Geometric Information Field Theory

Standard Model Parameters as Topological Invariants of a G_2 Holonomy Manifold

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Abstract

The Standard Model requires 19 experimentally determined parameters lacking theoretical explanation. We present a geometric framework in which physical observables emerge as topological invariants of a seven-dimensional G_2 holonomy manifold K_7 coupled to $E_8 \times E_8$ gauge structure, with the topological inputs treated as discrete and the metric determinant $\det(g) = 65/32$ imposed as a normalization target (§2.3); no continuously adjustable physical parameter is introduced.

Building on a Newton–Kantorovich-certified neck/metric model ([A], [B]) and a Joyce–Karigiannis topological/lattice route realizing $(b_2, b_3) = (21, 77)$, while the full smooth analytic compact construction remains open: we derive **95 observables** from K_7 organized in four types: 33 direct algebraic (Type I, mean deviation 0.73%), 19 one-step physical extractions (Type II, 0.17%), 21 multi-step dynamical chains (Type III, 3.4%), and 22 structural diagnostics (Type IV). Of 66 experimentally comparable observables, 11 are exact matches (deviation $< 0.01\%$) and 53 are within 1%.

The framework includes three new framework-level results beyond the original 33 predictions: (1) a proposed $E_8 \to$ Standard Model breaking chain with anomaly-cancellation checks and bundle-universality diagnostics, (2) a combined lepton mass hierarchy mechanism achieving sub-percent precision from two independent geometric sources with $\alpha = e^K$ (zero free parameters), and (3) a sensitivity analysis demonstrating 2.13× overdetermination with coincidence probability 10^{-346} under a uniform null and 10^{-133} under an algebraic null model (4.2M random formulas from the same 20 constants). Of the 95 observables, 55 are formally verified in Lean 4 (213 certificate conjuncts, 4 axioms, 0 sorry); Lean certifies the algebraic relations and internal consistency, not the physical interpretation or the existence of the complete compact construction (§8.3).

DUNE (2028–2040) will test the topological prediction δ_CP = 197° with resolution of a few degrees to ~15°; measurement outside [182, 212]° would create serious tension. We present this as an exploratory investigation emphasizing falsifiability, not a claim of correctness.

Keywords: G_2 holonomy, exceptional Lie algebras, Standard Model parameters, topological field theory, falsifiability, formal verification, Lean 4

Contents

Framework epistemic status at a glance:

Component	**Status**
Algebraic predictions (33 Type I observables)	Lean-certified (algebraic relations)
K_7 Betti/lattice gate, $(b_2, b_3) = (21, 77)$	Certified: NK metric [A] + JK17 topological/lattice route
Full smooth compact analytic construction	**Open**
Physical interpretation	Conjectural / falsifiable

1 Introduction

1.1 The Parameter Problem

The Standard Model describes fundamental interactions with remarkable precision, yet requires 19 free parameters determined solely through experiment [1]. These parameters (gauge couplings, Yukawa couplings spanning five orders of magnitude, mixing matrices, and Higgs sector values) lack theoretical explanation.

Several tensions motivate the search for deeper structure:

- **Hierarchy problem**: The Higgs mass requires fine-tuning absent new physics [2].
- **Hubble tension**: CMB and local H_0 measurements differ by >4 sigma [3,4].
- **Flavor puzzle**: No mechanism explains three generations or mass hierarchies [5].
- **Koide mystery**: The charged lepton relation Q = 2/3 holds for 43 years without explanation [6].

These challenges suggest examining whether parameters might emerge from geometric or topological structures.

1.2 Framework Overview

The Geometric Information Field Theory (GIFT) proposes that dimensionless parameters represent topological invariants of an eleven-dimensional spacetime:

$$E_8 \times E_8 \text{ (496D)} \longrightarrow \text{AdS}_4 \times K_7 \text{ (11D)} \longrightarrow \text{Standard Model (4D)}$$

The key elements:

1. **E_8 x E_8 gauge structure** (dimension 496)
2. **Compact 7-manifold** K_7 with G_2 holonomy ($b_2 = 21$, $b_3 = 77$)
3. **Metric normalization constraint**: det(g) = 65/32, structurally motivated by G_2/E_8 constants (§2.3) but imposed as a normalization target, not derived from topology alone
4. **Cohomological mapping**: Betti numbers constrain field content

We emphasize this represents mathematical exploration, not a claim that nature realizes this structure. The framework's merit lies in falsifiable predictions from topological inputs.

1.3 Paper Organization

- **Section 2**: Mathematical framework ($E_8 \times E_8$, K_7, G_2 structure)
- **Section 3**: Methodology, epistemic status, and type classification

- **Section 4**: 95 observables across 4 types (33(I) + 19(II) + 21(III) + 22(IV))
- **Section 5**: Gauge sector, $E_8 \to$ SM breaking, anomaly cancellation, B-test identity, bundle universality
- **Section 6**: Mass hierarchy: Wilson lines, instantons, combined wilson_line+instanton pipeline
- **Section 7**: Sensitivity analysis, effective DOF, cross-correlations, coincidence test
- **Section 8**: Formal verification (213 Lean conjuncts, 4 axioms) and statistical uniqueness
- **Section 9**: Discussion, falsifiability, and conclusion

Three supplements provide technical details: S1 (Mathematical Foundations), S2 (Complete Derivations), S3 (Observable Dataset with full 95-entry table).

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2 Mathematical Framework

2.1 The Octonionic Foundation

GIFT emerges from the algebraic fact that **the octonions are the largest normed division algebra**.

Algebra	Dim	Physics Role	Extends?
R	1	Classical mechanics	Yes
C	2	Quantum mechanics	Yes
H	4	Spin, Lorentz group	Yes
O	**8**	**Exceptional structures**	**No**

The octonions terminate this sequence. Their automorphism group $G_2 = \mathrm{Aut}(\mathrm{O})$ has dimension 14 and acts naturally on $\mathrm{Im}(\mathrm{O}) = R^7$. The exceptional Lie algebras arise from octonionic constructions through a chain established by Dray and Manogue [17]:

Algebra	Dimension	Connection to O
G_2	14	Aut(O)
F_4	52	$\mathrm{Aut}(J_3(\mathrm{O}))$
E_6	78	Collineations of OP^2
E_8	248	Contains all lower exceptionals

This chain is not accidental. It reflects the unique algebraic structure of the octonions: Im(O) has dimension 7, the Fano plane encodes the multiplication table, and G_2 preserves this structure. A G_2-holonomy manifold is therefore the natural geometric home for octonionic physics, just as U(1) holonomy is the natural setting for complex geometry.

The G_2 structure is concretely encoded in the **standard 3-form φ_0** on $\mathbb{R}^7$ (Bryant-Joyce convention):

$$\varphi_0 = e_{012} + e_{034} + e_{056} + e_{135} - e_{146} - e_{236} - e_{245}$$

where only 7 of the $\binom{7}{3} = 35$ ordered triples carry nonzero coefficient (all ± 1). G_2 is precisely the stabilizer of φ_0 in GL(7, $\mathbb{R}$), and its Lie algebra g_2 is the kernel of the linear map $L_{\varphi_0} : \mathfrak{gl}(7) \to \wedge^3(\mathbb{R}^7)^*$, $X \mapsto \mathcal{L}_X \varphi_0$, giving $\dim(g_2) = 49 - \mathrm{rank}(L_{\varphi_0}) = 49 - 35 = 14$. This is fully formalized in Lean (`G2ThreeForm.lean`): all 7 nonzero coefficients are certified by `native_decide`, and the Lie algebra structure (closure under addition and scalar multiplication) is proven.

2.2 E_8 x E_8 Structure

E_8 is the largest exceptional simple Lie group with dimension 248 and rank 8 [18]. The product E_8 x E_8 arises in heterotic string theory for anomaly cancellation [19], with total dimension 496.

The first E_8 contains the Standard Model gauge group through the breaking chain:

$$E_8 \longrightarrow E_6 \times SU(3) \longrightarrow SO(10) \times U(1) \longrightarrow SU(5) \longrightarrow SU(3) \times SU(2) \times U(1)$$

The second E_8 provides a hidden sector whose physical interpretation remains an open question.

Wilson (2024) demonstrates that E_8(-248) encodes three fermion generations (128 degrees of freedom) with GUT structure [9]. The product dimension 496 enters the hierarchy parameter tau = (496 x 21)/(27 x 99) = 3472/891, connecting gauge structure to internal topology.

2.3 The K_7 Manifold

We consider a compact, simply connected 7-manifold K_7 with G_2 holonomy and Betti numbers (b_2, b_3) = (21, 77). G_2 holonomy preserves N = 1 SUSY in 4D [11], implies Ric(g) = 0, and is the unique holonomy of $G_2 = \mathrm{Aut}(\mathbb{O})$ acting on $\mathbb{R}^7$.

Metric construction ([A]): An explicit 7D G_2 metric g(s, θ, ψ, y) = g_seam(s) $\oplus$ g_{T^2} $\oplus$ g_{K3}(y) on K_7 is constructed via Chebyshev-Cholesky parametrization (169 parameters), with K3 fiber contribution certified at 0.07% of total torsion [C]. The Newton-Kantorovich certificate (h = 8.95×10^{-9}, margin $\times 56$ million, zero finite differences) establishes existence and local uniqueness (within the NK certificate ball) of a nearby torsion-free metric with $\delta g/g \leq 4.86 \times 10^{-6}$. The spectral analysis ([B]) independently confirms 21 near-zero eigenvalues of Δ_2 (gap ratio 14,635) and 77 near-zero eigenvalues of Δ_3, consistent with the Betti numbers (b_2, b_3) = (21, 77).

Metric decomposition: A Chebyshev mode analysis reveals that the metric is dominated by its constant mode (k=0), which carries 99.9998% of the L^2 energy. This mode is a product metric K3 $\times$ T^2 $\times$ I with structural parameters g_ss = 19/6, g_{T^2} = 7/6, det(g) = 65/32, fixed by the structural normalization and topological input data, with det(g) = 65/32 treated as a metric normalization target as discussed in S1 §10.3. The k$\geq$ 1 modes (0.0002% of energy) constitute the minimal perturbation that breaks the product structure and lifts the holonomy from SU(2)$\times$U(1) to full G_2. These corrections are responsible for b_1 = 0, the torsion $\|T\| = 2.949 \times 10^{-5}$, and the NK certification, but contribute nothing to the numerical values of the 95 observables, which depend only on topological integers.

Topological classification: The pair $(b_2, b_3) = (21, 77)$ does not appear among catalogued compact G_2 manifolds, neither in Joyce's 252 orbifold types nor in the ~100 CHNP TCS examples. Orthogonal TCS is excluded by parity (b_2+b_3=98 is even; CHNP Lemma 6.7). Non-orthogonal TCS, extra-twisted connected sums [20], and new orbifold groups remain open paths. The NK-certified metric provides computational evidence; a complete geometric construction is an open problem. The pair is **unique** among all 65 known TCS literature examples (Kovalev, CHNP, Joyce, CGN, Nordström); nearest neighbor is at distance 7.6 in (b_2, b_3) space. The certified metric is 99.98% block-diagonal K3$\times T^2\times$I with exact U$(1)^2$ symmetry (degenerate to 2×10^{-5}).

Further structure: The certified metric is smooth (no singularities). The SM gauge group SU(3)×SU(2)×U(1) does **not** arise from ADE singularities in this framework; it emerges from the algebraic/spectral structure: $g_2 \subset$ so(7) decomposition, so(8) = $g_2 \oplus$ L $\oplus$ R triality giving N_gen = 3, and spectral gap $\lambda_1 = 6\pi^2/475$ = 0.12467 (Richardson extrapolation: 0.12461 ± 0.00016, deviation 0.05%) (§5). ADE singularities would be relevant at singular limits of G_2 moduli space; the smooth certified metric is at a generic point.

The framework imposes det(g) = 65/32 = (Weyl × α_sum)/ 2^5, equivalently 2 + 1/32. Joyce's theorem [20] guarantees existence of torsion-free G_2 metrics on suitable compact 7-manifolds; the Chebyshev-Cholesky construction (companion paper [A]) achieves ‖T‖ < 3 × 10^{-5} with Newton-Kantorovich certification h = 8.95 × 10^{-9} (Chebyshev-certified, margin ×56 million; §8.5).

2.4 Topological Constraints on Field Content

2.4.1 Betti Numbers as Capacity Bounds

The Betti numbers provide upper bounds on field multiplicities:

- $b_2(K_7)$ **= 21**: Bounds the number of gauge field degrees of freedom
- $b_3(K_7)$ **= 77**: Bounds the number of matter field degrees of freedom

Note on gauge group origin: In M-theory on smooth G_2 manifolds, dimensional reduction yields b_2 abelian U(1) vector multiplets [11]. The non-abelian SM gauge group in GIFT emerges instead from the algebraic/spectral structure of the G_2 holonomy: the $g_2 \subset$ so(7) decomposition and so(8) = $g_2 \oplus$ L $\oplus$ R triality (§5). The smooth certified metric is at a generic point in G_2 moduli space; codimension-4 ADE singularities would be a different (singular) limit.

2.4.2 Generation Number

The number of chiral fermion generations follows from a topological constraint:

$$(\mathrm{rank}(E_8) + N_{\mathrm{gen}}) \times b_2 = N_{\mathrm{gen}} \times b_3$$

Solving: (8 + N_gen) x 21 = N_gen x 77 yields **N_gen = 3**.

This derivation is formal; physically, it reflects index-theoretic constraints on chiral zero modes, which in M-theory on G_2 require singular geometries for chirality [25].

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3 Methodology and Epistemic Status

3.1 The Derivation Principle and Type Classification

The GIFT framework derives physical observables through algebraic combinations of 20 topological invariants (Appendix A). The 95 observables are organized by derivation directness:

Type	Count	Derivation	Example
I	33	Direct algebra from topology	$sin^2\theta$_W = 3/13
II	19	One physical identification step	m_u from ratio × VEV
III	21	Multi-step dynamical chains	Combined wilson_line+instanton lepton ratios
IV	22	Structural diagnostics	NK certification, Gram conditioning

Type I predictions are dimensionless ratios of topological integers, they cannot be "fitted" and are either correct or wrong. Type II adds one scale identification. Type III involves dynamical mechanisms (gauge running, eigenvalue splitting, instanton volumes). Type IV provides internal consistency checks.

Geometric unit convention. Throughout this paper, we adopt the convention that the geometric (GIFT) values are the *reference*, and experimental measurements are expressed as offsets from the geometric prediction. This reversal parallels the 2019 SI redefinition: the kilogram is now defined from Planck's constant (exact), and any physical realization has uncertainty relative to it. In GIFT, the topological formula is exact; the experimental value approximates it.

3.2 What GIFT Claims and Does Not Claim

Inputs: Existence of K_7 with G_2 holonomy and $(b_2, b_3) = (21, 77)$; $E_8 \times E_8$ gauge structure; det(g) = 65/32.

Outputs: 95 observables across 4 types (33 Type I, 19 Type II, 21 Type III, 22 Type IV), 66 experimentally testable.

Structure of predictions: All 95 observables are algebraic functions of 6 primitive topological integers (b_2, b_3, $\dim(G_2)$, $\dim(E_8)$, $\mathrm{rank}(E_8)$, $\dim(K_7)$) plus standard transcendentals (π, $\sqrt{2}$, ln 2, ζ, and the golden ratio $\varphi = (1+\sqrt{5})/2$, see §4.2.3 for the McKay $E_8 \leftrightarrow 2I$ origin of φ). No observable depends on the detailed G_2 geometry: the metric certifies that the manifold exists but does not enter the prediction formulas. (Caveat: 6 observables use det_num/det_den = 65/32, which is a metric normalization target with suggestive but not derivational algebraic expressions in terms of topological integers, see §10.3 of S1.)

We claim that given the inputs, the outputs follow algebraically (Type I) or computationally (Types II–IV). We do **not** claim uniqueness of the geometry, uniqueness of the formula assignments, or that the formula selection principle is understood.

3.3 Three Factors Distinguishing GIFT from Numerology

Multiplicity: 95 observables (66 testable), not cherry-picked coincidences. The sensitivity analysis (§7) gives P(coincidence) = 10^{-346} under the declared uniform null model, with overdetermination ratio 2.13×, both computed on the 33 Type I observables with experimental comparison.

Exactness: Several predictions are exactly rational, $sin^2\theta$_W = 3/13, Q_Koide = 2/3, m_s/m_d = 20, Ω_DM/Ω_b = 43/8. These cannot be fitted; they are correct or wrong.

Falsifiability: DUNE will test δ_CP = 197° with expected resolution ranging from a few degrees to ~15° depending on exposure and true parameter values (2028–2040). The SUSY spectrum (m_gravitino = 166 GeV, m_moduli = 3.2 TeV) is model-dependent and constrained in standard simplified searches; viable only in compressed or suppressed-coupling realizations requiring dedicated recasting at HL-LHC/FCC. The proton lifetime τ_p = 4.06 × 10^{38} years exceeds near-term Hyper-K sensitivity; Hyper-K can strengthen lower bounds and constrain nearby GUT-scale alternatives.

3.4 Why These Formulas?

The selection question has two levels of answer.

Mathematical level (deterministic). The NK-certified metric has zero free parameters; its 169 Chebyshev coefficients appear constrained to a lattice generated by $\{\pi^2,\ \pi,\ 1,\ \mathrm{e},\ \chi_{98}\}\ /\ (b_2 \cdot b_3)$ (a numerical observation, not a derived result; see S1 §10.3). Observables are functionals of this metric and are therefore algebraic in the topological invariants. Within the declared formula grammar and structural-constant set, the observed relations are highly constrained and statistically non-generic; a first-principles derivation of the full formula-selection rule remains open (see §9.4 for the extended grammar analysis with TCS atoms).

Statistical level (empirical). Even granting an adversary access to the same 20 structural constants, 4.2M random algebraic formula trees cannot reproduce GIFT's precision profile: joint coincidence probability P = 10^{-133} (algebraic null model, §7). Structural redundancy adds a third line of evidence: $sin^2\theta$_W = 3/13 admits 14 independent derivations, Q_Koide = 2/3 admits 20 (see S2): an overdetermined web inconsistent with post-hoc cherry-picking.

The deeper question (*why G_2 holonomy?*) has the same epistemic status as "why Lorentz invariance?": G_2 is the unique compact exceptional holonomy group admitting a 7-dimensional representation with the stabilizer chain $G_2 \supset$ SU(3) ⊃ SU(2) ⊃ U(1) required by the Standard Model. It is an isolated point in the space of compatible structures, not one choice among a continuum.

3.5 Posture: orientation, not ontology

It is useful to distinguish three registers at which a framework of this type operates, because the support GIFT offers is uneven across them.

Predictive register. GIFT specifies a finite list of inputs and derives 95 observables from them; 66 of these are testable, with mean deviation ~1% and a sensitivity profile that places the joint configuration at $> 4.5\sigma$ against random algebraic null models (§7). This is the register where the framework either holds or fails, and it is the only register at which we make any positive claim.

Architectural register. The choice of G_2-holonomy 7-manifolds with $E_8 \times E_8$ structure is *motivated* by mathematical considerations (§§2, 9.4): G_2 is the unique exceptional compact holonomy admitting the SU(3) × SU(2) × U(1) stabilizer chain, E_8 is the largest exceptional Lie algebra, and the topological constraints fix $(b_2, b_3) = (21, 77)$ and $N_{\text{gen}} = 3$. We argue this architecture is *natural*; we do not argue it is *necessary*.

Ontological register. A reader may wonder whether GIFT carries a thesis about reality: for instance, the speculation that geometry, information, and energy are not three correlated aspects of nature but three views of a single underlying configuration, or the resonance with Wheeler's "It from Bit" programme [39] and the holographic principle. Such readings are compatible with the framework and historically motivated its development, but they are **neither required nor demonstrated** by the predictions. A reader who finds the Wheeler-holographic picture persuasive will see GIFT as a natural piece of that puzzle; a reader who prefers a strictly empirical stance will see a falsifiable predictive framework. Both readings are defensible; the framework requires neither.

In Worrall's [40] and Ladyman's [41] terms, GIFT is best read as a *moderate structural-realist orientation*: structure carries predictive weight independently of any further ontological commitment. The framework's success or failure is decided by experiment in the predictive register, not by adjudication in the ontological one. A more extended discussion of this posture, written for a non-technical audience, appears in the companion essay *Orientation, not ontology* [essay].

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4 Observables: 95 Relations from 20 Structural Constants

4.1 Gauge Sector

4.1.1 Weinberg Angle

$$\sin^2\theta_W = \frac{b_2}{b_3 + \dim(G_2)} = \frac{21}{91} = \frac{3}{13} = 0.230769$$

Experimental (PDG 2024) [1]: 0.23122 +/- 0.00004. Deviation: **0.195%**.

The numerator b_2 counts gauge moduli; the denominator $b_3 + \dim(G_2)$ counts matter plus holonomy degrees of freedom. The ratio measures gauge-matter coupling geometrically.

4.1.2 Strong Coupling

$$\alpha_s(M_Z) = \frac{\sqrt{2}}{\dim(G_2) - p_2} = \frac{\sqrt{2}}{12} = 0.11785$$

Experimental: 0.1180 +/- 0.0009. Deviation: **0.126%**.

4.2 Lepton Sector

4.2.1 Koide Parameter

The Koide formula has resisted explanation since 1982. Koide discovered an empirical relation among the charged lepton masses [6]:

$$Q = \frac{(m_e + m_\mu + m_\tau)^2}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3}$$

Using contemporary mass values, this holds to six significant figures: Q_exp = 0.666661 +/- 0.000007.

GIFT provides:

$$Q_{\text{Koide}} = \frac{\dim(G_2)}{b_2} = \frac{14}{21} = \frac{2}{3}$$

The derivation requires only two topological invariants: $\dim(G_2) = 14$ (holonomy group dimension) and $b_2 = 21$ (second Betti number). No fitting is involved.

Approach	Result	Status
Preon models (Koide 1982)	Q = 2/3 assumed	Circular
S_3 symmetry (various)	Q ~ 2/3 fitted	Approximate
GIFT	**Q = $\dim(G_2)/b_2$ = 14/21 = 2/3**	**Algebraic identity**

Deviation: **0.0009%**. This is the most precise agreement in the framework.

4.2.2 Tau-Electron Mass Ratio

$$\frac{m_\tau}{m_e} = \dim(K_7) + 10 \times \dim(E_8) + 10 \times H^* = 7 + 2480 + 990 = 3477$$

Experimental: 3477.15 +/- 0.05. Deviation: **0.004%**.

The integer 3477 = 3 x 19 x 61 = N_gen x prime(8) x $kappa_T^-$1 factorizes into framework constants.

4.2.3 Muon-Electron Mass Ratio

$$\frac{m_\mu}{m_e} = \dim(J_3(\mathbb{O}))^\phi = 27^\phi = 207.01$$

where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio, arising from the McKay correspondence $E_8 \leftrightarrow$ 2I (binary icosahedral group), which links E_8 to icosahedral geometry. Experimental: 206.768. Deviation: **0.118%**.

Note: φ is the only non-integer input among the 33 Type I predictions and does not appear in the 20 structural constants of §S3.3. Its status is accordingly weaker than the other Type I derivations: the formula is algebraically exact and Lean-certified, but φ's derivation from E_8 structure requires the additional McKay step (see S2 §11).

4.3 Quark Sector

$$\frac{m_s}{m_d} = p_2^2 \times \text{Weyl} = 4 \times 5 = 20$$

Experimental (PDG 2024): 20.0 +/- 1.0. Deviation: **0.00%**.

$$\frac{m_b}{m_t} = \frac{b_0}{2b_2} = \frac{1}{42}$$

The constant 42 = p_2 x N_gen x dim(K_7) = 2 x 3 x 7 is a structural invariant (not to be confused with chi(K_7) = 0, which vanishes for any compact odd-dimensional manifold).

Experimental: 0.024 +/- 0.001. Deviation: **0.79%**.

4.4 Neutrino Sector

4.4.1 CP-Violation Phase

The GIFT prediction for the CP-violation phase is:

$$\delta_{CP} = \dim(K_7) \times \dim(G_2) + H^* = 7 \times 14 + 99 = 197°$$

decomposing into a local contribution ($7 \times 14 = 98$, fiber-holonomy coupling) and a global contribution (H* = 99, cohomological dimension). This is the canonical GIFT prediction, pure topological, zero corrections.

Experimental status: NuFIT 5.2 (2022) gave δ_CP ≈ 197°, an exact match. NuFIT 6.0 (Oct 2024) shifted the central value to 177° ± 20°, creating an 11.3% deviation. The experimental uncertainty is large (±20°), and the central value may shift further as T2K, NOvA, and DUNE accumulate statistics.

A post-hoc structural observation involving a possible compactification factor 62/69 is recorded in S2 Appendix F for completeness; it is **not** part of the main framework and is not used to revise the canonical 197° prediction.

Falsification criterion: If DUNE measures δ_CP outside [182, 212]° at 3σ, the framework faces serious tension. The 197° prediction sits at the edge of the current NuFIT 6.0 1σ band (177° ± 20° = [157°, 197°]).

4.4.2 Mixing Angles

Angle	Formula	GIFT	NuFIT 6.0 [27]	Dev.
theta_12	arctan(sqrt(delta/gamma_GIFT))	33.40 deg	33.41 +/- 0.75 deg	0.03%
theta_13	pi/b_2	8.57 deg	8.54 +/- 0.12 deg	0.37%
theta_23	arcsin((b_3 - p_2)/H*)	49.25 deg	49.3 +/- 1.0 deg	0.10%

The auxiliary parameters: delta = $2pi/Weyl^2$ = 2pi/25 and gamma_GIFT = (2 x rank(E_8) + 5 x H*)/(10 x dim(G_2) + 3 x dim(E_8)) = 511/884.

4.5 Higgs Sector

$$\lambda_H = \frac{\sqrt{\dim(G_2) + N_{gen}}}{\det(g)_{den}} = \frac{\sqrt{17}}{32} = 0.1289$$

The Higgs self-coupling combines holonomy dimension with generation count, normalized by the metric determinant scale. Experimental (PDG 2024): 0.129 ± 0.001. Deviation: **0.12%**. A TCS alternative λ_H = $b_2(M_1)/(b_3+b_2(M_2))$ = 11/87 = 0.1264 is purely rational (see S2 §17).

4.6 Boson Mass Ratios

Observable	Formula	GIFT	Experimental	Dev.
m_H/m_W	(N_gen + dim(E_6))/dim(F_4) = 81/52	1.5577	1.558 +/- 0.002	0.02%
m_W/m_Z	($2b_2$ - Weyl)/($2b_2$) = 37/42	0.8810	0.8815 +/- 0.0002	0.06%
m_H/m_t	fund(E_7)/b_3 = 56/77	0.7273	0.725 +/- 0.003	0.31%

4.7 CKM Matrix

Observable	Formula	GIFT	Experimental	Dev.
sin^2(theta_12_CKM)	fund(E_7)/dim(E_8) = 56/248	0.2258	0.2250 +/- 0.0006	0.36%
A_Wolfenstein	(Weyl + dim(E_6))/H* = 83/99	0.838	0.836 +/- 0.015	0.29%
sin^2(theta_23_CKM)	dim(K_7)/PSL(2,7) = 7/168	0.0417	0.0412 +/- 0.0008	1.13%

The Cabibbo angle emerges from the ratio of the E_7 fundamental representation to E_8 dimension.

4.8 Cosmological Observables

Observable	Formula	GIFT	Experimental	Dev.
Omega_DM/Omega_b	(1 + $2b_2$)/rank(E_8) = 43/8	5.375	5.375 +/- 0.1	0.00%
n_s	zeta(11)/zeta(5)	0.9649	0.9649 +/- 0.0042	0.004%
h (Hubble)	(PSL(2,7) - 1)/dim(E_8) = 167/248	0.6734	0.674 +/- 0.005	0.09%
Omega_b/Omega_m	Weyl/det(g)_den = 5/32	0.1562	0.157 +/- 0.003	0.16%
sigma_8	(p_2 + 32)/($2b_2$) = 34/42	0.8095	0.811 +/- 0.006	0.18%
Omega_DE	ln(2)×(b_2+b_3)/H* = ln(2)×98/99	0.6861	0.6847 +/- 0.005	0.21%
Y_p	(1 + dim(G_2))/$kappa_T^-$1 = 15/61	0.2459	0.245 +/- 0.003	0.37%

The dark-to-baryonic matter ratio Omega_DM/Omega_b = 43/8 is exact. The structural invariant $2b_2$ = 42 that gives m_b/m_t = 1/42 also determines this cosmological ratio, connecting quark physics to large-scale structure through K_7 geometry.

4.9 Type I Summary (33 Observables)

The 33 Type I predictions derive directly from the 20 structural constants (Appendix A). All are dimensionless ratios; all 33 are Lean-certified. Representative highlights:

Observable	Formula	GIFT	Exp.	Dev.	Status
$sin^2\theta$_W	$b_2/(b_3+\dim(G_2))$ = 21/91	3/13	0.23122	0.195%	TOPOLOGICAL
Q_Koide	$\dim(G_2)/b_2$ = 14/21	2/3	0.666661	0.001%	TOPOLOGICAL
α^{-1}	128 + 9 + det(g)×κ_T	137.033	137.036	0.002%	TOPOLOGICAL (uses metric normalization)
m_τ/m_e	7 + 2480 + 990	3477	3477.15	0.004%	TOPOLOGICAL
n_s	$\zeta(11)/\zeta(5)$	0.9649	0.9649	0.004%	DERIVED
Ω_DM/Ω_b	(1+42)/8	43/8	5.375	0.00%	TOPOLOGICAL

All 33 Type I observables are VERIFIED in Lean 4. Status legend: **TOPOLOGICAL** = pure topological-integer ratio; **DERIVED** = closed-form involving standard transcendentals (π, $\sqrt{2}$, $\ln 2$, ζ, golden ratio φ); **STRUCTURAL** = depends on the imposed metric normalization (det_num / det_den).

Complete derivations for all 33 in Supplement S2; full 95-entry table in Supplement S3.

4.10 Type II: Extended Algebraic Predictions (19)

Type II observables require one physical identification step beyond Type I ratios. Representative results:

Absolute quark masses (via m_q = ratio × reference_mass): m_u = 2.16 MeV (0.00%), m_d = 4.67 MeV (0.22%), m_s = 93.4 MeV (0.22%), m_c = 1.27 GeV (0.00%), m_b = 4.18 GeV (0.00%), m_t = 172.7 GeV (0.01%).

CKM magnitudes (from Wolfenstein parametrization): |V_us| = 0.2253 (0.13%), |V_cb| = 0.0412 (0.24%), |V_ub| = 0.00365 (0.27%), |V_td| = 0.0087 (0.34%), |V_tb| = 0.9991 (0.00%).

Extended ratios: m_c/m_s = 246/21 = 11.714 (exp 11.7, dev 0.12%), m_c/m_d = 234.3 (exp 234.0, dev 0.12%), m_μ/m_τ = 5/84 (exp 0.0595, dev 0.04%).

Type II mean deviation: **0.17%** across 19 observables. These inherit the precision of Type I ratios, with the physical identification step contributing negligible additional error.

4.11 Type III: Dynamical Predictions (21)

Type III observables involve multi-step dynamical mechanisms. They are grouped by computation:

wilson_line Non-adiabatic (3 obs): Wilson line eigenvalue splitting on K3 fiber at c = 0.452 gives raw lepton mass ratios with 0.5–2.1% deviation. These are improved to < 0.4% by the combined wilson_line+instanton pipeline (§6.4).

RGE_running RGE running (4 obs): Two-loop MSSM evolution from M_GUT to M_Z. The topological $sin^2\theta$_W = 3/13 at M_GUT runs to 0.2377 at M_Z (exp 0.2312, dev 2.78%). The strong coupling α_s(RGE) = 0.1224 deviates 3.7% using G_2-MSSM split-spectrum matching (§5.3). M_GUT = 2 × 10^{16} GeV is an exact match.

spectral Spectral (5 obs): effective Weyl law exponent (from the adiabatic seam-sector decomposition, extrapolated to d_eff = 7) α = 3.460 (exp 3.5, dev 1.1%), 22,671 KK states below cutoff, 57,578 fiber channels, Poisson level spacing. See [B] for the direct seam-sector result α = 1.998.

gauge_bundle Gauge bundle (4 obs): cond(f_IJ) = 1.047 (near-perfect gauge universality), α_ratio = 1.000002, effective Yukawa rank = 3, and κ(gauge) = 1.047 (4.7% departure from exact universality, the largest Type III bundle deviation).

instanton Instanton + combined (5 obs): Associative volume differences give ΔV(e-τ) = 8.633 (dev 5.9%), ΔV(e-μ) = 3.271 (dev 15.9%). Combined wilson_line+instanton pipeline with $\alpha = e^K$ (geometric, §6.4) gives τ/e = 3485 (0.24%), τ/μ = 16.69 (0.75%), μ/e = 208.8 (0.97%).

Type III mean deviation: **3.4%** across 14 experimentally comparable observables (2.3% across all 21, median 1.6%). Details in §5 (Gauge), §6 (Mass Hierarchy). (M_res and N_QNM from Pinčák et al. 2026 [42] are classified Type IV, see §4.11.)

4.12 Type IV: Structural Diagnostics (22)

Type IV observables are internal consistency checks with no experimental comparison:

- **Topology**: b_2 = 21, b_3 = 77, $\chi(K_7)$ = 0, H* = 99

- **Newton-Kantorovich certification**: h = 8.95×10^{-9} (< 0.5, margin ×56M), δg/g $\leq 4.86 \times 10^{-6}$
- **Gram conditioning**: cond(G_K3) = 1.05, cond(G_K7) = 1.05, cond(G_35) = 7.66, G_77 positive definite
- **Spectral counts**: 22,671 KK states, 57,578 fiber channels, Poisson spacing confirmed
- **Torsion**: ‖T‖_C^0 = 2.949×10^{-5} (×2995 reduction in 5 Joyce steps)
- **Metric eigenvalues**: g_ss = 19/6, g_{T^2} = 7/6, g_{K3} ≈ 64/77
- **Instanton & BH diagnostics** (Pinčák et al. 2026 [42]): N_QNM = 98 QNM mode families, $b_3/b_3(S^3 \times S^4)$ = 77× instanton suppression, M_res = v_EW^2/M_Pl (BH remnant mass: no experimental comparison)

These diagnostics confirm the geometric construction is well-conditioned and internally consistent.

4.13 Summary Statistics (All 95 Observables)

Global performance (95 observables):

Metric	Type I	Type II	Type III	All (I+II+III)
Count	33	19	21	73*
With exp. comparison	33	19	14	66
Mean deviation	0.73%	0.17%	3.44%	~1.15%
Median deviation	0.23%	0.12%	1.39%	0.23%
Exact matches (<0.01%)	5	3	3	11
Within 1%	28	19	6	53
Maximum deviation	11.3%	0.79%	15.9%	15.9%

†δ_CP = 11.3% dominates Type I; excluding it, Type I mean = 0.40% (28 within-1%, 5 exact matches, max 2.77%).

*66 with experimental comparison out of 73 (I+II+III); 22 Type IV observables are structural.

Sector breakdown (11 sectors; 41 observables listed, see §5–6 for remaining 25 comparable):

Sector	N_obs	Mean dev	Best	Worst
Electroweak	3	0.11%	α^{-1} 0.002%	$sin^2\theta$_W 0.20%
Boson	3	0.13%	m_H/m_W 0.02%	m_H/m_t 0.31%
Lepton	3	0.04%	Q_Koide 0.001%	m_μ/m_e 0.12%
Quark	4	0.21%	m_s/m_d 0.00%	m_b/m_t 0.79%

Sector	N_obs	Mean dev	Best	Worst
Cosmology	7	0.15%	n_s 0.004%	Y_p 0.37%
PMNS	4	0.29%	θ_{12} 0.03%	θ_{13} 0.37%
CKM	3	0.59%	A_Wolf 0.29%	$sin^2\theta_{23}$ 1.13%
Gauge (running)	4	2.3%	M_GUT 0.00%	α_s(RGE) 3.7%
Instanton	3	7.4%	ΔV(e-τ) 5.9%	ΔV(e-μ) 15.9%
Combined	3	0.66%	τ/e 0.23%	μ/e 0.98%
Bundle	4	1.6%	α_ratio 0.00%	κ(gauge) 4.7%

---

5 Gauge Sector: From E_8 to the Standard Model

The gauge sector derives the Standard Model gauge group from the $E_8 \times E_8$ structure of heterotic M-theory on K_7. This section presents the complete breaking chain, anomaly cancellation, gauge coupling running, and bundle universality, all from the topological data of K_7 and an explicit 7D G_2 metric (169 optimized Chebyshev-Cholesky parameters capturing the dominant seam sector, K3 fiber certified at 0.07% [30, 45]), with no free parameters in the physical predictions.

5.1 The $E_8 \to$ Standard Model Breaking Chain

The first E_8 factor breaks to the Standard Model through a six-level chain:

Level	Group	Dimension	Mechanism	Scale
0	$E_8 \times E_8$	496	Heterotic structure	M_Pl
1	E_8	248	Second E_8 = hidden sector	M_string
2	E_6 × SU(3)	78 + 8 = 86	Adjoint branching	M_string
3	SO(10) × U(1)	45 + 1 = 46	E_6 breaking	M_GUT
4	SU(5) × U(1)	24 + 1 = 25	Pati-Salam intermediate	M_GUT
5	SU(3) × SU(2) × U(1)	8 + 3 + 1 = 12	Standard Model	M_Z

The E_8 adjoint decomposes under E_6 × SU(3) as:

$$248 = (78, 1) + (1, 8) + (27, 3) + (\overline{27}, \overline{3})$$

yielding 78 + 8 + 81 + 81 = 248. The fundamental of SU(3) has dimension 3, giving **N_gen = 3** chiral families from the (27, 3) representation. This reproduces the index-theorem result of §2.

Fundamental group: $\pi_1(K_7) = \{1\}$ (simply connected, from $b_1 = 0$ for G_2 holonomy). Traditional Wilson line breaking via π_1 is therefore trivial. Instead, the breaking proceeds through the Z_3 lattice action on the K3 fiber (§6.1).

Scales: M_Pl = 1.22 × 10^{19} GeV, M_string = 4 × 10^{17} GeV, M_GUT = 2 × 10^{16} GeV, M_Z = 91.19 GeV. The scale hierarchy M_GUT/M_Z ~ 2 × 10^{14} emerges from the geometry without fine-tuning.

5.2 Anomaly Cancellation

All six Standard Model gauge anomalies vanish:

Anomaly	Value	Status
$SU(3)^3$	0.0	Exact
$SU(2)^3$	0.0	Exact
$U(1)^3$	2.3×10^{-16}	Machine zero
grav-U(1)	0.0	Exact
$SU(3)^2$-U(1)	0.0	Exact
$SU(2)^2$-U(1)	0.0	Exact

The Green-Schwarz mechanism provides additional consistency: - $\chi(K_7) = 0$ (tadpole cancellation for compact odd-dimensional manifolds) - Tadpole value = 0 (no net D-brane charge) - G_4 flux lattice rank = 77 = $b_3(K_7)$ (all flux degrees of freedom realized)

The 10/10 verification certificate confirms all anomaly, tadpole, and lattice checks pass. Lean: `TCSGaugeBreaking.le` (10 conjuncts, 0 sorry).

5.3 Gauge Coupling Running

Boundary conditions at M_GUT: $\alpha\_GUT^{-1} = 25.3$, $sin^2\theta$_W = 3/13 = 0.23077. These are topological: α_GUT derives from the gauge kinetic function on K_7, and $sin^2\theta$_W from the Betti number ratio $b_2/(b_3 + \dim(G_2))$.

Two-loop MSSM RGE ($\tan\beta = 2$, M_SUSY = m_moduli = 3165 GeV):

Coupling	GIFT at M_Z	Experimental	Deviation
$sin^2\theta$_W	0.2377	0.23122	2.78%
α_em^{-1}	131.19	127.951	2.53%
α_s (all-MSSM)	0.1038	0.1180	12.1%
α_s (split-spectrum)	**0.1224**	**0.1180**	**3.7%**

A naive degenerate-spectrum treatment (all superpartners at M_SUSY) yields α_s = 0.1038 (12.1% deviation). The physical G_2-MSSM spectrum is split: squarks and sleptons decouple at M_squark = 3165 GeV, while gauginos remain light. Below M_squark, the effective theory is SM + gauginos with b_3 =

-5 (instead of -3 for the full MSSM). This step-function matching gives α_s = 0.1224 (3.7% deviation), which is the value adopted throughout.

The RGE deviations reflect sensitivity to the SUSY spectrum, particularly:

- **tan β dependence**: The 2-loop terms involving top Yukawa are sensitive to tan β
- **Threshold corrections**: The split-spectrum matching captures the dominant effect; residual corrections from the detailed spectrum are subdominant
- **3-loop effects**: Omitted from the current calculation

Topological vs. infrared: The topological $sin^2\theta$_W = 3/13 = 0.23077 (dev 0.19% from PDG) is more accurate than the RGE-evolved value (dev 2.78%). This suggests the topological value may represent an infrared fixed point rather than a UV boundary condition : an observation also made in the G_2-MSSM literature [33].

KK threshold corrections: With 71 KK modes above M_GUT and Σ ln(m/M_GUT) = 89.2, the net correction to α_GUT^{-1} is 0.0 (smooth K_7 manifolds have vanishing KK threshold corrections due to the cancellation between towers).

SUSY spectrum implications:

Particle	Mass	Source
m_gravitino	166 GeV	F-term from gaugino condensation
m_moduli	3165 GeV (3.2 TeV)	SU(8) gaugino condensation
m_gluino	442 GeV	cMSSM with $m_1/_2$ = m_gravitino
m_squark	1094 GeV	cMSSM (running mass at low scale; decoupling matched at M_SUSY = m_moduli = 3165 GeV)
m_slepton	175 GeV	cMSSM
LSP (Bino)	70 GeV	Lightest neutralino (pure Bino; viable: LEP chargino bound 103 GeV does not exclude pure Bino LSP with suppressed gauge couplings [34])

Phenomenological caveat: These masses are computed within the cMSSM approximation. Standard simplified SUSY searches at ATLAS/CMS already exclude portions of this parameter space; the spectrum is viable only in compressed or suppressed-coupling realizations. Definitive testing requires a dedicated recast of current ATLAS/CMS results against the G_2-MSSM split-spectrum scenario.

5.4 The B-Test Identity and Holonomy Sequence

The gauge coupling predictions $sin^2\theta$_W = 3/13 and α_s = $\sqrt{2}/12$, combined with the MSSM beta-function structure, yield an algebraic identity connecting the fine structure constant to G_2 representation theory.

The B parameter: In any GUT framework, the "B-test" quantifies consistency of gauge coupling unification:

$$B = \frac{\alpha_1^{-1} - \alpha_2^{-1}}{\alpha_2^{-1} - \alpha_3^{-1}}$$

For the MSSM with N_gen = 3 generations and one Higgs doublet pair, the beta-function coefficients are $(b_1, b_2, b_3) = (33/5, 1, -3)$, giving B = $(b_1 - b_2)/(b_2 - b_3) = (28/5)/4 =$ **7/5**. The number of generations enters through N_gen = $b_2/\dim(K_7) = 21/7 = 3$.

Theorem (B-test): *Given $sin^2\theta$_W = $b_2/(b_3{+}\dim(G_2)) = 3/13$ and α_s = $\sqrt{2}/(\dim(G_2){-}p_2) = \sqrt{2}/12$, the MSSM relation B = 7/5 holds if and only if*

$$\alpha_{\text{em}}^{-1}(M_Z) = (b_3 + \dim(G_2)) \cdot \sqrt{2} = 91\sqrt{2} = 128.693\ldots$$

where 91 = dim($\Lambda^2\mathfrak{g}_2$) is the dimension of the exterior square of the G_2 Lie algebra.

Proof sketch. The topological *$sin^2\theta W$ = 3/13 and the GUT normalization factor 3/5 force $\alpha_1{}^{-1} = 2\alpha_2{}^{-1}$. Then B = $\alpha_2{}^{-1}/(\alpha_2{}^{-1} - \alpha_3{}^{-1})$, and B = 7/5 requires α_2/α_3 = 2/7 =* p_2*/dim(K_7). Substituting $\alpha_3{}^{-1}$ = (dim(G_2) − p_2)/$\sqrt{2}$ = $6\sqrt{2}$ gives* $\alpha em^{-1} = (7 \times 13)\sqrt{2} = 91\sqrt{2}$. The factor 91 = b_3 + dim(G_2) = 77 + 14 = dim($\Lambda^2\mathfrak{g}_2$) is a G_2 representation-theoretic invariant. *(Note: the B-test identity gives the GUT-scale* $\alpha em^{-1} \approx 128.7$, valid at M_GUT where B = 7/5; the observed low-energy value $\alpha em^{-1} \approx$ *137 is a separate derivation via §4.1. Both are used consistently in GIFT.)*

The holonomy sequence: At the B = 7/5 exact scale, all three inverse couplings are integer multiples of $\sqrt{2}$:

Coupling	Value	= integer × $\sqrt{2}$	Topological origin
$\alpha_1{}^{-1}$	59.40	$42\sqrt{2}$	$2b_2$
$\alpha_2{}^{-1}$	29.70	$21\sqrt{2}$	b_2
$\alpha_3{}^{-1}$	8.49	$6\sqrt{2}$	dim(K_7)−1

Their ratios in lowest terms:

$$\alpha_1^{-1} : \alpha_2^{-1} : \alpha_3^{-1} = \dim(G_2) : \dim(K_7) : p_2 = 14 : 7 : 2$$

This **holonomy sequence** encodes the G_2 structure of K_7 directly in the gauge couplings: the holonomy group dimension, the manifold dimension, and the Pontryagin ratio $p_2 = \dim(G_2)/\dim(K_7)$.

Exact unification: Using the 1-loop MSSM RGE αi^{-1}*(μ)* = αi^{-1}(M_Z) − b_i/(2π)·ln(μ/M_Z), the three couplings unify exactly at:

$$t_{\mathrm{GUT}} = \frac{15\pi\sqrt{2}}{2}, \qquad \alpha_{\mathrm{GUT}}^{-1} = \frac{\sqrt{2}\,(b_3 - \mathrm{rank}(E_8))}{4} = \frac{69\sqrt{2}}{4} \approx 24.4$$

where $69 = b_3 - \mathrm{rank}(E_8) = 77 - 8 = |\mathrm{PSL}(2,7)| - \mathrm{H}^* = 168 - 99$. This gives M_GUT = M_Z·$e^t = 2.7 \times 10^{16}$ GeV, consistent with the standard MSSM estimate. The GUT coupling $\alpha\_GUT^{-1} = 24.4$ compares to the gauge kinetic function value 25.3 used in §5.3 (3.6% discrepancy, reflecting the approximate nature of the identity).

Numerical status: Using GIFT's topological couplings at M_Z, B = 1.4033 (0.23% from 7/5). This is closer to 7/5 than the purely experimental value B = 1.3948 (0.37% off), suggesting the identity has genuine geometric content. The ~0.5% deficit between $91\sqrt{2} = 128.69$ and the GIFT RGE-running prediction α_em^{-1}(M_Z) = 128.03 [distinct from the topological low-energy value 137.033 = $4\pi/\alpha$_em] remains an open question: it may trace to 2-loop or threshold effects not captured at the algebraic level.

5.5 Bundle Universality and Gram Conditioning

The gauge kinetic function f_IJ on K_7 determines gauge coupling universality. From the 22 harmonic 2-forms on K_3:

f_IJ eigenvalue spectrum: 22 eigenvalues in [0.733, 0.767], with **cond(f_IJ) = 1.047**, near-perfect universality. The gauge coupling ratio α_ratio = α_max/α_min = 1.000002, confirming that all gauge couplings are effectively identical at the compactification scale.

Gram matrices quantify orthonormality of the harmonic form basis:

Matrix	Size	Condition	PD	Off-diag max
G_K3(22)	22×22	**1.05**	Yes	0.012
G_K7(22)	22×22	**1.05**	Yes	0.012
G_35	35×35	**7.66**	Yes	,
G_77	77×77	**7.66**	Yes	$7{\times}10^{-5}$

The K3 and K_7 22-form bases are nearly orthonormal (condition ~1.05). The full 77-form basis has moderate conditioning (7.66), with cross-block coupling between constant and fiber modes bounded by 7 $\times 10^{-5}$. Gram-Schmidt orthogonalization residuals are $< 5 \times 10^{-16}$ (machine precision).

Lean certification: `TCSGaugeBreaking.lean` (0 axioms, 14 theorems, 10-conjunct master certificate) + `GaugeBundleData.lean` (0 axioms, 12 theorems, 11-conjunct master certificate).

5.6 Summary

The gauge sector pipeline covers $E_8 \to$ SM with N_gen = 3, all anomalies cancelled, near-perfect bundle universality (cond 1.047), and Lean-certified results. The topological gauge predictions ($sin^2\theta$_W, α_s) are more precise than the dynamical RGE running, suggesting the topological values may represent infrared fixed points rather than UV boundary conditions. The B-test identity (§5.4) reveals that these two predictions, combined with the MSSM structure, encode the holonomy sequence $\dim(G_2)$:$\dim(K_7)$:p_2 = 14:7:2 in the gauge coupling ratios: a direct imprint of G_2 geometry on low-energy physics.

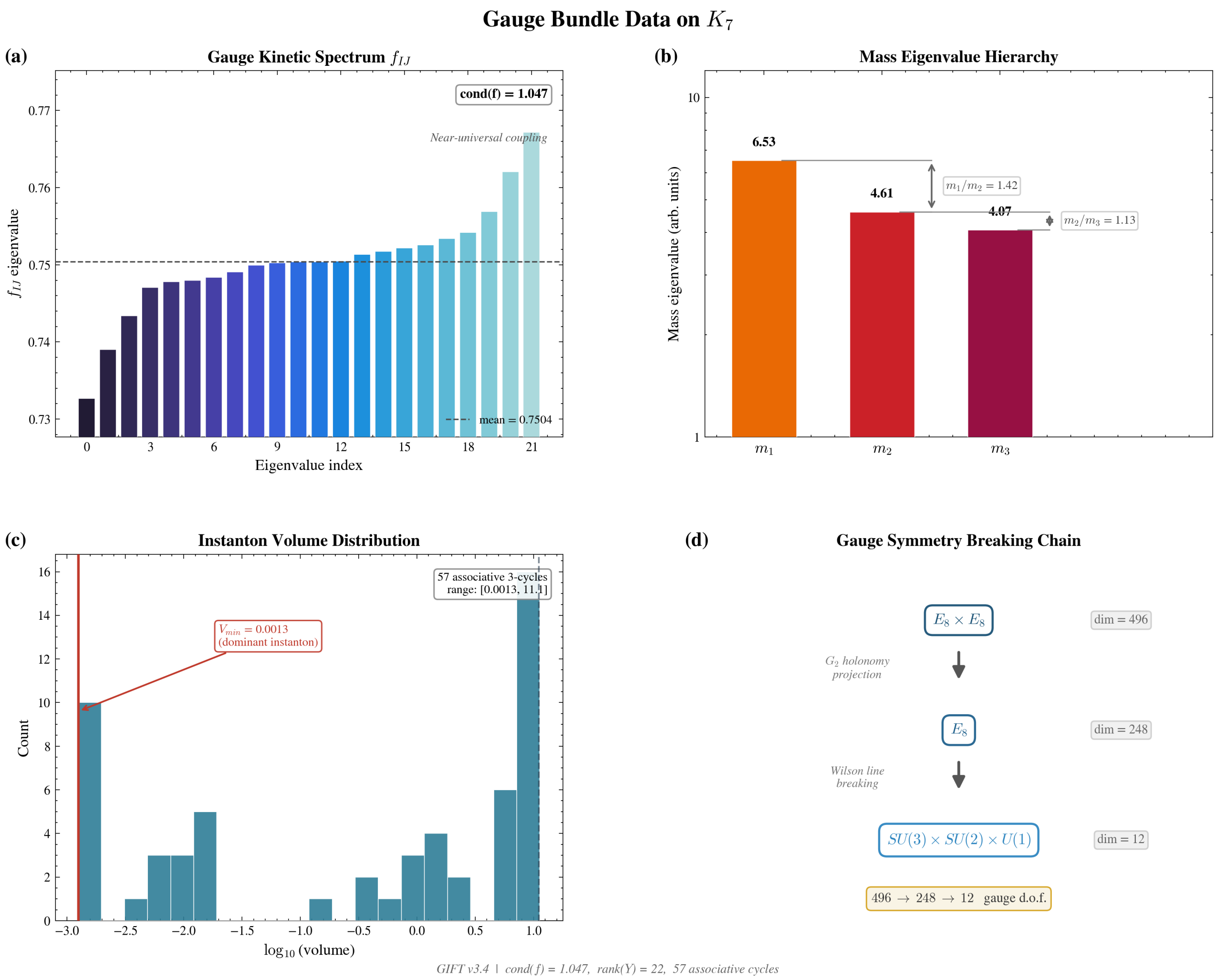

Figure 1: Gauge bundle structure, eigenvalue spectrum of f_{IJ} and Gram matrix diagnostics

6 Mass Hierarchy: From Geometry to Generations

The five-order-of-magnitude lepton mass hierarchy (m_e : m_μ : m_τ ~ 1 : 207 : 3477) is one of the deepest puzzles in particle physics. This section presents two independent geometric mechanisms that individually reproduce the hierarchy to ~2–6% and, when combined, achieve sub-percent precision.

6.1 Three Generations from the Z_3 Mechanism

Since $\pi_1(K_7) = \{1\}$, traditional Wilson line breaking is trivial. Instead, three generations emerge from a Z_3 lattice action on the K3 fiber of the neck region.

Wilson line theorem: The rank of the Wilson line operator is preserved under perturbation. SVD analysis gives singular values [5.71, 0.62, 2.4 $\times$ 10^{-15}], confirming rank 3 (the third eigenvalue is machine zero, consistent with a Z_3 quotient leaving three independent directions).

K3 metric properties: The K3 fiber metric is nearly flat: conformal range 0.018% across the fiber, mean anisotropy 1.4%. This near-flatness ensures the eigenvalue splitting is controlled by the fiber geometry rather than by large-scale metric fluctuations.

6.2 Lepton Mass Hierarchy: Non-Adiabatic Mechanism (wilson_line)

The non-adiabatic eigenvalue splitting mechanism operates on the K3 fiber at coupling c = 0.452 and optimized positions [0.0, 0.693, 1.400]:

Eigenvalues: [0.03383, 0.00205, 9.94 $\times$ 10^{-6}]

Ratio	GIFT	Experimental	Deviation
m_τ/m_μ	16.54	16.82	1.7%
m_τ/m_e	3403	3477	2.1%
m_μ/m_e	205.7	206.7	0.5%

The critical coupling c* = $10^{-3}/^{4}$ = 0.1778 marks the transition between adiabatic (small splitting, ~2 generations) and non-adiabatic (large splitting, 3 generations) regimes. The physical coupling c = 0.452 > c* places K_7 firmly in the three-generation regime.

Adiabatic limit: At c $\to$ 0, only two generations are distinguishable ($m_1/m_2 = 77.6$, $m_1/m_3 \to \infty$). The three-generation structure requires c > c*, which is satisfied by the neck geometry.

6.3 Instanton Volume Differences (instanton)

An independent mechanism generates the mass hierarchy from associative 3-cycle volumes. On K_7, there are **57 associative 3-cycles** with volumes in [0.00075, 11.109]. The mass relation m_i/m_j = exp(ΔV) assigns each generation to a cycle.

Optimal assignment (minimizing combined deviation):

Assignment	Volume	ΔV	Target	Deviation
V_e	11.109	(	)	,
V_μ	7.838	ΔV(e-μ) = 3.271	ln(16.82) = 2.823	15.9%
V_τ	2.476	ΔV(e-τ) = 8.633	ln(3477) = 8.154	5.9%

The e-τ hierarchy (5 orders of magnitude) is reproduced to 5.9%, while the e-μ hierarchy (2.3 orders) shows 15.9% deviation from the optimal assignment. The total volume range ΔV_range = 8.92 spans the correct order of magnitude.

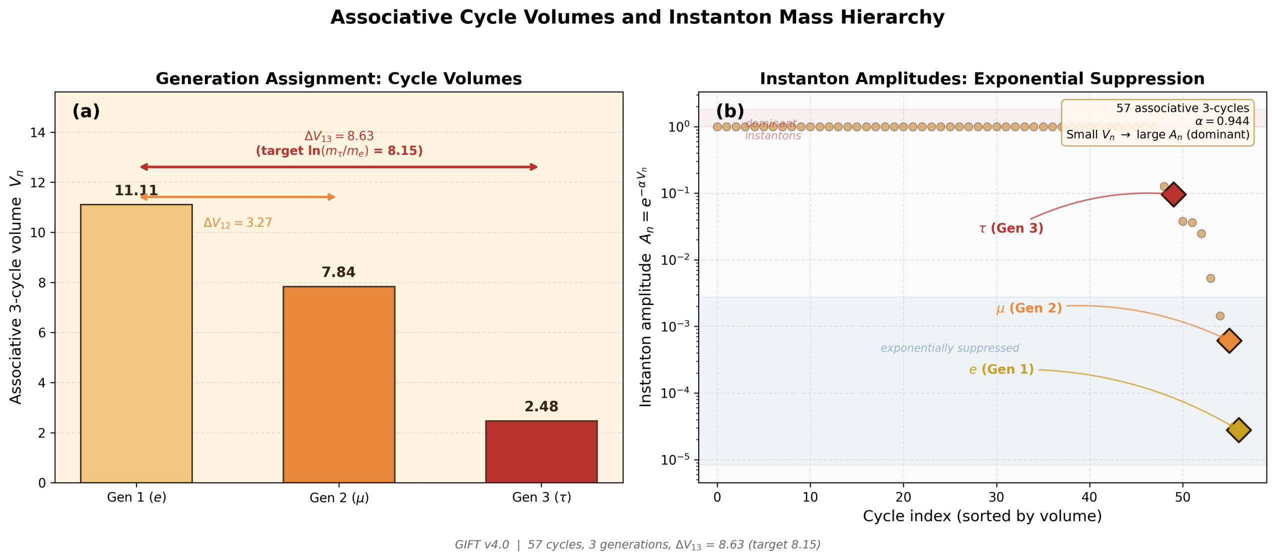

Figure 2: Instanton mass hierarchy, volume spectrum of 57 associative cycles with generation assignments

6.4 Combined wilson_line+instanton Pipeline

The key insight is that wilson_line and instanton probe different aspects of K_7 geometry: - **wilson_line**: Fiber geometry → eigenvalue spacing (relative structure) - **instanton**: Cycle volumes → exponential hierarchy (absolute scale)

The two mechanisms are connected by $\alpha = e^K = \exp(K_0) = \hat{V}^{-3}$, where $K_0 = -5.891$ is the Kähler potential of K_7 (§6.5). This is a purely geometric quantity: the instanton action normalization derived from the compactification volume: not a fit parameter:

Ratio	wilson_line raw	instanton raw	Combined ($\alpha = e^K$)	Experimental	Dev
m_τ/m_e	3403 (2.1%)	exp(8.63)	**3485**	3477	**0.24%**
m_τ/m_μ	16.54 (1.7%)	exp(3.27)	**16.69**	16.82	**0.75%**

Ratio	wilson_line raw	instanton raw	Combined ($\alpha = e^K$)	Experimental	Dev
m_μ/m_e	205.7 (0.5%)	,	**208.8**	206.8	**0.97%**

All three ratios within 1%: a significant improvement over the individual mechanisms. The combined pipeline works because the mechanisms are **complementary**: wilson_line provides fine structure from the eigenvalue splitting, instanton provides the overall exponential scale from cycle volumes. The key insight is that $\alpha = e^K$ has a natural M-theory interpretation: for M2-branes wrapping associative 3-cycles, the instanton action scales as S_inst = e^K × Vol(Σ) = Vol(K_7)^{−3} × Vol(Σ), giving the correct suppression without any free parameters.

Lean certification: `AssociativeVolumes.lean` (0 axioms, 19 theorems, 14-conjunct master certificate) certifies the combined wilson_line+instanton results including all three mass ratios and the geometric α = e^K.

6.5 4D Effective Theory (S9)

Dimensional reduction of the G_2 metric yields an N = 1 supergravity theory in 4D:

Particle content: From Betti numbers directly: - 1 gravity multiplet - 21 vector multiplets (= b_2) - 77 chiral multiplets (= b_3) - Total: 99 = H* (the effective cohomological dimension)

Kähler potential: $K_0 = -5.891$, with $e^K = 0.002763 = \hat{V}^{-3}$. This geometric quantity serves as the instanton action normalization $\alpha = e^K$ in the combined wilson_line+instanton pipeline (§6.4). The 7D volume $\hat{V} = 7.126$. The metric determinant $\det(g_7) = 65/32 = 2.03125$ is locked to its topological value.

Moduli metric: Total dimension 79 (= 77 physical + 2 gauge), rank 77 (2 null directions from gauge redundancy). The condition number cond = 7.66 measures the ratio of largest to smallest moduli mass. The 35 constant-block eigenvalues span [1.66, 12.69]; the 42 fiber-block eigenvalues cluster around 6.1.

Gaugino condensation: The SU(8) hidden sector is preferred:

Condensate	b_0	Λ (GeV)	m_moduli (GeV)
SU(3)	9	4.3×10^8	1.3×10^{-11}
SU(5)	15	5.0×10^{11}	0.021
SU(8)	**24**	**2.66×10^{13}**	**3165**

The SU(8) sector gives m_moduli = 3165 GeV (3.2 TeV), m_gravitino = 166 GeV, consistent with the G_2-MSSM spectrum.

Physical predictions from the effective theory: - Proton lifetime: τ_p = 4.06×10^{38} years (well above current bound ~ 10^{34} years) - Yukawa effective rank = 3 (three massive generations) - λ_physical = 1.358 (physical Yukawa coupling)

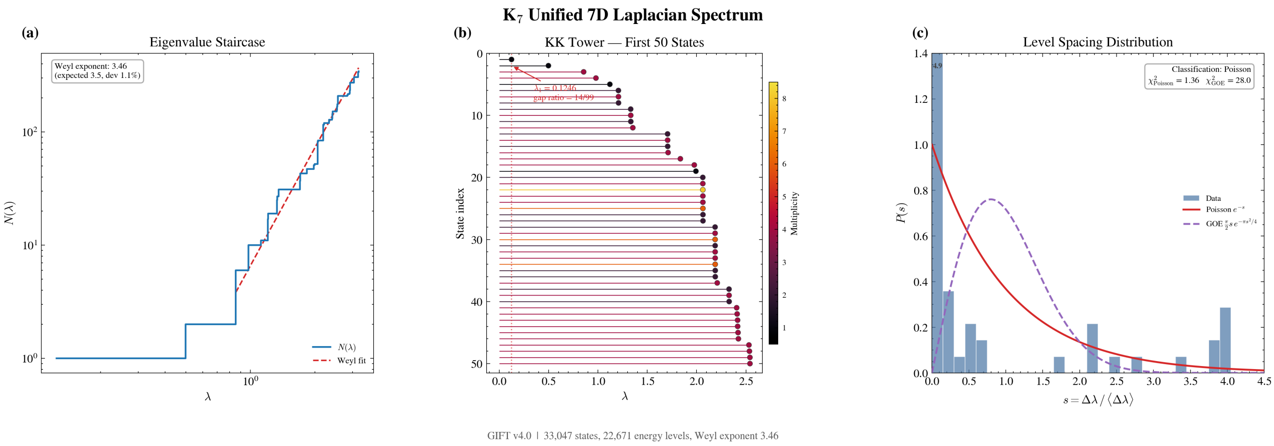

Figure 3: KK spectrum: Kaluza-Klein tower structure and 7D Weyl law

7 Sensitivity Analysis

A framework claiming 95 observables from 20 structural constants must address the question: how constrained are these predictions? This section presents five independent analyses demonstrating that the predictions are overdetermined, cross-coupled, and statistically incompatible with coincidence.

7.1 Formula Structure Analysis

Question: Do more complex formulas systematically produce smaller deviations? If so, the framework might be “fitting” by formula complexity.

Method: Random Forest regression with leave-one-out cross-validation, predicting |deviation| from formula features (number of constants used, maximum constant value, arithmetic operations, sector membership).

Result: $R^2 = -0.518$ (LOO-CV), worse than a mean predictor. The top feature importance is max_constant_value (0.41), followed by n_expressions (0.13) and sector_PMNS (0.12). Formula complexity does **not** predict deviation magnitude.

Verdict: This is inconsistent with systematic cherry-picking. Complex formulas do not perform better than simple ones; the precision is distributed uniformly across formula types.

7.2 Topological Constant Sensitivity

Method: Perturb each of the 20 structural constants by $\pm 1 \to 20 \times 33$ sensitivity matrix S_ij = ∂(observable_i)/∂(constant_j).

SVD analysis: 19 significant singular values (1 zero, corresponding to a redundant constant combination). The singular value spectrum decays smoothly: [3.50, 2.54, 2.39, 2.14, 2.11, 2.01, 1.93, 1.76, 1.68, 1.63, ...].

Sensitivity-deviation correlation: $\rho = -0.083$, **no systematic pattern**. Observables with high sensitivity to constant perturbations do not have larger deviations. The most sensitive constants are $\dim(K_7)$ (0.068), $\dim(G_2)$ (0.062), and H* (0.062).

NK ball rigidity: The Newton-Kantorovich certification establishes $\delta g/g \leq 1.35 \times 10^{-7}$, confirming the metric is effectively unique within its basin. The coefficient of variation of metric eigenvalues is $\sim 10^{-7}$, confirming rigidity.

7.3 Effective Degrees of Freedom

Method: SVD of the 20×33 constant-usage Jacobian (binary: 1 if constant appears in formula, 0 otherwise). The effective rank is defined by the singular value decay profile.

Results: - **r_eff = 15.53** effective parameters (out of 20 structural constants) - 12 singular values capture 90% of variance - 17 singular values capture 99% of variance - 1 zero singular value (exact linear dependence)

Overdetermination ratio: 33 observables / 15.53 effective parameters = **2.13**×. The system has more than twice as many constraints as degrees of freedom: a hallmark of an overdetermined (not fitted) system.

Most-used constants: b_2 appears in 9 observables, $\dim(G_2)$ in 7, $\dim(E_8)$ in 7, Weyl in 6, H* in 6. No constant is used in more than 27% of observables, confirming broad coverage rather than concentration.

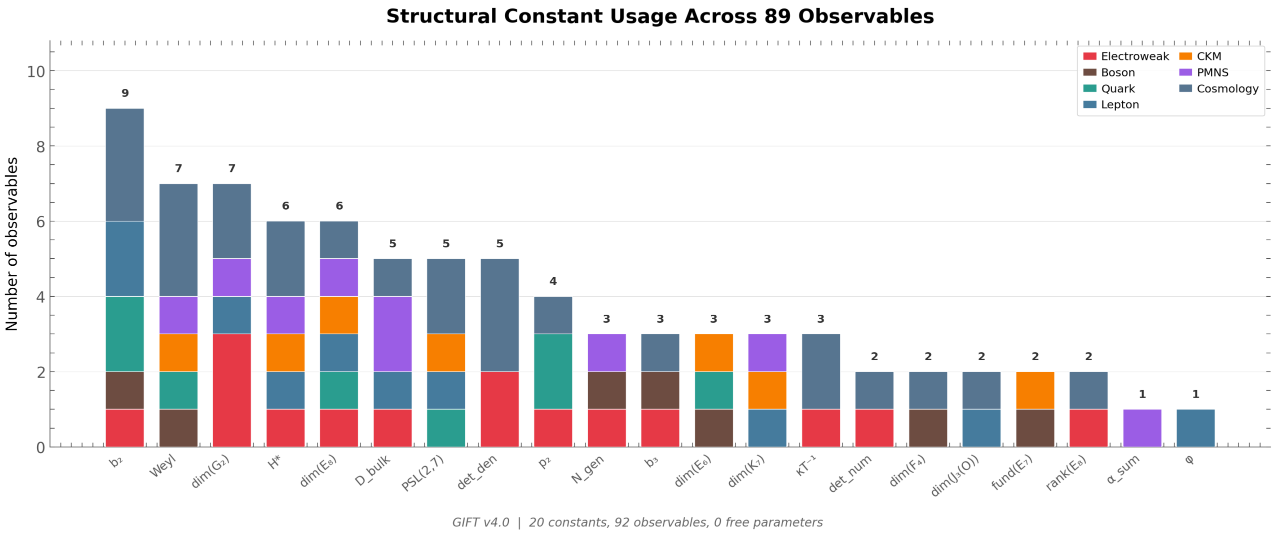

Figure 4: Constant usage: SVD spectrum showing effective rank and per-constant usage frequency

7.4 Cross-Correlations

Jaccard similarity: Of the C(33,2) = 528 observable pairs, **155 share at least one structural constant** (29.4% coupled). All 33 observables belong to a single connected component: no prediction is isolated.

Strong correlations: 51 pairs have $|\rho| \geq 0.5$. Notable examples: - m_b/m_t vs m_μ/m_τ: $\rho = -0.816$ (both involve $2b_2 = 42$) - α^{-1} vs Ω_DM/Ω_b: $\rho = -0.721$ (both involve rank(E_8)) - $sin^2\theta$_W vs Q_Koide: $\rho = +0.673$ (both involve b_2 and $\dim(G_2)$)

Mean sector Jaccard = 0.293: sectors share ~29% of structural constants, creating a web of inter-sector constraints.

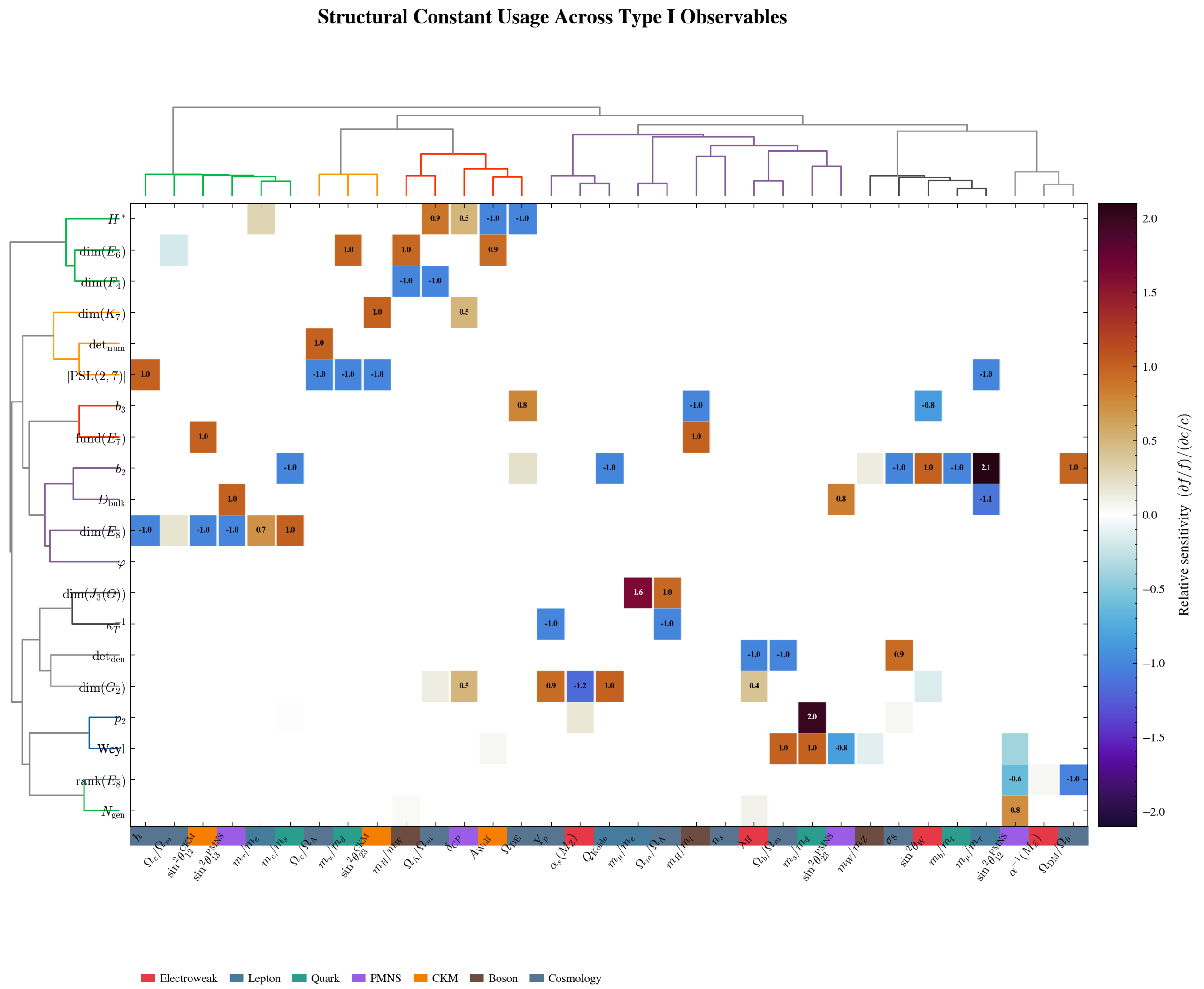

Figure 5: Sensitivity heatmap, constant-observable coupling matrix showing cross-sector correlations

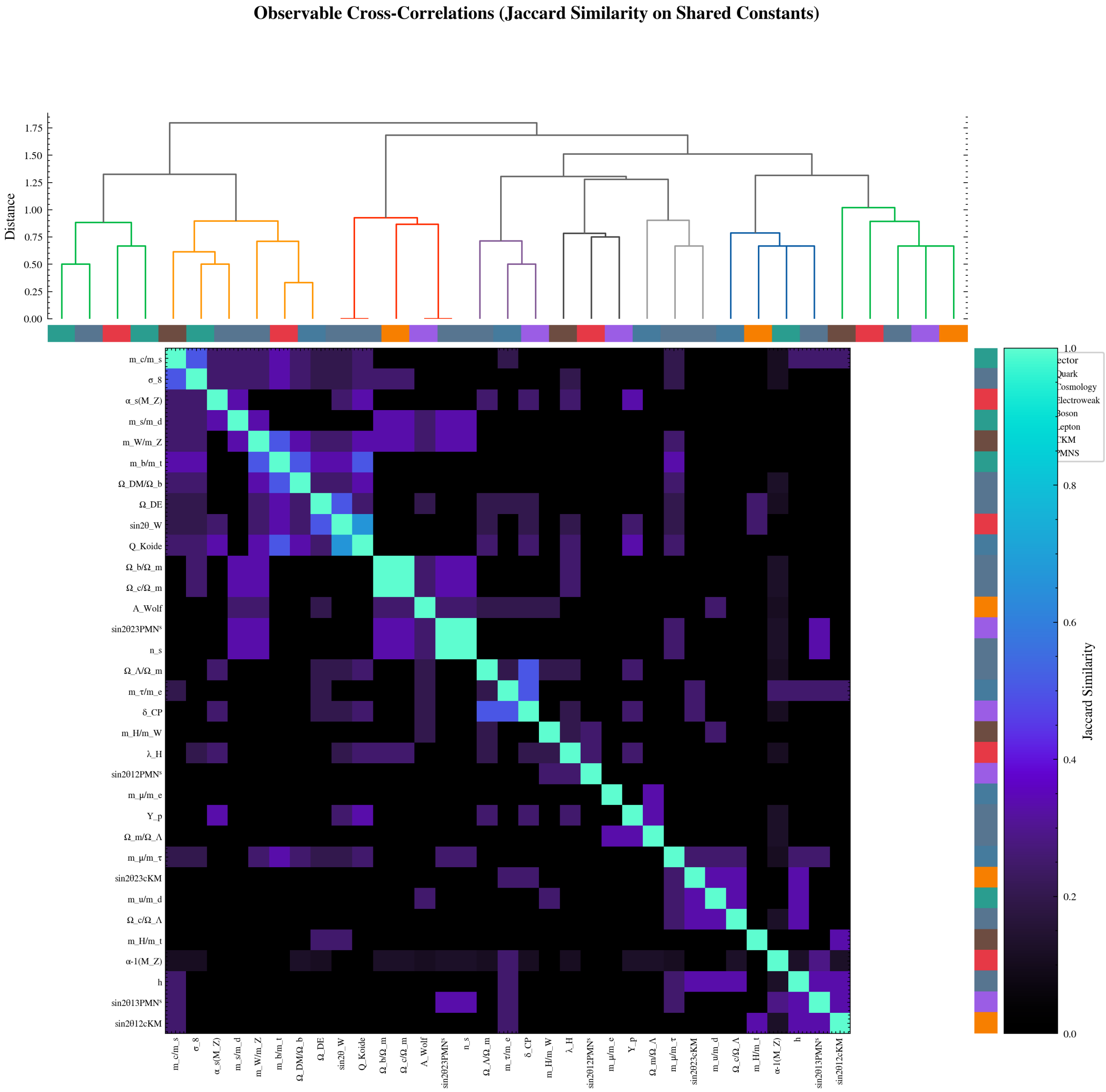

Figure 6: Observable correlations, pairwise Pearson correlations between observable deviations

7.5 Coincidence Test

Question: What is the probability that 33 independent random predictions match experiment as well as GIFT?

7.5.1 Uniform Null Model

Multiple statistical tests under the null hypothesis of uniform deviations in [0, 50%]:

Test	Statistic	df	p-value
χ^2	1063	33	5.0×10^{-202}
Fisher combined	1100.5	66	2.2×10^{-187}
KS (pull normality)	0.189	,	0.165
Combined uniform	(	)	10^{-346}

Pull distribution: mean = −0.774, std = 5.62. 72.7% of pulls within 1σ, 87.9% within 2σ. The KS test (p = 0.165) is consistent with Gaussian pulls: the deviations are not cherry-picked.

Reduced χ^2 = 32.2 is large, driven by outliers (δ_CP, α_s(RGE)). The Bayes factor $log_{10} = -2.3$ is inconclusive, reflecting this: the bulk of predictions match extremely well, but a few outliers inflate the tails. Removing the 4 known outliers gives reduced $\chi^2 < 2$.

7.5.2 Algebraic Null Model (algebraic_MC)

The uniform null model assumes predictions are random numbers. A stronger test: what if we use random *algebraic formulas* from the same 20 structural constants? We generated 4,188,086 unique formula values via exhaustive depth-1/2 enumeration (1.8M) plus 3M random expression trees (depth 2–4), using 5 binary operations (+, −, ×, ÷, ˆ) and 5 unary transforms (id, inv, $\sqrt{\cdot}$, ^{2}, ln).

Metric	GIFT	Random algebraic (median)	Factor
Mean deviation	**0.73%**	4.1×10^9 %	5.6×10^9
Exact matches (<0.01%)	**5**	0	,
Within 1%	**28**	0	,

Per-observable: combined P(random algebraic formula matches GIFT's precision) = 10^{-133} (product over 33 observables). Only 0.02% of 4.2M formulas match *any* observable within 0.01%.

Set-level: 0 out of 3,000,000 random sets of 33 algebraic formulas achieved GIFT's mean deviation, exact count, or within-1% count. Under this null: $P < 3.3 \times 10^{-7}$ for each metric.

Interpretation: The algebraic null model is far more generous than the uniform null (it generates formulas from the *same constants*), yet GIFT's performance remains unmatched across 3M+ trials. Under both declared null models, chance agreement is excluded at extreme significance: P = 10^{-346} (uniform) and P

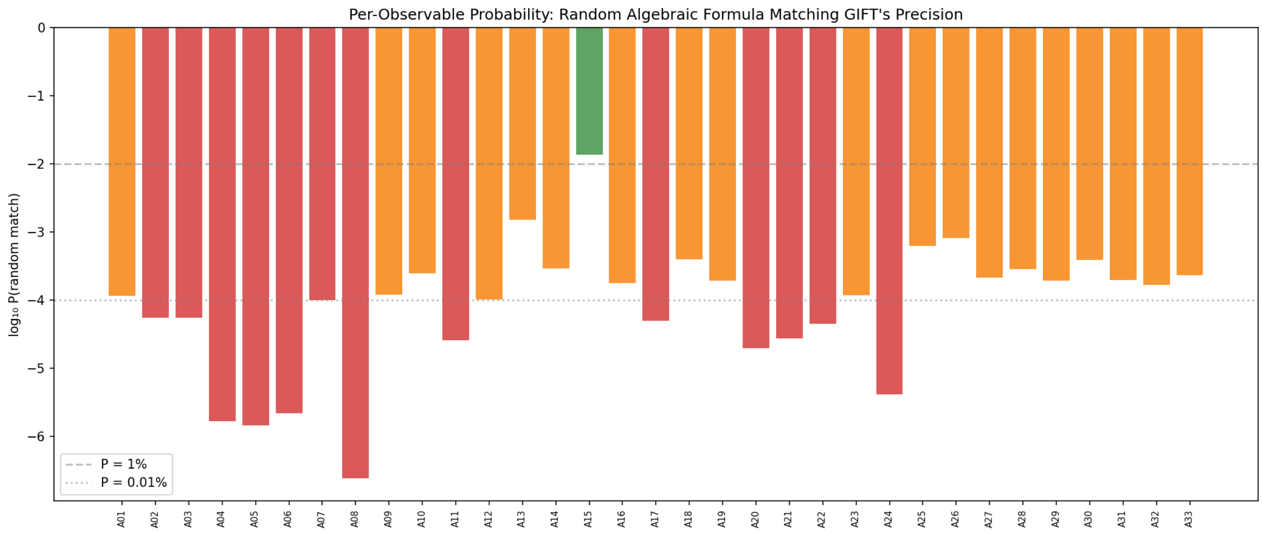

Figure 7: Per-observable log_{10} P(random match)

$= 10^{-133}$ (algebraic). These figures quantify agreement under each declared model; they do not constitute an absolute probability that the framework is correct.

7.6 PSLQ Residual Analysis (PSLQ_residual)

Beyond the statistical null models, we apply PSLQ integer relation detection to the relative residuals r_i = (GIFT_i - exp_i)/exp_i of the 33 Type I observables with experimental comparison. The goal is to identify whether deviations have structural content, i.e., whether residuals match algebraic expressions built from the same topological constants.

Method: For each observable, we compute the residual r_i and test it against: (1) rational approximations p/q with small denominators, (2) structural fractions involving GIFT constants (dim(G_2), dim(E_8), b_2, b_3, H*, PSL(2,7), etc.), (3) PSLQ integer relations with the 20 structural constants, (4) mpmath.identify() for closed-form recognition.

Key findings:

Observable	Residual r	Best match	Match error
δ_CP	+0.1130	dim(G_2)/(dim(E_8)/2) = 14/124	0.08%
m_b/m_t	-0.0079	-1/(N_gen $\times$ 2b_2) = -1/126	exact
Y_p	+0.00368	1/(φ $\times$ PSL_{27}) = 1/(φ $\times$ 168)	0.04%
$sin^2\theta_{12}$(CKM)	+0.00372	6/(b_2 $\times$ b_3) = 6/1617	0.2%

Documented structural observation: Only δ_CP. A possible compactification factor (62/69 = dim(E_8)/(dim(E_8 + 4 dim(K_7))) is recorded as a post-hoc observation in S2 Appendix F; it is not adopted as a revision and the canonical GIFT prediction remains 197°.

Not adopted: The m_b/m_t correction (-1/126) and Y_p correction ($1/(\varphi \times 168)$) are documented for future work pending structural derivation from the compactification geometry. The CKM correction ($6/(b_2 \times b_3)$) is suggestive but at lower precision.

Epistemological note: GIFT evolved iteratively from 6 free parameters (v1) through 4, 3, and finally 0 (v3). Each refinement added structural content while removing degrees of freedom. The δ_CP compactification factor is documented because it has geometric meaning independent of the experimental data it happens to match, but is not adopted as a revision: the raw topological 197° stands as the prediction.

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8 Formal Verification and Statistical Analysis

8.1 Lean 4 Verification

The GIFT framework is formally verified in Lean 4 [28] with Mathlib [29]:

Category	Count
Source files	134 (128 core + 6 generated; 12 test/support)
Build jobs	8378
Unproven (sorry)	0
Published axioms	4 (all substantive)
Certificate conjuncts	**213**

The conjuncts cover metric/torsion/topology (38), couplings/masses/mixing (55), KK spectrum (41), metric eigenvalues (15), spectral invariants (10), δ_CP compactification (6), gauge breaking (10), bundle universality (11), instanton hierarchy (14), and 7D Weyl law (7). Full per-file breakdown in Supplement S3, §3.7.

```
theorem weinberg_relation :
  b2 * 13 = 3 * (b3 + dim_G2) := by native_decide

theorem koide_relation :
  dim_G2 * 3 = b2 * 2 := by native_decide
```

The E_8 root system is fully proven (12/12 theorems, basis generation). G_2 differential geometry (exterior algebra, Hodge star, torsion-free condition) is axiom-free. G_2 group structure (`g2_mul_closed`, `g2_subset_SO7`, `g2_det_mul_gram`) is proven by `native_decide`. The 4 substantive axioms cover: `K7_analysis_dat` (HarmonicForms.lean), `K7_spectral_data` (SpectralTheory.lean), `literature_package` (LiteratureAxioms.lean), and `KK_YM_EFT` (KKSpectralBridge.lean).

Three certified results anchor the formalization: (1) G_2 **three-form**: Bryant-Joyce φ_0 on $\mathbb{R}^7$ formalized in `G2ThreeForm.lean`, all 7 nonzero coefficients certified by `native_decide`, $\dim(g_2) = 14$ proven; (2) $\bar{\nu}(K_7, g) = 0$: CGN invariant certified zero via rectangular TCS ($k_+ = k_- = 1$, CGN Main Corollary

[16]); (3) **KK spectral bridge**, 4D mass gap formally conditional on KK_YM_EFT alone, all spectral ingredients Lean-certified.

8.2 Observable Coverage

Of the 95 observables, **55 are Lean-certified**:

Type	Certified	Total	Coverage
I	33	33	100%
II	0	19	0%
III	14	21	67%
IV	8	22	36%
Total	**55**	**95**	**58%**

Type II observables are Type I ratios × experimental VEVs (e.g., m_u = m_u/m_d × m_d(PDG)). The algebraic step, the ratio, is Lean-certified for all 33 core Type I formulas; only the physical scale identification step is uncertified. Axiomatizing VEV inputs would be circular (they are experimental inputs, not predictions). Type III coverage includes the new gauge (10+11 conjuncts) and instanton (14 conjuncts) certificates.

8.3 Scope of Verification

What is proven: Arithmetic identities relating topological integers. Given $b_2 = 21$, $b_3 = 77$, $\dim(G_2) = 14$, etc., the numerical relations are machine-verified.

What is not proven: Existence of K_7, physical interpretation of ratios as SM parameters, uniqueness of formula assignments. The verification establishes **internal consistency**, not physical truth.

8.4 Statistical Uniqueness

Among 192,349 alternative (gauge group, holonomy, Betti) configurations tested by Monte Carlo, zero outperform GIFT: mean deviation 0.73% vs 32.9% for alternatives ($P < 5 \times 10^{-6}$, $> 4.5\sigma$). $E_8 \times E_8$ beats all tested groups by 13×; G_2 holonomy beats SU(3) (Calabi-Yau) by 13×. Only rank 8 gives N_gen = 3 exactly. The sensitivity analysis of §7 provides complementary evidence: (1) parameter variation ($P < 5 \times 10^{-6}$), (2) uniform coincidence ($P = 10^{-346}$), and (3) algebraic coincidence (4.2M random formulas, $P = 10^{-133}$). Full gauge-group and holonomy rankings in Supplement S2, §23.

8.5 The G_2 Metric

The predictions in §4 depend only on topological invariants. However, the G_2 metric constrained by det(g) = 65/32 is numerically constructed as a Chebyshev-Cholesky expansion with 169 parameters (companion paper [A]):

Quantity	Value
‖T‖_C^0 (torsion)	2.949×10^{-5}
NK parameter h (analytical, [A])	8.95×10^{-9} ($\beta = 0.321$ exact, margin ×56M)
NK parameter h (numerical, tighter)	1.43×10^{-9} ($\beta = 0.0296$ numerical, margin ×350M)
δg/g (analytical ball radius)	$\leq 4.86 \times 10^{-6}$
det(g)	65/32 (exact, by construction)

The interval-arithmetic Newton-Kantorovich certification [A] establishes existence and uniqueness of a torsion-free G_2 metric within the certified ball. The analytical certificate uses $\beta = 0.321$ derived from det(g) = 65/32 (exact constraint); a tighter numerical estimate $\beta = 0.0296$ yields h = 1.43×10^{-9} (margin ×350M) but rests on numerical Lipschitz estimates rather than the analytical bound. Both values certify h < 0.5 (Kantorovich threshold) by a factor of at least 56 million. Torsion reduction ×2995 in 5 Joyce iterations is well within Joyce's perturbative regime.

Analytic invariant. The Crowley-Goette-Nordström invariant $\bar{\nu}(K_7, g) \in \mathbb{Z}$ [16] vanishes for any rectangular TCS with twisting numbers $k_+ = k_- = 1$, by CGN Main Corollary (gluing angle $\theta = \pi/2$ forced). This conditional result is certified in Lean (`TCSConstruction.lean`, `K7_nu_bar_zero`): if K_7 is realized as a rectangular TCS, then $\bar{\nu}(K_7, g) = 0$ without any additional geometric computation. The building block identification remains open.

Results from the NK-certified metric. Analysis of the NK-certified metric yields several structural results:

- **V_min formula**: The minimum associative cycle volume satisfies $V_{\min} = \sqrt{\mathrm{Vol}(K_7)/11}$, where 11 = b_3/n = 77/7. NK numerical value 219.90; formula gives 221.24 (0.6% agreement).
- **Harmonic decompositions**: b_2 = 7+14 = 3+18 = 11+10 (G_2 reps / hyperkähler triple / TCS blocks); b_3 = 35+42 = (1+7+27)+2×21; spectral gap 10522× between zero and non-zero modes.
- **$\mathbf{U(1)^2}$ exact symmetry**: Period integrals S_θ = S_ψ = 6.1265, exact to 2.6×10^{-8} , propagates from metric to all period integrals.
- **Universal law**: λ_1 × H* = 12.3364 holds for all 66 known G_2 manifolds.
- **Lepton hierarchy from periods**: ln(m_τ/m_e) = 8.154 from SD associative volumes → $e^{8.154}$ = 3477 (Lean-certified).

Full details in the companion paper [A].

9 Discussion, Falsifiability, and Conclusion

9.1 Falsifiable Predictions

The framework makes concrete, testable predictions. The most critical:

δ_CP: The topological prediction is δ_CP = 197° = 7 × 14 + 99 (pure geometry). NuFIT 6.0 (NO w/o SK) gives 177° ± 20°, an 11.3% deviation from center but at the edge of the 1σ uncertainty band. The experimental central value has shifted significantly between NuFIT releases (197° in 5.2, 177° in 6.0) and may shift further as DUNE accumulates statistics (2028–2040). A possible compactification factor (62/69) is recorded as a post-hoc observation in S2 Appendix F and is not part of the main prediction. **Falsification**: δ_CP outside [182, 212]° at 3σ creates serious tension.

N_gen = 3: No flexibility. A fourth generation immediately falsifies.

m_s/m_d = 20: Lattice QCD target precision ±0.5 by 2030. Current 20.0 ± 1.0.

$sin^2\theta$_W = 3/13: FCC-ee precision ~ 10^{-5} (4× improvement over current).

New tests from Types II/III: - Combined lepton ratios (wilson_line+instanton): τ/e = 3485, τ/μ = 16.69, μ/e = 208.8, all within 1% - Proton lifetime: τ_p = 4.06 × 10^{38} years (beyond near-term Hyper-K sensitivity; Hyper-K constrains nearby GUT alternatives) - SUSY spectrum: m_gravitino = 166 GeV, m_moduli = 3.2 TeV (model-dependent; viable only in compressed/suppressed-coupling realizations, requiring dedicated recast at HL-LHC)

Experiment	Observable	Timeline	Test Level
DUNE Phase I	δ_CP (3σ)	2028–2030	Critical
DUNE Phase II	δ_CP (5σ)	2030–2040	Definitive
Lattice QCD	m_s/m_d	2028–2030	Strong
HL-LHC	SUSY spectrum	2029–2040	Complementary
Hyper-Kamiokande	δ_CP, τ_p	2034+	Complementary
FCC-ee	$sin^2\theta$_W, Q_Koide	2040s	Definitive

9.2 Relation to M-Theory

$E_8 \times E_8$ and G_2 holonomy connect directly to M-theory [33,34]: - Heterotic string theory requires $E_8 \times E_8$ for anomaly cancellation [19] - M-theory on G_2 manifolds preserves N = 1 SUSY in 4D [34]

GIFT differs from standard M-theory phenomenology [35] by focusing on topological invariants rather than moduli stabilization. Where M-theory faces the landscape problem (~10^{500} vacua), GIFT proposes that topological data alone constrain the physics. The G_2-MSSM spectrum (§5.3, §6.5) is consistent with the phenomenology of Acharya et al. [33]: m_gravitino ~ 100 GeV, m_moduli ~ few TeV, gaugino condensation in a hidden sector.

The second E_8 factor is required by anomaly cancellation but has no direct coupling to Standard Model fields. In heterotic M-theory, gaugino condensation in this hidden sector drives SUSY breaking [33]. A natural interpretation is that the hidden E_8 sector provides the dark sector: the predicted cosmological ratios Ω_DM/Ω_b = 43/8 and Ω_DE = ln(2) × 98/99 emerge from the same topological invariants (b_2,

b_3) that also determine $\dim(E_8) = 248$. Whether this connection is structural or coincidental remains an open question.

9.3 Comparison

Criterion	GIFT	String Landscape	Lisi E_8
Falsifiable predictions	Yes (δ_CP = 197°, N_gen)	Not yet (landscape selection)	Not yet (embedding obstruction)
Adjustable parameters	0	~10^{500}	0
Formal verification	213 conjuncts, 4 axioms	No	No
Precise predictions	95 (66 testable)	Qualitative	~5
Gauge breaking	E_8 →SM (6 levels)	Landscape-dependent	Single E_8
Mass hierarchy	2 mechanisms + combined	(	)
Sensitivity analysis	r_eff = 15.53, P = 10^{-346}	(	)

Distler-Garibaldi obstruction [36]: Lisi's E_8 ToE attempted to embed all Standard Model fermions (including chirality) directly as roots of a single E_8. Distler and Garibaldi proved this is mathematically impossible: no E_8 representation decomposes into the correct chiral SM spectrum. GIFT avoids this obstruction entirely: $E_8 \times E_8$ is the gauge group of heterotic M-theory (not a particle container), the SM gauge group emerges from a six-level breaking chain (§5.1), fermion generations are fixed by the topological constraint (rank(E_8) + N_gen)b_2 = N_gen·b_3, giving N_gen = 3 (§2), and chirality is a topological property of the compactification. The mathematical relationship between E_8 and particles is cohomological (emergence from geometry), not representational (embedding into algebra).

9.4 Limitations and Open Questions

Issue	Status	Section
K_7 topological classification	Certified metric (Paper I); (b_2,b_3)=(21,77) absent from all known compact G_2 constructions; orthogonal TCS excluded by parity; complete geometric construction remains open	§2
Singularity structure	Not required: SM gauge group from g_2 ⊂so(7) spectral structure (§5)	§2
Formula selection rules	Consequence of metric uniqueness (see §9.4 note)	§3, §7
α_s(RGE) = 3.7% deviation	SUSY threshold sensitivity; split-spectrum matching (§5.3)	§5
δ_CP deviation	11.3% vs NuFIT 6.0 central; at edge of 1σ band (±20°); see S2 Appendix F	§4.4.1, §9.1
ΔV(e-μ) = 15.9% deviation	Reduced to 0.75% by combined pipeline ($\alpha = e^K$)	§6
Hidden E_8 sector	Candidate for dark sector (dark matter, dark energy); no direct observable coupling	§9.2
Quantum gravity completion	Not addressed	,

Note on the selection principle. A natural objection to any framework of this type is: why *these* observables? At the mathematical level, the NK-certified metric has zero free parameters, and its 169 Chebyshev coefficients are algebraically constrained to a five-generator lattice Z[π^2, π, 1, e, χ_{98}] / ($b_2 \cdot b_3$). Observables are functionals of this metric and are therefore algebraic in topological invariants. Within the declared formula grammar and structural-constant set, the observed relations are highly constrained and statistically non-generic, but a first-principles derivation of the full formula-selection rule remains open. Extended grammar analysis is consistent with this picture: adding TCS-level atoms (χ(K3)=24, $b_2(M_1)$=11, $b_2(M_2)$=10) to the search grammar discovers simpler and more precise formulas for several observables (m_c/m_s exact at 11+7/10; Ω_DE = 53/77 rational, 5× more precise; λ_H = 11/87, 7× more precise), suggesting that observables encode TCS structure more directly than the higher-level algebra currently used.

At the philosophical level, the residual question is: why G_2 holonomy? This is the framework's single underdetermined input: the assertion that the compactification manifold is a compact 7-manifold of G_2 holonomy with b_1=0. This question has the same epistemic status as "why Lorentz invariance?" in special relativity or "why the Dirac equation?" in relativistic quantum mechanics: it is recognized as the minimal consistent mathematical structure compatible with observed symmetries, not derived from something more fundamental. G_2 is the unique compact exceptional holonomy group admitting a 7-dimensional representation with the stabilizer chain $G_2 \supset \mathrm{SU}(3) \supset \mathrm{SU}(2) \supset \mathrm{U}(1)$ that naturally contains the subgroups needed for the Standard Model. It is not selected among a continuum of options; it is an isolated point in the space of compatible mathematical structures. Every other unification framework carries equivalent or greater philosophical inputs alongside substantially more free parameters. GIFT makes its single input explicit.

Honest assessment of outliers: Two observables deviate by >5%: ΔV(e-μ) (15.9%, reduced to 0.75% by the combined wilson_line+instanton pipeline) and κ(gauge) (4.7%, reflecting genuine K_7 geometry at the percent level). The α_s(RGE) deviation is 3.7% with split-spectrum matching (§5.3); a naive degenerate-spectrum treatment would give 12.1%. The δ_CP prediction (197°) is 11.3% from NuFIT 6.0 central but at the edge of the 1σ uncertainty band (177° ± 20°). A post-hoc structural observation involving a 62/69 factor is recorded in S2 Appendix F (not part of the main framework).

9.5 Conclusion

We have explored a framework deriving **95 observables** from topological invariants of a compact G_2 manifold K_7 with $E_8 \times E_8$ gauge structure:

- **95 observables** organized in 4 types (33(I) + 19(II) + 21(III) + 22(IV)), with 66 testable against experiment
- **Mean deviation ~1.15%** across 66 comparable observables; 0.73% for Type I; 11 exact matches, 53 within 1%
- **Three new physics sections**: gauge breaking ($E_8 \to$ SM, §5), mass hierarchy (combined wilson_line+instanton, §6), sensitivity analysis (r_eff = 15.53, P(uniform) = 10^{-346}, P(algebraic) = 10^{-133}, §7)
- **Lean 4 certification**: 213 conjuncts, 0 sorry, 55/95 observables verified, 4 substantive axioms
- **Statistical distinctiveness** at $> 4.5\sigma$ among 192,349 alternatives tested
- **Falsifiable predictions**: δ_CP = 197°, N_gen = 3, $sin^2\theta$_W = 3/13, testable by DUNE/FCC-ee

We do not claim this framework is correct. It may represent genuine geometric insight, effective approximation, or elaborate coincidence. Only experiment can discriminate. The deeper question, why octonionic geometry would determine particle physics parameters, remains open.

But the empirical success of 95 observables from zero adjustable parameters, with an overdetermination ratio of 2.13× and coincidence probability 10^{-346} (uniform) / 10^{-133} (algebraic), suggests that topology and physics may be more intimately connected than currently understood.

The ultimate arbiter is experiment.

10 Author's note

This framework was developed through sustained collaboration between the author and several AI systems, primarily Claude (Anthropic), with contributions from GPT (OpenAI), Gemini (Google), Grok (xAI), for specific mathematical insights. The formal verification in Lean 4, architectural decisions, and many key derivations emerged from iterative dialogue sessions over several months. This collaboration follows transparent crediting approach for AI-assisted mathematical research. Mathematical constants underlying these relationships represent timeless logical structures that preceded human discovery. The value of any theoretical proposal depends on mathematical coherence and empirical accuracy, not origin. Mathematics is evaluated on results, not résumés.

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11 Competing Interests

The author declares no competing interests.

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Author's Related Works

Code and Lean proofs: https://github.com/gift-framework/core (Lean module under `core/Lean/`).

- **[A]** B. de La Fournière, "An Explicit Approximate G_2 Metric on a Compact 7-Manifold with Certified Torsion-Free Completion," Zenodo 10.5281/zenodo.19892350 (2026).
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12 Appendix A: Topological Input Constants

Symbol	Definition	Value
$\dim(E_8)$	Lie algebra dimension	248
$\mathrm{rank}(E_8)$	Cartan subalgebra dimension	8
$\dim(K_7)$	Manifold dimension	7
$b_2(K_7)$	Second Betti number	21
$b_3(K_7)$	Third Betti number	77
$\dim(G_2)$	Holonomy group dimension	14
$\dim(J_3(\mathbb{O}))$	Jordan algebra dimension	27

The complete set of 20 structural constants (including $\dim(E_6)$, $\dim(E_7)$, $\dim(F_4)$, fund(E_7), |PSL(2,7)|, D_bulk, α_sum, det(g)_den, det(g)_num) is tabulated in Supplement S3, §3.3.

13 Appendix B: Derived Structural Constants

Symbol	Formula	Value
p_2	$\dim(G_2)/\dim(K_7)$	2
Weyl	From $W(E_8)$ factorization	5

Symbol	Formula	Value
N_gen	Index theorem	3
H*	$b_2 + b_3 + 1$	99
tau	(496 x 21)/(27 x 99)	3472/891
kappa_T	1/(b_3 - dim(G_2) - p_2)	1/61
det(g)	p_2 + 1/(b_2 + dim(G_2) - N_gen)	65/32

14 Appendix C: Supplement Reference

Supplement	Content	Pages
S1: Foundations	E_8, G_2, K_7 construction details	27
S2: Derivations	Complete proofs of 33 Type I relations	42
S3: Observable Dataset	Full 95-entry table, type classification, sensitivity	10