A NEW DRAWING FOR SIMPLE VENN DIAGRAMS BASED ON ALGEBRAIC CONSTRUCTION

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Abstract. Venn diagrams are used to display all the relations between a finite number of sets. Recent research in this field deals with the mathematical aspects of these constructions, but are not directed towards the readability of the diagram. This article presents a new way to draw easy-to-read Venn diagrams, in which most of the regions tend to be drawn with the same size when the number of sets grows, and fit in a grid. Finally, using linear algebra, we prove that this construction yields a simple Venn diagram for any number of sets.

1 Introduction

As teachers, we use Venn diagrams with three curves to teach our students bases of the set theory. (Un)fortunately, because some students are more curious than others, we were asked to draw Venn diagrams with four or five sets. Here is now the problem: such diagrams cannot be drawn using circles. Some other constructions exist, using ellipses or other curves with no particular shape, but they have the same flaw: they are not really easy to read and to draw on the blackboard. The regions do not all have the same size and it can be hard to determine whether a region is included in a given set or not.

We became interested in the construction of an easy-to-read Venn diagram with more than four sets. The Carrol-Lewis diagram [3] persuades us because of its simplicity and its ease to be drawn on a blackboard. We present in this paper a construction which extends it to any number of sets.

Further research led us to the construction of Anderson and Cleaver [1] which was very close to ours. However, this construction seemed not to respect the actual definition of a simple Venn diagram and seemed to be more complex to read. Thus, we investigated a way to prove that our construction is valid in an algebraic way, and we proposed a study to verify which diagram was the easiest to read.

Let us now introduce some common definitions about Venn diagrams. We naturally begin with the formal definition of an n-Venn diagram. This definition is drawn from [13], itself based on [8].

Definition 1 (n-Venn Diagram)**.** An *n-Venn diagram* is a set $\mathcal{C} = \{C_0, C_1, \dots, C_{n-1}\}$ of simple closed curves intersecting at a finite number of points in the plane such that each of the 2^n set $\bigcap_{i=0}^{n-1} X_i$ is a non-empty connected set, with X_i denoting the interior or exterior of the i-th curve. Each intersection $\bigcap_{i=0}^{n-1} X_i$ is said to be a *region* of the diagram.

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An n-Venn diagram is said to be *extensible* if it is possible to add a curve giving rise to an $(n+1)$-Venn diagram. If this process can be repeated infinitely, the diagram is said to be *infinitely extensible*. A Venn diagram is *simple* if no more that two curves intersect at any point. Finally, a *symmetric* n-Venn diagram remains unchanged when applying a rotational symmetry of $2\pi/n$.

Following Definition 1, we present a new method to draw an n-Venn diagram whose main features are the following.

- It produces a simple Venn diagram.
- An algorithm allows construction for any number of sets.
- The areas of each region tend to be equal when the number of sets grows.
- The validity of this construction is proven using linear algebra.
- It is easy to identify if a region is included in a given set.

The rest of this paper is organized as follows. The next section exposes the existing constructions which are valid for any number of sets. Our construction is then explained by drawing examples for five, six and seven sets. The validity of the construction is proven in Section 2. The algorithm is detailed in Section 3. The readability study is given in Section 4.2, then Section 5 concludes this article.

1.1 Previous Work

The first construction of a Venn diagram was introduced by John Venn in [15] and is infinitely extensible. A few years later, Lewis Carroll introduced his diagram, up to ten sets [3, pp 178]. However, this kind of construction cannot be considered as a Venn diagram because some sets are not connected. Almost a hundred years later, another infinitely extensible construction of a simple diagram was proposed by Edwards [6]. The two infinitely extensible constructions are presented in Figure 1 for five sets.

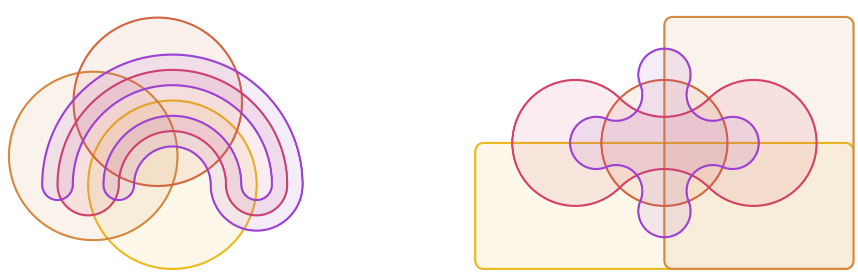

Figure 1: Venn's original diagram and Edwards' construction for five sets.

Other constructions employ only triangles [8] or ellipses (found by Venn in [15]), but they are valid only for a few values of n. Researchers have proven the existence of symmetric

11-Venn [9] and simple symmetric 13-Venn diagrams [11]. This last kind of construction is today a major interest for the scientific community because of the beauty of such diagrams. It is also worthwhile to note that Griggs et al. [7] proved that symmetric n-Venn diagrams exist if and only if n is a prime number. The implication was first addressed by Henderson in [10] and was further corrected in [16]. For further information about this kind of diagram, we invite the reader to refer to [13], which conducts a complete survey of Venn diagrams.

Again, these constructions cannot support an arbitrarily large finite number of sets. Now we focus on methods to create a simple Venn diagram for any number of sets. The most famous diagrams are probably the original Venn's and Edwards', represented in Figure 1. These constructions seem to be the only ones which produce a simple n-Venn diagram for any value of n.

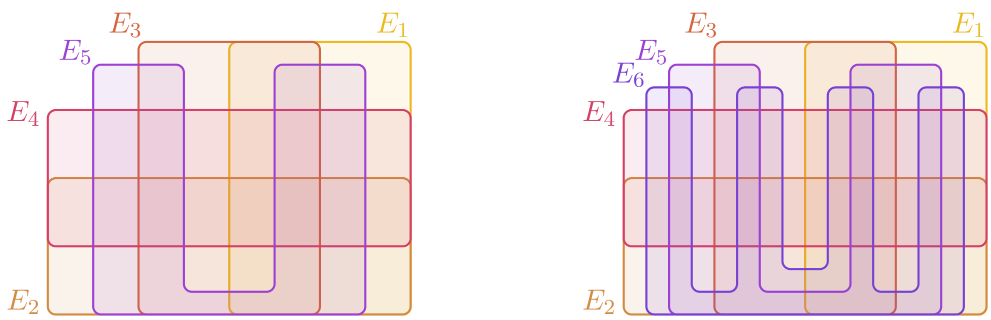

Figure 2: Anderson's construction for five (left) and six sets (right).

The construction most relevant to our work is the Anderson and Cleaver's [1] represented in Figure 2. As recommended by Venn, the authors created an inductive method for dividing each region into two new ones. The actual definition of a Venn diagram states that the curves delimiting the sets must intersect at a finite number of points which is in contradiction with this construction. Even if it seems obvious that an expansion of the sets would produce a valid Venn diagram, no proof was provided in their paper.

1.2 Informal Construction

Let us present our construction based on the Carroll diagram, represented in Figure 3 (left diagram). We extend this diagram for any number of sets and the resulting Venn diagrams are simple. It should be stressed that the Carroll construction is not exactly a Venn diagram because some parts of the curves run concurrently. It is also obvious that this problem can be solved by applying a small shift to these sets in order to avoid the combined lines as presented in Figure 3. The rest of this section details how the 5, 6 and 7-Venn diagrams are obtained using our method.

First, let us specify that to create an n-Venn diagram using this new method, the base is the same for all $n > 4$, i.e. the Carroll construction. To divide each region into two new regions, the set E_5 must be drawn over the Carroll construction. To do this, two new vertical rectangles are drawn aligned on the vertical sides of E_3, as presented in the left part

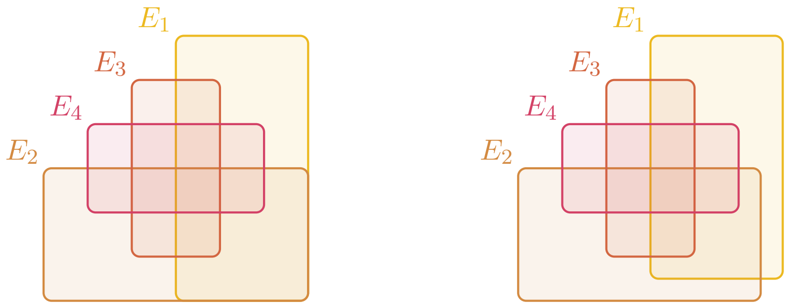

Figure 3: Carroll's construction for four sets (left), and the shifted one (right), resulting in a valid Venn diagram.

of Figure 4. The length of these rectangles must stay within the bounds of the previous diagram, and must be greater than the width of E_4. Thirty-two regions are now displayed, all distinct from each other. But this construction is not yet a Venn diagram because E_5 is disconnected. Thus, the two rectangles are connected, using a rectangle drawn from the bottom of E_5 to a line located between the first and the second sets encountered, as presented in the right diagram of Figure 4. The construction is now a valid 5-Venn diagram.

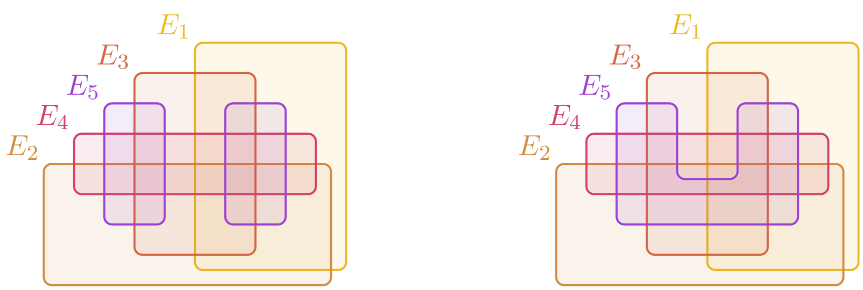

Figure 4: Our final construction for five sets (right) with the transition step (left).

It is worth noting that using a similar method to add a 6-th set horizontally around E_4 does not yield a 6-Venn diagram. Consequently, we start again from the transition step in Figure 4. The set E_6 is then represented by two horizontal rectangles aligned on the horizontal edges of E_4, as depicted in Figure 5. Then, the link is made as explained in the previous paragraph. This drawing now respects the formal definition of a 6-Venn diagram and is given on the right of Figure 5.

To draw the 7-Venn diagram, we start from the temporary six-sets diagram presented in the left of Figure 5. Divide each region into two new ones by adding four rectangles aligned on the vertical edges of E_5. Link this new set using the previous method. Finally, link the sets E_6 and E_5 as detailed previously to obtain the final diagram presented in Figure 6. Applying the same method leads to a generic construction for any number of sets.

This section has detailed the informal construction to help the reader understand the creation of the diagram. We have to point out that this method is not inductive because

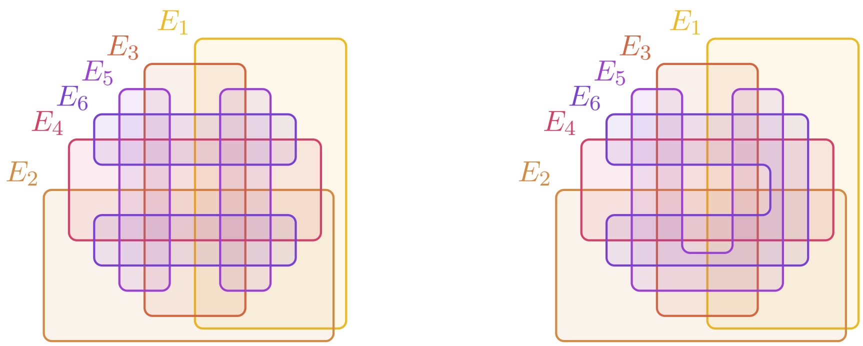

Figure 5: Our final construction for six sets (right) with the transition diagram (left).

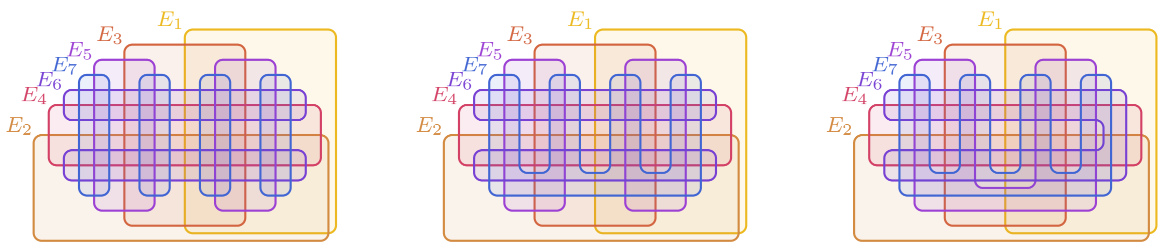

Figure 6: Our final construction for seven sets (right) with the transition diagrams (left and middle).

the n-Venn diagram is not derived from the $(n-1)$-Venn diagram. Nevertheless, that does not mean that the diagram produced is not extensible. In particular, it is still possible to divide each region of an n-Venn diagram by adding a new set using the method proposed by Venn. But because any n-Venn diagram can be drawn using our algorithm, the next section proves the validity of this construction. To the best knowledge of the authors, such a proof is the first one in the algebraic field for Venn diagrams. Firstly, some notations are introduced before explaining why these operations induce a valid construction.

2 Formal Proof of Our Construction

The goal of this section is to prove that our construction leads to a simple Venn diagram. To this end, we consider in a first part only the center of the diagram called the *central grid*, as presented in Figure 7 (used throughout this section to illustrate notations of the proof). We will prove that this part contains all possible regions of a Venn diagram only once. In this part, some sets are disconnected, and no vertical sets have common vertical borders (thus, the diagram is simple). We will next prove that this construction can be extended to avoid overlapping borders. Finally, each set is connected to give rise to a valid Venn diagram. But first, to explain the construction of the central grid, only the vertical rectangles (representing the odd sets) are considered and our proof details why it is sufficient to consider such a representation of the central grid.

We provide now an algebraic framework for our diagram. Let us begin with some notations. If x and y are vectors, $x \parallel y$ denotes the concatenation of x followed by y. The notation $\mathbb{F}_2$ denotes the Galois field of order two. Vectors are indexed in decreasing order, which means that vector x composed of n components is indexed as $(x_{n-1}, \ldots, x_1, x_0)$. The vertical set of index s is denoted by $\mathcal{V}_s$, and similarly, the horizontal set of index s is denoted by $\mathcal{H}_s$.

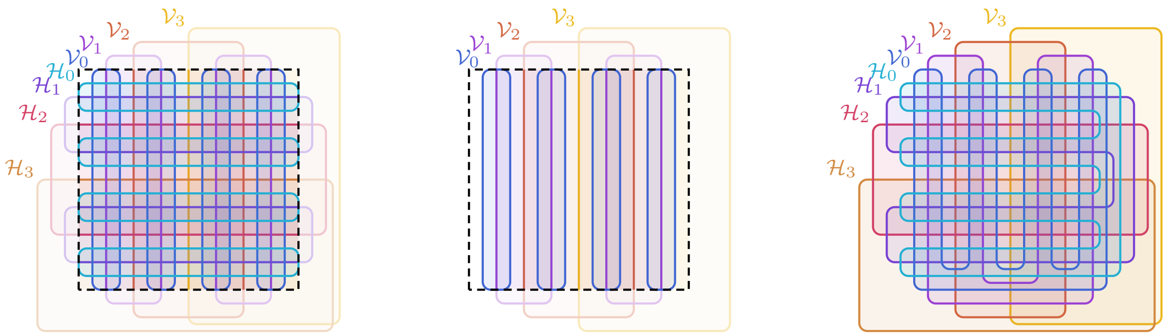

Figure 7: The central grid highlighted in the 8-Venn transition diagram (left) and the final 8-Venn diagram.

2.1 Vertical Sets of the Central Grid

In the remainder of this section, let n and m denote two fixed integers representing the number of vertical and horizontal sets respectively. Suppose the diagram is drawn on a grid, where each square region in the central grid is represented by a unit square, as displayed on the left of Figure 7. This grid is made up of 2^m rows of 2^n unit squares. Now we focus only on the vertical sets, as displayed in the middle of Figure 7. The 2^m rows of this new grid are all equal. Therefore, we need only to consider its first row, represented as a vector of 2^n components, without losing any information.

To represent the presence of $\mathcal{V}_s$ in position x of this row, we define the vector L_s of $\mathbb{F}_2^{2^n}$. We set $(L_s)_x = 1$ if the set $\mathcal{V}_s$ is present in position x, and $(L_s)_x = 0$ if not. Vectors L_s are then combined to form a final integer vector in which bit s of component x is $(L_s)_x$. Note that the sets are indexed in the "reverse" order, that is to say the set containing the most vertical rectangles is indexed 0, the following 1 etc. This indexing has been chosen only for the beauty of the following proofs.

We will now prove that the integer vector of size 2^n representing the first row of the matrix contains all possible values between 0 and $2^n - 1$. In other words, each possible region of the Venn diagram is represented on this row if only the vertical sets are considered. To this end, let us consider all the integers from 0 to $2^n - 1$ and arrange them in their binary form, in increasing order, following Figure 8. Let us observe that the rows labeled x_3 and L_3 are equal. Then, the row L_s is the XOR of the rows x_s and x_{s+1}. This observation explains the following definition.

$$
\begin{array}{c|cccccccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
x_2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
x_1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
x_0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}
$$

$$
\begin{aligned}
L_3 &= (\,0\;0\;0\;0\;0\;0\;0\;0\;\boxed{1\;1\;1\;1\;1\;1\;1\;1}\,) \\
L_2 &= (\,0\;0\;0\;0\;\boxed{1\;1\;1\;1\;1\;1\;1\;1}\;0\;0\;0\;0\,) \\
L_1 &= (\,0\;0\;\boxed{1\;1\;1\;1}\;0\;0\;0\;0\;\boxed{1\;1\;1\;1}\;0\;0\,) \\
L_0 &= (\,0\;\boxed{1\;1}\;0\;0\;\boxed{1\;1}\;0\;0\;\boxed{1\;1}\;0\;0\;\boxed{1\;1}\;0\,)
\end{aligned}
$$

Figure 8: Representation of the presence of each set $\mathcal{V}_s$ on the first row of the central grid.

Definition 2. For each s in $[\![0, n[\![$, define the element L_s of $\mathbb{F}_2^{2^n}$ for all x in $[\![0, 2^n[\![$ by

$$\text{if } s < n-1,\ (L_s)_x = x_s \oplus x_{s+1} \quad \text{and} \quad (L_{n-1})_x = x_{n-1}\ .$$

Let us define the following linear mapping over the finite field $\mathbb{F}_2$

$$
\begin{aligned}
\varphi : \mathbb{F}_2^n &\longrightarrow \mathbb{F}_2^n \\
(x_{n-1}, \ldots, x_0) &\longmapsto (x_{n-1}, x_{n-2} \oplus x_{n-1}, \ldots, x_0 \oplus x_1)\ .
\end{aligned}
$$

By definition, for any index x in $[\![0, 2^n[\![$ the equality $\varphi(x) = ((L_{n-1})_x, \ldots, (L_0)_x)$ holds. Thus, φ maps x to the region value at index x.

$$
\begin{array}{rcccccccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
(\varphi(x))_x = (& 0 & 1 & 3 & 2 & 6 & 7 & 5 & 4 & 12 & 13 & 15 & 14 & 10 & 11 & 9 & 8 \ )
\end{array}
$$

Figure 9: The first row of the central grid.

Example 1. Consider as an example the column 5 of Figure 8. The component $(L_3)_5$ is 0 which means $\mathcal{V}_3$ is not present in the associated cell of the first row of the central grid (and by extension absent from the column 5). The binary vector $((L_3)_5, (L_2)_5, (L_1)_5, (L_0)_5) = (0, 1, 1, 1)$ can thus be considered as the binary decomposition of the integer 7. This representation is presented in Figure 9 and helps to visualize the notations of Theorem 1.

Each region r of the central grid can thus be represented using a positive integer in which bit s is 1 if and only if the region belongs to set $\mathcal{V}_s$. The following theorem states that each region is represented once and only once in the first row.

Theorem 1. *For any region index r in $\mathbb{F}_2^n$, there exists a unique index x in $[\![0, 2^n[\![$ such that $(L_s)_x = r_s$ for all $0 \le s < n$.*

Proof. Let r be an element of $\mathbb{F}_2^n$. We have to prove that there exists a unique x in $[\![0, 2^n[\![$ such that $\varphi(x) = r$. This index is $\varphi^{-1}(r)$ since φ is an isomorphism. $\square$

Now, we have to prove that such a construction of the sets in the central grid leads to drawing sets with non overlapping vertical edges. The Vertical edges of $\mathcal{V}_s$ can be represented by a transition of bit values in the row L_s as presented in Figure 8. Thus, if two sets $\mathcal{V}_s$ and $\mathcal{V}_t$ have a common vertical border between the indices x and $x+1$, bit values $(L_s)_x$ and $(L_s)_{x+1}$ should be different, and so are $(L_t)_x$ and $(L_t)_{x+1}$.

It should be stressed that when x runs through the interval $[\![0, 2^n[\![$, the bit x_s switches between 0 and 1 every 2^s steps. It follows from this observation that for all x in $[\![0, 2^n[\![$, $x_s \neq x_{s+1}$ if and only if $x+1 \equiv 0\ [2^s]$. Let t be such that $t \leq s$. As 2^t divides 2^s, $x_s \neq (x+1)_s$ implies that $x_t \neq (x+1)_t$.

Theorem 2. *Let $s > t$ be two elements of $[\![0, n[\![$. There exists no index x in $[\![0, 2^n - 1[\![$ such that $(L_s)_x \neq (L_s)_{x+1}$ and $(L_t)_x \neq (L_t)_{x+1}$.*

Proof. By contradiction, suppose that there exists x in $[\![0, 2^n-1[\![$ such that $(L_s)_x = (L_s)_{x+1} \oplus 1$ and $(L_t)_x = (L_t)_{x+1} \oplus 1$. There are two cases.

- If $s = n-1$, then $x_{n-1} = (x+1)_{n-1} \oplus 1$.
- If $s < n-1$, then $x_s \oplus x_{s+1} = (x+1)_s \oplus (x+1)_{s+1} \oplus 1$. It must be the case that either $x_s \neq (x+1)_s$ or $x_{s+1} \neq (x+1)_{s+1}$.

Since $t < t+1 \leq s < s+1$, the previous discussion implies that $x_t = (x+1)_t \oplus 1$ and $x_{t+1} = (x+1)_{t+1} \oplus 1$. Thereby, $x_t \oplus x_{t+1} = \big((x+1)_t \oplus 1\big) \oplus \big((x+1)_{t+1} \oplus 1\big) = (x+1)_t \oplus (x+1)_{t+1}$. In other words, $(L_t)_x = (L_t)_{x+1}$. This is a contradiction, the result follows. □

2.2 Set Extension

Until now, the vertical sets do not have any overlapping vertical edges, but their horizontal border may be combined. The set extension avoids this problem. We first define how the vertical sets are extended. Let us consider the last row of the central grid and a particular vertical set denoted $\mathcal{V}_s$. As explained in the introduction of Section 2.1, all the rows are equal. A downward extension can thus be considered as multiple copies of this row. The number of copies is exactly the index s of this set. These copies are then placed in a matrix denoted M_s. The same process can be symmetrically applied above the central grid. We can now define such a construction more formally.

Definition 3. For each s in $[\![0, n[\![$, let us define the $n \times 2^n$ matrix M_s where the row of index i in $[\![0, n[\![$ is given by

$$(M_s)_i = \begin{cases} L_s & \text{if } i \leq s \ , \\ 0_{2^n} & \text{if } i > s \ . \end{cases}$$

Example 2. The set labeled 1 is extended vertically using one copy of the row L_1 previously defined. The produced matrix M_1 is then completed by zeros to form a 4×2^4 matrix. Each matrix M_s is given in Figure 9.

$$M_3 = \begin{array}{c} \begin{array}{cccccccccccccccc} 0&1&2&3&4&5&6&7&8&9&10&11&12&13&14&15 \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{pmatrix} \end{array} \quad M_2 = \begin{array}{c} \begin{array}{cccccccccccccccc} 0&1&2&3&4&5&6&7&8&9&10&11&12&13&14&15 \end{array} \\ \begin{pmatrix} 0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

$$M_1 = \begin{pmatrix} 0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0 \\ 0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \quad M_0 = \begin{pmatrix} 0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

Figure 10: Set extension.

For each i in $[\![0, n[\![$, we define the following linear mapping

$$\begin{aligned} \varphi_i : \mathbb{F}_2^n &\longrightarrow \mathbb{F}_2^n \\ (x_{n-1}, \dots, x_0) &\longmapsto (x_{n-1}; x_{n-2} \oplus x_{n-1}, \dots, x_i \oplus x_{i+1}) \parallel 0_m \ . \end{aligned}$$

Example 3. Consider as an example the row 1 (starting from 0), column 6 of each matrix in Figure 10. This value states that set $\mathcal{V}_3$ is absent on this cell, like $\mathcal{V}_1$ and $\mathcal{V}_0$, whereas $\mathcal{V}_2$ is present. The binary vector extracted from these four bits is thus

$$((M_3)_{1,6}, (M_2)_{1,6}, (M_1)_{1,6}, (M_0)_{1,6}) = (0, 1, 0, 0).$$

This vector can be considered as the binary decomposition of the integer 4, as displayed in Figure 11.

$$\big(\varphi_i(x)\big)_{i,x} = \begin{array}{c} \begin{array}{cccccccccccccccc} 0&1&2&3&4&5&6&7&8&9&10&11&12&13&14&15 \end{array} \\ \begin{pmatrix} 0&1&3&2&6&7&5&4&12&13&15&14&10&11&9&8 \\ 0&0&2&2&6&6&4&4&12&12&14&14&10&10&8&8 \\ 0&0&0&0&4&4&4&4&12&12&12&12&8&8&8&8 \\ 0&0&0&0&0&0&0&0&8&8&8&8&8&8&8&8 \end{pmatrix} \end{array}$$

Figure 11: Superposition of the matrices.

The mapping φ_i plays the same role for the M_s as φ for the L_s. Using linear algebra, it can be proven that $\mathrm{Im}(\varphi_i) = \mathbb{F}_2^{n-i} \times \{0_i\}$ and $\ker \varphi_i = \{0_{n-i}\} \times \mathbb{F}_2^i$. The first isomorphism theorem for vector spaces states that the quotient space $\mathbb{F}_2^n / \ker \varphi$ is isomorphic to $\mathrm{Im}\, \varphi_i$ by the application $\overline{\varphi_i}$ which maps the coset $x \oplus \ker \varphi_i$ to $\varphi_i(x)$. Let $\mathrm{Cl}_i(x)$ denote the coset $x \oplus \ker \varphi_i$ of an element x of $\mathbb{F}_2^n$. This theorem implies that φ_i is constant on each coset and associates different values in different cosets. Finally, it is worthwhile to notice that $\mathrm{Cl}_i(x) = [\![\bar{x}, \bar{x} + 2^i[\![$ with $\bar{x} = \sum_{k=i}^{n-1} x_k 2^k$. Thus, $\mathrm{Cl}_i(x)$ is an integer interval.

Until this point, the area of each region of the Venn diagram was only a unit square. Now, the regions are extended and some of them are composed of multiple unit squares. We have to prove that such regions are connected. To this end, we introduce some other notations. For each r in $\mathbb{F}_2^n$, we define

$$\mathcal{R}(r) = \{(i, x) \in [\![0, n[\![\times [\![0, 2^n[\![\mid \varphi_i(x) = r\} \ .$$

In the following, we also consider the subsets

$$\mathcal{R}_i(r) = \mathcal{R}(r) \cap (\{i\} \times \mathbb{F}_2^{2^n})\ , \quad \mathcal{R}_{<i}(r) = \bigcup_{0 \le j < i} \mathcal{R}_i(r) \quad \text{and} \quad \mathcal{R}_{\ge i}(r) = \bigcup_{i \le j < n} \mathcal{R}_i(r)\ .$$

Thus, $\mathcal{R}(r)$ gives the set of all coordinates (i,x) belonging to the region of index r. The subset $\mathcal{R}_i(r)$ considers only the coordinates whose row index is i. In the same way, the subsets $\mathcal{R}_{<i}(r)$ and $\mathcal{R}_{\ge i}(r)$ consider only the coordinates whose row indices are respectively smaller and greater than i.

Lemma 1. *Let i be an element of $[\![0,n[\![$ and r be a region index in $\operatorname{Im}\varphi_i$. Let x be an element of $\mathbb{F}_2^n$ such that $\varphi_i(x) = r$. The region $\mathcal{R}_i(r)$ is connected and the equality $\mathcal{R}_i(r) = \{(i,y) \mid y \in \mathrm{Cl}_i(x)\}$ holds. Moreover,*

- *if $r_i = 0$ and $i < n-1$, then $\mathcal{R}_{i+1}(r)$ is non empty and connected with $\mathcal{R}_i(r)$;*
- *if $r_i = 1$, then $\mathcal{R}_{\ge i+1}(r) = \varnothing$.*

Proof. The previous discussion states that for all y in $\mathrm{Cl}_i(x)$, $\varphi_i(y) = r$ and for all y in $\mathbb{F}_2^n \setminus \mathrm{Cl}_i(x)$, $\varphi_i(y) \ne r$. Hence, $\mathcal{R}_i(r) = \{(i,y) \mid y \in \mathrm{Cl}_i(x)\}$. Since $\mathrm{Cl}_i(x)$ is an interval, the region $\mathcal{R}_i(r)$ is connected. Now, consider the following two cases.

- Suppose $r_i = 0$ and $i < n-1$. We observe that $\varphi_{i+1}(x) = r$. Then $\mathcal{R}_{i+1}(r) = \{(i+1,y) \mid y \in \mathrm{Cl}_{i+1}\}$. Since $\mathrm{Cl}_i(x)$ is a subset of $\mathrm{Cl}_{i+1}(x)$, it follows that $\mathcal{R}_i(r)$ and $\mathcal{R}_{i+1}(r)$ are connected.
- Suppose $r_i = 1$. Let j be an element of $[\![i+1,n[\![$. By construction, the i-th bit of $\varphi_j(y)$ equals 0 for all y in $\mathbb{F}_2^n$. Thus, there exists no y in $\mathbb{F}_2^n$ such that $\varphi_j(y) = r$. In other words, $\mathcal{R}_{\ge i+1}(r) = \varnothing$.

The desired result is proven. □

Theorem 3. *For all r in $\mathbb{F}_2^n$, the region $\mathcal{R}(r)$ is non empty and connected.*

Proof. Let r be a region index in $\mathbb{F}_2^n$. Since $\operatorname{Im}\varphi_0 = \operatorname{Im}\varphi = \mathbb{F}_2^n$, r belongs to $\operatorname{Im}\varphi_0$. Thus $\mathcal{R}_0(r)$ is non empty and so $\mathcal{R}(r)$ is. Let j denote the smallest index such that $r_j = 1$ (and assume that $j = n-1$ if $r = 0_n$). It is easily seen that $\mathcal{R}_{<j+1}(r)$ is connected using Lemma 1 by induction. Then, $\mathcal{R}_{\ge j+1}(r) = \varnothing$ follows from the same Lemma. Finally, $\mathcal{R}(r) = \mathcal{R}_{<j+1}(r) \cup \mathcal{R}_{\ge j+1}(r)$ is connected. □

2.3 Set Connection

The previous section proved that all possible regions exist and the sets have no overlapping edges below the central grid. To form a valid Venn diagram, each set must now be connected. Such a connection is done by linking each set on the last row on which it is present. This section begins with some definitions.

Definition 4. For each s in $[\![0, n-2[\![$, let us define the $n \times 2^n$ matrix M'_s where the row of index i in $[\![0, n[\![$ is given by

$$(M'_s)_i = \begin{cases} L_s & \text{if } i < s \ , \\ 0_{2^i} \parallel 1_{2^n - 2^{i+1}} \parallel 0_{2^i} & \text{if } i = s \ , \\ 0_{2^n} & \text{if } i > s \ . \end{cases}$$

Further, we define $M'_{n-2} = M_{n-2}$ and $M'_{n-1} = M_{n-1}$.

$$M'_1 = \begin{array}{c} \begin{array}{cccccccccccccccc} 0&1&2&3&4&5&6&7&8&9&10&11&12&13&14&15 \end{array} \\ \begin{pmatrix} 0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0 \\ 0&0&1&1&1&1&1&1&1&1&1&1&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array} \quad M'_0 = \begin{array}{c} \begin{array}{cccccccccccccccc} 0&1&2&3&4&5&6&7&8&9&10&11&12&13&14&15 \end{array} \\ \begin{pmatrix} 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

Figure 12: Definitions of the matrices M'_s used to connect the vertical sets.

Example 4. Figure 12 shows how to connect a set below the center grid. As explained in definition 4, matrices M'_2 and M'_3 are respectively equal to M_2 and M_3. Above the central grid, only the extension is made. Thus, each set is well represented by a Jordan curve.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	3	2	6	7	5	4	12	13	15	14	10	11	9	8
0	0	2	2	6	6	4	4	12	12	14	14	10	10	8	8
0	0	0	0	4	4	4	4	12	12	12	12	8	8	8	8
0	0	0	0	0	0	0	0	8	8	8	8	8	8	8	8

Simple Extension

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	3	3	7	7	5	5	13	13	15	15	11	11	9	8
0	0	2	2	6	6	6	6	14	14	14	14	10	10	8	8
0	0	0	0	4	4	4	4	12	12	12	12	8	8	8	8
0	0	0	0	0	0	0	0	8	8	8	8	8	8	8	8

Connecting Extension

Figure 13: The region values below the central grid.

Example 5. Consider again as an example the coordinates (1,6). The binary vector representing region value in coordinates (1,6) was $((M_3)_{1,6}, (M_2)_{1,6}, (M_1)_{1,6}, (M_0)_{1,6}) = (0,1,0,0)$. After the connection, this vector becomes $((M'_3)_{1,6}, (M'_2)_{1,6}, (M'_1)_{1,6}, (M'_0)_{1,6}) = (0,1,1,0)$. Thus, this vector can be considered as the binary decomposition of integer 6, which is the region index in coordinates (1,6), as presented in Figure 13.

The rest of this section proves that the connection preserves the same number of regions and that each region is connected and appears only once.

Lemma 2. *Let i be in $[\![0, n-1[\![$ and x be in $\mathbb{F}_2^n$. There exist x^0 and x^1 in $\mathrm{Cl}_{i+1}(x)$ satisfying the following statements, with $r^0 = \varphi_i(x^0)$ and $r^1 = \varphi_i(x^1)$,*

- $\varphi_{i+1}(x^0) = \varphi_{i+1}(x^1) = \varphi_{i+1}(x) = r^0$,
- $r_i^0 = 0$ *and* $r_i^1 = 1$, *in particular* $\varphi_i(x^0) \neq \varphi_i(x^1)$,

- $\{\mathrm{Cl}_i(x^0), \mathrm{Cl}_i(x^1)\}$ *is a partition of* $\mathrm{Cl}_{i+1}(x)$.

Proof. Recall that

$$\begin{aligned}\varphi_{i+1}(y) &= (y_{n-1}; y_{n-2} \oplus y_{n-1}, \dots, y_{i+1} \oplus y_{i+2}, \quad 0 \quad, 0, \dots, 0)\\ \varphi_i(y) &= (y_{n-1}; y_{n-2} \oplus y_{n-1}, \dots, y_{i+1} \oplus y_{i+2}, y_i \oplus y_{i+1}, 0, \dots, 0)\end{aligned}$$

for all y in $\mathbb{F}_2^n$. Now, let us define the following elements of $\mathbb{F}_2^n$.

$$\begin{aligned}x^0 &= (x_{n-1}, \dots, x_{i+1}, \quad x_{i+1} \quad, 0, \dots, 0)\\ x^1 &= (x_{n-1}, \dots, x_{i+1}, x_{i+1} \oplus 1, 0, \dots, 0)\end{aligned}$$

Let r^0 and r^1 denote $\varphi_i(x^0)$ and $\varphi_i(x^1)$ respectively. It is not hard to check that $\varphi_{i+1}(x^0) = \varphi_{i+1}(x^1) = \varphi_{i+1}(x) = r^0$. By construction, $r_i^0 = 0$ and $r_i^1 = 1$. It remains to show that $\{\mathrm{Cl}_i(x^0), \mathrm{Cl}_i(x^1)\}$ is a partition of $\mathrm{Cl}_{i+1}(x)$. Since $\varphi_{i+1}(x^1) = \varphi_{i+1}(x^0) = \varphi_{i+1}(x)$, both x^0 and x^1 are elements of $\mathrm{Cl}_{i+1}(x)$, that is, $\mathrm{Cl}_{i+1}(x^0) = \mathrm{Cl}_{i+1}(x^1) = \mathrm{Cl}_{i+1}(x)$. It follows that both $\mathrm{Cl}_i(x^0)$ and $\mathrm{Cl}_i(x^1)$ are subsets of $\mathrm{Cl}_{i+1}(x)$. Furthermore, these two subsets are disjoint because $\varphi_i(x^0) \neq \varphi_i(x^1)$. Knowing that $\mathrm{Cl}_i(x^0)$ and $\mathrm{Cl}_i(x^1)$ have both 2^i elements and that the cardinal of $\mathrm{Cl}_{i+1}(x)$ is 2^{i+1}, we conclude that $\{\mathrm{Cl}_i(x^0), \mathrm{Cl}_i(x^1)\}$ is a partition of $\mathrm{Cl}_{i+1}(x)$, as desired. □

It should be highlighted that $(M_s)_i = (M_s')_i$ if $i \neq s$. For each region index r in $\mathbb{F}_2^n$, we define

$$\mathcal{R}'(r) = \{(i,x) \in [\![0,n[\![\times [\![0,2^n[\![\mid \forall s \in [\![0,n[\![, (M_s')_{i,x} = r_s\} \ .$$

Furthermore, we define the subsets $\mathcal{R}_i'(r)$, $\mathcal{R}_{<i}'(r)$ and $\mathcal{R}_{\geq i}'(r)$ in the same way as we did for the subsets of $\mathcal{R}(r)$.

Lemma 3. *Let i be in $[\![0,n[\![$ and r be an element of $\mathbb{F}_2^n \smallsetminus \operatorname{Im}\varphi_i$. The sets $\mathcal{R}_i(r)$ and $\mathcal{R}_i'(r)$ are empty.*

Proof. Since r does not belong to $\operatorname{Im}\varphi_i = \mathbb{F}_2^{n-i} \times \{0_i\}$, there exists $s < i$ such that $r_s = 1$. However, $(M_s)_i = (M_s')_i = 0_{2^n}$ hence $\mathcal{R}_i(r) = \mathcal{R}_i'(r) = \varnothing$. □

Lemma 4. *Let i be an element of $[\![0,n-2[\![$. For each r in $\mathbb{F}_2^n \smallsetminus \{\varphi(0), \varphi(2^n-1)\}$, the following statements hold:*

- *either $\mathcal{R}_{<i}'(r) \neq \varnothing$, or $\mathcal{R}_{\geq i}(r) \neq \varnothing$;*
- *$\mathcal{R}_{<i}'(r)$ is connected.*

Proof. First, suppose that i equals 0 and let r be an element of $\mathbb{F}_2^n \smallsetminus \{\varphi(0), \varphi(2^n-1)\}$. According to Theorem 3, $\mathcal{R}_{\geq 0}(r) = \mathcal{R}(r)$ is non empty. On the other hand, $\mathcal{R}_{<0}'$ is clearly empty. Thus, the result holds for $i = 0$.

Now, let i be in $[\![0,n-3[\![$ and assume that this statement holds for this integer. If r does not belong to $\operatorname{Im}\varphi_i$, then Lemma 3 ensures that $\mathcal{R}_i(r) = \mathcal{R}_i'(r) = \varnothing$. Thus $\mathcal{R}_{<i+1}'(r) = \mathcal{R}_{<i}'(r)$ and $\mathcal{R}_{\geq i+1}(r) = \mathcal{R}_{\geq i}(r)$ and the result holds for $i+1$ by induction. The same remains to be shown for all region indices in $\operatorname{Im}\varphi_i$.

Let x be an element of $\mathbb{F}_2^n$ such that $x \notin \mathrm{Cl}_{i+1}(0)$ and $x \notin \mathrm{Cl}_{i+1}(2^n - 1)$. Let x^0, x^1, r^0 and r^1 be as in Lemma 2. According to Lemma 1, one has

$$\begin{cases} \mathcal{R}_i(r^0) = \{(i,y) \mid y \in \mathrm{Cl}_i(x^0)\} , \\ \mathcal{R}_i(r^1) = \{(i,y) \mid y \in \mathrm{Cl}_i(x^1)\} , \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{R}_{i+1}(r^0) = \{(i+1,y) \mid y \in \mathrm{Cl}_{i+1}(x)\} , \\ \mathcal{R}_{\geq i+1}(r^1) = \varnothing \quad \text{since } r_i^1 = 1 . \end{cases} \tag{$*$}$$

Next, for all x in $\mathrm{Cl}_i(x^0)$, replace the coefficient of $(M_i)_{i,x}$ with one to obtain $(M_i')_{i,x}$. We have $\mathcal{R}_i'(r^0) = \varnothing$ and $\mathcal{R}_i'(r^1) = \{(i,y) \mid y \in \mathrm{Cl}_i(x^0) \cup \mathrm{Cl}_i(x^1)\}$ as $r_i^0 = 0$ and $r_i^1 = 1$. Again, Lemma 2 ensures that $\mathrm{Cl}_i(x^0) \cup \mathrm{Cl}_i(x^1) = \mathrm{Cl}_{i+1}(x)$. Hence,

$$\begin{cases} \mathcal{R}_i'(r^0) = \varnothing , \\ \mathcal{R}_i'(r^1) = \{(i,y) \mid y \in \mathrm{Cl}_{i+1}(x)\} , \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{R}_{i+1}(r^0) = \{(i+1,y) \mid y \in \mathrm{Cl}_{i+1}(x)\} , \\ \mathcal{R}_{\geq i+1}(r^1) = \varnothing . \end{cases}$$

Observe that the sets $\mathcal{R}_{\geq i}(r^0)$ and $\mathcal{R}_{\geq i}(r^1)$ are non empty. Therefore, $\mathcal{R}_{<i}'(r^0)$ and $\mathcal{R}_{<i}'(r^1)$ are empty by induction hypothesis. It follows that,

$$\mathcal{R}_{<i+1}'(r^0) = \varnothing \quad \text{and} \quad \mathcal{R}_{<i+1}'(r^1) = \mathcal{R}_i'(r^1) = \{(i,y) \mid y \in \mathrm{Cl}_{i+1}(x)\} .$$

Thus, $\mathcal{R}_{<i+1}'(r^1)$ is connected.

It remains to show that the result still holds if x belongs to $\mathrm{Cl}_{i+1}(0)$ or to $\mathrm{Cl}_{i+1}(2^n - 1)$. Suppose that x is an element of $\mathrm{Cl}_{i+1}(0)$. Again, define x^0, x^1, r^0 and r^1. The statement ($*$) still holds. However, as $r^0 = \varphi(0)$, we are no longer concerned with r^0. By induction hypothesis, $\mathcal{R}_{<i}'(r^1) = \varnothing$. Thus $\mathcal{R}_{<i+1}'(r^1)$ is connected. A similar reasoning can be conduced if x lies in $\mathrm{Cl}_{i+1}(2^n - 1)$. The result is proven by induction. □

Theorem 4. *For all r in $\mathbb{F}_2^n$, the region $\mathcal{R}'(r)$ is non empty and connected.*

Proof. Let r be a region index in $\mathbb{F}_2^n$. If $r = \varphi(0)$ or $r = \varphi(2^n - 1)$, then by construction $\mathcal{R}'(r) = \mathcal{R}(r)$ and Theorem 1 implies that this region is non empty and connected.

Now, suppose that r does not belong to $\{\varphi(0), \varphi(2^n - 1)\}$. Note that $(M_s)_{n-2} = (M_s')_{n-2}$ and $(M_s)_{n-1} = (M_s')_{n-1}$ for all s in $[\![0, n[\![$. It follows that, $\mathcal{R}_{\geq n-2}(r) = \mathcal{R}_{\geq n-2}'(r)$. Consequently, $\mathcal{R}'(r) = \mathcal{R}_{<n-2}'(r) \cup \mathcal{R}_{\geq n-2}(r)$. According to Lemma 4, one and only one of the following two cases happens.

- $\mathcal{R}_{<n-2}'(r) \neq \varnothing$. Then the same Lemma states that $\mathcal{R}_{<n-2}'(r)$ is connected.
- $\mathcal{R}_{\geq n-2}(r) \neq \varnothing$. Then Lemma 1 implies that $\mathcal{R}_{\geq n-2}(r)$ is connected.

Thus, $\mathcal{R}'(r)$ is non empty and connected. □

In this section we have proven that the extension and the connection of the sets lead to have all 2^n possible regions for an n-Venn diagram. This assertion is actually not exactly true, because the connections brought some overlapping segments. These segments are the top (or lower) part of rectangles which have been connected. But it is easy to be convinced that such a problem can be avoided by creating rectangles of width one and a half unit square instead of one as displayed previously in Figure 4. Such a construction can be formally explained using a similar reasoning to the connection.

2.4 The Central Grid

In this section, V_s and H_s are binary matrices that denote the vertical (resp. horizontal) disposition of the set labeled s in the central grid using the previous definition of L_s. The aim of this section is to prove that all possible regions of a Venn diagram are in this grid, and the sets have no common border.

Definition 5. For each s in $[\![0,n[\![$, let us define the $2^m \times 2^n$ matrices V_s and H_s where the coefficient (i,j) is given by $(V_s)_{i,j} = (L_s^n)_j$ and $(H_s)_{i,j} = (L_s^m)_i$.

Let (v,h) be an element of $\mathbb{F}_2^n \times \mathbb{F}_2^m$. Theorem 1 states that there exists unique x and y in $[\![0,2^n[\![$ and $[\![0,2^m[\![$ respectively, such that $\forall s \in [\![0,n[\![$, $(L_s^n)_x = v_s$ and $\forall s \in [\![0,m[\![$, $(L_s^m)_y = h_s$. Using notations introduced in the previous definition, this result can be restated as follows.

Corollary 1. *For any (v,h) in $\mathbb{F}_2^n \times \mathbb{F}_2^m$, there exists a unique (x,y) in $[\![0,2^n[\![\times[\![0,2^m[\![$ such that $\forall s \in [\![0,n[\![$, $(V_s)_{x,y} = v_s$ and $\forall s \in [\![0,m[\![$, $(H_s)_{x,y} = h_s$.*

This attests that our method gives rise to a valid Venn diagram. It is also obvious that such a construction leads to a simple Venn diagram. Indeed, by definition and according to Theorem 2, the verticals sets do not have any common border. Furthermore, vertical and horizontal sets do not have any common border again thanks to the extension. Because the same applies for horizontal sets, the diagram is thus a valid simple Venn diagram.

3 Algorithm

In this section, we expose the algorithm to create a simple $(n+m)$-Venn diagram using our method, composed of n vertical sets and m horizontal. The different sets are labeled as explained in the proof, that is to say, the set represented by the greatest number of rectangles in the central grid has index 0. Vertical sets are denoted $\mathcal{V}_i$, $i \in [\![0,n[\![$ and horizontal sets $\mathcal{H}_j$, $j \in [\![0,m[\![$. We suppose that the diagram is drawn in the center of a grid of width $2^n + 2(n-1)$ and height $2^m + 2(m-1)$.

- Draw the center of the diagram using the equations of the central grid given in Section 2.4.
- For i from 1 to $n-1$, extend the set $\mathcal{V}_i$ vertically by i unit squares at the top and bottom.
- For j from 1 to $m-1$, extend the set $\mathcal{H}_j$ horizontally by j unit squares at the left and right sides.
- For i from 0 to $n-3$, connect the set $\mathcal{V}_i$ using a band of 1.5 unit squares at the bottom.
- For j from 0 to $m-3$, connect the set $\mathcal{H}_j$ using a band of 1.5 unit squares at the right side.

4 Comparison With the Previous Work

It is interesting to compare this construction with the three other constructions mentioned earlier, that is to say, the very first construction proposed by Venn [15], Edwards [6] and Anderson and Cleaver [1]. Even if Venn's and Edwards' constructions are not really close to ours, the first aim of this section is to test whether they are isomorphic or not. The second is to determine which one is the easiest to read.

4.1 Similarities Between the Constructions

We study now diagrams produced by the different constructions. Each construction is seen as an Euler graph in the following way: the curves and their intersections of the construction become respectively the edges and vertices of the graph. Because Venn's, Edwards' and our construction lead to simple diagrams, they are isomorphic (as Venn diagrams) if and only if they are isomorphic as planar graphs. The open-source software SageMath [5] was used to test if two planar graphs are isomorphic. Anderson's construction can also be seen as a simple diagram if the curves in the bottom of the diagram are shifted.

For the 3 and 4-Venn diagrams, the four constructions lead to isomorphic diagrams. Indeed, the addition of the fourth set is done following the advices of John Venn for each construction. Concerning the 5-Venn diagrams, the method is again identical for Venn's, Anderson's and our construction. These three diagrams are hence isomorphic. Only the Edwards' diagram is not isomorphic to the other diagrams. Finally, the four constructions lead to non-isomorphic diagrams for $n = 6$.

4.2 Readability Study

A study was created to evaluate the readability of our diagram compared with the three other Venn diagrams mentioned above. All of them were 6-Venn diagrams. We have conducted a study over 103 students at bachelor's level with these four constructions, according to the following procedure.

For each diagram, we evaluated the time taken by each student to identify nine given regions and we took note of the number of correctly identified regions per diagram. These nine regions were identical for each diagram and the list can be found in Appendix. Because Anderson's construction is similar to ours, half of the students began by completing this diagram while the other half completed our construction first. The same process was used for Venn's and Edwards' diagrams. Finally, two lists of diagrams were produced as detailed in Table 1.

In list A, students had to complete Anderson's diagram before ours, and in list B they had to complete our diagram before Anderson's. Fifty-one students answered the questions of list A and fifty-two the questions of list B. For each student, the total number of correctly identified regions is stored and an average is computed over all the answers for each diagram. The average success rate for list A is 79% and 76% for list B. The other success rates are displayed in Table 2.

Order	List A	List B
1	Edwards	Venn
2	Anderson	Our construction
3	Venn	Edwards
4	Our construction	Anderson

Table 1: Diagram order in the two lists.

	Venn	Edwards	Anderson	Our construction
List A	53%	84%	78%	89%
List B	50%	92%	87%	91%
Average	52%	87%	83%	90%

Table 2: Success rate for each diagram.

Our construction presents a better success rate than Edwards' and Anderson's. It is worth noting that the success rate for Anderson's construction is higher if the students had to complete our construction before (87% against 78%). An ANOVA shows that the null hypothesis (the readability of Anderson's and our diagrams are the same) is rejected with $F = 13.45 > F(1, 204, p = 0.05) = 3.89$ with a p-value less than 0.005. This fact may be explained by the lines running concurrently at the bottom of the diagram which may hinder the readability of the boundary of each set as already stated in [12]. Conversely, our construction remains easy to read even if students see this kind of diagram for the first time.

Let us now focus on the time used by students to correctly identify the different regions. We have gathered the times taken to complete diagrams in which all the regions were correctly identified. On 103 tests, this represents 0 Venn's diagram, 47 Edwards' diagrams, 33 Anderson's diagrams and 42 diagrams produced by our construction. Table 3 presents the average time (in minutes : seconds) obtained for these diagrams, for each of the two lists.

	Venn	Edwards	Anderson	Our construction
List A	–	3:13	3:37	2:28
List B	–	2:42	3:18	3:09
Total	–	2:57	3:27	2:49

Table 3: Average time to correctly identify the nine regions in each diagram.

Because there was no perfect score for Venn's diagram, the average time is not displayed. Students completed our construction almost 20% faster than Anderson's, and as fast as Edwards'. An ANOVA reveals that the results are significantly better if the students see our construction for the first time regarding the same experiment with Anderson's. The obtained F value is 28.33 which is higher than the critical value of $F(1, 30, p = 0.05) = 4.17$. It is also interesting to note that for both Anderson's and our constructions, the diagrams were completed faster if the students saw the other diagram before.

As conclusion, our construction is significantly faster to complete, and leads to fewer errors in the identification of regions. Some other statistics are provided in Appendix and lead to the same observations.

5 Conclusion and Perspective

In this article, we have introduced a new construction for simple n-Venn diagrams. The scheme produced tends to display a grid when the number of sets grows. There exists another type of (almost) Venn diagrams called *polyVenn*, introduced in [14]. The curves representing the sets are the perimeters of polyominoes. If all the regions have the same size such as a unit square, the diagram is called *minimum area polyVenn*. These constructions cannot be properly called Venn diagrams because many of their edges overlap. However, such a drawing can be interesting to display each region with the same size. As an example, polyominoes are used in biology [4] for display purposes. In [2], the authors found a generic construction for an even number of sets, and their scheme excludes the empty set.

We believe that our construction can lead to a minimal area polyVenn for any number of sets because of the particular structure of the central grid of our diagram. As proven in Section 2, the central grid contains each possible region produced by the intersection of the sets. Since the sets are not all connected, this construction is not a formal polyVenn. We surmise that it is possible to produce a valid minimum area polyVenn by switching some unit squares. A future work could be to prove this conjecture.

It is worthwhile to notice that such a construction seems to form a fractal when the regions are colored according to the number of sets in which they are included. Indeed, if we consider only the central grid of the diagram, it is possible to move inside this grid and find equivalent constructions. As an example, consider our construction with as many vertical sets as horizontal ones. If we divide the central grid into four squares of the same size, the top left square seems to be identical to the complete drawing, without the two first sets. A future work could prove if a fractal really appears and, in this case, study the main properties of this construction.

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A Readability Statistics

A.1 List of the Nine Regions to Identify

Each diagram displayed six sets, named from E_1 to E_6 . For each diagram, the student had to identify the nine following regions:

- the region r_1 included exclusively in the set E_1;
- the region r_2 included exclusively in the set E_3;
- the region r_3 included exclusively in the set E_6;
- the region r_4 included exclusively in the sets E_1 and E_2;
- the region r_5 included exclusively in the sets E_4 and E_6;
- the region r_6 included exclusively in the sets E_1, E_3 and E_5;
- the region r_7 included exclusively in the sets E_2, E_3, E_5 and E_6;
- the region r_8 included exclusively in the sets E_1, E_2, E_3, E_4 and E_6.
- the region r_9 included in every set.

A.2 Complete Results of the Study

Table 4 (resp. Table 5) displays the main statistics extracted from the readability study of list A (resp. list B). For each diagram, the column *Time* indicates the average time taken by the students to identify all the regions in minutes : seconds. Note that this average includes the students who made some mistakes. The following nine columns indicate the total of correct answers for the region concerned (over 51 answers for list A and 52 for list B). The last column gives the average success rate for each diagram. Detailed times and scores are given in Tables 6 and 7.

	Time	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	Success rate
Edwards	03:39	50	46	45	50	44	36	34	36	44	0,84
Anderson	04:09	51	51	51	44	50	31	22	25	32	0,78
Venn	04:41	51	51	43	47	5	5	10	8	24	0,53
Our construction	02:29	51	49	51	48	51	45	33	36	44	0,89

Table 4: Average time to correctly identify the nine regions in each diagram in list A.

	Time	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	Success rate
Venn	04:16	51	52	39	49	6	1	5	7	25	0,50
Our construction	03:21	52	52	52	49	49	47	35	40	48	0,91
Edwards	02:46	52	50	45	51	49	40	38	45	50	0,92
Anderson	03:11	52	51	52	52	50	44	29	34	45	0,87

Table 5: Average time to correctly identify the nine regions in each diagram in list B.

List A	
Our construction	Anderson
163	214
150	122
161	136
182	151
148	150
270	170
207	230
209	232
183	238
186	174
227	303
210	220
232	115
240	254
139	252
146	294
147	255
149	180
238	236
234	159
142	
151	
247	

List B	
Anderson	Our construction
214	102
181	91
218	112
297	115
164	132
191	82
210	135
234	149
258	142
198	120
254	183
258	180
178	203
138	
174	
165	
185	
220	
147	

Table 6: Time taken in seconds to correctly identify all the regions in Anderson's and our construction for list A (left) and B (right).

List A			
Edwards	Venn	Anderson	Bannier
9	4	9	9
8	3	7	9
8	5	6	7
6	4	8	6
9	6	8	9
7	4	8	8
8	6	9	9
9	6	9	7
9	4	5	8
9	5	7	8
9	5	7	9
7	5	8	9
8	7	8	8
7	6	4	8
9	6	9	9
9	5	5	6
9	6	9	9
8	4	9	9
9	5	6	8
6	5	8	8
6	4	7	7
5	3	7	9
4	5	4	6
9	4	7	7
9	3	6	9
6	4	5	7
2	3	5	7
9	6	9	9
5	4	5	8
9	4	6	8
8	6	7	8
3	2	5	6
9	5	9	9
8	3	6	8
9	7	9	9
8	3	4	7
9	5	6	9
9	5	8	8
7	5	9	9
9	6	7	8
9	6	9	8
8	7	8	8
7	4	4	8
6	4	5	7
9	5	7	8
9	6	9	9
8	4	9	8
2	3	4	7
3	5	5	7
9	4	8	8
9	8	9	9

List B			
Venn	Edwards	Bannier	Anderson
7	9	9	9
3	8	9	6
4	9	9	9
4	9	8	9
5	9	9	7
4	8	8	5
7	9	7	8
7	9	9	9
6	8	8	9
5	8	8	8
5	9	8	9
6	6	7	6
5	9	7	6
5	7	7	5
4	7	9	7
3	9	9	8
5	9	9	8
5	9	7	8
3	7	7	7
2	6	7	7
4	8	7	6
4	9	9	8
5	8	9	8
2	6	7	8
5	8	9	8
3	7	7	6
5	9	9	9
5	9	9	9
5	8	7	9
3	9	9	7
4	7	8	6
3	8	7	8
6	8	6	9
3	4	7	9
6	9	8	9
5	8	8	8
6	8	9	9
4	9	9	9
5	8	8	8
6	9	8	9
3	4	8	7
5	9	9	7
4	8	8	8
4	8	9	8
5	9	9	9
4	8	8	7
3	9	9	9
5	9	8	9
4	9	8	8
6	9	9	9
4	7	9	9
4	8	9	7

Table 7: Number of correct answers for list A (left) and B (right). Each row gives the number of correctly identified regions out of a total of 9 in each diagram.