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Seven-Set Venn Diagrams with Rotational and Polar Symmetry

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A class of completely symmetric simple Venn diagrams for seven sets is constructed and displayed.

The geometrical properties of Venn diagrams have long attracted attention in their own right. Venn himself observed [9, 10] that although it was impossible to construct a four-set diagram with congruent circles, a diagram was possible with congruent ellipses, and Grünbaum [5] exhibited a five-set diagram also consisting of congruent ellipses. This diagram showed that more than one kind of simple Venn diagram exists for five sets, since the four-set figure which results on removing any one of the ellipses is not itself a Venn diagram. (By a simple Venn diagram we shall follow Grünbaum and mean one in which every intersection involves only two set boundaries; the removal of this restriction has been studied by, *inter alia*, Edwards and Smith [4].)

The present paper is concerned with simple Venn diagrams exhibiting certain symmetries which are best described if the diagrams are thought of as existing on the surface of a sphere, as suggested to Venn by H. J. S. Smith (see [2]). A diagram is then said to possess n-fold rotational symmetry if it is unchanged by a rotation of $2\pi/n$ radians about the axis of the sphere, and polar symmetry if the hemispheres (north and south) are congruent. When projected into the plane by a stereographic projection from one of the poles, which is the form in which the results of this paper are presented, the rotational symmetry is preserved whilst the polar symmetry becomes a kind of inside-out symmetry. We shall refer to diagrams that possess both rotational and polar symmetry as *completely symmetric*. We here present six completely symmetric simple Venn diagrams for seven sets.

The idea that rotationally symmetric Venn diagrams might exist for prime numbers of sets is due to Henderson [8] and arises because of Leibniz's divisibility theorem that every binomial coefficient $\binom{n}{r}$ in the n-row of Pascal's triangle, except the terminal 1s $\binom{n}{0}$ and $\binom{n}{n}$, is divisible by n if and only if n is prime. For the history of this theorem see [1]. Now a Venn diagram for n sets is a mapping of the 2^n ways in which r things can be selected from n different things, for $r = 0$ to n. The n sets correspond to the n things, the interior of a set indicating the selection of the corresponding thing and the exterior its omission. Each of the 2^n regions of the diagram consequently corresponds to the inclusion of those things for whose sets it is part of their interior, and to the exclusion of the rest. The single area which is internal to all the n sets, corresponding to the selection of all n things, is therefore surrounded by a 'necklace' of $\binom{n}{n-1}$ regions or 'beads' which correspond to the selections of $n-1$ of the n things, since crossing any set boundary is equivalent to leaving out the corresponding thing. By a similar argument, this '$\binom{n}{n-1}$' necklace might be able to be surrounded by a necklace of $\binom{n}{n-2}$ beads, and so on through the binomial coefficients until the single area corresponding to the inclusion of none of the things is reached.

It follows from Leibniz's theorem that if n is prime, and only then, each necklace corresponding to the inclusion of r things might be able to be arranged in a string of $\binom{n}{r}/n$ beads repeated n times. Since this might be true for all r from 1 to $n-1$, it is possible that a simple Venn diagram with n-fold rotational symmetry could thus be constructed. Such a diagram would evidently consist of n congruent sets, each rotated $2\pi/n$ with respect to its predecessor. In addition, polar symmetry might be possible. These observations neither guarantee the existence of such completely symmetric diagrams for prime numbers of sets, nor exclude the possibility of their construction by other means. It is, however, immediately obvious that Venn's three-circle diagram in the plane is such a completely symmetric diagram and that it is the only solution for $n = 3$; on the sphere it corresponds to an equator and two meridians at an angle of $\pi/2$ to each other (see [2]). It may also easily be shown that, when suitably drawn, Grünbaum's [5] five-set diagram is the only such solution for $n = 5$, though Henderson [8] had already exhibited two rotationally symmetric five-set diagrams that were not simple.

Attention now turns to the existence of rotationally symmetric seven-set diagrams. Henderson [8] claimed to have found a solution using seven irregular hexagons, but did not publish it, and by 1975 Grünbaum [5] had come to doubt the possibility. However, in October 1992 he himself published [6] two solutions for seven sets found early in that year by trial and error, though they lacked polar symmetry. The first of these interesting diagrams does not conform to the above 'necklace' construction, for it contains triangular regions that are *outside* the three sets whose lines form their boundaries, so that if the triangular region itself is inside r sets, say, the three regions accessed by passing through the 'revolving doors' at the vertices are each inside $r+2$ sets, leaving the r-set triangular region isolated and not part of an r-set necklace. The task of finding others of this kind seems daunting. Grünbaum [7] had also found a third diagram early in 1992, which he did not publish, but which will be described below. In addition he informs me that in 1993 he received a different seven-set symmetric diagram from C. Savage and P. Winkler which they had found in August 1992. Of these, Grünbaum's three were the first to be discovered (unless Henderson's diagram should be substantiated in the future).

I was unaware of the discovery of these diagrams when, on a visit to the University of Adelaide in November 1992, I embarked on a programme to seek a completely symmetric simple seven-set Venn diagram by making use of the idea of necklaces described above. The number of set intersections needed in each ring of intersections is determined by the binomial coefficients, and by permuting the order in which they occur it is possible to generate candidate diagrams. In this way I found a solution by hand which I christened the *Adelaide Rose Window* and which was published in the *Adelaidean* of December 14th 1992 (Vol 1 No 11 of the University newspaper). Continuing to work by hand I found a second solution, also in December 1992, which I christened *Hamilton* after the New Zealand town in which I was then working [3]. Grünbaum's third diagram, mentioned above, turned out to be isomorphic to *Adelaide*, so, during 1992, five seven-set symmetric Venn diagrams had been discovered, two of them being completely symmetric, one of these having been independently discovered twice.

Completion of a computer program that generates all the possible permutations of the set intersections has since revealed four further completely symmetric diagrams, *Palmerston North*, *Massey* and *Manawatu*, named after the New Zealand town, university and province in which they were all found, and *Victoria*, named after the capital of British Columbia, Canada. *Victoria*, though found by the computer program on its original run, was inadvertently omitted from the final list which was compiled by hand, and was rediscovered by F. Ruskey of the University of Victoria. The six diagrams are given in Figure 1. Copies of the computer program may be obtained from the author.

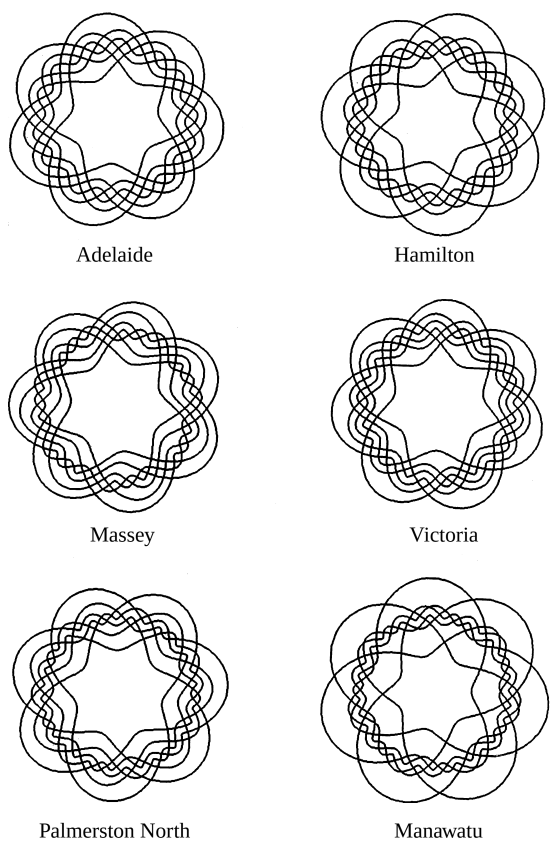

Figure 1

It may be noted that each r-necklace in each n-diagram defines an ordering of all the $\binom{n}{r}$ selections of n things taken r at a time, thus generating further 'revolving-door' algorithms of combinatorial theory which may be expected to be distinct from each other in general and from the revolving-door algorithm defined by the normal Venn diagram for n sets (see [2]).

Nothing in this account precludes the possibility of there being completely symmetric simple Venn diagrams for prime numbers of sets which do not conform to the necklace pattern, but the enumeration here reported was exhaustive for that pattern. The number of simple Venn diagrams of the various kinds for a specified number of sets remains an open question, as does the possibility of constructing a symmetric example for $n = 11$, or for prime numbers in general. The computer program is currently being extended to attack the case of $n = 11$ as well as the seven-set case without polar symmetry.

Acknowledgements

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References

[1] Edwards, A. W. F. (1987) *Pascal's Arithmetical Triangle*, 103. Charles Griffin, High Wycombe, and Oxford University Press, New York.

[2] Edwards, A. W. F. (1989) Venn diagrams for many sets. *New Scientist* **121** 51–56.

[3] Edwards, A. W. F. (1994) The sevenfold symmetric Venn diagrams. Hamilton, Ontario: *XVIIth International Biometric Conference Proceedings* **2** 238.

[4] Edwards, A. W. F. and Smith, C. A. B. (1989) New 3-set Venn diagram. *Nature* **339** 263.

[5] Grünbaum, B. (1975) Venn diagrams and independent families of sets. *Math. Mag.* **48** 12–22.

[6] Grünbaum, B. (1992) Venn diagrams II. *Geombinatorics* **2** 25–32.

[7] Grünbaum, B. (1992) Personal communication.

[8] Henderson, D. W. (1963) Venn diagrams for more than four classes. *Amer. Math. Month.* **70** 424–426.

[9] Venn, J. (1880) On the diagrammatic and mechanical representation of propositions and reasonings. *London, Edinburgh and Dublin Phil. Mag.* (*5th series*) **9** 1–18.

[10] Venn, J. (1881) *Symbolic Logic*, 106. Macmillan, London.