary in numerical addition "to go beyond our conceptions and have recourse to an intuition". We must represent before ourselves five fingers or five points, adding five to the number seven "by means of this material image, and by this process we at length see the number twelve arise," just as by representing to ourselves on paper the figure of Euclid's thirty-second proposition we can conclude that the sum of the interior angles of a triangle is two right angles. In neither case does Kant mean that the material image is necessary: it is only an empirical means of aiding us to reach the pure non-empirical intuition which lies behind. We should very much like to defend Kant's analogy between pure and empirical intuition, inasmuch as the many attacks directed against it in this connexion seem to be both ill-founded and inconclusive, but to do so would lead us far beyond our immediate task of exegetical exposition. And therefore we take Kant's argument to be quite sufficient of itself to warrant his conclusion, that "Arithmetical propositions are invariably synthetical," and ask only how much Kant here claims to have proved. He expressly says that such propositions as $7 + 5 = 12$ are not axioms but "numerical formulæ," giving (in 1781) the ostensible reasons that we cannot have an infinite number of axioms and that axioms must be universal as in geometry. In 1783 they bear the simple name of "propositions" (Sätze), and no mention is made of their non-axiomatic nature. They are treated as corresponding to the principles on which pure Geometry founds such principles as: "The straight line between two points is the shortest," but although Kant usually speaks of this example in language which implies that he himself regards it as an Axiom, it nevertheless ranks as a proved Proposition in Euclid, and diverse views of its real nature have always been common among mathematicians. For this reason it would be hazardous to attempt any elaborate comparison of these Arithmetical propositions with other instances of synthetic *a priori* propositions, but yet, when

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Kant has proved the synthetic nature of these "numerical formulæ," it would be quite as superfluous to point out that they are involved in all Arithmetical procedure, as it formerly was that the Definitions, Axioms and Principles of Euclid are involved in all Geometry. In both cases Kant has penetrated to what is obviously fundamental in Mathematics; it is immediately clear that all mathematical arguments, whether in Geometry or in Arithmetic, do ultimately depend upon an intuitive and synthetic groundwork. And if we admit that the most elementary processes in each branch of Mathematics are strictly synthetic, it is certain that this synthetic character will cling to all judgments that follow them. "For although a synthetic principle can certainly be discerned by means of the principle of Contradiction, yet this is possible only when another synthetical proposition precedes, from which it can be deduced." When the synthetical proposition that 'two straight lines do not enclose a space' is given, we can deduce directly from it, that two particular straight lines AC and AB, meeting at A, do not meet in any other point, but this conclusion, though analytically derived from our general principle, is none the less itself purely synthetical.

But before Kant could honestly formulate his comprehensive statement that 'mathematical judgments are one and all synthetic,' it was necessary that his view of the fundamental principles of Mathematics should be absolutely uniform, recognising no exceptions, and in 1781 it was just at this point that Kant failed. In the first edition of the *Critique* (p. 163) he deliberately styles a large number of Euclid's "Axioms" analytical, and inasmuch as it would be folly to exclude them from the category of mathematical judgments, he was fully justified in postponing the announcement of his final opinion. "As regards the quantity of a thing (*quantitas*)," he says in the passage referred to, "namely in answering the question, How great? we have at hand various propositions synthetical and immediately certain (*indemonstrabilia*), but we have in the proper sense of the term no Axioms. For example, the propositions, 'If equals be added to equals the wholes are equal,' 'If equals be taken from equals the remainders are equal,' are analytical, for I am immediately conscious of the identity of the production of the one quantity with the production of the other, whereas Axioms must be synthetic propositions *a priori*." Now this remark applies to no fewer than nine of Euclid's twelve axioms, and were we to judge of all of the axioms *de quantitate* by the two examples Kant gives, we might infer

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naturally enough, though perhaps rashly, that such axioms really were general principles applicable not only in Geometry and Arithmetic, but also, as we have already remarked, in all human reasoning, and therefore we might conclude that their proof was not completely attained if we attempted it mathematically. ' We might decide, and, I firmly believe, Kant's opinion in 1781 would support our decision, that Mathematics is here borrowing general principles proved analytically by Philosophy and applying them for her own particular purposes, not as axioms, for, as Kant states in his *Logic* (§ 36), "analytical propositions never are axioms (*axiomata*) but depend upon the identity of conceptions (*acroamata*)". But, as a matter of fact, if any one should try upon Kantian principles to defend Kant's statement, that we have no "axioms" *de quantitate*, we could at once deny this to be true within the sphere of Mathematics, by producing a positive instance of such an axiom, an instance which, so far as I see, admits of no doubt, namely the eighth axiom of Euclid: "Magnitudes which can be made to coincide are equal". Its very phraseology indicates that we are to superpose two magnitudes in pure intuition in order to see their equality; it certainly is an axiom *de quantitate* (I should say it is the fundamental axiom of quantity in Geometry), and no analysis of conceptions could ever produce it. Now I do not know what the test for Equality in general may be, I have serious doubt whether any such exist, but in Geometry I have a truly geometrical and spatial test of geometrical equality in the coincidence of superposed figures. And if a pure spatial intuition of the coincidence of two figures be the only means by which we can know them to be equal, then all axioms as to geometrical equality directly involve an intuition of this nature and therefore are like the eighth axiom synthetic. Indeed, whenever we compare any two geometrical figures, whether with regard to equality or inequality, we must in the end resort to superposition and an intuitive perception of their relation to each other, and for this reason I consider all axioms *de quantitate* to be perfectly synthetic.

But Kant's subsequent silence on this point makes it for ever impossible to discover whether such considerations had any real influence upon him or not. True, the passage we have opposed was neither obliterated nor altered in the second edition of the *Critique*, and yet I think there is in that work quite sufficient indication that Kant was well aware of the deficiencies of his earlier position. In the fifth paragraph of the Introduction, which appeared for the first

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time in the *Prolegomena* and was literally transferred from it to the *Critique*, Kant seems to repeat his former doctrine. "Some few fundamental principles," he says, "which are preposited by geometricians, are indeed really analytic and depend upon the law of contradiction, but they serve, after the fashion of identical propositions, only as links in the chain of method, and not as principles,—for example $a=a$, the whole is equal to itself; or $a + b > a$, the whole is greater than its part." Here the old examples are dropped and of the two substituted the latter alone is a recognised axiom of Euclid. There certainly is an axiom which runs, Things which are equal to the same thing are equal to one another; but I know of no geometrician who ever thought it necessary to preposit the law of Identity $a=a$ to his works. But let that pass and let us understand Kant to mean that two analytical propositions are actually used by mathematicians. Then read the next sentence, "And yet even these same propositions, although they are valid according to pure conceptions, *are admitted in Mathematics* only for the reason that they can be presented in intuition". Now these words mean, as plainly as words can, that in Mathematics these same principles are synthetic, which in the previous sentence were called analytic. I do not believe that Kant could quote the law of contradiction in one sentence and then proceed to break it with his next; some explanation of the disagreement must be found, and I think a sufficient explanation has already been shadowed forth in our own discussion of axioms *de quantitate*. The general statement of any of these "axioms" may be analytic, just because when formulated most generally it is intended to be applicable to all parts of experience, but in Mathematics it has its application confined to questions of mathematical magnitude, which, as we have said, always depend ultimately upon direct intuition. The logical or analytic proposition, the whole is greater than its part, may have a vast variety of different meanings under different circumstances, but in Mathematics it has a meaning it can have nowhere else. If in Geometry we wish to compare any two quantities as to their relative magnitude, they must be superposed and their equality or inequality intuitively perceived; before we can say that one quantity is greater than another, this same intuition must have preceded our judgment, that is to say, we must regard the two as superposed one upon the other. Thus it comes about that the proposition, $a + b > a$, is "admitted in Mathematics only for this reason that it can be presented in intuition".

Moreover the general analytical expression of this axiom

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could never be applied in Mathematics without leading at once to perfect absurdities. The circumference is 'a part' of the circle if we use the term 'part' in the general non-mathematical sense, and yet for geometry it is absolutely inane to say, that the circle is greater than the circumference, for a plane surface of two dimensions can never be superposed upon a curved line of one and therefore can never be compared to it in magnitude. If we borrow Kant's language concerning Space generally we might say, that a geometrical whole contains all its parts *in se, non sub se;* the parts of a line are lines and not points, a plane is divided by geometry into planes, a solid into solids. But the general or philosophic conception of a whole, from which is derived the analytic expression corresponding to Euclid's ninth axiom, contains its parts *sub se* and therefore the analysis of it can never give us a principle applicable as a geometrical axiom. The words 'whole' and 'part' have two quite distinct significations in Philosophy and Mathematics, while Mathematics has simply nothing to do with the philosophic meaning of 'greater' and 'less'. Accordingly in his next line Kant continues, "What commonly causes us to believe here, that the predicate of such apodeictic judgments is already contained in our conception, and that the judgment is therefore analytic, is merely the equivocal nature of the expression". Every word in Euclid's ninth axiom has a meaning peculiar to Mathematics and foreign to the ordinary usage, so that, unless we clearly distinguish the two applications of our one principle, we are apt to term it analytic in circumstances, under which its analytic meaning is at once false and absurd. And yet it is easy to see the danger of error, which must always confront the mathematician, when he expresses a geometrical axiom in words identical with those of a logical and analytical formula. "We must," says Kant, "join a certain predicate in thought to a certain given conception, and this necessity cleaves already to the conception. But the question is, not what we must join in thought to the given conception, but what we really think in conceptions though only obscurely. Then it becomes manifest that the predicate pertains to these conceptions, necessarily indeed, yet not immediately, but only by the mediation of an intuition, which must be added to the conceptions." In these axioms we use certain conceptions, which, to borrow a modern phrase, already have a certain connotation, but for purely mathematical purposes this connotation is quite valueless. Even though it be settled that a certain conception is necessarily joined to certain predicates, this junction

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may be effected either analytically or synthetically, that is, either immediately or mediately. Now when in Mathematics we ask the question, What do we really think in a conception? we must add an intuition to that conception, and apart from such an intuition no real mathematical judgment arises. Kant's meaning in the paragraph, from which we have quoted, I take to be perfectly plain. He has already pointed out that all mathematical judgments are synthetical with the exception of "certain principles which serve as links in the chain of method". But on a further examination of these apparently analytical principles, Kant finds that they are admitted in Mathematics only conditionally; they are not accepted simply because Philosophy claims to guarantee their validity from pure conceptions, they must from the first have their truth made self-evident by means of an intuitive representation of a particular sensible and concrete instance, just as is necessary in geometry for the acceptance of the axiom, that two straight lines cannot enclose a space. Inside Mathematics these axioms are synthetic and their analytic appearance is a mere delusion brought about by the fact that the words, in which they are expressed, have another distinct and non-mathematical interpretation that really is analytical. And thus in 1783 the universal affirmative proposition is possible for the first time: "All mathematical judgments, without exception, are synthetical," for the former supposed exception is now finally removed.

At this point we intend to discuss shortly a few of the criticisms that have been passed upon Kant's view of mathematics, not so much because we wish to defend Kant's doctrine in a way which must necessarily lead us beyond the study of Kant's own writings, but because so many of these criticisms seem to imply a different interpretation from ours of the passages which discuss axioms *de quantitate*. We have neither time nor inclination to treat of those arguments which object to Kant's doctrine of the synthetic nature of mathematical definitions, and by producing particular instances seek to cast doubt upon his whole theory. The statement, that the conception of a triangle contains the mark of having three sides, is a perfect type of them, and I confess it is difficult to regard such arguments seriously. I wonder how often in his works Kant lays down the maxim that Mathematics proceeds by construction of conceptions, always assuring itself by intuition of the possibility of the object to which they correspond. Without hoping ever to convince any one, to whom such an argument as that mentioned above seems conclusive, I would merely suggest for

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his consideration a true example of a proposition, which is analytical in Mathematics, namely, "A Duangle is a plane figure bounded by two straight lines"; and I would ask what part of Mathematics this definition occurs in? But in those criticisms of Kant's doctrine which discuss the nature of axioms *de quantitate* we have to deal with objections of quite another class. These arguments have their weight considerably increased by the fact that Kant's own words can so often be quoted in such a way as apparently to support them. Indeed we might safely say that it is universally considered to be Kant's final opinion of axioms *de quantitate* that they are analytic, and of course it is then easy to deny the statement that mathematical judgments are always synthetical. But such a view can only be held by those who are prepared either to ignore or explain away the glaring contradiction we have pointed out. One well-known commentator[1] has published a theory that the sentences in which Kant in our view of the matter declares the mathematically synthetic nature of axioms *de quantitate* have no immediate reference to those axioms. Therefore there must be a displacement of the order of this paragraph due either to a printer's error or to the author's negligence. It is supposed that Kant agrees to consider axioms *de quantitate* as analytic, and that the sentences which seem to contradict this view were originally penned in some quite different connexion from that in which they now stand. We have already tried to point out that such an artificial explanation is unnecessary and would indeed leave the opening statement of the general nature of mathematical judgments ill-founded. Moreover a displacement of a paragraph, so serious as that suggested here, might perhaps have been possible on one occasion, but that an author should have copied the same section literally into a work four years later, without correcting its error, seems to me an untenable hypothesis. The argument must be taken as it stands and we venture to claim that our explanation of it is the only one possible.

Of the many particular instances with which critics seek to confirm or illustrate their interpretation of Kant's views on those axioms we shall take only one example and endeavour to show, on mathematical grounds, that it is a fallacy. It is said that the Law of Identity, a thing is equal to itself, is largely used in Geometrical proofs after the fashion of an

[1] The objection here is not to Prof. Vaihinger's theory of "a leaf-displacement in the Prolegomena" but to the use made of the theory in explaining this passage in his "Commentar," vol. i.

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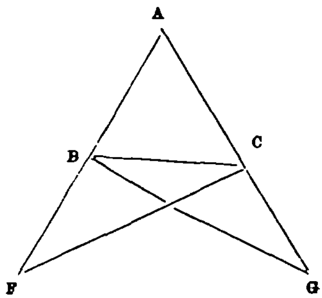
analytical proposition. And the illustration given is the use of an angle, which being "common to two triangles," often helps us to prove them equal in every respect. Now in Euclid the first case of such procedure occurs in the proof of the fifth proposition, where we have the following argument. (We presuppose Euclid's construction.)

In the triangles FAC, GAB,
FA = GA,
and AC = AB,
and the angle A is common to the two triangles,



therefore they are equal in every respect.

Here the conclusion follows directly from the fourth proposition which proves the equality of two triangles by superposition. But it must be noticed that, when we superpose the triangle FAC upon the triangle GAB, we must place the angle BAC in the entirely new position CAB. In fact we must invert the angle or, putting it with empirical crudeness, we must turn the paper over, and then we see that the angle is not changed. I should distinctly deny it to be an analytic proposition, following from the law of Contradiction, that the angle A is equal to itself inverted in this way.[1] That A = A inverted is a proposition eminently synthetic but in Geometry undoubtedly true. While no geometrician would ever think of denying the Law of Identity, yet I would assure any of Kant's critics that an instance of an identical proposition openly formulated in Mathematics cannot be found.

In his *Doctrine of Method* Kant lays down that "the ground-evidence of Mathematics depends upon Definitions, Axioms and Demonstrations"; and we have traced the gradual recognition by Kant of the synthetic and intuitive nature of all three of these 'moments' of Mathematics. It is true that Kant appears rather to neglect the Arithmetical part of Mathematics, preferring to discuss the nature of

[1] A clear apprehension of this fact would go far towards removing the objection commonly felt by mathematical purists to the simple proof of the fifth Proposition which is obtained by inverting the triangle BAC and superposing it upon itself without any further construction.

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geometrical procedure; yet, when need be, he is able to draw a good analogy between the science of Space and the science of symbols. "By means of a symbolical construction of quantity," he says, "Algebra arrives at results, which discursive cognition cannot hope to reach by the aid of mere conceptions, and in this way it resembles Geometry with its ostensive and geometrical construction (of the objects themselves)." And if there be any inequalities in Kant's analysis of Algebra and Geometry, it is perfectly clear that to the author himself they seemed unimportant. He speaks of his estimate of mathematical judgments as if he knew it to be true of Mathematics throughout and honestly proved in the case of Definition, Axiom and Demonstrated Proposition, in Geometry, in Arithmetic and Algebra alike. The various distinct steps of this proof we have already mentioned in detail, the conclusion Kant draws is summed up in the words "Mathematical judgments are one and all synthetic"; we have decided that not a single type of mathematical judgments was left out of Kant's consideration, and we only wish that, when critics examine the fitness of the premises to produce Kant's conclusion, they would abandon the old fallacy that the view taken of Mathematics in the *Critique* is carelessly partial or incomplete.

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