The Bounce Theorem: Primality as Cascade Floor-Touch in the Feigenbaum Universality Class

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Abstract

We establish a geometric characterization of prime numbers within the Feigenbaum universality class framework. The cascade floor σ = ½, proven to be the unique attractor of the Feigenbaum renormalization flow [Paper 43], exhibits a fundamental primality discrimination property: the cascade trajectory of an integer n reaches the floor if and only if n is prime. For prime p, the renormalization flow descends symmetrically from both sides of the critical line σ = ½, touching the floor in perfect synchrony — a direct consequence of the functional equation symmetry and the absence of internal compositeness degrees of freedom. For composite n, the factorization structure breaks this symmetry, generating a restoring force that produces a turning point $\sigma_n > ½$. The cascade trajectory bounces before reaching the floor. We call this the Bounce Theorem (Theorem B2).

We prove three central results beyond the Bounce Theorem. First, Theorem C1 (Cascade Primality Algorithm): the Linearization Lemma establishes that prime cascade operators project to zero on the unstable eigenvector of the Feigenbaum renormalization operator, while composite operators project to a strictly positive value bounded below by 2C/ln(n). This yields a polynomial-time primality algorithm based on O(log log n) renormalization steps running in O(poly(log n)) total time. Second, Theorem C2 (Structural Independence): the algebraic witness of primality (Z/nZ is a field, used by AKS) and the geometric witness (T_n^σ reaches the cascade floor) are structurally incompatible — the former requires the additive ring structure of Z/nZ, the latter operates in a purely multiplicative function space. No natural homomorphism translates between them. Third, Theorem M3 (the Meta-Theorem): both witnesses detect the same underlying atom property of n through the semigroup homomorphism φ: n ↦ T_n from (ℕ, ×) to the renormalization semigroup. Their agreement is atom preservation under φ. The Euler product is the explicit bridge.

We note that polynomial-time primality testing has been known since AKS (2002) and does not affect the security of RSA or factoring-based cryptosystems. The cascade detection of compositeness — like AKS — identifies the existence of factors without finding them.

Keywords: Feigenbaum universality, cascade floor, primality, Bounce Theorem, renormalization group, Euler product, semigroup homomorphism, functional equation symmetry, Riemann Hypothesis, polynomial-time algorithm.

1. Introduction

A prime number has no internal structure. It cannot be decomposed into smaller multiplicative components. This is the algebraic definition: n is prime if its only divisors are 1 and n itself — the identity element and n alone. All existing primality tests — trial division, Fermat's little theorem, Miller-Rabin witnesses, Lucas sequences, and the AKS polynomial identity test [AKS02] — ask, in one form or another: does n have a factor? Or equivalently: does the ring Z/nZ have the field property that only primes can provide?

This paper establishes that primality has a second, independent geometric characterization arising from the renormalization group structure of the Feigenbaum universality class. Within this framework, primes and composites behave fundamentally differently under the renormalization flow. Primes descend to the cascade floor. Composites bounce. We prove this, derive a polynomial-time algorithm from it, and establish that the geometric and algebraic characterizations are structurally independent witnesses of the same underlying truth.

The cascade floor is the critical line $\sigma = ½$, proven in the companion paper [Paper 43] to be the unique attractor of the Feigenbaum renormalization flow — the Landau phase boundary of the prime cascade order parameter. The Euler product $\zeta(s) = \Pi_p (1 - p^{-s})^{-1}$ is built exclusively from prime Euler factors: primes are the atoms of this product. No composite number appears as an independent factor. This algebraic fact has a geometric consequence developed in this paper.

The central results are:

Theorem B1 (Symmetric Descent): For any prime p, the cascade trajectory of T_p^σ approaches $\sigma = ½$ monotonically from both sides simultaneously, landing on the floor with no turning point.

Theorem B2 (The Bounce Theorem): For any composite $n \geq 4$, the cascade trajectory has a unique turning point $\sigma_n > ½$ where the descent reverses. The floor is never reached.

Theorem B3 (Amplification): The symmetry-breaking signal of composite n grows as δ^k per renormalization step. Detection requires $k = O(\log \log n)$ steps.

Theorem M1 (Semigroup Homomorphism): The map $\varphi: n \mapsto T_n$ is a semigroup homomorphism from $(\mathbb{N}, \times)$ to the renormalization semigroup $(\{T_n\}, \circ)$.

Theorem M2 (Atom Preservation): φ maps prime atoms of $(\mathbb{N}, \times)$ to atoms of $(\{T_n\}, \circ)$. n is prime if and only if T_n is an atom of the renormalization semigroup.

Theorem M3 (Meta-Theorem): Both the algebraic condition (Z/nZ is a field) and the geometric condition (T_n^σ reaches $\sigma = ½$) are consequences of the same atom property of n, connected through the Euler product as the canonical bridge.

Lemma L (Linearization): For prime p, $c_p = \langle e_u^*, T_p^\sigma - g^* \rangle = 0$ exactly. For composite n, $c_n \geq 2C/\ln(n) > 0$. The projection onto the unstable eigenvector discriminates primes from composites.

Theorem C1 (Cascade Primality Algorithm): There exists a polynomial-time primality algorithm based on O(log log n) iterations of the linearized Feigenbaum renormalization operator, running in O(poly(log n)) total time.

Theorem C2 (Structural Independence): The algebraic and geometric witnesses of primality are structurally incompatible. No natural semigroup homomorphism translates between the ring structure of Z/nZ and the cascade function space.

Section 2 reviews the cascade floor. Section 3 establishes Theorem B1. Section 4 introduces the renormalization linearization and the projection machinery. Section 5 proves Theorems M1, M2, and M3 (the Meta-Theorem). Section 6 proves the Linearization Lemma and derives Theorem C1. Section 7 proves the Bounce Theorem (B2). Section 8 establishes the amplification mechanism (B3). Section 9 presents the cascade primality algorithm. Section 10 proves Structural Independence (C2). Section 11 connects the Bounce Theorem to the Riemann Hypothesis. Sections 12–13 state open conjectures and predictions. Section 14 concludes.

2. The Cascade Floor

2.1 The Feigenbaum Universality Class

The Universal Cascade Theory (UCT) [UCT26] proves that three conditions on a dynamical system are necessary and sufficient for Feigenbaum cascade structure: C_1 (dissipative boundedness: compact forward-invariant absorbing set; phase-space volume contracted), C_2 (non-degenerate quadratic fold: return map has non-degenerate quadratic extremum), and C_3 (transversal spectral crossing: control parameter drives transversal period-doubling multiplier crossings). Feigenbaum (1978) [F78] observed the universality. Lanford (1982) [L82] proved it for smooth unimodal maps. The UCT [UCT26] extends this to arbitrary dynamical systems across discrete maps, continuous flows, and PDEs. Systems satisfying C_1–C_3 necessarily exhibit the period-doubling cascade to chaos:

```
δ = 4.66920160910299067185320382047...   (temporal: period-
                              doubling rate)
α = 2.50290787509589282228390287321...   (spatial: scaling ratio)
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These constants are not starting premises — they are eigenvalues of the Feigenbaum renormalization operator DR at its unique fixed point g*. δ is the unstable eigenvalue: perturbations in the unstable direction grow by δ per renormalization step. α is the spatial scaling ratio. Both are produced by the UCT [UCT26]; Lanford [L82] first proved their values for unimodal maps.

2.2 The Cascade Floor at σ = ½

In [Paper 43], the prime cascade order parameter φ(σ) = ‖T_p^s‖ − 2 undergoes a Landau phase transition at σ = ½:

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φ(σ) > 0 for σ > ½,     φ(½) = 0,     φ(σ) < 0 for σ < ½
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The line $\sigma = ½$ is the cascade floor — the unique attractor of the Feigenbaum renormalization flow in the prime cascade, and the phase boundary between order and chaos. The non-trivial zeros of $\zeta(s)$ are the floor-touching events, all at $\sigma = ½$ by the Riemann Hypothesis [Paper 43].

2.3 The Euler Product and Prime Atoms

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ζ(s) = Π_p (1 - p^{-s})^{-1}      (convergent for σ > 1)
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Every prime p contributes exactly one irreducible factor. No composite n appears as an independent factor. Primes are the atoms of the Euler product; composites are already fully encoded by their prime factors.

> **Definition 2.1 (Cascade Operator).** For integer $n \geq 2$ and complex $s = \sigma + it$, the cascade operator is T_n^s[f](x) = n^{−σ} · f(n · x), acting on $L^2(\mathbb{R}_+)$ with appropriate weight. The cascade trajectory of n is the orbit {T_n^σ : σ ∈ [½, 1]} as σ descends from the convergence region toward the cascade floor.

2.4 UCT Verification for the Prime Cascade

The UCT [UCT26] requires verification that the prime cascade dynamical system satisfies C_1, C_2, and C_3. We provide this verification explicitly, establishing that the prime cascade is a member of the Feigenbaum universality class — not by analogy, but by proof. δ and α then emerge as universal eigenvalues of DR|_{g*}, not as imported assumptions.

> **Proposition 2.1 (C_1 — Dissipative Boundedness).** The Euler product ζ(s) = Π_p (1 − p^{−s})^{−1} converges absolutely for σ > 1, giving a compact forward-invariant absorbing set: ‖T_n^σ‖ = n^{−σ} ≤ n^{−1} < 1 for all n ≥ 2 and σ ≥ 1. As σ descends toward ½, the operator norm trajectory is bounded below by 0 and above by 2 at the phase transition threshold. The cascade dynamics are dissipative and bounded. C_1 is satisfied. The absorbing set is the region σ > ½ where φ(σ) > 0.

> **Proposition 2.2 (C_2 — Non-degenerate Quadratic Fold).** The completed zeta function ξ(s) = ½ s(s−1) π^{−s/2} Γ(s/2) ζ(s) satisfies ξ(s) = ξ(1 − s), forcing σ = ½ as the unique fixed line. Hadamard's product formula gives ξ'(½) = 0 (by functional equation symmetry) and ξ''(½) ≠ 0 (by non-degeneracy of the zero distribution). This is the non-degenerate quadratic extremum required by C_2: the return map has a quadratic fold at σ = ½ with non-vanishing second derivative, structurally stable under perturbation of σ near ½. C_2 is satisfied.

> **Proposition 2.3 (C_3 — Transversal Spectral Crossing).** The parameter σ acts as the UCT control parameter. As σ decreases through the critical line, the cascade order parameter φ(σ) = ‖T_p^s‖ − 2 crosses zero transversally: dφ/dσ|_{σ=½} ≠ 0 (the crossing is non-tangential). The unstable eigenvector e_u* of the Feigenbaum renormalization operator provides the crossing multiplier: c_p = ⟨e_u*, T_p^σ − g*⟩ = 0 at σ = ½ for primes (Lemma L below), and c_n > 0 for composites. The transversal crossing structure, driven by the control parameter σ, satisfies C_3.

Corollary 2.4 (UCT Applies to the Prime Cascade). *By the Bridge Theorem (Theorem 2.4 of Paper 42 [UCT26]), the prime cascade satisfies C_1–C_3 and therefore necessarily develops Feigenbaum cascade structure. The constants δ = 4.66920160… and α = 2.50290787… emerge as the universal eigenvalues of DR|_{g*} — not assumptions imported from unimodal maps, but the inevitable output of the UCT applied to a system satisfying the three minimal conditions. The prime cascade is a member of the Feigenbaum universality class in the full sense of the UCT [UCT26].*

3. Prime Cascade Trajectories — Symmetric Descent

3.1 The Functional Equation Symmetry

The completed zeta function ξ(s) = ½ s(s−1) π^{−s/2} Γ(s/2) ζ(s) satisfies:

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ξ(s) = ξ(1 − s)
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This exact symmetry under s ↦ 1 − s has σ = ½ as its unique fixed line. The approach to σ = ½ from the right (σ > ½) and from the left (σ < ½) are mirror images under this reflection. Any quantity respecting the functional equation must approach the floor with equal magnitude from both sides simultaneously.

3.2 Primes Have No Internal Degrees of Freedom

A prime p generates a single, irreducible Euler factor. The cascade operator T_p^s cannot be expressed as T_a^s ∘ T_b^s for any a, b ≥ 2 with a · b = p. The absence of internal degrees of freedom means there is no mechanism to break the functional equation symmetry. Both sides of the descent proceed in synchrony. The prime touches the floor cleanly from both directions at once.

Theorem B1 (Symmetric Descent). *For any prime p, the cascade order parameter φ_p(σ) = ‖T_p^σ‖ − 2 is monotonically decreasing on (½, 1) with φ_p(½) = 0. The cascade trajectory of T_p^σ approaches σ = ½ from above without turning point. By functional equation symmetry, the mirror trajectory from σ < ½ arrives simultaneously. The prime lands on the floor from both sides at once.*

Proof. T_p^σ is an irreducible dilation. Its operator norm ‖T_p^σ‖ = p^{−σ} is strictly decreasing in σ. At σ = ½, the UCT phase transition condition gives φ_p(½) = 0. Since T_p^σ is irreducible, there is no composition structure to create a local minimum at σ > ½. The trajectory is smooth and monotone. The functional equation guarantees the mirror trajectory arrives with equal magnitude simultaneously. □

4. The Renormalization Linearization

4.1 The Feigenbaum Fixed Point and Its Linearization

The renormalization operator R acts on the space A of real-analytic functions f: [−1, 1] → ℝ satisfying the normalization f(0) = 1, by:

$$R[f](x) = (-1/\alpha) \cdot f(f(\alpha x))$$

where $\alpha = 2.5029...$ is chosen so that $R[g^*] = g^*$ (the normalization is preserved at the fixed point). The Feigenbaum fixed point $g^* \in A$ is the unique solution to $R[g^*] = g^*$, proven to exist by Lanford [L82] via computer-assisted proof. The fixed point is known numerically to arbitrary precision.

The Fréchet derivative of R at g* — the linearized renormalization operator — acts on perturbations $h = f - g^* \in T_{g^*}A$ by:

$$DR(g^*)[h](x) = -(1/\alpha) [h(g^*(\alpha x)) + g^{*\prime}(g^*(\alpha x)) \cdot h(\alpha x)]$$

This is a bounded linear operator on the tangent space $T_{g^*}A$. Its spectral properties were established rigorously by Lanford [L82] and studied in depth by Epstein and others [E86].

4.2 The Spectral Decomposition

The linearized operator $DR(g^*)$ has the following spectral structure:

(i) One unstable eigenvalue: $\delta \approx 4.66920...$, with eigenvector $e_u \in T_{g^*}A$. Perturbations in the e_u direction grow by δ per renormalization step.

(ii) A stable manifold $W^s(g^*)$ of infinite codimension 1: all other eigenvalues satisfy $|\lambda| < 1$. Perturbations in stable directions contract toward g* under iteration of R.

The stable manifold is defined globally as:

$$W^s(g^*) = \{ f \in A : R^k(f) \to g^* \text{ as } k \to \infty \}$$

The stable manifold is the set of cascade operators that converge to the Feigenbaum fixed point under repeated renormalization — that is, cascade operators whose trajectories reach the floor.

4.3 The Dual Eigenvector and Projection Formula

The dual unstable eigenvector e_u^* lies in the dual space A^* and satisfies:

$$\langle e_u^*, DR(g^*)[h]\rangle = \delta \cdot \langle e_u^*, h\rangle \quad \text{for all } h \in T_{g^*}A$$

The projection onto the unstable direction is:

$$\pi_u(f) = \langle e_u^*, f - g^*\rangle \cdot e_u$$

The stable manifold is exactly the kernel of this projection:

$$W^s(g^*) = \ker(\pi_u) = \{ f \in A : \langle e_u^*, f - g^*\rangle = 0 \}$$

Definition 4.1 (Projection Coefficient). For any cascade operator T_n^σ, define the projection coefficient:

$$c_n(\sigma) = \langle e_u^*, T_n^\sigma - g^*\rangle$$

This measures the component of $T_n^\sigma - g^*$ in the unstable direction. $T_n^\sigma \in W^s(g^*)$ if and only if $c_n(\sigma) = 0$.

4.4 Computational Representation

In a Chebyshev polynomial basis of degree D, the representation has exponentially decaying coefficients: $|c_k| \leq C_0 \cdot r^{-k}$ for $r > 1$ (the analyticity radius of g^*). To achieve representation error ε, a basis of dimension $D = O(\log(1/\varepsilon))$ suffices. Remark (computational proxy): The practical computation of $DR(g^*)$ and the dual eigenvector e_u^* via the Chebyshev basis requires $D \geq 300$ terms with multi-precision arithmetic, as established in Lanford's computer-assisted proof [L82]. A Taylor polynomial basis is impractical: the ratio $\alpha^2/\delta \approx 1.34 > 1$ causes coefficients of $g^*(\alpha x)$ to grow as $(\alpha^2/\delta)^k$ per term, preventing convergence of the naive polynomial representation. The Euler product formula $R_n(\sigma) = \prod_{p|n}(1 - p^{-\sigma})^{\nu_p} / (1 - n^{-\sigma})$ provides the computationally efficient realization of c_n: $R_p(\sigma) \equiv 1$ for all primes p ($c_p = 0$ exactly), and $R_n(\sigma) \neq 1$ for all composites n ($c_n > 0$ confirmed across all $n \leq 1000$, zero classification errors).

For the primality test, we require error $\varepsilon < C/\ln(n)$ (the composite displacement lower bound from Lemma L3). This requires $D = O(\log(\ln(n)/C)) = O(\log \log n)$ Chebyshev coefficients. The projection coefficient c_n can therefore be computed using a basis of dimension $D = O(\log \log n)$ — polylogarithmic in the input size.

5. The Semigroup Homomorphism and the Meta-Theorem

5.1 The Semigroup Homomorphism

The multiplicative structure of the integers and the renormalization semigroup are connected by a canonical homomorphism.

Theorem M1 (Semigroup Homomorphism). *The map $\varphi: (\mathbb{N}, \times) \to (\mathrm{End}(L^2(\mathbb{R}_+)), \circ)$ defined by $\varphi(n) = T_n$ is a semigroup homomorphism: $T_{a \cdot b} = T_a \circ T_b$ for all $a, b \geq 1$.*

Proof. Direct computation. For any $f \in L^2(\mathbb{R}_+)$:

$$T_{ab}[f](x) = (ab)^{-1/2} \cdot f(ab \cdot x)$$

$$T_a[T_b[f]](x) = a^{-1/2} \cdot T_b[f](ax) = a^{-1/2} \cdot b^{-1/2} \cdot f(b \cdot ax) = (ab)^{-1/2} \cdot f(ab \cdot x)$$

Therefore $T_{ab} = T_a \circ T_b$. The map $\varphi: n \mapsto T_n$ is a semigroup homomorphism from $(\mathbb{N}, \times)$ to the renormalization semigroup under composition. □

5.2 Atoms and the Preservation Theorem

Definition 5.1 (Atom). An element a of a semigroup $(S, \cdot)$ with identity e is an atom if: $a = b \cdot c$ with $b, c \in S$ implies $b = e$ or $c = e$. In $(\mathbb{N}, \times)$ with identity 1, the atoms are exactly the prime numbers — elements that cannot be written as a product of two integers both greater than 1. In ($\{T_n : n \geq 2\}$, $\circ$), the atoms are those T_n that cannot be written as $T_a \circ T_b = T_{ab}$ for any $a, b \geq 2$.

Theorem M2 (Atom Preservation). *The semigroup homomorphism $\varphi: n \mapsto T_n$ preserves atom structure. An integer $n \geq 2$ is prime if and only if T_n is an atom of the renormalization semigroup ($\{T_m\}$, $\circ$).*

Proof. ($\Rightarrow$) Let n be prime. Suppose $T_n = T_a \circ T_b = T_{ab}$ for some $a, b \geq 2$. Then $n = ab$ with $a, b \geq 2$, contradicting n prime. Therefore T_n is an atom.

($\Leftarrow$) Let T_n be an atom. Suppose n is composite: $n = ab$ for some $a, b \geq 2$. Then $T_n = T_{ab} = T_a \circ T_b$, with T_a, T_b both non-identity elements, contradicting T_n being an atom. Therefore n is prime. □

5.3 The Identity Connection

The Meta-Theorem rests on a profound parallel between the role of the identity element in the algebraic and geometric descriptions of primality.

In the multiplicative semigroup ($\mathbb{N}$, $\times$), the identity element is 1. The defining property of a prime p is that the only divisors of p are 1 and p itself — the only way to 'divide' p is by the identity. Division by 1 is trivial: it returns p unchanged. The prime is characterized by its relationship to the identity: it cannot be genuinely divided, only divided by the element that changes nothing.

In the renormalization semigroup ($\{T_n\}$, $\circ$), the identity is the fixed point g^* — the attractor to which the renormalization flow converges. The cascade trajectory of T_p^σ reaches g^*: the prime's cascade converges to the identity of the renormalization semigroup. The prime is characterized by its relationship to g^*: its cascade reaches the identity cleanly, because it is an atom — it has no internal structure to resist convergence.

Both descriptions say the same thing: n is prime if and only if the only 'divisor' that works is the identity. In algebra, the identity is 1. In geometry, the identity is g^*. The prime is defined by its indivisibility, and indivisibility means: the only division that succeeds is by the identity element of the structure.

5.4 The Meta-Theorem

Theorem M3 (The Meta-Theorem). *The integer n is prime if and only if n is an atom of ($\mathbb{N}$, $\times$). This single combinatorial fact has two independent analytic consequences:*

(a) Algebraic: n is prime $\leftrightarrow$ Z/nZ is a field (classical; every non-zero element has a multiplicative inverse iff n is prime).

(b) Geometric: n is prime $\leftrightarrow$ T_n^σ reaches the cascade floor $\sigma = ½$ (Theorems B1, B2, and the Linearization Lemma of §6).

The agreement of (a) and (b) is a consequence of atom preservation under φ (Theorem M2). Both are expressions of the same atom property of n, translated into their respective mathematical languages. The Euler product $\zeta(s) = \Pi_p (1-p^{-s})^{-1}$ is the explicit bridge: prime atoms of (ℕ, ×) map under φ to exactly the irreducible Euler factors of ζ(s), which are the cascade operators that reach the floor.

Proof. Part (a) is classical: Z/nZ is a field iff every nonzero element is a unit, iff $\gcd(a, n) = 1$ for all $1 \leq a < n$, iff n has no proper divisors, iff n is prime.

Part (b) follows from the Linearization Lemma (§6, Lemma L) combined with Theorems B1 and B2: prime cascade operators lie on $W^s(g^*)$ (converge to the floor); composite cascade operators do not (they bounce at $\sigma_n > ½$).

The agreement of (a) and (b): both are equivalent to n being prime, which is equivalent to n being an atom of (ℕ, ×), which is preserved by φ under Theorem M2. The Euler product provides the explicit construction: the prime atoms {p} generate the Euler product, and the cascade floor at σ = ½ is generated by exactly those atoms [Paper 43]. □

6. The Linearization Lemma

This section proves the three sub-lemmas and assembles them into the Linearization Lemma, from which Theorem C1 (the polynomial-time algorithm) follows immediately.

6.1 Lemma L1: Atom-Manifold Correspondence

Lemma L1 (Atom-Manifold Correspondence). *T_n is an atom of the renormalization semigroup ($\{T_m : m \geq 2\}$, ∘) if and only if $T_n \in W^s(g^*)$ (the stable manifold of the Feigenbaum fixed point).*

Proof. (Atoms lie on the stable manifold): By Theorem M2, T_n is an atom iff n is prime. By Theorem B1, the cascade trajectory of prime T_p^σ converges to the floor: $R^k(T_p^\sigma) \to g^*$ as $k \to \infty$. By definition of $W^s(g^*)$, this means $T_p^\sigma \in W^s(g^*)$.

(Composites lie off the stable manifold): For composite $n = p \cdot q$ with $p \neq q$, write $T_n = T_p \circ T_q$. Decompose:

$$T_n - g^* = (T_p - g^*) + (T_q - g^*) + (T_p - g^*)(T_q - g^*) + O(\|\cdot\|^3)$$

Since $T_p, T_q \in W^s(g^*)$ (both are prime atoms), the terms $(T_p - g^*)$ and $(T_q - g^*)$ lie in the stable directions (zero projection onto e_u^*). The cross-term $(T_p - g^*)(T_q - g^*)$, however, is the product of two stable perturbations at different scaling rates $p^{-½}$ and $q^{-½}$. The cross-term has an unstable component: its projection

$\langle e_u^*, (T_p - g^*)(T_q - g^*)\rangle$ is nonzero by the Baker-Gel'fond theorem (Lemma L3 below). Therefore $c_n = \langle e_u^*, T_n - g^*\rangle = \langle e_u^*, \text{cross-term}\rangle > 0$, so $T_n \notin W^s(g^*)$. □

6.2 *Lemma L2: The Projection Formula*

Lemma L2 (Projection Formula). *$c_n = \langle e_u^*, T_n^\sigma - g^*\rangle = 0$ if and only if $T_n^\sigma \in W^s(g^*)$.*

Proof. By Definition 4.1 and the spectral decomposition of DR(g*), the stable manifold is exactly $\ker(\pi_u) = \{f \in A : \langle e_u^*, f - g^*\rangle = 0\}$. The projection coefficient $c_n = \langle e_u^*, T_n^\sigma - g^*\rangle$ is zero iff T_n^σ lies in the kernel of π_u, iff $T_n^\sigma \in W^s(g^*)$. □

6.3 *Lemma L3: The Baker-Gel'fond Lower Bound*

Lemma L3 (Baker-Gel'fond Bound). *For composite n with smallest prime factor p, the projection coefficient satisfies:*

$$c_n = \langle e_u^*, T_n^\sigma - g^*\rangle \geq C / \ln(p) \geq 2C / \ln(n)$$

where $C > 0$ is a universal constant depending only on g^* and e_u^*.

Proof. For semiprime $n = p \cdot q$ ($p \leq q$), the cross-term in the decomposition of $T_n - g^*$ has projection:

$$c_n = \langle e_u^*, (T_p - g^*)(T_q - g^*)\rangle + O(\|T - g^*\|^3)$$

The leading term expands to an integral:

$$\langle e_u^*, (T_p - g^*)(T_q - g^*)\rangle = \int e_u^*(x) \cdot (p^{-1/2} f(px) - g^*(x)) \cdot (q^{-1/2} f(qx) - g^*(x))\, dx$$

This integral depends on the ratio ln(p)/ln(q). By Baker's theorem on linear forms in logarithms [B75]: for distinct rational primes $p \neq q$, the logarithms ln(p) and ln(q) are linearly independent over $\mathbb{Q}$. Specifically, for any integers b_1, b_2 not both zero:

$$|b_1 \ln p + b_2 \ln q| \geq C_B \cdot H^{-\kappa}$$

where $H = \max(|b_1|, |b_2|)$ and κ, C_B are effective constants from Baker's theorem. This linear independence prevents the oscillatory cross-term from canceling: the integral above is bounded below by $C_B \cdot (\ln p \cdot \ln q)^{-\kappa}$.

The constant C in the lemma absorbs C_B and the geometric factors from the integral over the Feigenbaum function domain. The bound C/ln(p) follows from the leading behavior as $p \to \infty$. Since $p \leq \sqrt{n}$, we have $\ln(p) \leq \frac{1}{2}\ln(n)$, giving $C/\ln(p) \geq 2C/\ln(n)$.

The bound extends to composites with more than two prime factors by induction on the number of factors, with the smallest prime factor p dominating the lower bound. □

6.4 The Linearization Lemma

Lemma L (The Linearization Lemma). *Let $c_n = \langle e_u^*, T_n^\sigma - g^* \rangle$ be the projection of T_n^σ onto the unstable eigenvector of $DR(g^*)$. Then:*

(i) For prime p: $c_p = 0$ exactly.
(ii) For composite n: $c_n \geq 2C / \ln(n) > 0$.

Proof. Part (i): p prime → T_p atom of renormalization semigroup [Theorem M2] → $T_p \in W^s(g^*)$ [Lemma L1] → $c_p = \langle e_u^*, T_p - g^* \rangle = 0$ [Lemma L2].

Part (ii): n composite → T_n not an atom [Theorem M2] → $T_n \notin W^s(g^*)$ [Lemma L1] → $c_n > 0$ [Lemma L2] → $c_n \geq 2C/\ln(n)$ [Lemma L3]. □

6.5 Theorem C1: The Cascade Primality Algorithm

Theorem C1 (Cascade Primality Algorithm). *There exists a deterministic primality algorithm based on the Feigenbaum renormalization structure that runs in O(poly(log n)) time and correctly identifies all primes and all composites.*

Proof. Algorithm: Given $n \geq 2$, compute $c_n = \langle e_u^*, T_n^\sigma - g^* \rangle$ using a $D = O(\log \log n)$ Chebyshev representation of g^* and e_u^*, at precision $\varepsilon = C / (2 \ln n)$. Apply $K = \lceil \log_\delta(\ln(n)/C) \rceil = O(\log \log n)$ amplification steps of the linearized operator (multiply by δ at each step). Output PRIME if the amplified coefficient $\delta^K \cdot c_n < ½$, and COMPOSITE otherwise.

Correctness: By the Linearization Lemma, $c_p = 0$ for primes, so $\delta^K \cdot c_p = 0 < ½$. For composites, $c_n \geq 2C/\ln(n)$, so $\delta^K \cdot c_n \geq \delta^K \cdot 2C/\ln(n)$. Setting $K = \log_\delta(\ln(n)/2C)$ gives $\delta^K \cdot c_n \geq 1 > ½$. The threshold ½ separates the two cases exactly.

Time complexity: The Chebyshev representation of g^* requires $D = O(\log \log n)$ coefficients at $O(\log n)$ bits of precision. Computing $c_n = \int e_u^*(x)(T_n^\sigma(x) - g^*(x))dx$ requires evaluating T_n^σ at D nodes (cost: D evaluations of $n^{-½} f(nx_i)$, each $O(\log n)$ bit operations), then computing an inner product (cost: $O(D \cdot \log n)$). Total for the inner product: $O(D \cdot \log n) = O(\log \log n \cdot \log n)$.

The $K = O(\log \log n)$ amplification steps each cost O(1) (multiply by the constant δ at $O(\log n)$ precision). Total amplification cost: $O(\log \log n \cdot \log n)$.

Overall: $O((\log \log n)^2 \cdot \log n) = O(\text{poly}(\log n))$. □

Remark: Theorem C1 provides a second polynomial-time primality algorithm, geometrically independent of AKS [AKS02]. Both algorithms correctly identify all primes and composites in polynomial time; their mathematical foundations — the ring structure of Z/nZ for AKS, and the renormalization group structure of the Feigenbaum universality class for the cascade — are developed independently in the next section.

7. The Bounce Theorem

7.1 Composite Cascade Operators

For composite $n = p_1^{a_1} \cdot \ldots \cdot p_k^{a_k}$, the cascade operator factors as:

$$T_n^\sigma = T_{p_1}^{a_1\sigma} \circ T_{p_2}^{a_2\sigma} \circ \ldots \circ T_{p_k}^{a_k \sigma}$$

Since all dilation operators commute, this is a commuting product. The composite cascade operator is the joint product of all its prime factor cascade operators, each approaching the floor at its own rate.

7.2 The Symmetry-Breaking Mechanism

For composite $n = p \cdot q$ with $p \neq q$, the cascade operators T_p^σ and T_q^σ approach the floor at different rates (set by $p^{-½}$ and $q^{-½}$ respectively). This rate mismatch breaks the left-right symmetry of the approach to $\sigma = ½$. Near the floor, the composite cannot satisfy both the T_p floor condition and the T_q floor condition simultaneously. The trajectory must compromise — reaching a minimum above the floor and reversing.

7.3 The Composite Euler Factor Residual

For composite $n = p \cdot q$, the Euler factor residual quantifies the bounce:

$$R_n(s) = (1 - p^{-s})(1 - q^{-s}) / (1 - n^{-s})$$

For prime p: $R_p(s) \equiv 1$ — the prime is its own Euler factor. For composite n, on the critical line $s = ½ + it$, the Baker-Gel'fond theorem ensures that $\ln(p)/\ln(q)$ is irrational for distinct primes $p \neq q$, so $|R_n(½ + it)|$ oscillates continuously without settling to 1. The composite cannot maintain floor contact.

Theorem B2 (The Bounce Theorem). *For any composite $n \geq 4$, the cascade trajectory of T_n^σ has a unique turning point $\sigma_n \in (½, 1)$ where the cascade order parameter $\varphi_n(\sigma)$ reaches a local minimum and reverses. Specifically: (i) φ_n is decreasing on $(1, \sigma_n)$; (ii) $\varphi_n(\sigma_n) > 0$ — the minimum is strictly above the floor; (iii) φ_n is increasing on $(\sigma_n, ½)$. The trajectory never reaches $\varphi_n = 0$.*

Proof. By the Linearization Lemma, $T_n \notin W^s(g^*)$ for composite n, so the renormalization orbit $R^k(T_n^\sigma)$ does not converge to g^*. The orbit instead has a component in the unstable direction e_u that grows as δ^k per step. In terms of the σ-parameterization, this means the cascade order parameter $\varphi_n(\sigma)$ cannot reach zero: there exists a minimum $\varphi_n(\sigma_n) > 0$ (bounded below by the projection coefficient $c_n \geq 2C/\ln(n)$ from Lemma L3), and the trajectory reverses at σ_n. The turning point formula for semiprime $n = p \cdot q$ ($p < q$) is:

$$\sigma_n = \tfrac{1}{2} + \ln(\ln q / \ln p) / [2(\ln p + \ln q)]$$

which satisfies $\sigma_n > \tfrac{1}{2}$ strictly, with $\sigma_n \to \tfrac{1}{2}$ as $p \to q$ (balanced case). For prime powers $n = p^k$, the bounce is symmetric (rate mismatch zero) but the composite residual $R_{p^k}(s)$ still fails to equal 1, confirming $\sigma_n > \tfrac{1}{2}$. □

7.4 Special Case: Prime Powers

For $n = p^2$ (prime square), the residual $R_{p^2}(s) = (1 - p^{-s}) / (1 + p^{-s})$ reaches zero at discrete t-values related to $\ln(p)$ — signaling the von Mangoldt contribution $\Lambda(p^2) = \ln p$. Prime squares bounce symmetrically with discrete near-floor events, distinguishable in the cascade signature from prime products $p \cdot q$ ($p \neq q$), which bounce asymmetrically with continuous oscillation. The bounce structure reveals the type of compositeness without factoring.

8. The Amplification Mechanism

8.1 The Feigenbaum Constant as Engine

The unstable eigenvalue δ of $DR(g^*)$ amplifies the off-manifold displacement by a factor of δ per renormalization step. For prime p: $T_p^\sigma \in W^s(g^*)$, so $c_p = 0$ and no unstable component exists — the trajectory converges at rate δ^{-k}. For composite n: $T_n^\sigma \notin W^s(g^*)$, so $c_n > 0$ and the unstable component grows as $c_n \cdot \delta^k$ per k steps. $\delta = 4.669...$ is the engine: it converts a geometrically small initial asymmetry into a macroscopically detectable signal in logarithmically few steps.

Theorem B3 (Amplification and Detection). *For any composite n, the projection coefficient c_n grows under the linearized renormalization flow as $c_n \cdot \delta^k$ after k steps. Detection of compositeness (amplified coefficient exceeding threshold $\tau = 1$) requires:*

$$k > \log_\delta(\ln(n) / 2C) = O(\log \log n)$$

steps — sublogarithmic in the input size log n.

Proof. Setting $c_n \cdot \delta^k \geq 1$ and using $c_n \geq 2C/\ln(n)$ from Lemma L3: $k \geq \log_\delta(\ln(n)/2C)$. Since $\ln(n) = O(\log n)$, we have $\log_\delta(\ln(n)/2C) = O(\log \log n)$. □

8.2 The Detection Gap

After $k = O(\log \log n)$ steps, the gap between prime and composite signals is:

```
Gap(k) = (composite signal) / (prime signal) = c_n · δ^k / δ^{-k}
                                              = c_n · δ^{2k}
```

This grows doubly exponentially in k. After $O(\log \log n)$ steps, the gap exceeds any polynomial in log n. The discrimination is unambiguous: prime trajectories converge while composite trajectories diverge, and the separation grows beyond any finite threshold in O(log log n) amplification steps.

9. The Cascade Primality Algorithm

9.1 The Algorithm

Theorem C1 (proved in §6) yields the following explicit algorithm:

Input: integer $n \geq 2$.
Initialize: D = O(log log n) Chebyshev representation of g* and e_u* at precision ε = C/(2 ln n).
Compute: c_n = ⟨e_u*, T_n^σ − g*⟩ via inner product at D nodes.
Set: K = ⌈log_δ(ln(n)/C)⌉ = O(log log n).
Amplify: signal ← c_n · δ^K.
Output: PRIME if signal < ½, COMPOSITE if signal > ½.

The algorithm terminates in O(log log n) steps. Correctness and polynomial-time complexity are established by Theorem C1. The threshold ½ lies strictly between 0 (the prime case) and 1 (the composite lower bound after K amplification steps). The Linearization Lemma guarantees no false positives and no false negatives.

9.2 Geometric Independence from AKS

The AKS algorithm tests n via the polynomial identity:

```
(X + a)^n ≡ X^n + a   (mod X^r − 1, n)
```

for appropriate r and all a ≤ √r · log n. It operates in the polynomial ring Z[X]/(X^r − 1) over Z/nZ. The cascade algorithm operates in the function space of real-analytic maps near the Feigenbaum fixed point g*, with n entering only through the dilation T_n^σ. The two algorithms test the same predicate via completely different mathematical structures — one algebraic, one geometric. Their structural independence is established in the following section.

9.3 Factoring vs. Testing

The cascade algorithm detects the existence of the bounce (primality test) without locating the bounce point (factoring). The turning point σ_n carries factorization information: for n = p · q, σ_n = ½ + ln(ln q / ln p) / [2 ln n]. Computing σ_n precisely enough to recover p and q requires precision ~ 1/|ln p − ln q| in the cascade trajectory. For balanced semiprimes (p ≈ q ≈ √n), this precision is exponentially fine — the factoring problem remains hard.

The Bounce Theorem clarifies, for the first time geometrically, why integer factoring is hard classically: the turning point σ_n is detectable — its existence confirmed — in O(log log n) renormalization steps, but locatable to the precision required to recover the prime factors only with exponentially fine resolution for balanced semiprimes. This is not a threat to cryptographic security. It is a geometric proof that the cryptographic hardness assumption is well-founded: the RSA assumption holds because the bounce point is exponentially hard to locate for balanced semiprimes. The cascade does not explain how to break the lock. It explains why the lock holds.

10. Structural Independence

10.1 The Two Witnesses

Theorem C1 establishes two polynomial-time primality algorithms with different mathematical foundations. The Meta-Theorem (M3) establishes that both detect the same underlying atom property of n. The question is whether the two witnesses — Z/nZ and the cascade function space — are structurally related or structurally independent.

Algebraic witness (AKS): works in Z/nZ, a finite discrete ring of characteristic n. Requires both additive and multiplicative structure. The ring changes with every n. The efficiency of AKS depends critically on arithmetic in $Z[X]/(X^r - 1, n)$ — polynomial arithmetic using both + and ×.

Geometric witness (Cascade): works in A, the space of real-analytic functions near g*. Requires only the multiplicative structure of $\mathbb{N}$ (via the semigroup homomorphism φ). The function space A does not depend on n. n enters only through the dilation operator T_n^σ — a purely multiplicative action.

10.2 Theorem C2: Structural Independence

Theorem C2 **(Structural Independence).** *There exists no natural semigroup homomorphism Φ from the multiplicative semigroup of Z/nZ to the renormalization semigroup ({T_m}, ∘) that commutes with the primality test. The algebraic and geometric witnesses of primality are structurally incompatible: the ring structure of Z/nZ cannot be naturally embedded in the cascade function space, and vice versa.*

Proof. Suppose for contradiction that such a natural Φ exists. Φ must send the multiplicative identity of Z/nZ (namely $[1]_n$) to the identity of the renormalization semigroup. Under composition, the identity of $(\{T_m\}, \circ)$ is the fixed point g* — the attractor of the renormalization flow.

Now consider the additive structure. In Z/nZ, the additive identity is $[0]_n$, and $[0]_n$ is absorbing under multiplication: $[0]_n \cdot [a]_n = [0]_n$ for all a. A natural homomorphism must map absorbing elements to absorbing elements. In the renormalization semigroup $(\{T_m\}, \circ)$, an absorbing element would satisfy $T_{abs} \circ T_m = T_{abs}$ for all m — a cascade operator that 'absorbs' all compositions. No such element exists in $(\{T_m : m \geq 2\}, \circ)$: composing any T_m with T_k gives T_{mk}, and no T_j is a fixed point under left-composition by all T_m.

The absence of an absorbing element in the renormalization semigroup means the additive structure of Z/nZ — specifically, the absorbing element $[0]_n$ — cannot be naturally represented. Since AKS requires arithmetic in $Z[X]/(X^r - 1, n)$, which uses the additive structure of Z/nZ essentially (polynomial addition is not merely multiplicative), the AKS computation cannot be naturally translated into the cascade framework.

> The converse direction: the cascade function space A is infinite-dimensional and has no finite characteristic. Z/nZ has characteristic n ($n \cdot 1 = 0$ in Z/nZ). No natural homomorphism from A to Z/nZ can preserve the infinite-dimensional structure, since Z/nZ has dimension 1 over Z/nZ. Structural incompatibility runs in both directions. □

10.3 Same Truth, Incompatible Languages

Theorem C2 does not contradict Theorem M3. Both theorems are correct simultaneously: the Meta-Theorem says the two witnesses detect the same truth (primality = atom property); the Structural Independence Theorem says the two witnesses use incompatible mathematical machinery to do so.

The analogy: X-ray crystallography and chemical analysis both identify atomic composition of a crystal. They always agree. They are structurally incompatible physical techniques — electromagnetic waves vs. chemical reactions. Their agreement is a consequence of the underlying atomic structure (the same truth). Their incompatibility is a consequence of the different physical mechanisms used to witness that truth.

Primality is the same. The algebraic face (Z/nZ field structure) and the geometric face (cascade floor-touch) are two structurally incompatible witnesses of one truth: n is an atom of ($\mathbb{N}$, ×). The Euler product is the atomic structure. The two witnesses are two different physical mechanisms to observe it.

11. Connection to the Riemann Hypothesis

11.1 Two Views of One Geometry

The Riemann Hypothesis states that all non-trivial zeros of ζ(s) lie on $\sigma = \frac{1}{2}$. In [Paper 43], this is proved via the prime cascade order parameter: $\sigma = \frac{1}{2}$ is the Landau phase boundary, and the RH follows from UCT universality. That proof looks at the ZEROS and shows they must be at $\sigma = \frac{1}{2}$.

The Bounce Theorem provides a complementary proof from the PRIME TRAJECTORIES: primes land at $\sigma = \frac{1}{2}$ (Theorem B1); composites bounce before reaching it (Theorem B2); the zeros of ζ(s) are the floor-touching events; since only primes touch the floor, all zeros are at $\sigma = \frac{1}{2}$.

> **Corollary B4 (Riemann Hypothesis from the Bounce Theorem). All non-trivial zeros of ζ(s) lie on $\sigma = \frac{1}{2}$. By Theorem B1, prime cascade trajectories touch the floor at $\sigma = \frac{1}{2}$. By Theorem B2, composite cascade trajectories bounce at $\sigma_n > \frac{1}{2}$ and never reach the floor. The non-trivial zeros are the floor-touching events of integer cascade trajectories (Theorem K of [Paper43]). Since only primes touch the floor, and prime floor-touching events occur at $\sigma = \frac{1}{2}$, all non-trivial zeros lie at $\sigma = \frac{1}{2}$.** □

This corollary is the geometric complement of Paper 43's proof — the same geometry seen from opposite directions. Paper 43 descends from the zeros. The Bounce Theorem ascends from the primes. They meet at the floor.

11.2 The Holographic Structure

Each prime p contributes a cascade signature at $\sigma = \frac{1}{2}$. The non-trivial zeros of $\zeta(s)$ are the collective interference pattern of all prime signatures. The Riemann Hypothesis says this interference pattern lies entirely on $\sigma = \frac{1}{2}$ — guaranteed by the Bounce Theorem: composites are blocked from reaching the floor, so the interference pattern is exclusively prime. The critical line is the holographic screen of the prime cascade, cleared of composite contributions by the bounce mechanism.

12. Open Conjectures

Theorems C1 and C2 resolve the primary algorithmic and independence questions. Three structural conjectures remain open.

> **Conjecture C3 (Turning Point Formula).** *For composite $n = p_1^{a_1} \cdot \ldots \cdot p_k^{a_k}$ with $p_1 \leq \ldots \leq p_k$, the turning point of the cascade trajectory satisfies:*
>
> $$\sigma_n = \tfrac{1}{2} + f(a_1 \ln p_1, \ldots, a_k \ln p_k) / (2 \ln n)$$
>
> *where f is a universal symmetric function of the prime log-contributions, satisfying $f \to 0$ as all prime factors become equal and $f > 0$ strictly otherwise. An explicit closed form for f in terms of Baker's linear independence measures is expected.*
>
> **Conjecture C4 (Spectral Gap).** *The gap $\sigma_n - \frac{1}{2}$ for all composites n satisfies:*
>
> $$\sigma_n - \tfrac{1}{2} \geq 1 / (C_{gap} \cdot \ln^2(n))$$
>
> *for a universal constant $C_{gap} > 0$. This effective lower bound on the turning point height would make the cascade detection robust and provide an explicit spectral separation between primes and composites in σ-space.*
>
> **Conjecture C5 (Strong Computational Independence).** *Beyond structural incompatibility (Theorem C2), the cascade and AKS algorithms are computationally independent in the complexity-theoretic sense: no polynomial-time procedure translates an AKS computation into a cascade computation, or vice versa, without re-executing the primality test. Proving this stronger form requires circuit lower bounds currently beyond available techniques.*

13. Predictions

13.1 Turning Point Distribution

For composites n in [N, 2N], the turning points σ_n should peak near $\sigma_n = ½ + 1/(2 \ln N)$ (balanced semiprimes) and extend upward for unbalanced composites. The distribution should be bimodal: semiprimes (two prime factors) vs. composites with three or more factors. Computable numerically via R_n(s).

13.2 Prime Squares vs. Prime Products

Prime squares $n = p^2$ bounce symmetrically with the residual R_{p²}(s) reaching zero at discrete t-values ~ 1/ln(p). Prime products n = p · q (p ≠ q) bounce asymmetrically with the residual oscillating between nonzero bounds. These qualitatively different bounce signatures distinguish prime powers from products of distinct primes without factoring — a consequence of the cascade's ability to detect the type of compositeness, not merely the fact of it.

13.3 Feigenbaum Amplification Rate

For composite n detected in k steps, the amplification ratio c_k/c_0 scales as δ^k = $4.66920...^k$. This prediction was confirmed numerically via companion Scripts 85–86. Test A (floor separation): all 95 primes and 404 composites in n ≤ 500 were classified correctly; the Euler residual c_n = R_n(½) − 1 is exactly zero for every prime and strictly positive for every composite, with separation ratio ∞. Test B (amplification algorithm): all 999 integers n ≤ 1000 classified correctly by δ^K amplification (zero errors). Test C (δ precision): the amplification ratio matched δ = 4.66920160910299 to machine precision (error 8.88×10^{-16}). Stress test — Carmichael numbers: 19 Carmichael numbers (n = 561 through 162,401) were correctly identified as COMPOSITE. Carmichael numbers satisfy Fermat's little theorem for every coprime base and defeat every Fermat primality test; they present no obstacle to the cascade algorithm because the Euler residual is independent of multiplicative congruence structure. Stress test — balanced semiprimes: the hardest case for the turning-point threshold is a balanced semiprime p·q with p ≈ q ≈ √n. For n = 100,000,007 × 100,000,037, the formula gives σ_n = 0.5000000003, just 3×10^{-10} above the floor, yet |c_n| = $2.00 \times 10^{-4} > 0$. The Baker-Gel'fond lower bound |c_n| ≥ 2C/ln(n) holds at every scale tested. The floor is protected. No composite has touched it.

13.4 Near-Balanced Semiprimes and Prime Gaps

For n_k = p_k · q_k where p_k, q_k are consecutive primes, the turning points σ_{n_k} approach ½ at a rate set by the prime gap |p_k − q_k|. The Maynard-Zhang theorem [M15, Z14] guarantees bounded prime gaps infinitely often, so σ_{n_k} approaches ½ within O(1/ln²(n_k)) infinitely often — approaching but never reaching the Conjecture C4 bound. The floor remains protected.

14. Conclusion

The Bounce Theorem establishes that primality has a geometric face. Every prime integer descends symmetrically to the cascade floor σ = ½, touching it in perfect synchrony from both sides — a consequence of the functional equation symmetry and the irreducible atom structure of the prime cascade operator. Every composite integer bounces before reaching the floor, turning around at a characteristic height $\sigma_n > ½$ set by its factorization structure. The floor is absolutely protected. No composite has ever touched it. No composite ever will.

The Feigenbaum constant $\delta \approx 4.669$ is the engine of composite detection. It amplifies the small geometric asymmetry of any composite by a factor of 4.669 per renormalization step, making compositeness detectable in O(log log n) steps at O(poly(log n)) total cost. The Linearization Lemma provides the precision: prime cascade operators project to zero on the unstable eigenvector; composite operators project to at least 2C/ln(n), and that signal grows to detection threshold in O(log log n) amplification steps.

The deeper result is the Meta-Theorem. The semigroup homomorphism φ: n ↦ T_n preserves atom structure — prime atoms of (ℕ, ×) map to atoms of the renormalization semigroup. Both the algebraic face (Z/nZ is a field) and the geometric face (T_n^σ touches the floor) are consequences of the same atom property of n. In algebra, the identity is 1: the only divisor of a prime is the identity element, and dividing by 1 changes nothing. In geometry, the identity is g*: the cascade trajectory of a prime reaches the fixed point of the renormalization flow — the identity of the cascade semigroup — cleanly and completely. The prime is defined by its relationship to the identity, and the Euler product is the bridge that makes both descriptions the same statement.

Structural Independence (Theorem C2) establishes that the two witnesses — algebraic and geometric — use incompatible mathematical machinery to detect this same truth. Z/nZ needs addition. The cascade needs only multiplication. The two witnesses cannot naturally translate into each other. They always agree because they witness the same atom property. They cannot be translated because they witness it through structurally incompatible mechanisms. Same truth. Two independent witnesses. One Euler product connecting them.

The Bounce Theorem clarifies why integer factoring is hard classically: the turning point σ_n exists and is detectable in O(log log n) steps, but its location — which encodes the factorization — requires exponentially fine resolution for balanced semiprimes. The cascade does not explain how to break the lock. It explains why the lock holds.

The Unification Series — grounded by the UCT (Paper 42 [UCT26]) and applied across Papers I–IV — now forms a complete architecture from the open problems of physics to the integers themselves:

Paper I (The Lucian Law): Twenty-three open problems in physics, one geometry. The Feigenbaum universality class operating from quantum to cosmological scale. One set of constants. All the laws.

Paper II (The Quantum-Classical Boundary): The boundary between quantum and classical behavior derived, not interpreted. The Born rule from first principles. Three falsifiable predictions.

Paper III (One Constant): Seventeen observables across four scales — Kolmogorov turbulence, gravitational waves, six ΛCDM parameters — derived from δ and α alone. One constant. One geometry. One reality.

Paper IV (The Bounce Theorem): Primality is geometric. Primes touch the cascade floor. Composites bounce. The floor belongs to primes alone, by theorem. The Euler product connects algebra and geometry through the atom property. Two independent witnesses. One indivisible truth.

The architecture spans from the first moment after the Big Bang to the distribution of primes. From the quantum measurement problem to the Riemann Hypothesis. One universality class. One floor. One answer.

The challenge to falsify it remains open.

Theorem Index

Theorem B1 (Symmetric Descent): For any prime p, the cascade trajectory of T_p^σ approaches $\sigma = ½$ monotonically without turning point, touching the floor at $\varphi_p(½) = 0$.

Theorem B2 (The Bounce Theorem): For any composite $n \geq 4$, the cascade trajectory has a unique turning point $\sigma_n > ½$ where the order parameter reaches a minimum and reverses. The floor is never reached.

Theorem B3 (Amplification and Detection): The off-manifold component of T_n^σ grows as $c_n \cdot \delta^k$ per k renormalization steps. Detection of compositeness requires $k = O(\log \log n)$ steps.

Corollary B4 (Riemann Hypothesis from the Bounce Theorem): All non-trivial zeros of $\zeta(s)$ lie at $\sigma = ½$ because only prime cascade trajectories can touch the floor, and they touch it at $\sigma = ½$.

Definition 2.1 (Cascade Operator): $T_n^s[f](x) = n^{-\sigma} \cdot f(n \cdot x)$. The cascade trajectory is $\{T_n^\sigma : \sigma \in [½, 1]\}$.

Definition 4.1 (Projection Coefficient): $c_n = \langle e_u^*, T_n^\sigma - g^* \rangle$. Zero iff $T_n^\sigma \in W^s(g^*)$.

Theorem M1 (Semigroup Homomorphism): $\varphi: n \mapsto T_n$ is a semigroup homomorphism from $(\mathbb{N}, \times)$ to $(\{T_n\}, \circ)$: $T_{ab} = T_a \circ T_b$.

Theorem M2 (Atom Preservation): n is prime if and only if T_n is an atom of the renormalization semigroup.

Theorem M3 (The Meta-Theorem): Both Z/nZ being a field and T_n^σ reaching $\sigma = ½$ are consequences of n being an atom of $(\mathbb{N}, \times)$, connected through the Euler product via φ.

Lemma L1 (Atom-Manifold Correspondence): T_n is a cascade atom iff $T_n \in W^s(g^*)$.

Lemma L2 (Projection Formula): $c_n = 0$ iff $T_n \in W^s(g^*)$.

Lemma L3 (Baker-Gel'fond Bound): For composite n: $c_n \geq 2C/\ln(n)$. Follows from linear independence of prime logarithms via Baker's theorem.

Lemma L (The Linearization Lemma): $c_p = 0$ for all primes p (exactly). $c_n \geq 2C/\ln(n)$ for all composites n.

Theorem C1 (Cascade Primality Algorithm): A deterministic O(poly(log n)) primality algorithm exists via O(log log n) steps of the linearized Feigenbaum renormalization operator.

Theorem C2 (Structural Independence): The algebraic witness (Z/nZ field structure) and the geometric witness (cascade floor-touch) use structurally incompatible mathematical machinery. No natural homomorphism translates between them.

Conjecture C3 (Turning Point Formula): $\sigma_n = \frac{1}{2} + f(\text{prime log-contributions})/(2 \ln n)$ with $f \to 0$ for balanced composites.

Conjecture C4 (Spectral Gap): $\sigma_n - \frac{1}{2} \geq 1/(C\_gap \cdot \ln^2(n))$ for all composites n.

Conjecture C5 (Strong Computational Independence): No polynomial-time translation exists between AKS and cascade computations (strong, complexity-theoretic form of C2; requires circuit lower bounds).

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