HYPERCOMPLEX STRUCTURES IN RELATIVISTIC SPACE-TIME GEOMETRY

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1 Introduction

General relativity describes gravitation as the geometric manifestation of space-time curvature. In its standard formulation, the geometry is entirely determined by the metric $g_{\mu\nu}$ and by the Levi–Civita connection derived from it. The trajectories of free particles are then described by the geodesics associated with this connection.

However, several geometric extensions of general relativity have been studied over the last century. Among them are Weyl geometries, which introduce a non-metricity associated with a dilation field, as well as Einstein–Cartan theories, in which the affine connection has a torsion coupled to the spin of matter. These approaches share a common idea: the affine structure of space-time could be richer than the one described by the Levi–Civita connection alone.

In this work, we explore—and I emphasize the exploratory character of this approach—another geometric possibility, based on the existence of an internal hypercomplex structure associated with each point of space-time. More precisely, we consider a total space endowed with a quaternionic fiber $\mathbb{H}$, so that the fundamental geometric structure is described on a fibered space of the form

$$E = M_4 \times \mathbb{H},$$

where M_4 denotes observable space-time and $\mathbb{H}$ the internal hypercomplex fiber.

In this perspective, the observable geometry is no longer the fundamental structure, but appears as the projection of a geometry defined on the total space. The effective affine connection of space-time may then be interpreted as the projection of a more general connection acting on the hypercomplex fibered space.

Such a construction naturally leads to the introduction of an extended connection combining several geometric sectors, in particular the relativistic spin connection, a Weyl-type conformal field, and an internal connection associated with the quaternionic structure.

We postulate the following minimal affine decomposition:

$$\Gamma^{\rho}{}_{\mu\nu,\text{eff}} = \Gamma^{\rho}{}_{\mu\nu}(g) + \Delta\Gamma^{\rho}{}_{\mu\nu}(W) + \Delta\Gamma^{\rho}{}_{\mu\nu}(H),$$

where the additional terms respectively represent the conformal and hypercomplex contributions.

The decomposition

$$\Gamma^{\rho}{}_{\mu\nu,\text{eff}} = \Gamma^{\rho}{}_{\mu\nu}(g) + \Delta\Gamma^{\rho}{}_{\mu\nu}(W) + \Delta\Gamma^{\rho}{}_{\mu\nu}(H)$$

should be understood as a tensorial decomposition of the effective connection relative to the Levi–Civita connection. By contrast, the explicit form of the hypercomplex contribution

$$\Delta\Gamma^{\rho}{}_{\mu\nu}(H) = \frac{\lambda}{2}\left(A^I_\mu \Sigma^{\rho}{}_{\nu I} + A^I_\nu \Sigma^{\rho}{}_{\mu I}\right)$$

is not derived here from a complete fundamental geometry, but is introduced as the minimal local ansatz compatible with symmetry in μ, ν and with the existence of a coupling between the internal quaternionic connection and the tangent structure of space-time.

In this framework, the internal quaternionic structure does not simply behave as an additional gauge field, but may also modify the affine structure of observable space-time. Geodesic trajectories and effective curvature may then receive contributions from the dynamics of the hypercomplex fiber.

The purpose of this article is to explore this possibility within a minimal geometric framework and to clarify the physical implications of such a structure. To this end, we introduce a simple affine ansatz linking the internal quaternionic connection to the effective connection of space-time, and discuss the resulting geometric and physical consequences.

This work is situated in an exploratory perspective at the interface between differential geometry and gravitational physics. Its main goal is to highlight the geometric coherence of such a hypercomplex extension and to identify possible directions for future developments.

2 Quaternionic algebra and geometric interpretation

The algebra of quaternions $\mathbb{H}$ provides a natural framework for describing three-dimensional rotational structures as well as internal degrees of freedom associated with the group $SU(2)$. A quaternion is defined by

$$q = a + b\,i + c\,j + d\,k,$$

where $a, b, c, d \in \mathbb{R}$ and the imaginary units satisfy the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

The quaternionic algebra is associative but non-commutative. In particular,

$$ij = k, \qquad jk = i, \qquad ki = j,$$

whereas

$$ji = -k.$$

Any quaternion can be decomposed into a real part and a vector imaginary part,

$$q = \mathrm{Re}(q) + \vec{q},$$

with

$$\vec{q} = (b, c, d).$$

The imaginary components (i, j, k) therefore generate a three-dimensional internal space. This observation establishes a natural correspondence between the imaginary directions of quaternions and the generators of three-dimensional rotations.

An important subset of quaternionic algebra consists of unit quaternions satisfying

$$|q| = 1.$$

These elements form a group isomorphic to $SU(2)$,

$$SU(2) \simeq \{q \in \mathbb{H} \mid |q| = 1\}.$$

This relation plays a central role in many areas of mathematical physics, because $SU(2)$ is the double cover of the rotation group $SO(3)$ and appears naturally in the description of spin and of certain internal symmetries.

In the geometric framework considered here, the algebra of quaternions provides the internal fiber associated with each point of the space-time manifold. The imaginary directions (i, j, k) may then be interpreted as internal directions attached to space-time, while the subgroup of unit quaternions describes internal rotational transformations.

This interpretation motivates the introduction of a quaternionic fiber structure

$$E = M_4 \times \mathbb{H},$$

where M_4 denotes the observable space-time manifold and $\mathbb{H}$ the internal hypercomplex fiber.

In such a framework, the internal quaternionic structure may influence the effective geometry of space-time through a generalized connection defined on the total bundle.

3 Hypercomplex fibered structure

The quaternionic algebra introduced in the previous section naturally suggests a geometric interpretation in terms of a fibered structure. In this framework, space-time is described by a base manifold M_4, while each point of this manifold is endowed with an internal hypercomplex structure represented by a quaternionic fiber.

The total geometric space is therefore described by a fiber bundle of the form

$$E = M_4 \times \mathbb{H},$$

where M_4 denotes the observable space-time manifold and $\mathbb{H}$ the internal hypercomplex fiber.

A point of the total space can thus be represented by

$$(x^\mu, q),$$

where x^μ are the space-time coordinates and $q \in \mathbb{H}$ is a quaternion describing the internal configuration.

In this framework, observable space-time corresponds to the projection

$$\Pi_{\text{obs}} : (x^\mu, q) \to x^\mu.$$

The internal hypercomplex structure is therefore not directly observable, but it may influence the effective geometry of space-time through the connection defined on the total bundle.

In order to describe parallel transport in the total space, we introduce a generalized connection

$$\mathcal{D}_\mu.$$

This connection contains several geometric contributions associated with the different symmetry structures present in the theory. In the minimal construction considered here, it can be written as

$$\mathcal{D}_\mu = \partial_\mu + \omega_\mu + W_\mu + A_\mu.$$

The first term ω_μ corresponds to the relativistic spin connection associated with the local Lorentz symmetry $Spin(1,3)$.

The field W_μ represents a Weyl-type conformal contribution associated with local scale transformations.

Finally, A_μ denotes the internal quaternionic connection acting on the hypercomplex fiber. Its components may be written as

$$A_\mu = A_\mu^I T_I,$$

where T_I are the generators associated with the quaternionic imaginary directions (i, j, k).

The connection $\mathcal{D}_\mu$ therefore describes parallel transport on the total hypercomplex bundle, combining the geometric structure of space-time and the internal degrees of freedom.

In the next section, we show how the effective affine connection of space-time may be obtained from this extended connection by projection onto the observable manifold.

4 Projection toward the observable affine connection

The connection defined on the bundle introduced in the previous section acts on the total hypercomplex space $E = M_4 \times \mathbb{H}$. In order to recover an effective space-time geometry, it is necessary to specify how this extended connection projects onto the observable manifold M_4.

The central hypothesis explored in this work is that the affine structure perceived in space-time is not purely primitive, but can be interpreted as the projection of a richer connection defined on the total hypercomplex bundle. In this perspective, the observable geometry of space-time appears as an effective sector of a broader geometric structure.

We therefore postulate that the effective affine connection of space-time has the form

$$\Gamma^{\rho}{}_{\mu\nu,\text{eff}} = \Gamma^{\rho}{}_{\mu\nu}(g) + \Delta\Gamma^{\rho}{}_{\mu\nu}(W) + \Delta\Gamma^{\rho}{}_{\mu\nu}(H),$$

where $\Gamma^{\rho}{}_{\mu\nu}(g)$ is the Levi–Civita connection associated with the observable metric $g_{\mu\nu}$, $\Delta\Gamma^{\rho}{}_{\mu\nu}(W)$ represents a Weyl-type correction, and $\Delta\Gamma^{\rho}{}_{\mu\nu}(H)$ corresponds to the contribution induced by the internal hypercomplex sector.

The hypercomplex correction is assumed to arise from the connection defined on the quaternionic fiber through a local base–fiber projection mechanism. At the minimal level, we introduce the following ansatz:

$$\Delta\Gamma^{\rho}{}_{\mu\nu}(H) = \frac{\lambda}{2}\left(A^I_\mu \Sigma^{\rho}{}_{\nu I} + A^I_\nu \Sigma^{\rho}{}_{\mu I}\right),$$

where A^I_μ are the components of the quaternionic connection, $\Sigma^{\rho}{}_{\nu I}$ is a tensor describing the local projection of internal quaternionic directions toward space-time, and λ is a geometric coupling constant.

This expression should be understood as a minimal effective ansatz. Its aim is not to provide a complete fundamental theory, but to capture the possibility that the internal hypercomplex structure directly modifies the affine geometry, and not only through a contribution to the energy–momentum tensor.

From a geometric point of view, this construction may be interpreted as the space-time projection of a connection defined on the total bundle $M_4 \times \mathbb{H}$. The tensor $\Sigma^{\rho}{}_{\nu I}$ then plays the role of a local soldering object relating the internal quaternionic fiber to the tangent bundle of space-time.

Once the effective affine connection is modified, the geodesic equation becomes

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^{\rho}{}_{\mu\nu,\text{eff}}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0.$$

This implies that the internal hypercomplex structure may influence the motion of test particles even in regions where ordinary matter is weakly present. In this sense, the quaternionic sector contributes not only to the effective energy–momentum content, but also to the very definition of parallel transport in space-time.

Such a mechanism is especially interesting in situations where the observed gravitational field appears stronger than what visible matter alone would allow one to explain. In these regimes, the hypercomplex modification of the affine structure could provide an effective geometric contribution to the observed dynamics.

The framework presented here remains deliberately minimal. A more complete formulation would require deriving the projection tensor $\Sigma^{\rho}{}_{\nu I}$ from a fully specified bundle geometry and clarifying its relation to tetrads, internal triads, or more generally to Cartan-type soldering structures.

$$E \ = \ M_4 \times \mathbb{H}$$
Total hypercomplex space

$$\downarrow$$

$$\mathcal{D}_\mu \ = \ \partial_\mu + \omega_\mu + W_\mu + A_\mu$$
Extended connection

$$\downarrow$$

$$\Gamma^\rho{}_{\mu\nu,\text{eff}} \ = \ \Gamma^\rho{}_{\mu\nu}(g) + \Delta\Gamma^\rho{}_{\mu\nu}(W) + \Delta\Gamma^\rho{}_{\mu\nu}(H)$$
Effective affine connection

$$\downarrow$$

$$\ddot{x}^\rho + \Gamma^\rho{}_{\mu\nu,\text{eff}}\dot{x}^\mu\dot{x}^\nu \ = \ 0$$
Modified geodesics

Figure 4.1. Conceptual chain of hypercomplex general relativity. The observable geometry is obtained as the projection of the structure defined on the hypercomplex bundle $E = M_4 \times \mathbb{H}$.

5 Geodesics and physical implications

The modification of the affine connection discussed in the previous section directly affects the geodesic structure of space-time. Since the motion of freely falling particles is governed by the affine connection, any correction to it may lead to observable effects.

Using the effective connection introduced above, the geodesic equation takes the form

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho{}_{\mu\nu,\text{eff}}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0.$$

Substituting the decomposition

$$\Gamma^\rho{}_{\mu\nu,\text{eff}} = \Gamma^\rho{}_{\mu\nu}(g) + \Delta\Gamma^\rho{}_{\mu\nu}(W) + \Delta\Gamma^\rho{}_{\mu\nu}(H),$$

one obtains

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho{}_{\mu\nu}(g)\dot{x}^\mu\dot{x}^\nu + \Delta\Gamma^\rho{}_{\mu\nu}(W)\dot{x}^\mu\dot{x}^\nu + \Delta\Gamma^\rho{}_{\mu\nu}(H)\dot{x}^\mu\dot{x}^\nu = 0.$$

The first term corresponds to the usual geodesic motion in general relativity. The additional terms represent the deviations induced by the Weyl and hypercomplex sectors.

The hypercomplex contribution

$$\Delta\Gamma^\rho{}_{\mu\nu}(H) = \frac{\lambda}{2}\left(A^I_\mu\Sigma^\rho{}_{\nu I} + A^I_\nu\Sigma^\rho{}_{\mu I}\right)$$

introduces into the geodesic equation a term analogous to an effective force, whose strength depends on the internal quaternionic connection A^I_μ and on the projection tensor $\Sigma^\rho{}_{\nu I}$.

From a physical point of view, this term can modify the trajectories of particles even in regions where the ordinary energy–momentum tensor is small. In other words, the internal hypercomplex structure can contribute directly to the effective gravitational dynamics.

Such a mechanism could potentially generate additional gravitational effects without requiring a large amount of visible matter. In this perspective, the hypercomplex modification of the affine structure can act as an additional geometric contribution to the observed gravitational field.

It is important to emphasize that the framework presented here remains exploratory. The ansatz introduced should be regarded as a minimal effective description capturing the possibility that internal quaternionic degrees of freedom influence the space-time connection.

A more complete treatment would require deriving the internal projection tensor and the dynamics of the quaternionic connection from a geometric action defined on the full hypercomplex bundle.

5.1 Motivation for an internal quaternionic structure. The choice of an internal quaternionic fiber is strongly motivated by the algebraic structure underlying three-dimensional rotations and their spinorial extensions.

The group of spatial rotations is described by the Lie group $SO(3)$, whose Lie algebra is generated by three generators J_i satisfying

$$[J_i, J_j] = \epsilon_{ij}{}^k J_k.$$

However, the symmetry group acting correctly on spinorial degrees of freedom is not directly $SO(3)$, but its double cover $Spin(3)$, which is isomorphic to $SU(2)$,

$$Spin(3) \simeq SU(2).$$

An important observation is that the group of unit quaternions also forms a group isomorphic to $SU(2)$,

$$SU(2) \simeq \{q \in \mathbb{H} \mid |q| = 1\}.$$

The quaternionic imaginary units

$$(i, j, k)$$

therefore provide a natural basis for the Lie algebra $\mathfrak{su}(2)$, which governs the infinitesimal structure of three-dimensional rotations and their spinorial lifting.

From a geometric point of view, the imaginary subspace of quaternionic algebra

$$\mathrm{Im}(\mathbb{H}) = \mathrm{Span}_{\mathbb{R}}(i, j, k)$$

forms a three-dimensional internal vector space naturally associated with the generators of rotations.

These observations motivate the use of the algebra of quaternions $\mathbb{H}$ as an internal geometric structure attached to each point of space-time. In this framework, the quaternionic imaginary directions may be interpreted as internal rotational degrees of freedom, while the subgroup of unit quaternions reproduces the spinorial symmetry $SU(2)$.

This therefore provides a natural algebraic foundation for the introduction of a quaternionic fibered structure in certain geometric extensions of space-time.

6 Extended algebraic structure: $Cl(1,3) \otimes \mathbb{H}$

The introduction of an internal quaternionic fiber can be motivated not only by the structure of three-dimensional rotations and of the group $SU(2)$, but also by the very algebraic form of the relativistic spinorial sector.

In local relativity, the natural algebraic structure associated with Minkowski space-time is the Clifford algebra $Cl(1,3)$, generated by elements γ_μ satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}.$$

This algebra encodes the local metric structure, the generators of the Lorentz group, and the description of relativistic spinor fields.

On the other hand, as discussed above, the algebra of quaternions $\mathbb{H}$ provides a natural realization of internal rotational degrees of freedom, through the identification of the imaginary subspace $\mathrm{Im}(\mathbb{H})$ with a structure of $\mathfrak{su}(2)$ type.

These two structures suggest introducing an extended algebra of the form

$$\mathcal{A}_{\text{ext}} = Cl(1,3) \otimes \mathbb{H},$$

in which the Clifford part describes the observable relativistic sector, while the quaternionic factor describes an internal hypercomplex structure attached to each point of space-time.

A generic element of this extended algebra may then be written schematically as

$$X = a\,\mathbf{1} + b^\mu \gamma_\mu + \frac{1}{2} c^{\mu\nu} \gamma_{\mu\nu} + d^I T_I + e^{\mu I} \gamma_\mu \otimes T_I + \cdots ,$$

where T_I denote the generators associated with the quaternionic imaginary directions (i, j, k).

This expression shows that the extended algebra contains simultaneously:

- the relativistic vector and spinorial structure,
- the internal quaternionic-type generators,
- as well as mixed sectors coupling space-time structure to internal directions.

From a conceptual point of view, this suggests that the observable geometry could correspond to an effective projection of a richer algebraic structure, in which space-time degrees of freedom and internal degrees of freedom are not strictly separated at the fundamental level.

In this perspective, the total connection introduced above,

$$\mathcal{D}_\mu = \partial_\mu + \omega_\mu + W_\mu + A_\mu,$$

may be viewed as a connection taking its values in an extended algebra compatible with this $Cl(1,3) \otimes \mathbb{H}$ structure. The sector ω_μ then acts on the relativistic spinorial component, while A_μ acts on the internal quaternionic component. In a more complete formulation, possible mixed terms may be responsible for the projection mechanism linking the hypercomplex fiber to the effective affine connection of space-time.

Such a structure does not yet constitute a closed theory. It nevertheless provides a natural algebraic framework in which the hypercomplex geometry explored in this work can be formulated coherently. In this framework, the real or complex observable part of the geometry may be interpreted as an effective sector arising from a more fundamental hypercomplex-type structure.

This observation reinforces the idea that the quaternionic extension is not an arbitrary addition to relativistic geometry, but may instead fit into a unified algebraic structure relating spin, local geometry, and internal degrees of freedom.

7 Minimal geometric action

In order to provide a dynamical basis for the geometric structure introduced above, it is natural to ask whether the extended connection

$$\mathcal{D}_\mu = \partial_\mu + \omega_\mu + W_\mu + A_\mu$$

can be associated with an effective geometric action.

In standard general relativity, the dynamics of geometry is governed by the Einstein–Hilbert action

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, R,$$

where R is the curvature scalar built from the Levi–Civita connection.

In the hypercomplex framework considered here, the total connection also contains an internal quaternionic contribution A_μ as well as a Weyl-type conformal sector W_μ. It is therefore natural to introduce an effective action of the form

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G} R + \alpha F^I_{\mu\nu} F_I^{\mu\nu} + \beta W_{\mu\nu} W^{\mu\nu} \right),$$

where:

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + \epsilon^{IJK} A^J_\mu A^K_\nu$$

is the curvature associated with the internal quaternionic connection, while

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$$

represents the curvature field associated with the conformal sector.

The constants α and β are coupling parameters determining the relative strength of the hypercomplex and conformal contributions.

In this framework, the dynamics of the system results from varying the total action with respect to the metric $g_{\mu\nu}$, the quaternionic field A^I_μ, and the conformal field W_μ.

The proposed action should be understood as a minimal effective description compatible with the geometric structure introduced in the previous sections. A more complete formulation may require additional terms explicitly linking the internal quaternionic connection to the effective affine structure of space-time.

8 Discussion

The framework presented in this work explores the possibility that the observed affine structure of space-time is the effective projection of a richer geometry defined on a hypercomplex bundle.

In this approach, each point of space-time is associated with an internal quaternionic fiber $\mathbb{H}$, and the total connection contains both the usual relativistic contributions, a Weyl-type conformal component, and an internal quaternionic connection.

The projection of this structure onto observable space-time leads to an effective modification of the affine connection, introducing an additional term depending on the internal connection. This correction can modify geodesic trajectories and the effective gravitational dynamics.

It is important to emphasize that the construction presented here should be regarded as a minimal geometric framework. The projection tensor $\Sigma^{\rho}{}_{\nu I}$ introduced in the affine ansatz has not been derived from a complete bundle geometry, but represents an effective parametrization of the coupling between internal quaternionic directions and the tangent structure of space-time.

Several development directions can be envisaged.

First, a more fundamental description of the geometry of the hypercomplex bundle would make it possible to explicitly derive the projection tensor as well as the compatibility relations between the internal connection and the space-time geometry.

Second, the proposed minimal geometric action could be extended to include direct coupling terms between the quaternionic and gravitational sectors. Such an extension could lead to modified field equations and to new gravitational solutions.

Finally, it would be interesting to explore the phenomenological consequences of this structure, in particular in regimes where effective gravitation appears stronger than what visible matter can explain. In such cases, the geometric contribution arising from the hypercomplex structure could play a role analogous to that of an additional effective gravitational component.

9 Conclusion

In this article, we have explored a geometric extension of general relativity based on the introduction of an internal hypercomplex structure associated with each point of space-time.

This construction rests on the idea that the observable geometry could be interpreted as the projection of a structure defined on a fibered space of the form

$$E = M_4 \times \mathbb{H}.$$

In this framework, the total connection combines the relativistic spin connection, a Weyl-type conformal contribution, and an internal quaternionic connection. The projection of this connection onto space-time leads to a modified effective affine connection, capable of directly influencing geodesic trajectories.

We have shown that a minimal ansatz can relate the internal quaternionic connection to the effective affine connection, thereby introducing an additional geometric contribution to gravitational dynamics.

The underlying algebraic structure also suggests the existence of a more general framework based on the extended algebra

$$Cl(1,3) \otimes \mathbb{H},$$

in which local relativistic geometry and internal quaternionic structure coexist at the fundamental level.

Although the model presented remains exploratory, it highlights the geometric coherence of a hypercomplex extension of gravitation and opens several research directions concerning the dynamics of internal connections, the complete structure of the hypercomplex bundle, and the physical consequences of such a geometry.

These results suggest that hypercomplex structures could provide a natural framework for extending the geometric description of gravitation beyond the classical formalism of general relativity.