A Mini Bi-Weyl Sector Compatible with HGR

Minimal action, field equations, and FLRW reduction

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Abstract

We propose here a minimal extension of the Weyl sector of Hypercomplex General Relativity (HGR) in which the usual scale connection is doubled into two independent Abelian fields, weakly mixed with one another. The goal is not to introduce a second metric, but rather a *two-scale structure*: one average mode, interpreted as the observable conformal sector, and one relative mode, interpreted as a hidden scale tension. We write a minimal action compatible with the symplectic spirit of HGR, derive the associated field equations, and then perform a homogeneous and isotropic FLRW reduction.

1 Statement of the problem

In the current symplectic version of HGR, the Weyl sector is carried by a single 1-form $\phi = \phi_\mu \mathrm{d}x^\mu$, with curvature

$$F^{(\phi)} = \mathrm{d}\phi,$$

and Maxwell-type dynamics. The formalism also admits mixed couplings between the Weyl sector, the internal hypercomplex sector, and the symplectic structure. The idea studied here is to replace the single Weyl field by two Abelian 1-forms

$$W^{(1)} = W^{(1)}_\mu \mathrm{d}x^\mu, \qquad W^{(2)} = W^{(2)}_\mu \mathrm{d}x^\mu,$$

with the aim of modeling a two-scale structure:

- a **visible** or *average* mode, generalizing the usual Weyl sector;
- a **hidden** or *relative* mode, measuring an internal difference between two local dilation structures.

2 Minimal hypotheses

2.1 Hypothesis H1: double Abelian conformal sector

We introduce two Abelian scale connections $W^{(1)}_\mu$ and $W^{(2)}_\mu$ with curvatures

$$\mathcal{F}^{(1)}_{\mu\nu} = \partial_\mu W^{(1)}_\nu - \partial_\nu W^{(1)}_\mu, \qquad \mathcal{F}^{(2)}_{\mu\nu} = \partial_\mu W^{(2)}_\nu - \partial_\nu W^{(2)}_\mu. \tag{1}$$

2.2 Hypothesis H2: a single observable metric

We keep a unique effective metric $g_{\mu\nu}$, interpreted as the emergent metric of HGR. The two Weyl fields therefore do not define two distinct metric geometries, but rather two *conformal connection modes* coupled to the same observable gravitational sector.

2.3 Hypothesis H3: weak mixed couplings

We allow:

- a bi-Weyl kinetic mixing term $\mathcal{F}^{(1)} \cdot \mathcal{F}^{(2)}$;
- a relative mass or geometric stiffness term for $W^{(1)} - W^{(2)}$;
- weak couplings to the hypercomplex sector A_μ and/or the symplectic sector Ω.

3 Minimal bi-Weyl–HGR action

We take as a starting point an effective action of the form

$$S = S_{\text{grav}} + S_\Omega + S_H + S_{\text{biW}} + S_{\text{mix}} + S_{\text{mat}}, \tag{2}$$

with

$$\begin{aligned} S_{\text{biW}} = \int \mathrm{d}^4x \sqrt{-g} \Big[&- \frac{1}{4} \mathcal{F}^{(1)}_{\mu\nu} \mathcal{F}^{(1)\mu\nu} - \frac{1}{4} \mathcal{F}^{(2)}_{\mu\nu} \mathcal{F}^{(2)\mu\nu} - \frac{\varepsilon}{2} \mathcal{F}^{(1)}_{\mu\nu} \mathcal{F}^{(2)\mu\nu} \\ &+ \frac{m_-^2}{2} (W^{(1)}_\mu - W^{(2)}_\mu)(W^{(1)\mu} - W^{(2)\mu}) \Big]. \end{aligned} \tag{3}$$

One may add minimal mixed couplings,

$$\begin{aligned} S_{\text{mix}} = \int \mathrm{d}^4x \sqrt{-g} \Big[&\zeta_1 \mathcal{F}^{(1)}_{\mu\nu} \mathcal{H}^{\mu\nu} + \zeta_2 \mathcal{F}^{(2)}_{\mu\nu} \mathcal{H}^{\mu\nu} + \eta (W^{(1)}_\mu + W^{(2)}_\mu) J^\mu_H \\ &+ \xi (W^{(1)}_\mu - W^{(2)}_\mu) J^\mu_{\text{rel}} \Big]. \end{aligned} \tag{4}$$

4 Diagonalization into average and relative modes

We introduce the orthogonal combinations

$$W^{(+)}_\mu = \frac{W^{(1)}_\mu + W^{(2)}_\mu}{\sqrt{2}}, \qquad W^{(-)}_\mu = \frac{W^{(1)}_\mu - W^{(2)}_\mu}{\sqrt{2}}, \tag{5}$$

with

$$\mathcal{F}^{(+)}_{\mu\nu} = \partial_\mu W^{(+)}_\nu - \partial_\nu W^{(+)}_\mu, \qquad \mathcal{F}^{(-)}_{\mu\nu} = \partial_\mu W^{(-)}_\nu - \partial_\nu W^{(-)}_\mu. \tag{6}$$

The kinetic sector becomes

$$\mathcal{L}_{\text{biW}} = -\frac{1}{4}(1+\varepsilon) \mathcal{F}^{(+)}_{\mu\nu} \mathcal{F}^{(+)\mu\nu} - \frac{1}{4}(1-\varepsilon) \mathcal{F}^{(-)}_{\mu\nu} \mathcal{F}^{(-)\mu\nu} + m_-^2 W^{(-)}_\mu W^{(-)\mu}. \tag{7}$$

If $|\varepsilon| < 1$, both kinetic terms keep the correct sign.

5 Field equations

Variation yields

$$\nabla_\mu \Big(\mathcal{F}^{(1)\mu\nu} + \varepsilon \mathcal{F}^{(2)\mu\nu} \Big) + m_-^2 (W^{(1)\nu} - W^{(2)\nu}) = J^\nu_{(1)}, \tag{8}$$

$$\nabla_\mu \Big(\mathcal{F}^{(2)\mu\nu} + \varepsilon \mathcal{F}^{(1)\mu\nu} \Big) - m_-^2 (W^{(1)\nu} - W^{(2)\nu}) = J^\nu_{(2)}. \tag{9}$$

In the diagonal basis,

$$(1+\varepsilon) \nabla_\mu \mathcal{F}^{(+)\mu\nu} = J^\nu_+, \tag{10}$$

$$(1-\varepsilon) \nabla_\mu \mathcal{F}^{(-)\mu\nu} + 2m_-^2 W^{(-)\nu} = J^\nu_-. \tag{11}$$

6 Energy–momentum tensor

The bi-Weyl sector contributes schematically as

$$T_{\mu\nu}^{(\mathrm{biW})} = T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)} + T_{\mu\nu}^{(12)} + T_{\mu\nu}^{(-m)}, \tag{12}$$

with the crossed term

$$T_{\mu\nu}^{(12)} = \varepsilon\Big(\mathcal{F}_{\mu\rho}^{(1)}\mathcal{F}^{(2)\rho}{}_{\nu} + \mathcal{F}_{\mu\rho}^{(2)}\mathcal{F}^{(1)\rho}{}_{\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{F}_{\rho\sigma}^{(1)}\mathcal{F}^{(2)\rho\sigma}\Big). \tag{13}$$

The sector therefore enters the effective Einstein equation as

$$G_{\mu\nu} = 8\pi G\Big(T_{\mu\nu}^{(\mathrm{mat})} + T_{\mu\nu}^{(H)} + T_{\mu\nu}^{(\Omega)} + T_{\mu\nu}^{(\mathrm{biW})} + T_{\mu\nu}^{(\mathrm{mix})}\Big). \tag{14}$$

7 Minimal FLRW reduction

To preserve background isotropy, we first take

$$W^{(1)} = w_1(t)\,\mathrm{d}t, \qquad W^{(2)} = w_2(t)\,\mathrm{d}t. \tag{15}$$

Then

$$\mathcal{F}_{\mu\nu}^{(1)} = 0, \qquad \mathcal{F}_{\mu\nu}^{(2)} = 0, \tag{16}$$

and the purely kinetic sector does not contribute directly to the background. However, the relative mode may still survive through the m_-^2 term and through mixed couplings.

In the diagonal basis,

$$W^{(+)} = w_+(t)\,\mathrm{d}t, \qquad W^{(-)} = w_-(t)\,\mathrm{d}t, \tag{17}$$

the relative mode contributes schematically as

$$\rho_{W_-}^{(0)} \sim +m_-^2 w_-^2, \qquad p_{W_-}^{(0)} \sim +m_-^2 w_-^2, \tag{18}$$

namely with a typically *stiff* behavior if no additional constraint is imposed.

If one enriches the ansatz by an effectively isotropized version, one obtains phenomenologically

$$\rho_{W_+}(a) \sim \frac{Q_+^2}{a^4}, \qquad \rho_{W_-}(a) \sim \frac{Q_-^2}{a^4} + m_-^2\,\Xi_-(a). \tag{19}$$

8 Effective Friedmann equation

The cosmological dynamics may then be written as

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\Big(\rho_{\mathrm{mat}} + \rho_H + \rho_\Omega + \rho_{W_+} + \rho_{W_-} + \rho_{\mathrm{mix}}\Big). \tag{20}$$

A minimal parametrization compatible with the HGR spirit is

$$\rho_{\mathrm{eff}}(a) = \frac{\alpha_+}{a^2} + \frac{\gamma_+}{a^4} + \frac{\gamma_-}{a^4} + \rho_{W_-}^{(m)}(a) + \rho_{\mathrm{mix}}(a) - \frac{\beta_H}{a^4}. \tag{21}$$

For a bounce scenario, one may condense this into

$$\rho_{\mathrm{eff}}(a) = \frac{\alpha_{\mathrm{biW}}}{a^2} - \frac{\beta_{\mathrm{RGH}}}{a^4}, \qquad \alpha_{\mathrm{biW}} > 0, \quad \beta_{\mathrm{RGH}} > 0, \tag{22}$$

with

$$a_{\mathrm{min}} = \sqrt{\frac{\beta_{\mathrm{RGH}}}{\alpha_{\mathrm{biW}}}}. \tag{23}$$

9 Conclusion

The mini bi-Weyl sector proposed here realizes a *two-scale structure without a second metric.* It provides:

1. a coherent action with two mixed Abelian fields;
2. coupled Maxwell-type equations;
3. a clear decomposition into average and relative modes;
4. a natural insertion into the HGR effective Friedmann dynamics;
5. a possible feedback on the bounce through mixed couplings.