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VENN-TYPE DIAGRAMS FOR ARGUMENTS OF N TERMS

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I

The attempt to find usable diagrams for n terms of the sort devised by John Venn[1] seems to have originated with Venn himself, who published diagrams for up to five classes (the fifth class, however, was shaped like a doughnut, and contained an area outside itself — like the hole in the doughnut). Venn then suggested that "if we wanted to use a diagram for *six* terms (x, y, z, w, v, u) the best plan would probably be to take *two* five-term figures, one for the u part and one for the non-u part of all the other combinations..."[2] Such a method would, as Venn admits, be somewhat confusing to the eye, and thus fail to fulfil one of the major purposes of the diagram; nevertheless, it is a method which could be extended to cover n classes.

Venn also suggests (but without illustration) another method for drawing diagrams for n terms: "... the rule of formation would be very simple. We should merely have to begin by drawing any closed figure, and then proceed to draw others subject to the one condition that each intersect once, and once only, all the existing subdivisions produced by those which had gone before."[3] Then, in a rather surprising footnote, he adds: "It will be found that when we adhere to continuous figures, instead of the discontinuous five-term figure given above, there is a tendency for the resultant outlines thus successively drawn to assume a comb-like shape after the first four or five. If we begin by circles or other rounded figures the teeth are curved, if by parallelograms then they are straight. Thus the fifth term figure will have two teeth, the sixth four, and so on, till the $(4+x)$th has 2^x teeth..."[4]

In general, although numerous methods have been devised for drawing

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[1] ***Symbolic logic,*** London (1881). The attempt to diagram logical arguments is of course much older, originating, perhaps among the ancients. William Hamilton (***Discussions on philosophy and literature*** (1866)) reproduces diagrams of the syllogism that date from the fifth century. Venn reproduces diagrams of class inclusion attributed to Ludovicus Vives, and to Maass. The last chapter of Venn's book contains a fairly thorough review of the early literature. A more up to date review is given in ***Logic machines and diagrams,*** Martin Gardner, New York (1958), ch. 2. Gardner reproduces many of the proposed diagrams which otherwise may only be found scattered through numerous journals — many of which are now difficult to obtain.

[2] *Ibid.*, n. 1, p. 108.

[3] *Ibid.*, p. 108.

[4] n. 2, pp. 108–9.

diagrams to represent the class calculus,[5] they have suffered from one or more of the following difficulties: a) the method of drawing the diagram cannot be extended to cover *n* terms; b) the method has required the use of discontinuous regions; or c) the diagrams produced are difficult to grasp visually.

The method presented here is in principle the same as that worked out by More and, previously, by Hocking.[6] The main advantages this method has over the others are that the diagrams lend themselves more easily to the insertion of stars and shading, and they are easier to grasp visually.

In working out the following method it was decided to draw the diagrams in two somewhat different ways. The first (Figures 2 and 3) more exactly meets the visual characteristics of the Venn diagram, and the second (Figure 4), though it is mathematically the same, simplifies the proof.

II

But for the fact that we have used rectangles instead of ellipses, the four-term diagram (Figure 1) is essentially the same as that which is in general use.[7] Region A is the vertical rectangle designated by the bracket 'A'; region B is the vertical rectangle designated by bracket 'B'; region C is the horizontal rectangle designated by bracket 'C'; region D is the horizontal rectangle designated by bracket 'D'. Each letter inside a region designates the area of the corresponding region which excludes the other three regions. Thus area 1 is the intersection of regions A, D, $\sim$B, $\sim$C; 2: of regions A, B, D, $\sim$C; 6: A, B, C, D; etc. Area 16 designates the intersection of $\sim$A, $\sim$B, $\sim$C, $\sim$D, — i.e., the class negate of the four classes. The sixteen cases together exhaust the universe of discourse.

In Figure 2, regions A, B, C, D are the same as in Figure 1. To them

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[5] More recently a method for constructing diagrams for *n* terms was devised by Trenchard More, Jr. (*On the construction of Venn diagrams*, this JOURNAL, vol. 24 (1959)). In November of 1953, M. Karnaugh (*Map method for synthesis of combinational logic circuits*, ***Transactions of the American Institute of Electrical Engineers, Communications and electronics*** Pt. I, No. 9) refined a chart originally designed by E. W. Veitch (*A chart method for simplifying truth functions*, ***Association for Computing Machinery. Proceedings,*** May 2, 3 (1952)) to accommodate up to six terms, but for more than six terms the regions become discontinuous. More's method (cited above) allows for the construction of diagrams for *n* terms in which the regions for all terms are continuous; but the adaptation of the diagram to allow for shading to indicate emptiness and asterisks to indicate non-emptiness is confusing to the eye and, for pedagogical purposes at least, not very useful.

[6] *Two extensions of the use of graphs in elementary logic*, William Ernest Hocking; ***University of California publications in philosophy,*** Vol. 2 (1909).

[7] Cf. C. I. Lewis, ***A survey of symbolic logic,*** Berkeley, (1918); W. V. Quine, ***Methods of logic,*** New York, (1950); H. N. Lee, ***Symbolic logic,*** New York (1961).

is added region E, which has been shaded in the drawing for ease of identification. It will be seen that region E intersects every small area of the four-term diagram once and only once, and that it intersects

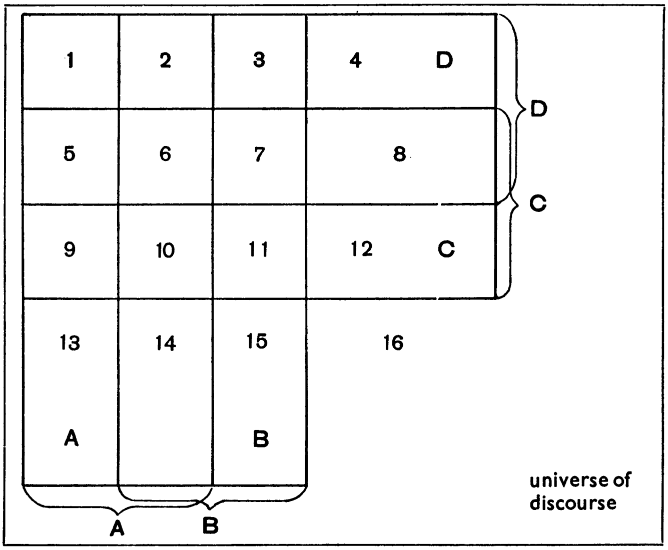

Figure 1

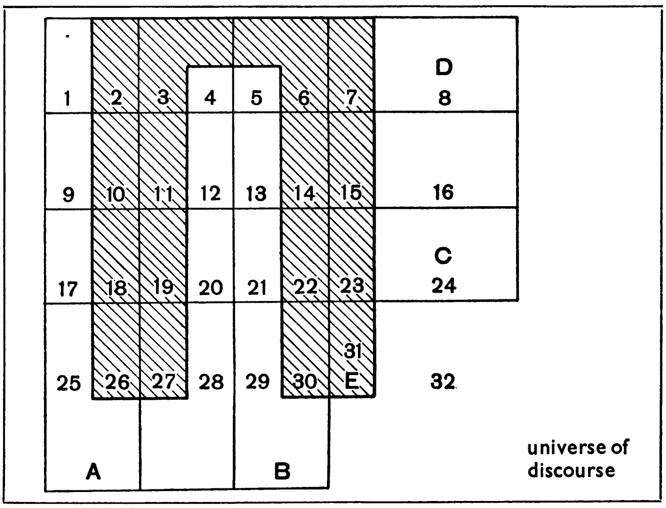

Figure 2

the area representing the class ∼A, ∼B, ∼C, ∼D at area 31 (i.e., area 31 represents the intersection of E, ∼A, ∼B, ∼C, ∼D). Area 32 represents the intersection of classes ∼A, ∼B, ∼C, ∼D, ∼E.

Thus, area 1 represents the intersection of regions A, D, ∼B, ∼C, ∼E; area 2: of regions A, D, E, ∼B, ∼C; 3: A, B, D, E, ∼C; 11: A, B, C, D, E; etc.

In Figure 3, regions A, B, C, D, and E remain the same as in Figure 2. A sixth region (F) has been added. This is the shaded area of Figure 3. It will be seen that the four "pant-legs" that constitute region F are connected across the top, and that they intersect every area of Figure 2 once and only once, in such a way as to present, as an area of the diagram,

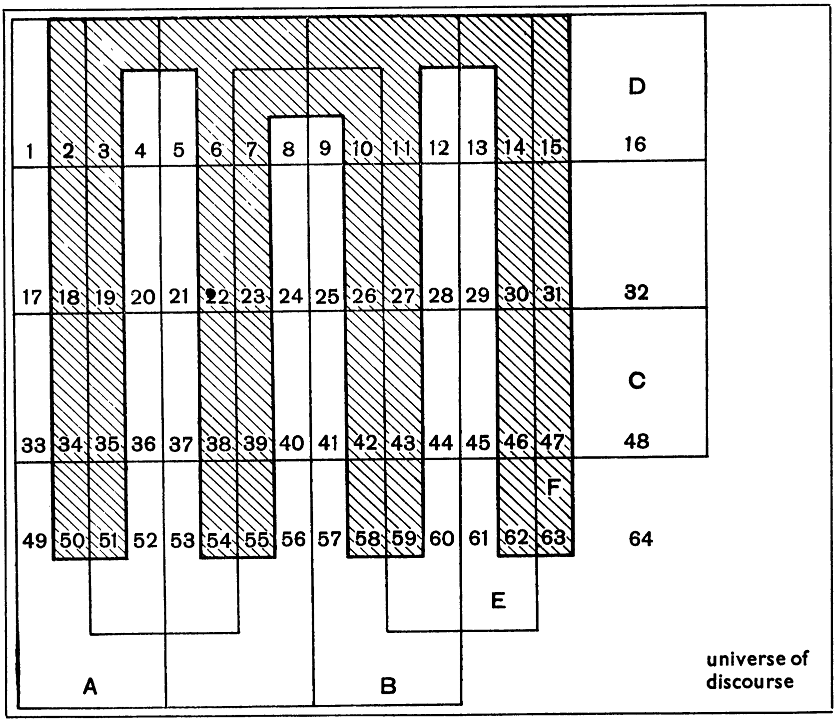

Figure 3

every possible combination of terms of a six-term argument. Area 22 represents the intersection of A, B, C, D, E, F; area 63 represents the intersection of F, ∼A, ∼B, ∼C, ∼D, ∼E; and area 64 represents the intersection of ∼A, ∼B, ∼C, ∼D, ∼E, ∼F.

It will be noted that in drawing region F, a pant-leg was drawn so that it covered the boundary between region E and areas 8, 16, 24, and 32 of Figure 2. The second pant-leg covers the boundary between E and areas 5, 13, 21, and 29; the third, the boundary between E and areas 4, 12, 20, and 28; and the fourth, the boundary between E and areas 1, 9, 17, and 25. That is, the vertical boundary of each pant-leg of region E was covered by one pant-leg of region F, while F remained connected across the top.

By covering the vertical boundary of each pant-leg of F with a pant-leg of a new region (G) which remains connected across the top, another diagram will be constructed that satisfies the requirements of a seven-term argument. It will be seen that this same method may be repeated an indefinite number of times, each repetition producing another region which satisfies the requirements for arguments of the next higher number of terms.[8]

To show that the construction explained above is mathematically feasible for n sets, we prove the following theorem. Let p and q denote two positive numbers and let $X = \{(x, y)\colon 0 \leqq x \leqq q \text{ and } 0 \leqq y \leqq p\}$, where X is considered as a topological space with the usual topology relative to the plane.

THEOREM. For any positive integer n there exist n connected subsets U_j of X for $j = 1, 2, \ldots, n$ such that each of the 2^n subsets $U_1^e U_2^e \ldots U_n^e$ are not empty for $e = 1$ or -1 where $U_j^1 = U_j$ and $U_j^{-1} = X - U_j$. Moreover, each of the 2^n subsets are connected.

Proof. The following construction actually gives a stronger result than the theorem states, since the interiors of the 2^n sets and the n sets U_j, with respect to the plane, are open subsets of the plane and these subsets are also simply connected subsets of the plane and of X.

Let $U_1 = \left\{(x, y)\colon \frac{p}{2} \leqq y \leqq p \text{ and } x \neq 0\right\}$ and $U_2 = \left\{(x, y)\colon \frac{p}{4} \leqq y \leqq \frac{3p}{4} \text{ and } x \neq 0\right\}$. Let $x_1^{(3)} = \frac{q}{2}$, $x_1^{(4)} = \frac{q}{4}$, and $x_2^{(4)} = \frac{3q}{4}$; then define $U_3 = \{(x, y)\colon 0 \leqq x \leqq x_1^{(3)} \text{ and } y \neq 0\}$ and $U_4 = \{(x, y)\colon x_1^{(4)} \leqq x \leqq x_2^{(4)} \text{ and } y \neq 0\}$. Choose numbers $x_i^{(5)}$ for $i = 1, 2, 3, 4$ such that $0 < x_1^{(5)} < x_1^{(4)} < x_2^{(5)} < x_1^{(3)} < x_3^{(5)} < x_2^{(4)} < x_4^{(5)} < q$ and choose $y^{(5)}$ such that $\frac{3p}{4} < y^{(5)} < p$. Define $U_5 = \{(x, y)\colon x_1^{(5)} \leqq x \leqq x_2^{(5)} \text{ and } y \neq 0\} \cup \{(x, y)\colon x_3^{(5)} \leqq x \leqq x_4^{(5)} \text{ and } y \neq 0\} \cup R_5$ where R_5 denotes the rectangle given by $\{(x, y)\colon x_1^{(5)} \leqq x \leqq x_4^{(5)} \text{ and } y^{(5)} \leqq y \leqq p\}$. Since U_5 is the union of two rectangles each of which meets the rectangle R_5 which is also contained in U_5, the set U_5 is connected.

The figure below illustrates these sets, where U_5 is the shaded area.

The theorem will be proved using induction, and the induction is started with $n = 5$. The successive sets are constructed by choosing points on the x-axis between each of the points already chosen and a point between $\frac{3p}{4}$ and $y^{(j)}$. For example, to construct U_6 choose 8 points $x_i^{(6)}$ for $i = 1, 2, \ldots, 8$ such that $0 < x_1^{(6)} < x_1^{(5)} < x_2^{(6)} < x_1^{(4)} < x_3^{(6)} < x_2^{(5)} < x_4^{(6)} < x_1^{(3)} < x_5^{(6)} < x_3^{(5)} < x_6^{(6)} < x_2^{(4)} < x_7^{(6)} < x_4^{(5)} < x_8^{(6)} < q$, and the point $y^{(6)}$ such that $\frac{3p}{4} < y^{(6)} < y^{(5)}$. Let $U_6 = \bigcup_{k=1}^{k=4} \{(x, y)\colon x_{2k-1}^{(6)} \leqq x \leqq x_{2k}^{(6)} \text{ and } y \neq 0\} \cup R_6$ where $R_6 = \{(x, y)\colon x_1^{(6)} \leqq x \leqq x_8^{(6)} \text{ and } y^{(6)} \leqq y \leqq p\}$.

[8] It is worth noting that this diagram fits exactly the description given in Venn's n. 2, pp. 108–9, quoted p. 113 above.

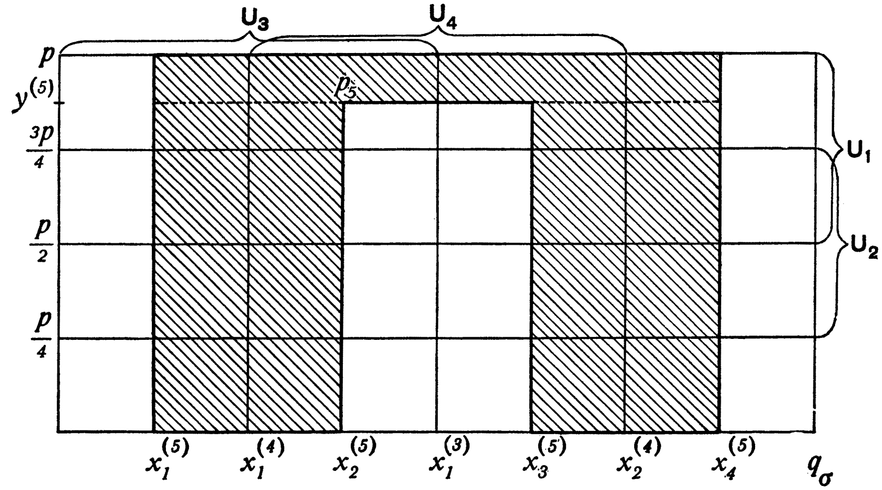

Figure 4

Assume the theorem is true for $n \geqq 5$, to show the theorem is true for $n+1$. Now U_n was constructed by choosing 2^{n-3} points and $U_n = \bigcup_{k=1}^{k=2^{n-4}} \{(x, y)\colon\ x_{2k-1}^{(n)} \leqq x \leqq x_{2k}^{(n)} \text{ and } y \neq 0\} \cup R_n$ where $R_n = \{(x, y)\colon x_1^{(n)} \leqq x \leqq x_{2^{n-3}}^{(n)} \text{ and } y^{(n)} \leqq y \leqq p\}$. To construct U_{n+1} choose $2^{(n+1)-3}$ points between the points already chosen and the point $y^{(n+1)}$ between $\dfrac{3p}{4}$ and $y^{(n)}$. Let $U_{n+1} = \bigcup_{k=1}^{k=2^{n-3}} \{(x, y)\colon\ x_{2k-1}^{(n+1)} \leqq x \leqq x_{2k}^{(n+1)} \text{ and } y \neq 0\}$ $\cup\, R_{n+1}$ where $R_{n+1} = \{(x, y)\colon\ x_1^{(n+1)} \leqq x \leqq x_{2^{n-2}}^{(n+1)} \text{ and } y^{(n+1)} \leqq y \leqq p\}$. U_{n+1} is connected since it is the union of rectangles each of which meets the rectangle R_5 which is included in U_{n+1}. The 2^{n+1} subsets $U_1^e \cap U_2^e \cap \ldots \cap U_{n+1}^e$ for $e = 1$ or -1 are all rectangles (thus connected and non-void) except for some of the sets in the band $B = \left\{(x, y)\colon \dfrac{3p}{4} \leqq y \leqq p\right\}$. The subsets that are in B are either rectangles or polygons with their sides parallel to the axes. In either event they are non-empty and connected. This completes the induction and the theorem follows.

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