Minimal Semi-Quantum Closure of H_0 in HGR

Autonomous LaTeX Document

Assisted drafting

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Abstract

This autonomous document formalizes a minimal closure of the present Hubble constant H_0 within the HGR framework (Hypercomplex General Relativity), by connecting: (i) the effective FLRW reduction, (ii) the quantized internal sector defined by an observable $X = \alpha_X Q^p$, and (iii) a unique late-time cosmological variable $Q_{\text{eff}}(a)$. The aim is not to present a definitive theorem, but rather a coherent, compilable, notation-stable architecture that can be directly reused as a sub-appendix in a larger manuscript.

Contents

1 Purpose and status of the present text

The goal is to provide an autonomous, coherent, and compilable version of the following logical chain:

$$Q_{\text{micro}} \longrightarrow X = \alpha_X Q^p_{\text{micro}} \longrightarrow W^{\text{eff}}_\star = \kappa_X \left\langle \hat{X}^2 \right\rangle \longrightarrow Q_0 \longrightarrow Q_{\text{eff}}(a) \longrightarrow H(a) \longrightarrow H_0.$$

Its precise status is the following.

- **Results already present in the HGR manuscript:** modified Friedmann equation, effective form $\rho_{\text{eff}}(a) = \alpha/a^2 - \beta/a^4$, definition of an internal observable $X = \alpha_X Q^p$, effective quantum threshold $W^{\text{eff}}_\star = \kappa_X \left\langle \hat{X}^2 \right\rangle$, and semi-classical approximation of $\left\langle \hat{X}^2 \right\rangle$.
- **Additions made in this document:** explicit separation between the microscopic Q and the cosmological Q, projection hypothesis $Q_0 = \zeta Q^s_{\text{cl}}$, late-time locking onto a unique mode $Q_{\text{eff}}(a)$, and explicit algebraic resolution of H_0.
- **Nature of the construction:** a coherent minimal closure, not yet an ultimate derivation in the sense of a fully closed microscopic theorem.

2 Effective cosmological basis

We start from the standardized FLRW form:

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}(\rho_{\text{mat}} + \rho_{\text{eff}}). \tag{1}$$

The effective geometric contribution is taken to be

$$\rho_{\text{eff}}(a) = \frac{\alpha}{a^2} - \frac{\beta}{a^4}. \tag{2}$$

The internal radiative term is parameterized by a comoving amplitude:

$$\beta = \frac{C_A}{\lambda^2} Q^2_{\text{cosmo}}. \tag{3}$$

In a more symplectic formulation, one may also write

$$\chi(t) = \frac{Q_{\text{cosmo}}}{a^2}, \qquad \rho_\Omega(a) = \frac{C_\Omega}{\kappa} \frac{Q^2_{\text{cosmo}}}{a^4}. \tag{4}$$

Comment

Equation (2) is the effective starting point. As long as α is defined in terms of H_0, one does not yet have an autonomous prediction of H_0. The purpose of the present text is precisely to replace this circular dependence by a connection to the quantized internal sector.

3 Quantized internal sector

We introduce the microscopic internal invariant

$$Q_{\text{micro}}(x) \equiv E^a_i E^i_a + B^a_i B^i_a. \tag{5}$$

The internal observable is then defined by

$$X(x) = \alpha_X \, Q_{\text{micro}}(x)^p. \tag{6}$$

After quantum promotion:

$$\hat{X}(x) = \alpha_X \left(\hat{E}^a_i \hat{E}^i_a + \hat{B}^a_i \hat{B}^i_a \right)^p. \tag{7}$$

We then define the effective threshold

$$W^{\text{eff}}_\star(x) = \kappa_X \, \left\langle \hat{X}^2(x) \right\rangle. \tag{8}$$

In the semi-classical regime, we set

$$Q_{\text{cl}} \equiv \left\langle \hat{Q}_{\text{micro}} \right\rangle, \qquad (\Delta Q)^2 \equiv \left\langle \hat{Q}^2_{\text{micro}} \right\rangle - \left\langle \hat{Q}_{\text{micro}} \right\rangle^2. \tag{9}$$

The first-order cumulant approximation gives:

$$\left\langle \hat{X}^2 \right\rangle \approx \alpha_X^2 \, Q_{\text{cl}}^{2p} \left[1 + \frac{(2p)(2p-1)}{2} \frac{(\Delta Q)^2}{Q_{\text{cl}}^2} \right]. \tag{10}$$

We introduce the abbreviation

$$\delta_q \equiv \frac{(2p)(2p-1)}{2} \frac{(\Delta Q)^2}{Q_{\text{cl}}^2}, \tag{11}$$

so that

$$\left\langle \hat{X}^2 \right\rangle \approx \alpha_X^2 Q_{\text{cl}}^{2p} (1 + \delta_q). \tag{12}$$

Therefore,

$$W^{\text{eff}}_\star = \kappa_X \alpha_X^2 Q_{\text{cl}}^{2p} (1 + \delta_q). \tag{13}$$

This can be inverted as

$$Q_{\text{cl}} = \left(\frac{W^{\text{eff}}_\star}{\kappa_X \alpha_X^2 (1 + \delta_q)} \right)^{1/(2p)}. \tag{14}$$

4 Separation of the two amplitudes Q

Hypothesis 1 (Notation separation)**.** *We explicitly distinguish:*

$$Q_{\text{micro}} : \textit{microscopic/quantized internal invariant}, \tag{15}$$

$$Q_{\text{cosmo}} : \textit{effective cosmological amplitude entering } \beta. \tag{16}$$

This separation is conceptually indispensable: using the same letter for these two objects is dangerous, even if their physical kinship is clear.

Hypothesis 2 (Minimal semi-classical projection)**.** *We introduce a projection relation*

$$Q_0 = \zeta\, Q_{\rm cl}^s, \tag{17}$$

where Q_0 is the late-time cosmological amplitude at $a = 1$, ζ is an effective normalization coefficient, and s is a coarse-graining exponent.

The minimal choice is

$$s = 1, \qquad Q_0 = \zeta\, Q_{\rm cl}. \tag{18}$$

Combining (14) and (17), one obtains

$$Q_0 = \zeta \left(\frac{W_\star^{\rm eff}}{\kappa_X \alpha_X^2 (1+\delta_q)} \right)^{s/(2p)}. \tag{19}$$

In the minimal case $s = 1$:

$$Q_0 = \zeta \left(\frac{W_\star^{\rm eff}}{\kappa_X \alpha_X^2 (1+\delta_q)} \right)^{1/(2p)}. \tag{20}$$

5 General late-time HGR–Weyl version

We consider an equation of the form

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} + \alpha\, \omega_J^2 + \beta\, \dot{\omega}_J^2 + \gamma H \omega_J + A\phi^2 + BH\phi + C\dot{\phi}. \tag{21}$$

This equation contains two additional sectors:

- an effective hypercomplex mode ω_J;
- a homogeneous Weyl mode ϕ.

The problem is to close this system in the most economical way possible.

6 Reduction to a unique effective mode

Hypothesis 3 (Minimal late-time locking)**.** *We assume that, in the late-time regime, the two sectors lock onto a single mode:*

$$\omega_J = u\, Q_{\rm eff}, \qquad \phi = v\, Q_{\rm eff}, \tag{22}$$

where u and v are projection constants.

We then take a late-time dilution ansatz

$$Q_{\rm eff}(a) = Q_0\, a^{-m}, \qquad m > 0. \tag{23}$$

Since $\dot{a} = Ha$, one has

$$\dot{Q}_{\rm eff} = -mHQ_{\rm eff}, \tag{24}$$

and hence

$$\dot{\omega}_J = -mH\omega_J, \qquad \dot{\phi} = -mH\phi. \tag{25}$$

Injecting (22)–(25) into (21), one obtains

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} + \alpha u^2 Q_{\rm eff}^2 + \beta m^2 u^2 H^2 Q_{\rm eff}^2 + \gamma u H Q_{\rm eff} + A v^2 Q_{\rm eff}^2 + B v H Q_{\rm eff} - C m v H Q_{\rm eff}. \tag{26}$$

We then regroup terms:

$$(1-\beta m^2 u^2 Q_{\rm eff}^2)H^2 - [\gamma u + (B - Cm)v]Q_{\rm eff} H - \left[\frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} + (\alpha u^2 + A v^2)Q_{\rm eff}^2\right] = 0. \tag{27}$$

Define

$$\mu \equiv \beta m^2 u^2, \tag{28}$$
$$\nu \equiv \gamma u + (B - Cm)v, \tag{29}$$
$$\sigma \equiv \alpha u^2 + A v^2. \tag{30}$$

The minimal closed cosmological equation then becomes

$$(1-\mu Q_{\rm eff}^2)H^2 - \nu Q_{\rm eff} H - \left[\frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} + \sigma Q_{\rm eff}^2\right] = 0. \tag{31}$$

7 Highlighted block: 10 central equations

The 10 structuring equations

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}(\rho_{\rm mat} + \rho_{\rm eff}) \tag{C1}$$
$$\rho_{\rm eff}(a) = \frac{\alpha}{a^2} - \frac{\beta}{a^4} \tag{C2}$$
$$\beta = \frac{C_A}{\lambda^2} Q_{\rm cosmo}^2 \tag{C3}$$
$$X = \alpha_X Q_{\rm micro}^p \tag{C4}$$
$$W_\star^{\rm eff} = \kappa_X \left\langle \hat{X}^2 \right\rangle \tag{C5}$$
$$\left\langle \hat{X}^2 \right\rangle \approx \alpha_X^2 Q_{\rm cl}^{2p}(1+\delta_q) \tag{C6}$$
$$Q_0 = \zeta Q_{\rm cl}^s \tag{C7}$$
$$Q_{\rm eff}(a) = Q_0 a^{-m} \tag{C8}$$
$$(1-\mu Q_{\rm eff}^2)H^2 - \nu Q_{\rm eff} H - \left[\frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} + \sigma Q_{\rm eff}^2\right] = 0 \tag{C9}$$
$$H_0 = \frac{\nu Q_0 + \sqrt{\nu^2 Q_0^2 + 4(1-\mu Q_0^2)(E_0 + \sigma Q_0^2)}}{2(1-\mu Q_0^2)} \tag{C10}$$
$$E_0 = \frac{8\pi G}{3}\rho_0 + \frac{\Lambda}{3} - k \tag{32}$$

8 General resolution for $H(a)$

Equation (31) is quadratic in H. Let

$$D(a) = 1 - \mu Q_{\rm eff}(a)^2, \tag{33}$$

$$\Gamma(a) = \nu Q_{\text{eff}}(a), \tag{34}$$

$$\Xi(a) = \frac{8\pi G}{3}\rho(a) + \frac{\Lambda}{3} - \frac{k}{a^2} + \sigma Q_{\text{eff}}(a)^2. \tag{35}$$

Then

$$D(a)H^2 - \Gamma(a)H - \Xi(a) = 0. \tag{36}$$

The expanding-branch solution is

$$H(a) = \frac{\Gamma(a) + \sqrt{\Gamma(a)^2 + 4D(a)\Xi(a)}}{2D(a)}. \tag{37}$$

9 Explicit formula for H_0

At the present epoch $a = 1$:

$$Q_{\text{eff}}(1) = Q_0, \qquad \rho(1) = \rho_0 = \rho_{m0} + \rho_{r0}. \tag{38}$$

We define

$$D_0 = 1 - \mu Q_0^2, \tag{39}$$

$$\Gamma_0 = \nu Q_0, \tag{40}$$

$$\Xi_0 = \frac{8\pi G}{3}\rho_0 + \frac{\Lambda}{3} - k + \sigma Q_0^2. \tag{41}$$

Then

$$D_0 H_0^2 - \Gamma_0 H_0 - \Xi_0 = 0, \tag{42}$$

which yields

$$H_0 = \frac{\Gamma_0 \pm \sqrt{\Gamma_0^2 + 4D_0\Xi_0}}{2D_0}. \tag{43}$$

The physical expanding branch is

$$H_0 = \frac{\Gamma_0 + \sqrt{\Gamma_0^2 + 4D_0\Xi_0}}{2D_0}. \tag{44}$$

Replacing explicitly,

$$H_0 = \frac{\nu Q_0 + \sqrt{\nu^2 Q_0^2 + 4(1 - \mu Q_0^2)\left(\frac{8\pi G}{3}\rho_0 + \frac{\Lambda}{3} - k + \sigma Q_0^2\right)}}{2(1 - \mu Q_0^2)}. \tag{45}$$

10 Injection of the quantum threshold into H_0

From (20), one has

$$Q_0^2 = \zeta^2 \left(\frac{W_\star^{\text{eff}}}{\kappa_X \alpha_X^2 (1 + \delta_q)}\right)^{1/p}. \tag{46}$$

Hence

$$D_0 = 1 - \mu\zeta^2 \left(\frac{W_\star^{\text{eff}}}{\kappa_X \alpha_X^2 (1 + \delta_q)}\right)^{1/p}, \tag{47}$$

$$\Gamma_0 = \nu\zeta \left(\frac{W_\star^{\text{eff}}}{\kappa_X \alpha_X^2 (1+\delta_q)} \right)^{1/(2p)}, \tag{48}$$

$$\Xi_0 = \frac{8\pi G}{3}\rho_0 + \frac{\Lambda}{3} - k + \sigma\zeta^2 \left(\frac{W_\star^{\text{eff}}}{\kappa_X \alpha_X^2 (1+\delta_q)} \right)^{1/p}. \tag{49}$$

Replacing these expressions into (44) gives an explicit dependence of H_0 on the effective quantum threshold.

To lighten the notation, let

$$Y \equiv \left(\frac{W_\star^{\text{eff}}}{\kappa_X \alpha_X^2 (1+\delta_q)} \right)^{1/(2p)}, \qquad E_0 \equiv \frac{8\pi G}{3}\rho_0 + \frac{\Lambda}{3} - k. \tag{50}$$

Then

$$H_0 = \frac{\nu\zeta Y + \sqrt{\nu^2\zeta^2 Y^2 + 4(1-\mu\zeta^2 Y^2)(E_0 + \sigma\zeta^2 Y^2)}}{2(1-\mu\zeta^2 Y^2)}. \tag{51}$$

11 Late-time perturbative regime

Assume

$$\mu Q_0^2 \ll 1, \qquad |\nu Q_0| \ll \sqrt{E_0 + \sigma Q_0^2}. \tag{52}$$

Then a first-order expansion gives

$$H_0 \approx \sqrt{E_0 + \sigma Q_0^2} + \frac{\nu Q_0}{2} + \frac{\mu Q_0^2}{2}\sqrt{E_0 + \sigma Q_0^2}. \tag{53}$$

Replacing Q_0 by (20), one obtains:

$$\begin{aligned} H_0 \approx {} & \sqrt{E_0 + \sigma\zeta^2 \left(\frac{W_\star^{\text{eff}}}{\kappa_X \alpha_X^2 (1+\delta_q)} \right)^{1/p}} \\ & + \frac{\nu\zeta}{2} \left(\frac{W_\star^{\text{eff}}}{\kappa_X \alpha_X^2 (1+\delta_q)} \right)^{1/(2p)} \\ & + \frac{\mu\zeta^2}{2} \left(\frac{W_\star^{\text{eff}}}{\kappa_X \alpha_X^2 (1+\delta_q)} \right)^{1/p} \sqrt{E_0 + \sigma\zeta^2 \left(\frac{W_\star^{\text{eff}}}{\kappa_X \alpha_X^2 (1+\delta_q)} \right)^{1/p}}. \end{aligned} \tag{54}$$

12 Consistency conditions

For the branch to be physically acceptable, one at least requires:

(i) Absence of singularity in the kinetic coefficient

$$D_0 = 1 - \mu Q_0^2 > 0. \tag{55}$$

(ii) Reality of the solution

$$\Gamma_0^2 + 4D_0\Xi_0 \geq 0. \tag{56}$$

(iii) Quantum version of the first condition

From (47):

$$1 - \mu\zeta^2 \left(\frac{W_\star^{\mathrm{eff}}}{\kappa_X \alpha_X^2 (1+\delta_q)} \right)^{1/p} > 0. \tag{57}$$

13 Physical interpretation

The minimal reading of the construction is the following.

1. The quantized internal sector defines an observable $X = \alpha_X Q_{\mathrm{micro}}^p$.
2. This observable controls an effective threshold $W_\star^{\mathrm{eff}}$.
3. This threshold determines a late-time amplitude Q_0 after projection/coarse-graining.
4. The amplitude Q_0 governs an effective mode $Q_{\mathrm{eff}}(a) = Q_0 a^{-m}$.
5. This mode backreacts on the closed Friedmann equation through three couplings:

$$\mu : \text{renormalization of the } H^2 \text{ term}, \tag{58}$$
$$\nu : \text{linear correction in } H, \tag{59}$$
$$\sigma : \text{additional geometric density}. \tag{60}$$

6. The present value H_0 then becomes an *output* of the effective model, rather than merely a constant inserted as an external normalization.

14 What is established, and what remains open

Established in this document

We have constructed an explicit dependence of the form

$$H_0 = H_0\Big(W_\star^{\mathrm{eff}}, \delta_q, \zeta, p, \mu, \nu, \sigma, \rho_0, \Lambda, k\Big). \tag{61}$$

Still open

The point not yet derived microscopically is the exact projection law

$$Q_0 = \zeta Q_{\mathrm{cl}}^s. \tag{62}$$

Three physical readings remain possible:

- direct projection of the semi-classical mean;
- coarse-grained domain variable;
- saturated amplitude.

Remark 1. *In other words, the logical structure is closed, but the detailed micro-foundation of the passage $Q_{\mathrm{cl}} \mapsto Q_0$ still needs to be clarified if one wants to turn this coherent closure into an entirely derived theorem.*

15 Final summary

The full chain may be summarized as

$$
\begin{aligned}
Q_{\text{micro}} &\longrightarrow X = \alpha_X Q_{\text{micro}}^p \longrightarrow W_\star^{\text{eff}} = \kappa_X \left\langle \hat{X}^2 \right\rangle \longrightarrow Q_0 = \zeta \left(\frac{W_\star^{\text{eff}}}{\kappa_X \alpha_X^2 (1+\delta_q)} \right)^{1/(2p)} \\
&\longrightarrow Q_{\text{eff}}(a) = Q_0 a^{-m} \longrightarrow (1-\mu Q_{\text{eff}}^2) H^2 - \nu Q_{\text{eff}} H - \left[\frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} + \sigma Q_{\text{eff}}^2 \right] = 0 \\
&\longrightarrow H_0 = \frac{\nu Q_0 + \sqrt{\nu^2 Q_0^2 + 4(1-\mu Q_0^2)(E_0 + \sigma Q_0^2)}}{2(1-\mu Q_0^2)}.
\end{aligned} \tag{63}
$$

This document therefore provides an autonomous LaTeX basis for incorporating the minimal semi-quantum closure of H_0 into a larger HGR manuscript.