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How many Venn diagrams are there?

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Mathematics teachers seem to have an ambivalent attitude towards Venn diagrams. Basically, lecturers acknowledge that these pictorial representations can clarify thought; but the profession will not countenance their use for actually proving propositions. The difficulty appears to be that, when using these aids, one is likely to overlook certain special cases; and such omission invalidates proof. This point of view is queried.

1. Introduction

Any rule of thumb is dangerous unless one is clearly aware of the exceptions. Thus, the fact that Venn diagrams are frequently used to confirm (or even derive) results suggests that either this is bound sometimes to lead to error (if contrary possibilities are not perceived), or (if properly managed) Venn diagram considerations may be made equivalent to 'formal' argument. I believe the latter. First base then is to establish, for a given context, exactly how many Venn representations are needed for 'proof'.

In what follows, we will usually denote intersection by product, and union by sum.

2. Classifying Venn diagrams

The problem of the pictorial approach is generally seen as stemming from the reality that, as one increases the number of sets involved, the number of 'topologically' distinct possibilities increases rapidly. As an example, given two sets (A and B say), the orthodox approach is to say that there are four cases to consider. See figure 1.

(However, some questions arise immediately. For instance, what about the case when $A=B$, or when either A or B (or both) are null? And what about the distinction between situations where $\{A, B\}$ covers and does not cover the universal set? Finally, is a complete enumeration of possible representations necessary anyway—are not, for instance, (c) and (d) frequently equivalent by symmetry?)

I would maintain that all two-set possibilities are subsumed in a clarification of (a) (see figure 2). For, when $AB=\varnothing$, this yields diagram (b) of figure 1; with $A\bar{B}=\varnothing$, we get (c); whilst, with $\bar{A}B=\varnothing$, we have (d). And, similarly, we can dispatch any other so-called special case that we care to think of; as we may always identify it uniquely as one of the twelve combinations of the four labelled 'mutually exclusive and exhaustive areas' of figure 2. (For example, $A\bar{B}\oplus AB=A$, $A\bar{B}\oplus AB\oplus\bar{A}B$ $(=A\cup B)=\overline{\bar{A}\bar{B}}$, and $A\bar{B}\oplus AB\oplus\bar{A}B\oplus\bar{A}\bar{B}=U$, the universal set; where we use the symbol $\oplus$ to emphasise the union of *mutually exclusive* 'areas'.

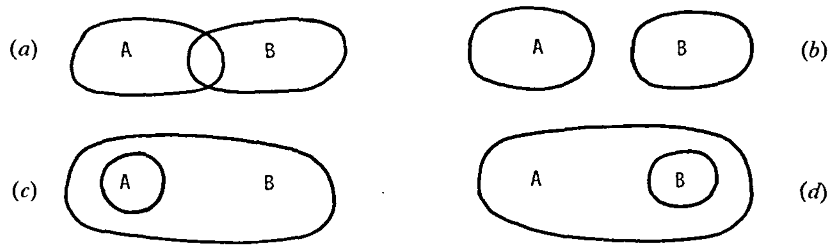

Figure 1. The four orthodox two-set Venn diagrams.

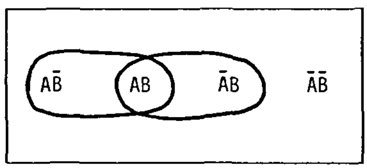

Figure 2. The all-embracing two-set Venn diagram.

Note that this use of $\oplus$ does not contradict the more usual one for denoting the symmetric difference of not necessarily mutually exclusive sets—the 'exclusive OR' of logic.)

Then, say, to demonstrate that $A-B=A\bar{B}$ (where $A-B$ denotes the set of all elements in A which are not also in B), one does not need to look individually at each of the diagrams (*a*)–(*d*) of figure 1 (and whatever other special cases one reckons there to be), getting $A-B$ as, respectively, $A\ominus AB$, $A\ominus\varnothing=A$, $A\ominus A=\varnothing$, $A\ominus B$, and so on (where $\ominus$ is an operator that removes *all* the elements of the second set from the first set, and is used here to emphasise the 'complete subtraction' which is indicated when the minuend contains the subtrahend), and then show that, in every case, these expressions equal $A\bar{B}$. Rather, we can just consider the general situation of our single diagram (figure 2), as all cases are clearly subsumed by the deduction there that $A-B=(A\bar{B}\oplus AB)\ominus AB=A\bar{B}$. For instance, the results for diagrams (*b*)–(*d*) of figure 1 are retrieved from figure 2 by, respectively, putting $AB=\varnothing\Rightarrow A-B=A\bar{B}=A$, $A\bar{B}=\varnothing\Rightarrow A-B=\varnothing$, and $\bar{A}B=\varnothing\Rightarrow A-B=A\bar{B}$. As elsewhere in mathematics, it is clear that geometric and algebraic argument can be made equivalent.

Thus we maintain that, contrary to popular belief, there is only *one* Venn diagram for any collection of sets.

3. The three-set case and beyond

Any three sets A, B and C can be uniquely represented by figure 3, provided we understand that some or all of the displayed 'sub-areas' may be taken as null. (And, similarly, a single Venn diagram can be used as an aid to 'rigorously' prove any proposition on n sets, where n is any positive integer. Proof is by induction.)

It is frequently helpful to use a shorthand notation, suggested by the Venn diagram aid. For the situation of *three* sets (A, B and C, say), this is: in each *triple*

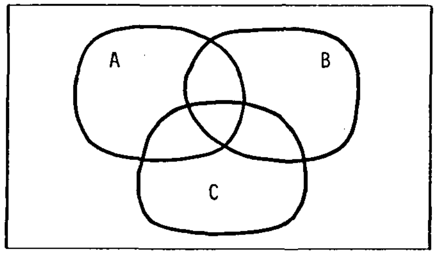

Figure 3. The general three-set Venn diagram.

intersection expression, replace A by α, B by β and C by γ, and omit any complement terms $\bar{A}$, $\bar{B}$ or $\bar{C}$. Thus: α denotes $A\bar{B}\bar{C}$; $\alpha\beta$ denotes $AB\bar{C}$; $\alpha\beta\gamma$ denotes ABC; and so on (and, of course, some of these Greek expressions may actually be null). Then, for instance, A has a four block partition $\{\alpha, \alpha\beta, \alpha\gamma, \alpha\beta\gamma\}$, and the Venn diagram can be labelled as in figure 4; where η denotes the otherwise unlabelled region, $\bar{A}\bar{B}\bar{C}$.

In manipulating these 'Greek areas', we will let the context distinguish between 'union' and 'exclusive union', and thus allow $+$ to denote both.

4. Examples

Example 1. To show $\overline{A+B+C}=\bar{A}\bar{B}\bar{C}$.

$$\text{LHS}=\overline{\Sigma\alpha+\Sigma\alpha\beta+\alpha\beta\gamma}=\eta=\text{RHS}$$

It is instructive to also look at the 'proof' of (the basic De Morgan law) $\overline{A+B}=\bar{A}\bar{B}$, again using the three-set diagram. Here

$$\text{LHS}=\overline{\alpha+\alpha\beta+\beta+\gamma\alpha+\alpha\beta\gamma+\beta\gamma}$$
$$=\gamma+\eta=\text{RHS}$$

Example 2. Prove $(A\cup B)\cap C=(A\cap C)\cup(B\cap C)$.

$$\text{LHS}=\gamma\alpha+\alpha\beta\gamma+\beta\gamma$$
$$=(\gamma\alpha+\alpha\beta\gamma)+(\alpha\beta\gamma+\beta\gamma)=\text{RHS}$$

Example 3. Simplify $\bar{A}(B+C)+\bar{B}(C+A)+\bar{C}(A+B)$.

$$\text{LHS}=(\beta+\beta\gamma+\gamma)+(\gamma+\gamma\alpha+\alpha)+(\alpha+\alpha\beta+\beta)$$
$$=(\beta+\beta\gamma)+(\gamma+\gamma\alpha)+(\alpha+\alpha\beta)$$
$$=\bar{A}B+\bar{B}C+\bar{C}A \qquad \text{(or, by symmetry, } \bar{A}C+\bar{B}A+\bar{C}B\text{)}$$

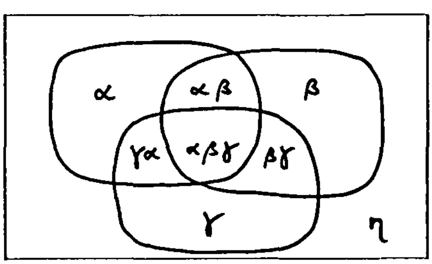

Figure 4. Three-set Venn diagram demonstrating shorthand notation.

Note that the way to achieve this simplification, using set algebra without recourse to a Venn diagram, is not at all obvious (even if one is given the result). However, ideally, one keeps an open mind to both approaches, when one could immediately notice the alternative simplification of $(\bar{A}+\bar{B}+\bar{C})(A+B+C)$. Both simplifications involve five simple operations (unions or intersections), whilst the LHS requires eight.

The second simplification also follows from the Venn diagram (which of course it must do), but is less obvious that way. A hybrid approach is instructive

$$\begin{aligned}\text{LHS} &= \overline{\eta+\alpha\beta\gamma}\\ &= \bar{\eta}\cdot\overline{\alpha\beta\gamma}\\ &= (A+B+C)(\bar{A}+\bar{B}+\bar{C})\end{aligned}$$

where the first use of De Morgan's laws could be geometric, but the second is algebraic (at least for someone with the usual bias in training to that approach). Although note that a geometric proof is implied by example 1.

Example 4. Prove $A\bar{B}+AC+BC=A\bar{B}+BC$ (the Race-Hazard theorem).

$$\text{LHS}=(\alpha+\gamma\alpha)+(\cancel{\gamma\alpha}+\alpha\cancel{\beta\gamma})+(\alpha\beta\gamma+\beta\gamma)=\text{RHS}$$

Thus the theorem arises from obvious simplification of the LHS using the Venn diagram. Simplifying the LHS algebraically is far less obvious; although, if one knows the RHS required, a little ingenuity suggests the trick of writing AC as $A\bar{B}C+ABC$ which provides two redundant terms ($A\bar{B}C\subseteq A\bar{B}$, $ABC\subseteq BC$).

5. Discussion

We are not denying the value of teaching 'formal' set-theoretic proof. This indeed has a place in the teaching of engineers and scientists, as well as mathematicians. But, mainly, because it helps to provide insight (as does the use of Venn diagrams) into set theory and is thus a useful aid to learning about sets. If you like, it gives a further angle on manipulating sets. Whether it is intrinsically any more rigorous than using Venn diagrams is open to doubt. (It seems that we have the typical duality of approach—geometric and algebraic—and neither way of thinking is fundamentally more correct than the other; although one may be more fashionable or more highly developed—that is: thought about, used, and assimilated.)

Certainly, there is little justification for generation after generation of mathematics students (let alone those studying science and technology) being so consistently biased in favour of algebra over geometry. And I can see no defence for teachers categorially stating that only 'formal' (algebraic) arguments are acceptable, and that 'Venn diagrams do not constitute a proof'.

Of course, what they really mean is that mathematical fashion favours one approach, so that both they, and their own teachers before them, have given scant thought to the other angle. But I maintain that one can think just as tightly in terms of Venn diagrams as one can in terms of set-notation, and that indeed the two perspectives are exactly analogous.

The clue of course, to a manageable geometric approach, is to rephrase the original question, so that it reads: 'How many *fundamentally different* Venn diagrams are there?' And I believe, for any finite number of sets, the answer is just one. That this is indeed transparent follows from the isomorphism which exists between sets and their Venn representations.

Of course, given any n sets, the number of special cases that can be constructed is evidently $N=2^{2^n}=4^n$, since each symbol η, α, $\alpha\beta$ etc. can be either null or non-null; and there are 2^n of these symbols, as we have the choice of either including or excluding each of the n Greek letters to obtain symbols—or, alternatively, one may note how many terms there are in the expansion of

$$(1+\alpha)(1+\beta)(1+\gamma)\ldots=1+\Sigma\alpha+\Sigma\alpha\beta+\Sigma\alpha\beta\gamma+\ldots.$$

However, if we really needed to consider all these N cases individually, then we would also need to consider all the analogous N subproofs in the algebraic proof as well.

6. Conclusion: the universal Venn diagram

At the risk of labouring the point, any three-set Venn diagram can be mapped onto or from a 'universal' three-set diagram, like figure 5. Here, each of the eight disparate domains may be either null or non-null (although, if all are null, then so of course is the universal set under consideration).

This then yields $2^8=256$ possibilities, which are rather too many to illustrate individually. Rather, we just show all the cases for $n=2$, with universal diagram shown in figure 6. The $2^4=16$ special cases are then as shown in figure 7.

Some slight variations are possible within this classification. For instance, (8) of figure 7 can be displayed equally well as in figure 8—as either (*a*) or even (*b*).

Clearly, it is not generally necessary to consider the distinction between $\eta=\varnothing$ and $\eta\neq\varnothing$; which reduces the effective choice of alternatives to 2^{2^n-1} ($=8$, here). That is, one can usually ignore any exterior to the union of all the n sets, and need not worry about whether the universal set is completely covered or not. We effectively

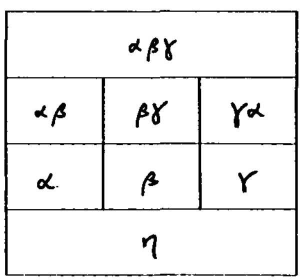

Figure 5. Universal three-set Venn diagram.

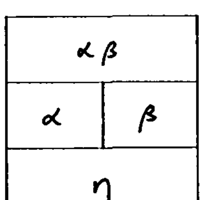

Figure 6. Simpler universal two-set Venn diagram.

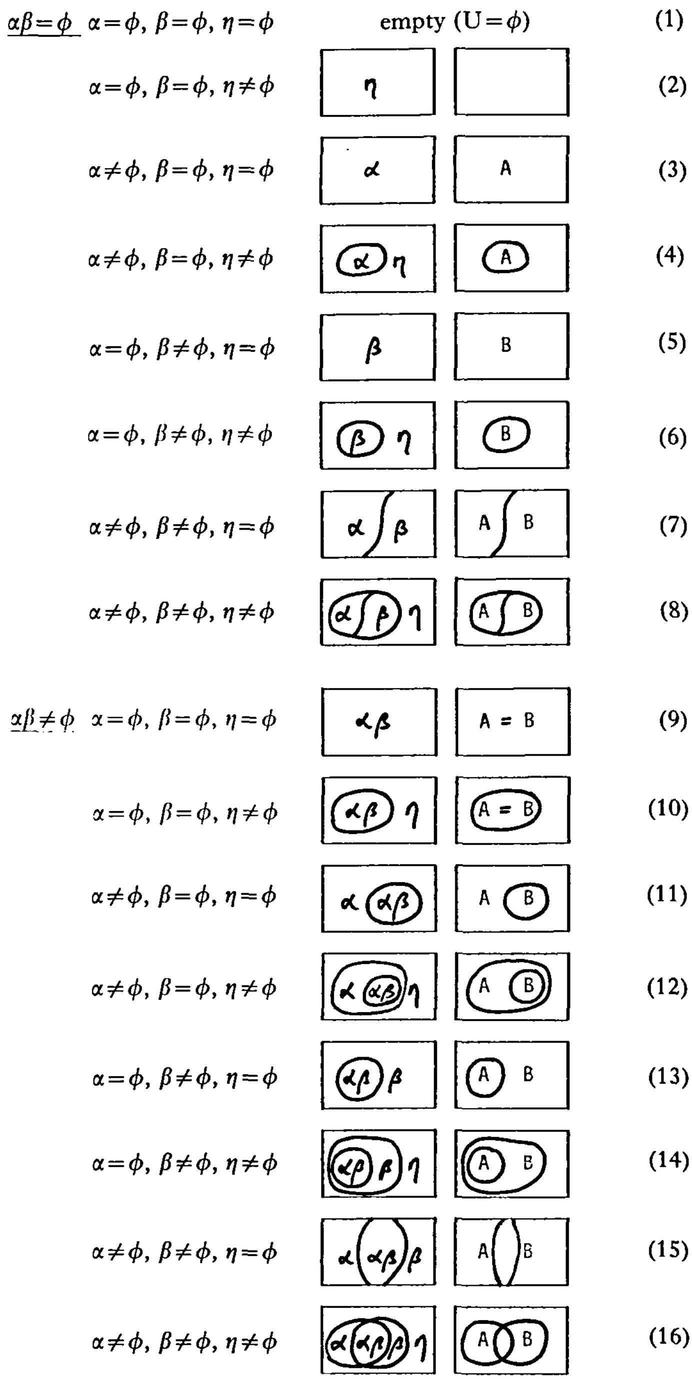

Figure 7. All sixteen special cases for a two-set Venn diagram.

(a)  (b)

Figure 8. Variations to alternative (8) of figure 7.

discard that part of U which does not interest us, and work with a smaller universal set, the union of the n sets.

However, such enumerations of the special cases, although instructive, are quite unnecessary in practice. For, allowing null domains, any n-set special case Venn diagram can be mapped onto or from the corresponding universal (n-set) diagram; which, for smaller n at any rate, can be conveniently represented by n intersecting 'circles', to provide all the 2^n regions of interest. Thus the 4^n (or $4^n/2$) alternative diagrams are really all equivalent.