Supplement S2: Complete Derivations

Mathematical Proofs for All 33 Type I Predictions Brieuc de La Fournière
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Contents

This supplement provides mathematical derivations for the 33 Type I dimensionless predictions in the GIFT framework (Relations #1–#33, all Lean-certified). Each derivation proceeds from topological definitions to numerical predictions. Two additional BH remnant topological predictions (Relations #34–#35 in §24.3b) are included as an appendix; these are classified as Type IV structural diagnostics in the main 95-observable dataset.

Status: All 33 Type I observables Lean-certified. 213 certificate conjuncts, 4 axioms, 0 sorry.

Note on the 95-observable dataset: The main paper presents 95 observables across 4 types (33 I + 19 II + 21 III + 22 IV). This supplement covers the 33 Type I predictions (direct algebraic from topology). Type II extensions (19 one-step physical extractions), Type III dynamical results (21 multi-step chains), and Type IV structural diagnostics (22 internal consistency checks, including BH remnant M_res and N_QNM from Pinčák et al. 2026 [42]) are documented in Supplement S3 and the main text §4–§6.

The topological constants that determine these relations are described in S1.

0.1 Table of Contents

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1 Part 0: Derivation Philosophy

1.1 0. What "Derivation" Means in GIFT

Before presenting derivations, we clarify the logical structure:

1.1.1 0.1 Inputs vs Outputs

Inputs (taken as given): - The octonion algebra $\mathbb{O}$ and its automorphism group $G_2 = \mathrm{Aut}(\mathbb{O})$ - The $E_8 \times E_8$ gauge structure - The K_7 manifold (compact G_2 7-manifold with $b_2 = 21$, $b_3 = 77$; explicit metric certified in [A]; topological construction via Joyce-Karigiannis ${Z_2}^3$ orbifold route (S1 §8.4); explicit closed-form neck ansatz with Wirtinger certificate (S1 §8.4, [D]))

Outputs (derived from inputs): - The 33 Type I dimensionless predictions

1.1.2 0.2 What We Do NOT Claim

- That $\mathbb{O} \to G_2 \to K_7$ is the unique geometry for physics
- That the formulas are uniquely determined by geometric principles
- That the selection rule for specific combinations ($b_2/(b_3$ + dim_G_2) vs b_2/b_3) is understood

1.1.3 0.3 What We Observe

- Given the inputs, the outputs follow by algebra
- The outputs match experiment to 0.73% mean deviation (PDG 2024, NuFIT 6.0, 33 Type I core)
- No continuous parameters are fitted

1.1.4 0.4 Torsion Independence

Important: All 33 Type I predictions use only topological invariants. The torsion T does not appear in any formula. Therefore: - Predictions depend only on topology, not on the actual torsion value - The value κ_T = 1/61 is a topological bound, not a prediction ingredient

1.1.5 0.5 Results from Analysis of the NK-Certified Metric

Analysis of the NK-certified metric ([B]) computed all invariants directly from the explicit metric. Key results:

- **Topological realization**: The pair $(b_2, b_3) = (21, 77)$ is realized topologically by the Joyce-Karigiannis ${Z_2}^3$ orbifold route (S1 §8.4); an explicit closed-form neck ansatz with five-layer Wirtinger certificate is established in [D]. Orthogonal TCS is excluded by parity (b_2+b_3=98 even). Any Betti decomposition via Mayer-Vietoris is conditional on the building block identification, which is open.
- $\bar{\nu}$ **= 0**: Lean-certified in TCSConstruction.lean. For rectangular TCS (k_+=k_-=1), $\bar{\nu}$=0 automatically (CGN invariant); this is conditional on TCS realization.
- **New topological formula**: V_min = $\sqrt{}$(Vol(K_7)/11), where 11 = b_3/n = 77/7. NK numerical: 219.90; formula: 221.24 (0.6% ✓). Identity: V_min^2 × b_3/n = Vol(K_7): a universal G_2 relation.

- **U(1)^2 exact symmetry**: The T^2 directions θ and ψ are degenerate to 2×10^{-5} in the metric eigenvalues, propagating to S_θ = S_ψ = 6.1265 in all period integrals (exact to 2.6×10^{-8}).
- **SM gauge group origin**: NOT from ADE singularities (TCS is smooth). From $g_2 \subset so(7)$ spectral structure and $so(8) = g_2 \oplus L \oplus R$ triality → N_gen = 3.

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2 Part I: Foundations

2.1 1. Status Classification

Status	Criterion
VERIFIED	Complete mathematical proof, exact result from topology
VERIFIED (Lean 4)	Verified by Lean 4 kernel with Mathlib (machine-checked)
TOPOLOGICAL	Direct consequence of manifold structure

2.2 2. Notation

Symbol	Value	Definition
$\dim(E_8)$	248	E_8 Lie algebra dimension
$\mathrm{rank}(E_8)$	8	E_8 Cartan subalgebra dimension
$\dim(G_2)$	14	G_2 holonomy group dimension
$\dim(K_7)$	7	Internal manifold dimension
$b_2(K_7)$	21	Second Betti number
$b_3(K_7)$	77	Third Betti number
H*	99	Effective cohomology = $b_2 + b_3 + 1$
$\dim(J_3(O))$	27	Exceptional Jordan algebra dimension
N_gen	3	Number of fermion generations
p_2	2	Binary duality parameter
Weyl	5	Weyl factor: $(\dim(G_2)+1)$/N_gen = b_2/N_gen - p_2 = $\dim(G_2)$ - $\mathrm{rank}(E_8)$ - 1

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3 Part II: Foundational Theorems

3.1 3. Relation #1: Generation Number N_gen = 3

Statement: The number of fermion generations is exactly 3.

Classification: VERIFIED (three independent derivations)

3.1.1 Proof Method 1: Fundamental Topological Constraint

Theorem: For G_2 holonomy manifold K_7 with E_8 gauge structure:

$$(\text{rank}(E_8) + N_{\text{gen}}) \cdot b_2(K_7) = N_{\text{gen}} \cdot b_3(K_7)$$

Derivation:

$$(8 + N_{\text{gen}}) \times 21 = N_{\text{gen}} \times 77$$

$$168 + 21 \cdot N_{\text{gen}} = 77 \cdot N_{\text{gen}}$$

$$168 = 56 \cdot N_{\text{gen}}$$

$$N_{\text{gen}} = \frac{168}{56} = 3$$

Verification: - LHS: (8 + 3) × 21 = 231 - RHS: 3 × 77 = 231 ✓

3.1.2 Proof Method 2: Atiyah-Singer Index Theorem

$$\text{Index}(D_A) = \left(77 - \frac{8}{3} \times 21\right) \times \frac{1}{7} = 3$$

Status: VERIFIED ■

3.2 4. Relation #2: Hierarchy Parameter τ = 3472/891

Statement: The hierarchy parameter is exactly rational.

Classification: VERIFIED

3.2.1 Proof

Step 1: Definition from topological integers

$$\tau := \frac{\dim(E_8 \times E_8) \cdot b_2(K_7)}{\dim(J_3(\mathbb{O})) \cdot H^*}$$

Step 2: Substitute values

$$\tau = \frac{496 \times 21}{27 \times 99} = \frac{10416}{2673}$$

Step 3: Reduce

$$\gcd(10416, 2673) = 3$$

$$\tau = \frac{3472}{891}$$

Step 4: Prime factorization

$$\tau = \frac{2^4 \times 7 \times 31}{3^4 \times 11}$$

Step 5: Numerical value

$$\tau = 3.8967452300785634...$$

Status: VERIFIED ■

3.3 5. Relation #3: Torsion Capacity κ_T = 1/61

Statement: The topological torsion capacity equals exactly 1/61.

Classification: TOPOLOGICAL (structural parameter, not physical prediction)

3.3.1 Proof

Step 1: Define from cohomology

$$61 = b_3(K_7) - \dim(G_2) - p_2 = 77 - 14 - 2 = 61$$

Step 2: Formula

$$\kappa_T = \frac{1}{b_3 - \dim(G_2) - p_2} = \frac{1}{61}$$

Step 3: Geometric interpretation - 61 = effective degrees of freedom available for torsional deformation - 61 = dim(F_4) + N_gen^2 = 52 + 9

3.3.2 Clarification

Quantity	Definition	Value
κ_T	Topological capacity	1/61 (fixed)
T_base	Torsion for torsion-free metric (Joyce)	**0** (by theorem)
T_physical	Effective torsion for interactions	**Open question**

Role in predictions: κ_T appears in only one formula (α^{-1}, as a small correction term det(g)×κ_T ≈ 0.033). The other 32 Type I predictions are independent of torsion capacity. It is primarily a structural parameter characterizing K_7, not a directly measured observable.

Joyce's theorem: For compact G_2 manifolds satisfying perturbation bounds, guarantees existence of a torsion-free metric. The Newton-Kantorovich certificate (Paper I) establishes these bounds for K_7.

Status: TOPOLOGICAL (structural, not predictive) ■

3.4 6. Relation #4: Metric Determinant det(g) = 65/32

Statement: The K_7 metric determinant is exactly 65/32.

Classification: METRIC NORMALIZATION (not TOPOLOGICAL, see S1 §10.3)

3.4.1 Proof

Step 1: Define from structural normalization integers

$$\det(g) = p_2 + \frac{1}{b_2 + \dim(G_2) - N_{gen}}$$

Step 2: Compute denominator

$$b_2 + \dim(G_2) - N_{gen} = 21 + 14 - 3 = 32$$

Step 3: Compute determinant

$$\det(g) = 2 + \frac{1}{32} = \frac{65}{32}$$

Step 4: Alternative derivation

$$\det(g) = \frac{\text{Weyl} \times (\text{rank}(E_8) + \text{Weyl})}{2^5} = \frac{5 \times 13}{32} = \frac{65}{32}$$

Verification: The analytical metric g = (65/32)^{1/7} × I_7 has det(g) = [(65/32)^{1/7}]^7 = 65/32 exactly, consistent with the topological formula.

Epistemic note: Three independent algebraic paths converge to 65/32, which is suggestive but does not constitute a derivation from topology: the metric optimization was constrained to this normalization target, and the formulas were identified post-hoc. See S1 §10.3 for full discussion. Six observables depending on det(g)_num or det(g)_den carry this normalization dependence.

Status: STRUCTURAL (metric normalization, algebraically exact; see S1 §10.3) ■

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4 Part III: Gauge Sector

4.1 7. Relation #5: Weinberg Angle $\sin^2\theta$_W = 3/13

Statement: The weak mixing angle has exact rational form 3/13.

Classification: VERIFIED

4.1.1 Proof

Step 1: Define ratio from Betti numbers

$$\sin^2\theta_W = \frac{b_2(K_7)}{b_3(K_7) + \dim(G_2)} = \frac{21}{77+14} = \frac{21}{91}$$

Step 2: Simplify

$$\gcd(21, 91) = 7$$

$$\sin^2\theta_W = \frac{3}{13} = 0.230769...$$

Step 3: Experimental comparison

Quantity	Value
Experimental (PDG 2024)	0.23122 ± 0.00004
GIFT prediction	0.230769
Deviation	0.195%

Status: VERIFIED ■

4.2 8. Relation #6: Strong Coupling α_s = $\sqrt{}$2/12

Statement: The strong coupling at M_Z scale.

Classification: TOPOLOGICAL

4.2.1 Proof

Formula:

$$\alpha_s(M_Z) = \frac{\sqrt{2}}{\dim(G_2) - p_2} = \frac{\sqrt{2}}{14-2} = \frac{\sqrt{2}}{12}$$

Components: - $\sqrt{}$2: E_8 root length - 12 = dim(G_2) - p_2: Effective gauge degrees of freedom

Numerical value: α_s = 0.117851

Experimental comparison:

Quantity	Value
Experimental	0.1180 ± 0.0009
GIFT prediction	0.11785
Deviation	0.127%

Status: TOPOLOGICAL ■

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5 Part IV: Lepton Sector

5.1 9. Relation #7: Koide Parameter Q = 2/3

Statement: The Koide parameter equals exactly 2/3.

Classification: VERIFIED

5.1.1 Proof

Formula:

$$Q_{\text{Koide}} = \frac{\dim(G_2)}{b_2(K_7)} = \frac{14}{21} = \frac{2}{3}$$

Physical definition:

$$Q = \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2}$$

Experimental comparison:

Quantity	Value
Experimental	0.666661 ± 0.000007
GIFT prediction	0.666667
Deviation	0.0009%

Status: VERIFIED ■

5.2 10. Relation #8: Tau-Electron Mass Ratio m_τ/m_e = 3477

Statement: The tau-electron mass ratio is exactly 3477.

Classification: VERIFIED

5.2.1 Proof

Formula:

$$\frac{m_\tau}{m_e} = \dim(K_7) + 10 \cdot \dim(E_8) + 10 \cdot H^*$$
$$= 7 + 10 \times 248 + 10 \times 99 = 7 + 2480 + 990 = 3477$$

Prime factorization:

$$3477 = 3 \times 19 \times 61 = N_{gen} \times prime(8) \times \kappa_T^{-1}$$

Period integral cross-confirmation: Analysis of the NK-certified metric ([B], §5) extracts associative 3-cycle volumes directly. The SD associative volumes yield ln(m_τ/m_e) = 8.154 from the Fano-plane volume hierarchy, giving e^8.154 = 3477 ✓. This provides an independent geometric confirmation of the algebraic prediction.

Experimental comparison:

Quantity	Value
Experimental	3477.15 ± 0.05
GIFT prediction	3477 (exact)
Deviation	0.0043%

Status: VERIFIED ■

5.3 11. Relation #9: Muon-Electron Mass Ratio

Statement: m_μ/m_e = 27^φ

Classification: TOPOLOGICAL

5.3.1 Proof

Formula:

$$\frac{m_\mu}{m_e} = [\dim(J_3(\mathbb{O}))]^\phi = 27^\phi = 207.012$$

Components: - 27 = dim(J_3(O)): Exceptional Jordan algebra - $\varphi = (1+\sqrt{5})/2$: Golden ratio from McKay correspondence $E_8 \leftrightarrow$ 2I (binary icosahedral group)

Experimental comparison:

Quantity	Value
Experimental	206.768
GIFT prediction	207.01
Deviation	0.1179%

Note on φ: The golden ratio is the only non-integer constant among the 33 Type I predictions and does not appear in the 20 structural constants (S3.3). Its derivation from E_8 via the McKay correspondence is well-established mathematically but constitutes an additional identification step beyond the direct topological reasoning used for the other 32 predictions.

Status: TOPOLOGICAL (with caveat: φ external to the 20 structural constants) ■

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6 Part V: Quark Sector

6.1 12. Relation #10: Strange-Down Ratio m__s/m__d = 20

Statement: The strange-down quark mass ratio is exactly 20.

Classification: VERIFIED

6.1.1 Proof

Formula:

$$\frac{m_s}{m_d} = p_2^2 \times \text{Weyl} = 4 \times 5 = 20$$

Geometric interpretation: - $p_2^2 = 4$: Binary structure squared - Weyl = 5: Pentagonal symmetry

Experimental comparison:

Quantity	Value
Experimental	20.0 ± 1.0
GIFT prediction	20 (exact)
Deviation	0.00%

Status: VERIFIED ■

6.2 12b. Relation #10b: Charm-Strange Ratio m__c/m__s = 246/21

Statement: The charm-strange quark mass ratio.

Classification: TOPOLOGICAL

6.2.1 Proof

Formula:

$$\frac{m_c}{m_s} = \frac{\dim(E_8) - p_2}{b_2(K_7)} = \frac{248 - 2}{21} = \frac{246}{21} = 11.714...$$

Components: - 246 = $\dim(E_8)$ - p_2: Effective E_8 dimension - 21 = $b_2(K_7)$: Second Betti number

Experimental comparison:

Quantity	Value
Experimental	11.7 ± 0.3
GIFT prediction	11.714
Deviation	0.12%

TCS-level alternative: The charm-strange ratio also admits a decomposition at the TCS building-block level:

$$\frac{m_c}{m_s} = b_2(M_1) + \frac{\dim(K_7)}{b_2(M_2)} = 11 + \frac{7}{10} = \frac{117}{10} = 11.700$$

This additive form reflects the adiabatic decomposition of the KK mass operator across the two TCS halves. The M_1 contribution ($b_2(M_1) = 11$) dominates, with a geometric correction from M_2 ($\dim(K_7)/b_2(M_2) = 7/10$). Both formulas are within 0.05σ of the experimental central value.

Status: TOPOLOGICAL ■

6.3 12c. Relation #10c: Bottom-Top Ratio m_b/m_t = 1/42

Statement: The bottom-top quark mass ratio involves the constant $42 = p_2 \times N\_gen \times \dim(K_7)$.

Classification: TOPOLOGICAL

6.3.1 Proof

Step 1: Define the structural constant

$$42 = p_2 \times N_{gen} \times \dim(K_7) = 2 \times 3 \times 7$$

This constant 42 also equals $2 \times b_2 = 2 \times 21$.

Step 2: Formula

$$\frac{m_b}{m_t} = \frac{b_0}{42} = \frac{1}{42} = 0.02381...$$

Components: - $b_0 = 1$: Zeroth Betti number - 42: Structural constant from K_7 geometry

Experimental comparison:

Quantity	Value
Experimental	0.024 ± 0.001
GIFT prediction	0.02381
Deviation	0.79%

Geometric interpretation: The same constant 42 appears in the cosmological ratio $\Omega\_DM/\Omega\_b = (1 + 42)/8 = 43/8$ (Section 18b), connecting quark physics to cosmological structure through the K_7 geometry.

Status: TOPOLOGICAL ■

6.4 12d. Relation #10d: Up-Down Ratio m_u/m_d = 79/168

Statement: The up-down quark mass ratio.

Classification: TOPOLOGICAL

6.4.1 Proof

Formula:

$$\frac{m_u}{m_d} = \frac{b_0 + \dim(E_6)}{|PSL_2(7)|} = \frac{1+78}{168} = \frac{79}{168} = 0.4702...$$

Components: - $\dim(E_6) = 78$: Exceptional Lie algebra dimension - $|PSL_2(7)| = 168$: Order of the simple group $PSL_2(7) = \text{rank}(E_8) \times b_2$

Experimental comparison:

Quantity	Value
Experimental	0.47 ± 0.03
GIFT prediction	0.4702
Deviation	0.05%

Status: TOPOLOGICAL ■

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7 Part V-B: CKM Matrix

7.1 12e. Relation #10e: Cabibbo Angle $\sin^2\theta_{12}$(CKM) = 7/31

Statement: The CKM Cabibbo mixing angle.

Classification: TOPOLOGICAL

7.1.1 Proof

Formula:

$$\sin^2\theta_{12}^{CKM} = \frac{\dim(\text{fund}_{E_7})}{\dim(E_8)} = \frac{56}{248} = \frac{7}{31} = 0.2258...$$

Alternative expressions: - $(b_3 - b_2)/\dim(E_8) = (77 - 21)/248 = 56/248$ - $(2b_2 + \dim(G_2))/\dim(E_8) = (42 + 14)/248 = 56/248$

Experimental comparison:

Quantity	Value
Experimental	0.2250 ± 0.0006

Quantity	Value
GIFT prediction	0.2258
Deviation	0.36%

Status: TOPOLOGICAL ■

7.2 12f. Relation #10f: Wolfenstein A Parameter = 83/99

Statement: The Wolfenstein A parameter of the CKM matrix.

Classification: TOPOLOGICAL

7.2.1 Proof

Formula:

$$A_{\text{Wolf}} = \frac{\text{Weyl} + \dim(E_6)}{H^*} = \frac{5+78}{99} = \frac{83}{99} = 0.8384...$$

Alternative expression: - (b_3 + p_2 × N_gen)/H* = (77 + 6)/99 = 83/99

Experimental comparison:

Quantity	Value
Experimental	0.836 ± 0.015
GIFT prediction	0.8384
Deviation	0.29%

Status: TOPOLOGICAL ■

7.3 12g. Relation #10g: CKM θ_{23} Mixing $\sin^2\theta_{23}$(CKM) = 1/24

Statement: The CKM 23-mixing angle.

Classification: TOPOLOGICAL

7.3.1 Proof

Formula:

$$\sin^2\theta_{23}^{CKM} = \frac{\dim(K_7)}{|PSL_2(7)|} = \frac{7}{168} = \frac{1}{24} = 0.04167...$$

Experimental comparison:

Quantity	Value
Experimental	0.0412 ± 0.0008
GIFT prediction	0.04167
Deviation	1.13%

Status: TOPOLOGICAL ■

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8 Part VI: Neutrino Sector

8.1 13. Relation #11: CP Violation Phase δ_CP = 197°

Statement: The CP violation phase is exactly 197°.

Classification: VERIFIED (with NuFIT 6.0 tension discussed in Appendix F)

8.1.1 Proof

Formula:

$$\delta_{CP} = \dim(K_7) \cdot \dim(G_2) + H^* = 7 \times 14 + 99 = 98 + 99 = 197^\circ$$

Experimental comparison:

Quantity	Value
Experimental (NuFIT 5.2, 2022)	$197^\circ \pm 24^\circ$
Experimental (NuFIT 6.0, Oct 2024)	$177^\circ \pm 20^\circ$
GIFT prediction	197° (exact)
Deviation (vs NuFIT 5.2)	0.00%
Deviation (vs NuFIT 6.0)	11.3% (within 1σ)

NuFIT 6.0 tension: The central value shifted from 197° to 177° between NuFIT 5.2 and 6.0, but the NuFIT collaboration flags δ_CP as one of the least constrained parameters, with substantial dependence on reactor flux assumptions. The value 197° lies at the 1σ boundary (177+20=197). DUNE (2028–2040) will resolve this definitively.

The canonical GIFT prediction remains δ_CP = 197°. A potential compactification correction (62/69 factor, yielding 177.01°) is documented in Appendix F as contingent on experimental confirmation. See also §4.4.1 of the main paper.

Status: VERIFIED, original prediction, pending DUNE ■

8.2 14. Relation #12: Reactor Mixing Angle $\theta_{13} = \pi/21$

Statement: The reactor neutrino mixing angle.

Classification: TOPOLOGICAL

8.2.1 Proof

Formula:

$$\theta_{13} = \frac{\pi}{b_2(K_7)} = \frac{\pi}{21} = 8.571°$$

Experimental comparison:

Quantity	Value
Experimental (NuFIT 6.0)	8.54° ± 0.12°
GIFT prediction	8.571°
Deviation	0.37%

TCS-level refinement: At the building-block level, the reactor angle admits a purely rational formula:

$$\theta_{13} = \frac{\dim(E_8) - \dim(G_2) \cdot b_2(M_1)}{b_2(M_1)} = \frac{248 - 14 \times 11}{11} = \frac{94}{11} = 8.545°$$

The numerator 94 = $\dim(E_8) - \dim(G_2) \times b_2(M_1)$ is the residual E_8 gauge content after subtracting the G_2 holonomy contribution to M_1. This formula is 5.8× more precise (0.064% vs 0.37%) and uses only algebraic invariants (no π), but works in degrees rather than radians.

Status: TOPOLOGICAL ■

8.3 15. Relation #13: Atmospheric Mixing Angle θ_{23}

Statement: The atmospheric neutrino mixing angle.

Classification: TOPOLOGICAL

8.3.1 Proof

Formula:

$$\theta_{23} = \arcsin\left(\frac{b_3 - p_2}{H^*}\right) = \arcsin\left(\frac{75}{99}\right) = \arcsin\left(\frac{25}{33}\right) = 49.251°$$

Components: - $b_3 = 77$: Third Betti number (3-cycles of K_7) - $p_2 = 2$: Pontryagin class contribution (spin structure correction) - $H^* = 99$: Effective cohomology ($b_2 + b_3 + 1$)

Physical interpretation: The atmospheric mixing angle θ_{23} governs τ-μ flavor mixing. The formula (b_3 - p_2)/H^* represents the relative weight of spin-corrected 3-cycles in the total cohomology. This captures how the τ-μ sector couples through the 3-cycle topology of K_7, with the Pontryagin correction accounting for the spin structure that distinguishes fermionic generations.

Experimental comparison:

Quantity	Value
Experimental (NuFIT 5.3)	49.3° ± 1.0°
GIFT prediction	49.251°
Deviation	0.10%

Status: TOPOLOGICAL ■

8.4 16. Relation #14: Solar Mixing Angle θ_{12}

Statement: The solar neutrino mixing angle.

Classification: TOPOLOGICAL

8.4.1 Proof

Formula:

$$\theta_{12} = \arctan\left(\sqrt{\frac{\delta}{\gamma_{\text{GIFT}}}}\right) = 33.419°$$

Components: - $\delta = 2\pi/\text{Weyl}^2 = 2\pi/25$ - γ_GIFT = 511/884

Derivation of γ_GIFT:

$$\gamma_{\text{GIFT}} = \frac{2 \cdot \text{rank}(E_8) + 5 \cdot H^*}{10 \cdot \dim(G_2) + 3 \cdot \dim(E_8)} = \frac{511}{884}$$

Experimental comparison:

Quantity	Value
Experimental (NuFIT 5.3)	33.41° ± 0.75°
GIFT prediction	33.40°
Deviation	0.030%

Status: TOPOLOGICAL ■

8.5 16b. PMNS Matrix: $\sin^2$ Form

The PMNS mixing angles can also be expressed directly as $\sin^2$ values, providing alternative topological formulas.

8.5.1 Relation #14b: $\sin^2\theta_{12}$(PMNS) = 4/13

Formula:

$$\sin^2\theta_{12}^{PMNS} = \frac{b_0 + N_{gen}}{\alpha_{sum}} = \frac{1+3}{13} = \frac{4}{13} = 0.3077...$$

Components: - α_sum = 13: Anomaly coefficient sum - b_0 + N_gen = 4: Cohomological + generation count

Quantity	Value
Experimental	0.307 ± 0.013
GIFT prediction	0.3077
Deviation	0.23%

8.5.2 Relation #14c: $\sin^2\theta_{23}$(PMNS) = 6/11

Formula:

$$\sin^2\theta_{23}^{PMNS} = \frac{D_{bulk} - \text{Weyl}}{D_{bulk}} = \frac{11-5}{11} = \frac{6}{11} = 0.5455...$$

Alternative expression: - 42/b_3 = 42/77 = 6/11 (after reduction)

Quantity	Value
Experimental	0.546 ± 0.021
GIFT prediction	0.5455
Deviation	0.10%

8.5.3 Relation #14d: $\sin^2\theta_{13}$(PMNS) = 11/496

Formula:

$$\sin^2\theta_{13}^{PMNS} = \frac{D_{bulk}}{\dim(E_8\times E_8)} = \frac{11}{496} = 0.02218...$$

Quantity	Value
Experimental	0.0220 ± 0.0007
GIFT prediction	0.02218
Deviation	0.81%

Status: TOPOLOGICAL ■

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9 Part VII: Higgs & Cosmology

9.1 17. Relation #15: Higgs Coupling λ_H = √17/32

Statement: The Higgs quartic coupling is determined by G_2 holonomy parameters.

Classification: TOPOLOGICAL

9.1.1 Proof

Formula:

$$\lambda_H = \frac{\sqrt{\dim(G_2) + N_{gen}}}{\det(g)_{den}} = \frac{\sqrt{14+3}}{32} = \frac{\sqrt{17}}{32} = 0.12885$$

Components: - $\dim(G_2)$ + N_gen = 17: holonomy dimension plus generation count - det(g)_den = 32: metric determinant denominator (= $b_2 + \dim(G_2) -$ N_gen)

Physical interpretation: The Higgs self-coupling combines the holonomy group dimension with the generation count, normalized by the metric determinant scale. The square root reflects the quadratic nature of the Higgs potential.

Numerical value: λ_H = √17/32 = 0.128847…

Experimental comparison:

Quantity	Value
Experimental (PDG 2024)	0.129 ± 0.001
GIFT prediction	√17/32 = 0.12885
Deviation	0.12%

9.1.2 TCS Alternative

A purely rational TCS formula is also available:

$$\lambda_H^{TCS} = \frac{b_2(M_1)}{b_3 + b_2(M_2)} = \frac{11}{87} = 0.12644$$

where $b_2(M_1) = 11$ and $b_2(M_2) = 10$ are the Betti numbers of the two TCS building blocks. This formula is lower-complexity (cost 3 vs cost 6) and connects directly to the twisted connected sum decomposition. Against a running coupling value λ_H(μ_high) ≈ 0.126, it achieves 0.35% deviation, though against the PDG 2024 pole value (0.129) it gives 1.98%.

Status: TOPOLOGICAL ■

9.2 17b. Boson Mass Ratios

Statement: The ratios of electroweak boson masses have topological origins.

Classification: VERIFIED

9.2.1 Relation: m_W/m_Z = 37/42

Formula:

$$\frac{m_W}{m_Z} = \frac{2b_2 - \text{Weyl}}{2b_2} = \frac{42 - 5}{42} = \frac{37}{42}$$

Physical interpretation: - $2b_2 = 42$ is the structural constant ($= p_2 \times b_2$) - Weyl = 5 is the triple identity factor - The ratio involves (structural_const − Weyl) / structural_const

Note: The true Euler characteristic $\chi(K_7) = 0$ for odd-dimensional manifolds. The constant $42 = 2b_2$ is a distinct topological invariant.

Numerical value: m_W/m_Z = 0.8810

Experimental comparison:

Quantity	Value
Experimental	0.8815 ± 0.0002
GIFT prediction	0.8810
Deviation	**0.06%**

9.2.2 Relation: m_H/m_t = 56/77

Formula:

$$\frac{m_H}{m_t} = \frac{fund(E_7)}{b_3} = \frac{56}{77} = \frac{8}{11}$$

Numerical value: m_H/m_t = 0.7273

Quantity	Value
Experimental	0.725 ± 0.003
GIFT prediction	0.7273
Deviation	0.31%

9.2.3 Relation: m_H/m_W = 81/52

Formula:

$$\frac{m_H}{m_W} = \frac{N_{gen} + \dim(E_6)}{\dim(F_4)} = \frac{3 + 78}{52} = \frac{81}{52}$$

Numerical value: m_H/m_W = 1.5577

Quantity	Value
Experimental	1.558 ± 0.002
GIFT prediction	1.5577
Deviation	**0.02%**

Status: VERIFIED ■

9.3 18. Relation #16: Dark Energy Density Ω_DE = ln(2)×98/99

Statement: The dark energy density fraction emerges from information-theoretic structure of K_7 moduli.

Classification: TOPOLOGICAL

9.3.1 Proof

Formula:

$$\Omega_{DE} = \ln(2) \times \frac{b_2 + b_3}{H^*} = \ln(2) \times \frac{98}{99} = 0.6861$$

Components: - ln(2): binary entropy factor - $b_2 + b_3 = 98$: total non-trivial cohomological content - $H^* = 99 = b_2 + b_3 + 1$: full cohomological dimension

Physical interpretation: The dark energy fraction combines a binary information factor with the ratio of non-trivial to total cohomological content. The ln(2) reflects the fundamental binary structure of the moduli space partition.

Numerical value: Ω_DE = ln(2) × 98/99 = 0.68615…

Experimental comparison:

Quantity	Value
Experimental (Planck PR4)	0.6847 ± 0.005
GIFT prediction	ln(2)×98/99 = 0.6861
Deviation	0.21%

9.3.2 TCS Alternative

A purely rational TCS formula is also available:

$$\Omega_{DE}^{TCS} = 1 - \frac{\chi(K3)}{b_3} = 1 - \frac{24}{77} = \frac{53}{77} = 0.68831$$

where χ(K3) = 24 is the Euler characteristic of the K3 fiber. This formula is lower-complexity (rational, cost 2 vs cost 6), and the interpretation is transparent: Ω_matter = $\chi(K3)/b_3$ counts the K3-fiber fraction of the 3-cycle moduli, with the rest being vacuum energy. Against Planck 2018 ($\Omega\_\Lambda = 0.6889$), it achieves 0.085% deviation, though against Planck PR4 (0.6847) it gives 0.53%.

Status: TOPOLOGICAL ■

9.4 19. Relation #17: Spectral Index n__s

Statement: The primordial scalar spectral index.

Classification: VERIFIED

9.4.1 Proof

Formula:

$$n_s = \frac{\zeta(D_{bulk})}{\zeta(\text{Weyl})} = \frac{\zeta(11)}{\zeta(5)} = 0.9649$$

Components: - $\zeta(11)$: From 11D bulk spacetime - $\zeta(5)$: From Weyl factor

Experimental comparison:

Quantity	Value
Experimental (Planck 2020)	0.9649 ± 0.0042
GIFT prediction	0.9649
Deviation	0.004%

Status: VERIFIED ■

9.5 19b. Relation #17c: Dark Matter to Baryon Ratio Ω__DM/Ω__b = 43/8

Statement: The dark matter to baryon density ratio.

Classification: TOPOLOGICAL

9.5.1 Proof

Formula:

$$\frac{\Omega_{DM}}{\Omega_b} = \frac{b_0 + 42}{\text{rank}(E_8)} = \frac{1+42}{8} = \frac{43}{8} = 5.375$$

Components: - 42 = $p_2 \times$ N__gen $\times$ dim(K_7): The same constant appearing in m__b/m__t = 1/42 - rank(E_8) = 8: Cartan subalgebra dimension

Experimental comparison:

Quantity	Value
Experimental (Planck 2020)	5.375 ± 0.05
GIFT prediction	5.375
Deviation	0.00%

Status: TOPOLOGICAL ■

9.6 19c. Relation #17d: Reduced Hubble Parameter h = 167/248

Statement: The reduced Hubble parameter $H_0 = 100h$ km/s/Mpc.

Classification: TOPOLOGICAL

9.6.1 Proof

Formula:

$$h = \frac{|PSL_2(7)| - b_0}{\dim(E_8)} = \frac{168 - 1}{248} = \frac{167}{248} = 0.6734...$$

Experimental comparison:

Quantity	Value
Experimental (Planck 2020)	0.674 ± 0.005
GIFT prediction	0.6734
Deviation	0.09%

Status: TOPOLOGICAL ■

9.7 19d. Relation #17e: Baryon Fraction Ω__b/Ω__m = 5/32

Statement: The baryon fraction of total matter.

Classification: TOPOLOGICAL

9.7.1 Proof

Formula:

$$\frac{\Omega_b}{\Omega_m} = \frac{\text{Weyl}}{\det(g)_{den}} = \frac{5}{32} = 0.15625$$

Experimental comparison:

Quantity	Value
Experimental (Planck 2020)	0.156 ± 0.003
GIFT prediction	0.15625
Deviation	0.16%

Status: TOPOLOGICAL ■

9.8 19e. Relation #17f: Amplitude of Fluctuations $\sigma_8 = 17/21$

Statement: The amplitude of matter fluctuations at 8 h^{-1} Mpc.

Classification: TOPOLOGICAL

9.8.1 Proof

Formula:

$$\sigma_8 = \frac{p_2 + \det(g)_{den}}{42} = \frac{2+32}{42} = \frac{34}{42} = \frac{17}{21} = 0.8095...$$

Experimental comparison:

Quantity	Value
Experimental (Planck 2020)	0.811 ± 0.006
GIFT prediction	0.8095
Deviation	0.18%

Status: TOPOLOGICAL ■

9.9 19f. Relation #17g: Primordial Helium Fraction Y_p = 15/61

Statement: The primordial helium mass fraction from Big Bang nucleosynthesis.

Classification: TOPOLOGICAL

9.9.1 Proof

Formula:

$$Y_p = \frac{b_0 + \dim(G_2)}{\kappa_T^{-1}} = \frac{1+14}{61} = \frac{15}{61} = 0.2459...$$

Experimental comparison:

Quantity	Value
Experimental	0.245 ± 0.003
GIFT prediction	0.2459
Deviation	0.37%

Status: TOPOLOGICAL ■

9.10 20. Relation #17b: Matter Density Ω_m

Statement: The matter density fraction derives from dark energy via $\sqrt{}$Weyl.

Classification: DERIVED (from Weyl triple identity + Ω_DE)

9.10.1 Proof

Step 1: Establish $\sqrt{}$Weyl as structural

From the Weyl Triple Identity (S1, Section 2.3):

$$\text{Weyl} = \frac{\dim(G_2)+1}{N_{gen}} = \frac{b_2}{N_{gen}} - p_2 = \dim(G_2) - \text{rank}(E_8) - 1 = 5$$

Therefore $\sqrt{}$Weyl = $\sqrt{}$5 is a derived quantity.

Step 2: Matter-dark energy ratio

The cosmological density ratio:

$$\frac{\Omega_{DE}}{\Omega_m} = \sqrt{\text{Weyl}} = \sqrt{5}$$

Step 3: Compute Ω_m

Using Ω_DE = ln(2) × (b_2 + b_3)/H* = 0.6861 (Relation #16):

$$\Omega_m = \frac{\Omega_{DE}}{\sqrt{\text{Weyl}}} = \frac{\ln(2)\times 98/99}{\sqrt{5}} = \frac{0.6861}{2.236} = 0.3068$$

Step 4: Verify closure

$$\Omega_{total} = \Omega_{DE} + \Omega_m = 0.6861 + 0.3068 = 0.9929 \approx 1$$

Consistent with flat universe (Ω_total = 1).

Experimental comparison:

Quantity	Value
Experimental (Planck 2020)	0.3153 ± 0.007
GIFT prediction	0.3068
Deviation	2.7%

9.10.2 Interpretation

The $\sqrt{5}$ ratio between dark energy and matter densities emerges from the same structural constant (Weyl = 5) that determines: - det(g) = 65/32 (metric determinant) - |W(E_8)| factorization (group theory) - N_gen^3 coefficient in |W(E_8)| (topology)

Status: DERIVED (structural, 2.7% deviation) ■

9.11 21. Relation #18: Fine Structure Constant α^{-1}

Statement: The inverse fine structure constant.

Classification: TOPOLOGICAL

9.11.1 Proof

Formula:

$$\alpha^{-1}(M_Z) = \frac{\dim(E_8) + \text{rank}(E_8)}{2} + \frac{H^*}{D_{bulk}} + \det(g) \cdot \kappa_T$$
$$= 128 + 9 + \frac{65}{32} \times \frac{1}{61} = 137.033$$

Components: - 128 = (248 + 8)/2: Algebraic - 9 = 99/11: Bulk impedance - 65/1952: Torsional correction

Experimental comparison:

Quantity	Value
Experimental	137.035999
GIFT prediction	137.033
Deviation	0.002%

Status: TOPOLOGICAL ■

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10 Part VIII: Summary Table

10.1 21. The 33 Type I Dimensionless Relations (+ 2 Type IV BH Appendix)

Note: All predictions use only topological invariants (b_2, b_3, $\dim(G_2)$, etc.). None depend on the realized torsion value T. Relation #19 (Ω_m) is DERIVED from Ω_DE via the Weyl triple identity.

#	Relation	Formula	Value	Exp.	Dev.	Status
1	N_gen	Atiyah-Singer	3	3	exact	VERIFIED
2	τ	496×21/(27×99)	3472/891	-	-	VERIFIED
3	κ_T	1/(77-14-2)	1/61	-	-	STRUCTURAL*
4	det(g)	5×13/32	65/32	-	-	METRIC NORM.
5	$\sin^2\theta$_W	21/91	3/13	0.23122	0.195%	VERIFIED
6	α_s	$\sqrt{}$2/12	0.11785	0.1180	0.127%	TOPOLOGICAL
7	Q_Koide	14/21	2/3	0.666661	0.0009%	VERIFIED
8	m_τ/m_e	7+2480+990	3477	3477.15	0.0043%	VERIFIED
9	m_μ/m_e	27^φ	207.01	206.768	0.118%	TOPOLOGICAL
10	m_s/m_d	4×5	20	20.0	0.00%	VERIFIED
11	δ_CP	7×14+99	197°	177°±20°*	11.3% (1σ)	VERIFIED
12	θ_{13}	π/21 (TCS: 94/11)	8.57°	8.54°	0.37%	TOPOLOGICAL
13	θ_{23}	arcsin((b_3-p_2)/H*)	49.25°	49.3°	0.10%	TOPOLOGICAL
14	θ_{12}	arctan(...)	33.40°	33.41°	0.030%	TOPOLOGICAL
15	λ_H	$\sqrt{}$(dim(G_2)+N_gen)/det_den	$\sqrt{}$17/32	0.129	0.12%	TOPOLOGICAL
16	Ω_DE	ln(2)×(b_2+b_3)/H*	0.6861	0.6847	0.21%	TOPOLOGICAL
17	n_s	$\zeta(11)/\zeta(5)$	0.9649	0.9649	0.004%	VERIFIED
18	α^{-1}	128+9+corr	137.033	137.036	0.002%	TOPOLOGICAL
19	Ω_m	Ω_DE/$\sqrt{}$Weyl	0.3068	0.3153	2.7%	DERIVED

*NuFIT 6.0 (Oct 2024); ±20° uncertainty. The 197° prediction matched NuFIT 5.2 (2022) exactly. See §4.4.1 of main paper.

*κ_T is a structural parameter (capacity), not a physical prediction. It does not appear in other formulas.

10.2 22. Deviation Statistics

Range	Count	Percentage
0.00% (exact)	4	22%

Range	Count	Percentage
<0.01%	3	17%
0.01-0.1%	4	22%
0.1-0.5%	7	39%

Mean deviation: 0.73% (PDG 2024, NuFIT 6.0; 33 Type I observables with experimental comparison, canonical δ_CP=197°). For the full 92-observable dataset (66 with experimental comparison), see Supplement S3.

10.3 23. Statistical Uniqueness of (b_2=21, b_3=77)

A critical question for any unified framework is whether the specific topological parameters represent overfitting. We conducted comprehensive Monte Carlo validation to address this concern.

10.3.1 Methodology

- **Betti variations**: 100,000 random (b_2, b_3) configurations
- **Gauge group comparison**: $E_8 \times E_8$, $E_7 \times E_7$, $E_6 \times E_6$, SO(32), SU(5)×SU(5), etc.
- **Holonomy comparison**: G_2, Spin(7), SU(3), SU(4)
- **Full combinatorial**: 91,896 configurations varying all parameters
- **Local sensitivity**: ±10 grid around (b_2=21, b_3=77)

10.3.2 Results

Metric	Value
Total configurations tested	**192,349**
Configurations better than GIFT	**0**
GIFT mean deviation	0.73% (33 observables)
Alternative mean deviation	32.9%
P-value	$\mathbf{< 5 \times 10^{-6}}$
Significance	$\mathbf{> 4.5\sigma}$

10.3.3 Gauge Group Ranking

Rank	Group	Mean Dev.
1	$\mathbf{E_8 \times E_8}$	**0.73%**
2	$E_7 \times E_8$	8.80%
3	$E_6 \times E_8$	15.50%

E_8 x E_8 achieves approximately 10x better agreement than all tested alternatives.

10.3.4 Holonomy Ranking

Rank	Holonomy	Mean Dev.
1	$\mathbf{G_2}$	**0.73%**
2	SU(4)	1.46%
3	SU(3)	4.43%

G_2 achieves approximately 5x better agreement than Calabi-Yau (SU(3)).

10.3.5 Interpretation

Among 192,349 tested alternatives, the configuration (b_2=21, b_3=77) with E_8 x E_8 gauge group and G_2 holonomy achieves the lowest mean deviation. Zero alternatives outperform it.

Literature uniqueness: Analysis of the NK-certified metric ([B], §6) compared against all ~65 known compact G_2 manifolds from the literature (Kovalev, CHNP, CGN, Joyce, Nordström). The point (b_2=21, b_3=77) is UNIQUE: no known diffeomorphic example exists. The nearest neighbor in (b_2,b_3) space is the CHNP grid point (9,9), at distance 7.6. The topological pair is realized by the Joyce-Karigiannis $Z_2{}^3$ orbifold route (S1 §8.4), extending the known catalogue; an explicit closed-form neck ansatz is established in [D].

Complete methodology: docs/STATISTICAL_EVIDENCE.md

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11 Part IX: Observable Catalog

11.1 24. Structural Redundancy and Expression Counts

Each prediction admits multiple algebraically independent expressions that reduce to the same fraction. This multiplicity provides a measure of structural robustness: quantities arising from many paths through the topological invariants are less likely to represent numerical coincidence.

11.1.1 24.1 Classification Scheme

Classification	Expressions	Interpretation
CANONICAL	≥20	Maximally over-determined; emerges from algebraic web
ROBUST	10-19	Highly constrained; multiple independent derivations
SUPPORTED	5-9	Structural redundancy
DERIVED	2-4	Dual derivation minimum
SINGULAR	1	Unique path (possible coincidence)

11.1.2 24.2 Core 18 Type I Predictions with Expression Counts (subset of 33 certified)

#	Observable	Formula	Value	Exp.	Dev.	Expr.	Class
1	N_gen	Atiyah-Singer	3	3	0.00%	24+	CANONICAL
2	$\sin^2\theta$_W	$b_2/(b_3+\text{dim\_}G_2)$	3/13	0.2312	0.20%	14	ROBUST
3	α_s(M_Z)	$\sqrt{}2/12$	0.11785	0.11800	0.126%	9	SUPPORTED
4	λ_H	$\sqrt{}17/32$	0.1288	0.129	0.12%	4	DERIVED
5	α^{-1}	128+9+corr	137.033	137.036	0.002%	3	DERIVED
6	Q_Koide	$\text{dim\_}G_2/b_2$	2/3	0.6667	0.001%	20	CANONICAL
7	m_τ/m_e	7+10×248+10×99	3477	3477.2	0.004%	3	DERIVED
8	m_μ/m_e	27^φ	207.01	206.77	0.12%	2	DERIVED
9	m_s/m_d	$p_2{}^2$×Weyl	20	20.0	0.00%	14	ROBUST
10	m_b/m_t	$1/(2b_2)$	1/42	0.024	0.79%	21	CANONICAL
11	m_u/m_d	$(1+\text{dim\_}E_6)/PSL_{27}$	79/168	0.47	0.05%	1	SINGULAR
12	δ_CP	7×14+99	197°	177°±20°*	11.3% (1σ)	3	VERIFIED
13	θ_{13}	π/b_2	8.57°	8.54°	0.37%	3	DERIVED
14	θ_{23}	$\arcsin((b_3-p_2)/H^*)$	49.25°	49.3°	0.10%	2	DERIVED
15	θ_{12}	$\arctan(\sqrt{}(\delta/\gamma))$	33.40°	33.41°	0.03%	2	DERIVED
16	Ω_DE	$\ln(2)\times(b_2+b_3)/H^*$	0.6861	0.6847	0.21%	2	DERIVED
17	n_s	$\zeta(11)/\zeta(5)$	0.9649	0.9649	0.004%	2	DERIVED
18	det(g)	65/32	2.0313	-	-	8	SUPPORTED

Distribution: 4 CANONICAL (22%), 4 ROBUST (22%), 2 SUPPORTED (11%), 7 DERIVED (39%), 1 SINGULAR (6%).

11.1.3 24.3 Extended Certified Predictions (#19–#33, 15 entries)

#	Observable	Formula	Value	Exp.	Dev.	Expr.	Class
19	$\sin^2\theta_{12}$^PMNS	(1+N_gen)/α_sum	4/13	0.307	0.23%	28	CANONICAL
20	$\sin^2\theta_{23}$^PMNS	(D_bulk−Weyl)/D_bulk	6/11	0.546	0.10%	15	ROBUST
21	$\sin^2\theta_{13}$^PMNS	D_bulk/dim_$E_8$$_2$	11/496	0.022	0.81%	5	SUPPORTED
22	$\sin^2\theta_{12}$^CKM	7/31	0.2258	0.225	0.36%	16	ROBUST
23	A_Wolf	(Weyl+dim_E_6)/H*	83/99	0.836	0.29%	4	DERIVED
24	$\sin^2\theta_{23}$^CKM	dim_K_7/PSL_{27}	1/24	0.041	1.13%	3	DERIVED
25	m_H/m_t	8/11	0.7273	0.725	0.31%	19	ROBUST
26	m_H/m_W	81/52	1.5577	1.558	0.02%	1	SINGULAR
27	m_W/m_Z	($2b_2$−Weyl)/($2b_2$) = 37/42	0.8810	0.8815	**0.06%**	8	SUPPORTED
28	m_μ/m_τ	5/84	0.0595	0.0595	0.04%	9	SUPPORTED
29	Ω_DM/Ω_b	(1+42)/rank_E_8	43/8	5.375	0.00%	6	SUPPORTED
30	Ω_b/Ω_m	Weyl/det(g)_den	5/32	0.156	0.16%	7	SUPPORTED
31	Ω_Λ/Ω_m	(det_g_den−dim_K_7)/D_bulk	25/11	2.27	0.12%	6	SUPPORTED
32	h	(PSL_{27}−1)/dim_E_8	167/248	0.674	0.09%	3	DERIVED
33	σ_8	(p_2+det_g_den)/($2b_2$)	34/42	0.811	0.18%	4	DERIVED

11.1.4 24.3b Relations #34–#35: BH Remnant Topological Predictions: Type IV (Pinčák et al. 2026 [42])

*These are classified as **Type IV structural diagnostics** (D22 = M_res, D20 = N_QNM) in the 95-observable dataset, not as Type I. They are included here for completeness as they are topologically derived from the same G_2 structural constants.*

#	Observable	Formula	Value	Status
34	BH remnant mass M_res	v_EW^2/M_Pl (torsion VEV: τ_0 = v_EW)	~125 GeV^2 / M_Pl	Type IV; topological, not yet Lean-certified
35	QNM families N_QNM	b_3 + Weyl×dim(K_7) + p_2 (G_2 torsion modes)	98	Type IV; topological, not yet Lean-certified

These two predictions are topologically derived from G_2 torsion structure but have not yet been formally certified in Lean. M_res involves a physical scale identification (τ_0 = v_EW); N_QNM is a purely structural count. No experimental comparison is currently available.

11.1.5 24.4 Illustrative Examples of Multiple Expressions

$\sin^2\theta$_W = 3/13 (14 independent expressions):

#	Expression	Evaluation
1	N_gen / α_sum	3/13
2	N_gen / (p_2 + D_bulk)	3/(2+11) = 3/13
3	b_2 / (b_3 + dim_G_2)	21/91 = 3/13
4	dim(J_3O) / (dim_F_4 + det_g_num)	27/117 = 3/13
5	(b_0 + dim_G_2) / det_g_num	15/65 = 3/13
6	(p_2 + b_0) / α_sum	3/13
7	dim_K_7 / (b_2 + dim_K_7 + dim_G_2)	$7/42 \neq 3/13$ ×

(Expression 7 illustrates that not all combinations work; only those reducing to 3/13 are valid.)

Q_Koide = 2/3 (20 independent expressions):

#	Expression	Evaluation
1	p_2 / N_gen	2/3
2	dim_G_2 / b_2	14/21 = 2/3
3	dim_F_4 / dim_E_6	52/78 = 2/3
4	rank_E_8 / (Weyl + dim_K_7)	8/12 = 2/3
5	(dim_G_2 − rank_E_8) / (rank_E_8 + 1)	6/9 = 2/3

m_b/m_t = 1/42 (21 independent expressions):

#	Expression	Evaluation
1	b_0 / ($2b_2$)	1/42
2	(b_0 + N_gen) / PSL(2,7)	4/168 = 1/42
3	p_2 / (dim_K_7 + b_3)	2/84 = 1/42
4	N_gen / (dim(J_3O) + H*)	3/126 = 1/42
5	dim_K_7 / (dim_E_8 + dim(J_3O) + dim_K_7)	7/294 = 1/42

The ratio m_b/m_t = 1/42 = $1/(2b_2)$ illustrates structural redundancy: the bottom-to-top mass hierarchy equals the inverse of the structural constant $2b_2 = p_2 \times b_2$.

Note: The true Euler characteristic $\chi(K_7) = 0$ for G_2 manifolds (odd-dimensional). The constant 42 is the structural invariant $2b_2$.

11.1.6 24.5 The Algebraic Web

The topological constants satisfy interconnected identities:

Identity	Left side	Right side
Fiber-holonomy	dim(G_2) = 14	p_2 × dim(K_7) = 2 × 7
Gauge moduli	b_2 = 21	N_gen × dim(K_7) = 3 × 7

Identity	Left side	Right side
Matter-holonomy	b_3 + dim(G_2) = 91	dim(K_7) × α_sum = 7 × 13
Fano order	PSL(2,7) = 168	rank(E_8) × b_2 = 8 × 21
Fano order	PSL(2,7) = 168	N_gen × fund(E_7) = 3 × 56
Anomaly sum	α_sum = 13	rank(E_8) + Weyl = 8 + 5

These relations form a closed algebraic system. The mod-7 structure (dim(K_7) = 7 divides dim(G_2), b_2, b_3, PSL(2,7)) reflects the Fano plane underlying octonion multiplication.

11.1.7 24.6 Fibonacci-Lucas Embedding

The GIFT constants embed naturally into the Fibonacci (F_n) and Lucas (L_n) sequences:

n	F_n	GIFT Constant	Role
3	2	p_2	Pontryagin class
4	3	N_gen	Fermion generations
5	5	Weyl	Pentagonal symmetry
6	8	rank(E_8)	Cartan subalgebra
7	13	α^2_B sum	Structure coefficient
8	21	b_2	Second Betti number

This sequence propagates via the recurrence:

$$F_3 + F_4 = F_5 \quad \Rightarrow \quad p_2 + N_{gen} = \text{Weyl}$$

Lucas numbers also appear naturally:

L_n	Value	GIFT Role
L_4	7	dim(K_7)
L_5	11	D_bulk
L_8	47	Scale bridge exponent

The Lucas identity $L_8 = F_7 + F_9$ = 13 + 34 decomposes as:

$$L_8 = \alpha^B_{sum} + d_{hidden} = 13 + 34 = 47$$

This structure reflects the icosahedral geometry underlying the McKay correspondence $E_8 \leftrightarrow$ 2I, where icosahedral coordinates involve the golden ratio $\varphi = \lim(F_{n+1}/F_n)$.

Status: STRUCTURAL (mathematical pattern; physical significance unclear)

11.2 Appendix F: δ_CP: The 197° Prediction and the Compactification Correction

11.2.1 F.1 The Original Prediction

The GIFT prediction δ_CP = 197° = 7 × 14 + 99 = dim(K_7) × dim(G_2) + H* matches NuFIT 5.2 (2022: 197° ± 24°) exactly.

NuFIT 6.0 (October 2024) shifted the central value to 177° ± 20°. The 197° prediction is at the 1σ boundary of this measurement. The NuFIT collaboration notes that δ_CP remains one of the least constrained oscillation parameters, with significant sensitivity to reactor flux model assumptions and the mass ordering.

The canonical GIFT prediction remains δ_CP = 197°.

11.2.2 F.2 The Compactification Factor 62/69

PSLQ residual analysis (§7.6 of main paper) identifies a potential correction factor:

$$\frac{62}{69} = \frac{\dim(E_8)}{\dim(E_8) + 4\ \dim(K_7)} = \frac{248}{248 + 28} = \frac{248}{276}$$

This factor has a clean structural interpretation: the ratio of gauge degrees of freedom (dim(E_8) = 248) to total degrees of freedom (248 gauge + 4 × 7 gravitational = 276).

Applied to the original prediction:

$$197 \times \frac{62}{69} = \frac{12214}{69} = 177.01°$$

This matches NuFIT 6.0 to **0.008%**. The formula 12214/69 uses only GIFT topological integers and the compactification ratio.

11.2.3 F.3 Why We Do Not Adopt It (Yet)

1. **Post-hoc identification**: The factor 62/69 was found AFTER the NuFIT 6.0 shift. Adopting it would be fitting to the latest central value: the opposite of prediction.

2. **197° remains within 1σ**: The value 197° = 177° + 20° is AT the 1σ boundary. One-sigma deviations are expected ~32% of the time. This is not a falsification.

3. **Experimental instability**: δ_CP shifted 20° between NuFIT 5.2 and 6.0. The next update could shift back. DUNE will provide the definitive measurement with resolution of a few degrees to ~15° (2028–2040).

4. **Formula integrity**: GIFT predictions are derived from topology, not fitted to experiment. The 197° formula (3 constants, 2 operations) is cleaner than the 177.01° formula (5 constants, 4 operations).

11.2.4 F.4 Contingency Plan

If DUNE confirms δ_CP ≈ 177° (central value within ±5° of 177°): → Adopt 12214/69 = 177.01° as the primary prediction → Interpret 62/69 as a compactification factor (gauge/total DOF ratio) → The original 197° becomes the "bare" topological value before compactification

If DUNE confirms δ_CP ≈ 197° (central value within ±10° of 197°): → Original prediction vindicated → The 62/69 factor was a false lead (PSLQ artifact)

If DUNE gives a value inconsistent with both 177° and 197°: → Both formulas fail; the neutrino sector requires revision

11.2.5 F.5 Summary

Formula	Value	Status	Complexity
7×14+99	197°	**PRIMARY** (original prediction)	3 constants, 2 ops
12214/69	177.01°	Contingent (pending DUNE)	5 constants, 4 ops
Experimental	177°±20°	NuFIT 6.0 (1σ band: [157°, 197°])	,

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