HYPERCOMPLEX GENERAL RELATIVITY — COMPLETE SYMPLECTIC REWRITING

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SYMPLECTIC REWRITING: CHATGPT

Abstract. This document is a *complete rewriting* of the manuscript "Hypercomplex General Relativity" within a *fundamentally symplectic* framework. Instead of taking the metric as the primitive variable and directly "quaternionizing" the coordinates, we reformulate the whole set of ingredients (coordinates, derivatives, transport, curvatures, Lagrangians, field equations) in terms of a fibered manifold $\mathcal{M} = M^4 \times \mathcal{F}$ equipped with a global symplectic 2-form Ω, a dynamical internal hypercomplex structure (I, J, K), and a Weyl connection (scale gauge) ϕ. The metric becomes an emergent structure through an almost-Kähler compatibility condition in the Weyl sense.

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1 Introduction

General relativity (GR) describes gravitation as a geometry of spacetime. Quantum theory, on the other hand, is naturally formulated in terms of state spaces and non-commuting operators. A long-standing intuition (Hamilton, Pauli, Weyl, Cartan, and others) is that *non-commutativity* and/or *internal degrees of freedom* may be geometrically encoded.

The historical HGR manuscript followed the most direct route: making the coordinates of the four-vector *hypercomplex* and then tracking the consequences for the connection symbols and curvature, while reintroducing a Weyl scale gauge. The present version keeps the same spirit (hypercomplexity + Weyl + new tensors), but *changes the choice of primitive variables*:

- the primary structure is no longer the metric, but a *symplectic form* Ω (a phase-space structure);
- hypercomplexity is no longer a manually imposed "quaternionic coordinate", but an *internal fiber* $\mathcal{F} \simeq \mathbb{H}$ carrying a triplet (I, J, K);
- the Weyl scale gauge is a field ϕ related to the *conformal* compatibility between Ω and the emergent metric.

This "symplectic" rewriting is particularly well suited to: (i) a Hamiltonian/ADM reading, (ii) canonical quantization, (iii) a unified interpretation of the additional transport terms (formerly $H^{j}_{\mu i}$ and $\Phi^{i}_{\mu j}$) as connections on the internal fiber.

2 Definition

2.1 Definition of the fibered spacetime. We consider a total manifold

$$\mathcal{M} \;=\; M^4 \times \mathcal{F}, \tag{2.1}$$

where M^4 is spacetime (an oriented differentiable manifold of dimension 4) and $\mathcal{F}$ is an internal fiber of real dimension 4, locally isomorphic to the quaternions $\mathbb{H}$. A point is denoted by (x^μ, q^a), where x^μ are coordinates on M^4 and q^a are internal coordinates on $\mathcal{F}$.

2.2 Definition of the symplectic structure. The fundamental datum is a global symplectic 2-form

$$\Omega \in \Omega^2(\mathcal{M}), \qquad \mathrm{d}\Omega = 0, \qquad \Omega \text{ non-degenerate.} \tag{2.2}$$

We allow a natural decomposition into "spacetime", "internal", and "mixed" parts:

$$\Omega \;=\; \Omega_{\text{ext}} \;+\; \Omega_{\text{int}} \;+\; \Omega_{\text{mix}}. \tag{2.3}$$

Typically,

$$\Omega_{\text{ext}} = \frac{1}{2}\,\Omega_{\mu\nu}(x)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{2.4}$$

$$\Omega_{\text{int}} = \sum_{a=1}^{3} \omega^a(x)\,\sigma_a, \tag{2.5}$$

$$\Omega_{\text{mix}} = A^a{}_\mu(x)\,\mathrm{d}x^\mu \wedge \sigma_a, \tag{2.6}$$

where $(\sigma_1, \sigma_2, \sigma_3)$ is a triplet of 2-forms on $\mathcal{F}$ and $A^a{}_\mu$ encodes the "base–fiber" coupling which historically played the role of the new transport tensors.

2.3 Definition of the quaternions (internal reminder). The fiber $\mathcal{F}$ is locally modeled on $\mathbb{H} = \mathrm{Span}_{\mathbb{R}}(1, h_1, h_2, h_3)$ with

$$h_1 h_2 h_3 = -1, \qquad h_i^2 = -1 \ (i \neq 0), \tag{2.7}$$

and non-commutativity is encoded by the commutator

$$[h_i, h_j] \equiv h_i h_j - h_j h_i = 2\,\varepsilon_{ij}{}^{k}\, h_k. \tag{2.8}$$

In the symplectic formulation, this non-commutativity does not live in the coordinates x^μ, but in the internal geometry (and therefore in the connections on the fiber).

2.4 Einstein notation. We use Einstein summation convention over repeated indices (upper/lower) whenever no ambiguity arises.

3 Postulates

First postulate (equivalence). The equivalence principle remains valid: locally, to first order, one may choose a frame in which inertia and gravitation coincide.

Second postulate (internal hypercomplexity). The "hypercomplex" degrees of freedom are carried by an internal fiber $\mathcal{F} \simeq \mathbb{H}$ and interact with M^4 through the mixed part Ω_{mix} of the symplectic structure. Non-commutativity is therefore geometrized by an internal connection.

Third postulate (Weyl). Length comparison is defined up to a scale gauge: there exists a Weyl 1-form ϕ such that

$$\nabla_\lambda g_{\mu\nu} = \phi_\lambda g_{\mu\nu}, \tag{3.1}$$

where g is the *emergent* metric (defined below) and ∇ is an affine connection.

4 Various notation: covariant derivatives, commutators, etc.

4.1 Covariant derivative and base/fiber separation. Instead of writing $\partial/\partial(x^{\mu i} h_i)$ as in the historical version, we introduce a connection on the fibration $\mathcal{M} \to M^4$. We denote by ∇_μ the covariant derivative along ∂_μ (base), and by ∇_a the one along the fiber. Schematically, any infinitesimal variation decomposes as

$$\mathrm{d} = \mathrm{d}_{\mathrm{ext}} + \mathrm{d}_{\mathrm{int}}, \tag{4.1}$$

and the historical coefficients $H_{\mu i}{}^{j}$ are reinterpreted as internal connection coefficients:

$$\nabla_\mu h_i = H_{\mu i}{}^{j}\, h_j. \tag{4.2}$$

4.2 Hypercomplex structure as a field. We allow (I, J, K) to depend on x:

$$\nabla_\mu I = \mathcal{A}_\mu \cdot I, \qquad \nabla_\mu J = \mathcal{A}_\mu \cdot J, \qquad \nabla_\mu K = \mathcal{A}_\mu \cdot K, \tag{4.3}$$

where $\mathcal{A}_\mu$ is an internal connection (generalizing $A^a{}_\mu$).

4.3 Commutator of derivatives. The core of the construction remains the operator

$$[\nabla_\mu, \nabla_\nu]\,, \tag{4.4}$$

which produces an effective curvature made of a Riemannian part (base), an internal part (fiber), and coupling terms.

5 Tensors

The goal of this section is to redefine curvature and transport in symplectic terms, while recovering the historical manuscript's "sum of contributions" structure: Riemann/Ricci + Weyl/EM + internal field + crossed couplings.

5.1 Transport of a fibered vector.

5.2 Detailed double-transport computation (analogue of the historical manuscript).

To keep a *step-by-step* correspondence with the source document, we spell out here the computation of $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)X$ on a fibered field.

We start from a field of the form

$$X = x^\alpha \, \psi^i \, u_i \, e_\alpha, \tag{5.1}$$

where x^α are base components, u_i is an internal basis, and ψ^i are internal coefficients (denoted φ^i in the historical notation).

The covariant derivative along ∂_μ reads

$$\nabla_\mu X = \partial_\mu \Big(x^\alpha \psi^i \Big) \, u_i e_\alpha + x^\alpha \psi^i \, \nabla_\mu(u_i) \, e_\alpha + x^\alpha \psi^i u_i \, \nabla_\mu(e_\alpha) \tag{5.2}$$

$$= \partial_\mu x^\alpha \, \psi^i u_i e_\alpha + x^\alpha \, \partial_\mu \psi^i \, u_i e_\alpha + x^\alpha \psi^i \, H_{\mu i}{}^j u_j \, e_\alpha + x^\alpha \psi^i u_i \, \Gamma_{\mu\alpha}{}^\beta e_\beta. \tag{5.3}$$

As in the original manuscript, it is convenient to collect the connection terms into an effective quantity

$$\gamma_\mu \equiv \Phi_{\mu j}{}^i \, \psi^j u_i + \psi^i \, H_{\mu i}{}^j u_j + \psi^i u_i \, \Gamma_{\mu\alpha}{}^\alpha, \tag{5.4}$$

where the object $\Phi_{\mu j}{}^i$ is interpreted here as a Weyl connection and/or a scale connection in the internal sector (which corresponds to the "Weyl/EM" reading of the historical version).

We then obtain the compact form

$$\nabla_\mu X = \partial_\mu x^\alpha \, \psi^i u_i e_\alpha + x^\alpha \, \gamma_\mu \, e_\alpha. \tag{5.5}$$

Applying ∇_ν again and antisymmetrizing, one isolates the connection-order terms:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)X = x^\alpha \Big(\partial_\mu \gamma_\nu - \partial_\nu \gamma_\mu + [\gamma_\mu, \gamma_\nu] \Big) e_\alpha + (\text{base terms acting on } x^\alpha). \tag{5.6}$$

The commutator $[\gamma_\mu, \gamma_\nu]$ contains:

- the commutator of internal connections, responsible for quaternionic curvature (analogue of the tensor T),
- the commutator of scale connections (analogue of the tensor F),
- mixed terms with Γ, yielding crossed couplings (analogue of the block C).

In particular, by explicitly separating the contributions, one recovers an expression of the form

$$[\nabla_\mu, \nabla_\nu]X = R^{(\Gamma)}{}_{\sigma\mu\nu}{}^\rho \, X^\sigma e_\rho + \Big(\partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu + [\Phi_\mu, \Phi_\nu] \Big) \cdot X + \Big(\partial_\mu H_\nu - \partial_\nu H_\mu + [H_\mu, H_\nu] \Big) \cdot X + C_{\mu\nu} \cdot X, \tag{5.7}$$

which is the direct symplectic translation of the final decomposition in the source manuscript (Riemannian tensor + "Weyl/EM" + "quaternionic field" + couplings).

Let X be a section (field) of the total tangent bundle $T\mathcal{M}$. In an adapted basis, one may write symbolically

$$X = X^\alpha \, e_\alpha + X^i \, u_i, \tag{5.8}$$

where e_α is a basis on M^4 and u_i is an internal basis on $\mathcal{F}$ (related to the h_i). Parallel transport is defined by a total connection ∇ such that

$$\nabla_\mu e_\alpha = \Gamma_{\mu\alpha}{}^{\beta}\, e_\beta, \qquad \nabla_\mu u_i = H_{\mu i}{}^{j}\, u_j, \qquad \nabla_\mu(\text{scale}) = \Phi_\mu\,(\text{scale}), \tag{5.9}$$

where the object Φ plays the role of the Weyl connection (scale gauge) in the historical notation.

5.3 Curvature: decomposition into four blocks. We define the total curvature by

$$[\nabla_\mu, \nabla_\nu]X \;=\; \mathcal{R}_{\mu\nu}\, X. \tag{5.10}$$

In an adapted basis, it can be decomposed into contributions:

$$[\nabla_\mu, \nabla_\nu]X = R^{(\Gamma)}{}_{\sigma\mu\nu}{}^{\rho}\, X^\sigma\, e_\rho + F_{\mu\nu}\cdot X + T_{\mu\nu}\cdot X + C_{\mu\nu}\cdot X. \tag{5.11}$$

5.3.1 Riemannian block (standard Einstein/Riemann sector).

$$R^{(\Gamma)}{}_{\sigma\mu\nu}{}^{\rho} = \partial_\mu\Gamma_{\nu\sigma}{}^{\rho} - \partial_\nu\Gamma_{\mu\sigma}{}^{\rho} + \Gamma_{\mu\lambda}{}^{\rho}\Gamma_{\nu\sigma}{}^{\lambda} - \Gamma_{\nu\lambda}{}^{\rho}\Gamma_{\mu\sigma}{}^{\lambda}. \tag{5.12}$$

Its contraction gives the Ricci tensor and then the usual Einstein tensor.

5.3.2 Weyl block (scale gauge). The Weyl connection is carried by a 1-form $\phi = \phi_\mu \mathrm{d}x^\mu$. Its "field strength" is the 2-form

$$F^{(\phi)} = \mathrm{d}\phi, \tag{5.13}$$

and one recovers a Maxwell-like structure (possibly non-Abelian if the scale space is extended).

5.3.3 Internal hypercomplex block. The internal connection $\mathcal{A}_\mu$ (arising from Ω_{mix}) has curvature

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu], \tag{5.14}$$

where $[\cdot,\cdot]$ is the quaternionic commutator (internal non-commutativity). This is the symplectic reinterpretation of the historical objects $H_{\mu i}{}^{j}$ and $T^m_{n\mu\nu}$.

5.3.4 Mixed terms (crossed couplings). The terms $C_{\mu\nu}$ collect the gravity–Weyl–internal mixtures. They arise geometrically from the non-triviality of the fibration and from the compatibility conditions imposed between Ω, (I, J, K), and g. They may be regarded as the "torsion/coupling" part of the total curvature (5.11).

5.4 Symplectic interpretation: the curvature form. In the symplectic version, it is natural to encode the dynamics with forms:

$$\mathbb{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}, \qquad F^{(\phi)} = \mathrm{d}\phi. \tag{5.15}$$

The structure of Ω also suggests the topological invariant

$$\int_{\mathcal{M}} \Omega\wedge\Omega, \tag{5.16}$$

which plays a fundamental role in the action (next section).

6 Possible Lagrangians for HGR

The historical manuscript proposed a total Lagrangian $L = L_{\mathrm{grav}} + L_\Phi + L_H + L_{\mathrm{coup}} + L_{\mathrm{mat}}$. We keep this decomposition, but reinterpret it as an action built from forms.

6.1 Gravitational term (emergent). Instead of postulating the Einstein–Hilbert action *from the outset*, we start from the minimal symplectic action

$$S_\Omega = \int_{\mathcal{M}} \Omega \wedge \Omega. \tag{6.1}$$

The metric (and therefore the Ricci scalar R) enters only *after* g has been defined by almost-Kähler compatibility.

6.2 Lagrangian of the Weyl field. We keep a Maxwell-type kinetic term for the scale gauge:

$$S_\phi = -\frac{1}{4} \int_{M^4} F^{(\phi)}_{\mu\nu} F^{(\phi)\mu\nu} \sqrt{-g}\, \mathrm{d}^4x, \qquad F^{(\phi)}_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu. \tag{6.2}$$

6.3 Lagrangian of the internal hypercomplex field. For the internal connection $\mathcal{A}$ (valued in the internal quaternionic algebra), we take

$$S_{\mathcal{A}} = -\frac{1}{4} \int_{M^4} \mathrm{Tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}) \sqrt{-g}\, \mathrm{d}^4x, \qquad \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{6.3}$$

6.4 Coupling terms. The "crossed" couplings of the historical text are naturally represented by mixed invariants built from Ω_{mix}, ϕ, and the effective curvature of the base. A prototype is

$$S_{\mathrm{coup}} = \int_{M^4} \left(\kappa\, \mathrm{Tr}(\mathcal{F} \wedge \star \mathcal{F}) + \eta\, F^{(\phi)} \wedge \star F^{(\phi)} \right), \tag{6.4}$$

where the Hodge operator is defined by the emergent metric.

6.5 Total action. In summary:

$$S = S_\Omega + S_\phi + S_{\mathcal{A}} + S_{\mathrm{coup}} + S_{\mathrm{mat}}. \tag{6.5}$$

7 Derivation of the Field Equations from the Lagrangian

7.1 Explicit variations of the fundamental fields. We consider an action of the form

$$S[\Omega, \phi, A, \Psi] \;=\; S_\Omega[\Omega] + S_\phi[\phi; g] + S_A[A; g] + S_{\mathrm{coup}}[\Omega, \phi, A; g] + S_{\mathrm{mat}}[g, \Psi], \tag{7.1}$$

where g is the emergent metric obtained from (Ω, J) and Ψ denotes matter. As a first step, we perform the variations *at fixed metric* (i.e. $\delta\star = 0$). The corrections induced by $\delta g(\delta\Omega)$ are discussed afterward.

7.1.1 Variation with respect to the symplectic 2-form Ω*.* For the minimal symplectic term we take

$$S_\Omega[\Omega] = \frac{\kappa}{2} \int_{\mathcal{M}} \Omega \wedge \Omega. \tag{7.2}$$

Then

$$\delta S_\Omega = \kappa \int_{\mathcal{M}} \delta\Omega \wedge \Omega, \tag{7.3}$$

and, up to sources, the corresponding Euler–Lagrange equation is

$$\Omega = \Omega_{\mathrm{source}}. \tag{7.4}$$

To obtain local dynamics (rather than a purely topological term), one adds a kinetic term of the type

$$S_{\Omega,\mathrm{dyn}} = \frac{\kappa_\Omega}{2} \int_{\mathcal{M}} \mathrm{d}\Omega \wedge \star\, \mathrm{d}\Omega, \tag{7.5}$$

which yields

$$\delta S_{\Omega,\mathrm{dyn}} = \kappa_\Omega \int_{\mathcal{M}} \mathrm{d}(\delta\Omega) \wedge \star\, \mathrm{d}\Omega = -\kappa_\Omega \int_{\mathcal{M}} \delta\Omega \wedge \mathrm{d} \star \mathrm{d}\Omega \tag{7.6}$$

(after integration by parts and neglecting boundary terms), hence

$$\mathrm{d} \star \mathrm{d}\Omega = J_\Omega, \tag{7.7}$$

where J_Ω collects the contributions from couplings and/or matter.

7.1.2 Variation with respect to the Weyl field ϕ. We define $F^{(\phi)} = \mathrm{d}\phi$ and

$$S_\phi[\phi] = -\frac{1}{2}\int_{M^4} F^{(\phi)} \wedge \star F^{(\phi)}. \tag{7.8}$$

Then

$$\delta S_\phi = -\int_{M^4} \mathrm{d}(\delta\phi) \wedge \star F^{(\phi)} = \int_{M^4} \delta\phi \wedge \mathrm{d} \star F^{(\phi)}, \tag{7.9}$$

and the field equation is

$$\mathrm{d} \star F^{(\phi)} = J_\phi. \tag{7.10}$$

7.1.3 Variation with respect to the internal connection A. We define the internal curvature $\mathcal{F} = \mathrm{d}A + A \wedge A$. With

$$S_A[A] = -\frac{1}{2}\int_{M^4} \mathrm{Tr}(\mathcal{F} \wedge \star\mathcal{F}), \tag{7.11}$$

one has $\delta\mathcal{F} = D(\delta A)$, where $D = \mathrm{d} + [A, \cdot]$ is the covariant derivative. Thus

$$\delta S_A = -\int_{M^4} \mathrm{Tr}\Big(D(\delta A) \wedge \star\mathcal{F}\Big) = \int_{M^4} \mathrm{Tr}\Big(\delta A \wedge D \star \mathcal{F}\Big), \tag{7.12}$$

which gives the internal Yang–Mills equation

$$D \star \mathcal{F} = J_A. \tag{7.13}$$

7.1.4 Remark on metric dependence (the $\delta\star$ terms). Since $\star$ depends on g, and g depends on Ω through $g(\cdot,\cdot) = \Omega(\cdot, J\cdot)$, the complete variation $\delta S/\delta\Omega$ contains additional contributions of the type $\delta(\star) \neq 0$. In a systematic treatment, these terms may be collected into the effective current J_Ω.

The variation of S gives:

- Maxwell-like equations for ϕ,
- internal Yang–Mills equations for $\mathcal{A}$,
- and, through the emergent metric, an effective gravitational equation of the Einstein + corrections type.

7.2 Equations for the Weyl field. The variation $\delta S/\delta\phi = 0$ gives

$$\nabla_\mu F^{(\phi)\mu}{}_\nu = J^{(\phi)}_\nu, \tag{7.14}$$

where $J^{(\phi)}$ comes from the couplings and matter.

7.3 Equations for the internal field. The variation $\delta S/\delta\mathcal{A} = 0$ gives an internal Yang–Mills equation:

$$\nabla_\mu \mathcal{F}^\mu{}_\nu + [\mathcal{A}_\mu, \mathcal{F}^\mu{}_\nu] = J^{(\mathcal{A})}_\nu. \tag{7.15}$$

7.4 Effective gravitational equations. Once the metric is defined, one can write a modified Einstein equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\, T_{\mu\nu} + \Theta_{\mu\nu}(\phi, \mathcal{A}, \Omega_{\text{mix}}), \tag{7.16}$$

where $\Theta_{\mu\nu}$ is the effective stress-energy tensor of the Weyl and internal sectors.

8 Stability of the Lagrangian and Ostrogradsky's theorem

Ostrogradsky's theorem generally rules out non-degenerate Lagrangians depending on higher-order derivatives, because they lead to a Hamiltonian that is unbounded from below (instabilities and "ghosts").

8.1 Why the symplectic formulation helps. The symplectic rewriting has two structural advantages:

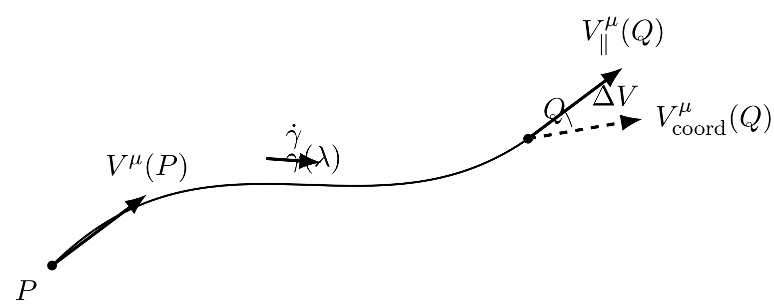
(1) The fundamental variable Ω is a *form*, and the minimal action $S_\Omega = \int \Omega \wedge \Omega$ is first-order/topological; dynamics arise through curvature terms of the type $\mathrm{Tr}(\mathcal{F} \wedge \star\mathcal{F})$ which remain *second order* in derivatives of the potentials.

(2) The internal fields $(\phi, \mathcal{A})$ appear with Maxwell/Yang–Mills type kinetic terms, known to be stable (at least at the classical level) and free from higher derivatives.

Thus, the theory naturally avoids Ostrogradsky pathologies while keeping the possibility of effective corrections in the gravitational sector through the emergent metric.

8.2 Remark on Weyl compatibility. The condition $\nabla g = \phi \otimes g$ does not require introducing higher-order derivatives: it simply modifies the connection and transport structure.

Appendix: the transport figure (TikZ)



$\nabla_{\dot\gamma} V^\mu = 0$
(parallel transport along γ)

In Weyl geometry: $\nabla_\alpha g_{\mu\nu} = -2\Phi_\alpha g_{\mu\nu}$,
transport may induce a scale factor (depending on the weight).

Figure 8.1. Schematic transport of a four-vector: $V^\mu(P)$ is transported along γ toward Q (parallel transport). It is compared with the vector defined by a coordinate identification.

9 FLRW cosmology in the symplectic reformulation

Convention choice. Throughout this section, the effective geometric contributions are expressed as *physical energy densities*, so that the Friedmann equation keeps its standard form $H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}(\rho_{\text{mat}} + \rho_{\text{eff}})$.

9.1 Emergent metric: minimal reminder. The emergent metric is defined by almost-Kähler compatibility:

$$g(X, Y) = \Omega(X, JY). \tag{9.1}$$

We then assume that the emergent metric (9.1) takes the FLRW form

$$ds^2 = -dt^2 + a(t)^2 \gamma_{ij} dx^i dx^j, \tag{9.2}$$

where γ_{ij} is the spatial metric of constant curvature $k \in \{-1, 0, +1\}$, and $H = \dot{a}/a$ is the expansion rate.

Homogeneity and isotropy require that *all scalar invariants* built from g, the Weyl gauge field ϕ, and the internal fields (generically denoted A / $\mathcal{A}$) depend only on cosmic time t.

9.2 Explicit FLRW ansatz for A and Ω (no isotropy ambiguity).
Key point: no strictly invariant spatial 1-form exists in FLRW.. There is no non-zero spatial 1-form invariant under the full isometry group of homogeneous and isotropic slices. Consequently, if one wants to preserve FLRW cosmology at the *background* level, one cannot take “A purely spatial and invariant” in the literal sense.
Minimal compatible choice (recommended in this manuscript). We impose on the background:

$$\phi = \phi_0(t)\, dt, \qquad A = A_0(t)\, dt, \qquad \mathcal{A}^a_i(t) = \psi(t)\, \delta^a_i, \qquad \mathcal{A}^a_0 = 0, \tag{9.3}$$

where $\mathcal{A}_\mu = \mathcal{A}^a_\mu\, h_a$ is the internal connection (triplet), and δ^a_i identifies the spatial index $i \in \{1, 2, 3\}$ with the internal index $a \in \{1, 2, 3\}$. This “triadic ansatz” (analogous to that of $SU(2)$ gauge fields in cosmology) is *isotropic on average* because a spatial rotation can be compensated by an internal rotation.[1]
FLRW coframe. We use the orthonormal basis

$$e^0 = dt, \qquad e^i = a(t)\, \tilde{e}^i, \tag{9.4}$$

where $\tilde{e}^i$ is a coframe on the constant-curvature 3-manifold (Σ_k, γ_{ij}).
Explicit background decomposition of Ω. We take a basis of internal 2-forms $(\sigma_1, \sigma_2, \sigma_3)$ on the fiber, normalized according to a fixed internal convention, and write the homogeneous/isotropic ansatz

$$\Omega_{\text{ext}} = 0, \qquad \Omega_{\text{int}} = \varpi(t) \sum_{a=1}^{3} \sigma_a, \qquad \Omega_{\text{mix}} = \chi(t) \sum_{i=1}^{3} e^i \wedge \sigma_i. \tag{9.5}$$

The choice $\Omega_{\text{ext}} = 0$ is the simplest one: any non-zero component Ω_{ext} tends to select a preferred direction, unless an additional structure is introduced or it is reparametrized in terms of the late-time sector θ already encoded in α.

[1] An internal triplet δ^a_i (or an equivalent structure) is allowed in homogeneous and isotropic cosmology because the spatial index i is compensated by an internal index a, so no preferred spatial direction is selected. This construction is standard in cosmological models with an internal $SU(2)$-type structure, for instance isotropic gauge fields, where isotropy is preserved by a diagonal space–internal identification.

Consequence: this is precisely the choice that makes the explicit part unambiguous. The equations (7.10) and (7.13) then reduce to ordinary differential equations for $(\phi_0, A_0, \psi, \varpi, \chi)$, and the identification of the scaling laws a^{-2} and a^{-4} becomes transparent (see §9.4).

The equations (7.10) and (7.13) reduce to ordinary differential equations for the compatible modes, while (7.7) induces an effective equation for $a(t)$ through the dependence of $\star$ on $g(a)$.

It is natural to write the cosmological evolution in the form of a modified Friedmann equation

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\,\rho_{\rm mat} + \rho_{\rm eff}(\Omega, \phi, A), \tag{9.6}$$

where $\rho_{\rm eff}$ collects the effective geometric contributions (Weyl, internal sector, and couplings). In the HGR framework, a late-time contribution of the θ type may be effectively parametrized by an a^{-2} law, which naturally fits into (9.6).

9.3 Bounce condition (prototype solution). A cosmological bounce corresponds to the existence of an instant t_b such that

$$H(t_b) = 0, \qquad \dot H(t_b) > 0, \qquad a(t_b) = a_{\min} > 0. \tag{9.7}$$

To exhibit a prototype solution, we parametrize the effective geometric contribution by a simple combination of power laws:

$$\rho_{\rm eff}(a) = \frac{\alpha}{a^2} - \frac{\beta}{a^4}, \qquad \alpha > 0,\ \beta > 0, \tag{9.8}$$

where the a^{-2} term represents a late-time sector (of θ type), while the a^{-4} term arises from an internal invariant associated with the symplectic/hypercomplex structure (effective contribution).

For $k = 0$ and neglecting ordinary matter near the bounce, (9.6) becomes

$$H^2 = \frac{8\pi G}{3}\left(\frac{\alpha}{a^2} - \frac{\beta}{a^4}\right) = \frac{8\pi G}{3}\frac{1}{a^4}\left(\alpha a^2 - \beta\right). \tag{9.9}$$

Thus $H^2 \ge 0$ imposes $a^2 \ge \beta/\alpha$, yielding a bounce at

$$a_{\min} = \sqrt{\frac{\beta}{\alpha}}. \tag{9.10}$$

The condition $\dot H(t_b) > 0$ is satisfied provided the effective contribution violates the standard equation of state near $a_{\min}$; here this is an emergent property of the geometric dynamics, rather than the introduction of a fundamental exotic fluid.

9.4 Explicit identification of α and β from the action (FLRW reduction). To remove any dimensional ambiguity, we rewrite (9.6) in the standard form

$$H^2 + \frac{k}{a^2} \;=\; \frac{8\pi G}{3}\Big(\rho_{\rm mat} + \rho_{\rm eff}\Big), \tag{9.11}$$

where $\rho_{\rm eff}$ is a *physical energy density.* We then parametrize

$$\rho_{\rm eff}(a) = \frac{\alpha}{a^2} - \frac{\beta}{a^4}, \tag{9.12}$$

with α and β physical constants (with dimensions of energy density).
Relation to the late-time component θ. In the late-time HGR version, one introduces an effective contribution of the form

$$\rho_\theta(a) = \alpha_W\,\frac{H_0^2}{a^2}. \tag{9.13}$$

Convention (normalization of ρ_θ). Two writings are common. (A) If ρ_θ is a physical energy density,

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left(\rho_{\rm mat} + \rho_\theta\right), \qquad \rho_\theta(a) = \alpha_W \frac{H_0^2}{a^2}, \tag{9.14}$$

then α in (9.12) is identified as

$$\alpha = \alpha_W\, \rho_{\rm crit,0} = \frac{3}{8\pi G}\alpha_W H_0^2. \tag{9.15}$$

(B) If, instead, ρ_θ is defined as a term already normalized in H^2,

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_{\rm mat} + \rho_\theta^{(H)}, \qquad \rho_\theta^{(H)} = \alpha_W \frac{H_0^2}{a^2}, \tag{9.16}$$

then the equivalent physical density is $\rho_\theta = \frac{3}{8\pi G}\rho_\theta^{(H)}$. In this manuscript, we adopt convention (A) (physical density).

To compare it with (9.12) in (9.11), one must express (9.13) as an energy density. This yields the direct identification

$$\boxed{\alpha \;=\; \frac{3}{8\pi G}\,\alpha_W\, H_0^2} \tag{9.17}$$

(up to normalization convention, i.e. depending on whether ρ_θ is defined as a physical density or as an already normalized term in H^2).

Origin of the a^{-4} term: internal sector (Yang–Mills type). The term $-\beta/a^4$ appears naturally whenever an internal invariant (connection curvature) contributes like a "radiative" fluid in an FLRW background. For instance, if the internal sector is described by

$$S_A = -\frac{1}{2\lambda^2}\int \mathrm{Tr}(\mathcal{F}\wedge \star\mathcal{F}), \tag{9.18}$$

then the effective energy associated with a homogeneous/isotropic mode (conserved comoving amplitude $\mathcal{Q}$) scales as

$$\rho_A(a) \;=\; \frac{C_A}{\lambda^2}\,\frac{\mathcal{Q}^2}{a^4}, \tag{9.19}$$

where C_A is a numerical factor (depending on the normalization of Tr and on the choice of isotropic ansatz), and $\mathcal{Q}$ is a comoving invariant (analogous to a "charge" or field amplitude).

In the bounce scenario, the internal contribution must enter with an *effective negative sign* in (9.12):

$$\boxed{\beta \;=\; \frac{C_A}{\lambda^2}\,\mathcal{Q}^2} \tag{9.20}$$

and the origin of the sign ("$-\beta/a^4$") is interpreted as an emergent property of the geometric dynamics (constraint/auxiliary behavior) of the hypercomplex sector, rather than as the introduction of a fundamental phantom fluid.

Role of κ (normalization of the symplectic sector). In this manuscript, κ denotes the normalization of the symplectic sector (written κ_Ω in (7.5) when local dynamics is added). At the cosmological level, κ sets the energy scale associated with homogeneous modes of Ω.

To make this point *explicit*, we start from the ansatz (9.5). At fixed internal normalization (i.e. $\int_{\mathcal{F}}\sigma_i\wedge\sigma_j \propto \delta_{ij}$), the square of the mixed component behaves like a radiation-type term:

$$\int_{\mathcal{F}} \Omega_{\rm mix}\wedge\star\,\Omega_{\rm mix} \;\propto\; \chi(t)^2\,\frac{1}{a^4}. \tag{9.21}$$

In other words, if the dynamics imposes a conserved comoving amplitude $\mathcal{Q}$ (for example through the equation of motion of an internal gauge sector), one may parametrize

$$\chi(t) = \frac{\mathcal{Q}}{a^2}, \tag{9.22}$$

and the effective energy inherited from the Ω sector (or, more generally, from the internal sector coupled to Ω) indeed takes the form

$$\rho_\Omega(a) = \frac{C_\Omega}{\kappa}\frac{\mathcal{Q}^2}{a^4}, \tag{9.23}$$

where C_Ω is a numerical factor depending on the normalization conventions for σ_i and on the precise definition of the action.

This dependence is compatible with (9.20): *at the effective level*, one may absorb C_Ω/κ into C_A/λ^2 (if the internal sector is Yang–Mills-like as in (9.18)), or distinguish the two contributions if a more microscopic expression is desired. In the present version, all radiative internal contributions are collected into β.

Finally, the a^{-2} contributions (parametrized by α) are primarily related here to the late-time sector θ through (9.17) (and therefore to α_W). A possible additional contribution of Ω to the a^{-2} term would depend on a richer ansatz for Ω_{ext}; it is not required for the minimal bounce and is therefore not activated in (9.5).

9.5 Effective pressure and equation-of-state parameter w_{eff} near the bounce. For an effective component scaling as $\rho \propto a^{-n}$, covariant conservation in FLRW implies

$$w = \frac{n}{3} - 1. \tag{9.24}$$

Thus, an a^{-2} term corresponds to $w_{(2)} = -1/3$, and an a^{-4} term corresponds to $w_{(4)} = +1/3$. Using (9.12), the effective pressure is therefore

$$p_{\text{eff}}(a) = -\frac{1}{3}\frac{\alpha}{a^2} + \frac{1}{3}\Big(-\frac{\beta}{a^4}\Big) = -\frac{\alpha}{3a^2} - \frac{\beta}{3a^4}. \tag{9.25}$$

The effective equation-of-state parameter is then

$$w_{\text{eff}}(a) = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = -\frac{\alpha a^2 + \beta}{3(\alpha a^2 - \beta)}. \tag{9.26}$$

Near the bounce, $a_{\min}^2 = \beta/\alpha$, the denominator tends to zero (with $\alpha a^2 - \beta \to 0^+$ just after the bounce), which leads to $w_{\text{eff}} \ll -1$ *effectively* near $a_{\min}$. In particular, the acceleration condition $w_{\text{eff}} < -1/3$ is satisfied in a range around the bounce, ensuring $\dot{H}(t_b) > 0$ in the effective description.

9.6 Order-of-magnitude estimate and falsifiability.

Toy numerical example and constraint on $T_{\max}$

We consider

$$\rho_{\text{eff}}(a) = \frac{\alpha}{a^2} - \frac{\beta}{a^4}, \qquad a_{\min} = \sqrt{\frac{\beta}{\alpha}}.$$

We set a *toy* example:

$$\alpha_W = 0.1, \qquad \alpha = \frac{3}{8\pi G}\alpha_W H_0^2 = \alpha_W\,\rho_{\text{crit},0},$$

where $\rho_{\mathrm{crit},0} = 3H_0^2/(8\pi G)$. For the internal contribution, we parametrize

$$\beta = \frac{C_A}{\lambda^2}\mathcal{Q}^2, \qquad C_A \sim \mathcal{O}(1),$$

and choose illustrative values $(\lambda, \mathcal{Q})$. The bounce occurs at

$$a_{\min} = \sqrt{\beta/\alpha}.$$

A simple estimate of the maximum temperature reached at the bounce is

$$T_{\max} \simeq \frac{T_0}{a_{\min}}, \qquad T_0 \simeq 2.725\,\mathrm{K}.$$

Falsifiability consists in constraining $(\alpha_W, \lambda, \mathcal{Q})$ so that $T_{\max}$ remains compatible (i) with the upper bound set by the Planck scale and (ii) with the requirement of standard cosmology before Big Bang nucleosynthesis (BBN), while reproducing observables (CMB/BAO/SNe) through CLASS+Cobaya.

The closed expression for the minimum

$$a_{\min} = \sqrt{\frac{\beta}{\alpha}} \tag{9.27}$$

makes the scenario falsifiable once α and β are related to measurable physical parameters.
Estimate of α from α_W. Combining (9.17) with $H_0 \simeq 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ gives a characteristic density of order

$$\alpha \sim \alpha_W\,\rho_{\mathrm{crit},0}, \qquad \rho_{\mathrm{crit},0} = \frac{3H_0^2}{8\pi G}. \tag{9.28}$$

In other words, α_W directly calibrates the effective density fraction associated with the late-time component (at $a = 1$).
Estimate of β from the internal sector. The identification (9.20) shows that β depends on a comoving amplitude $\mathcal{Q}$ and on the coupling λ. In a field-theory scenario, $\mathcal{Q}$ is fixed by initial conditions (or by effective quantization), and λ is constrained by the absence of observable deviations in local/perturbative tests. The ratio β/α is therefore testable through (9.27).
Physical interpretation of $a_{\min}$. Once α and β are fixed, $a_{\min}$ determines the minimum scale reached by the Universe in the bounce scenario. In physical units, this sets the maximum redshift and the maximum temperature reached before the bounce, which may be confronted with: (i) BBN constraints, (ii) constraints on the primordial spectrum, and (iii) CMB/LSS constraints through the numerical implementation (modified CLASS).
Observational comparison (roadmap). Falsifiability is then formulated in three steps:

(1) relate α to α_W through (9.17) and constrain α_W with CMB+BAO+SNe,
(2) relate β to $(\lambda, \mathcal{Q})$ through (9.20) while controlling the impact on perturbations,
(3) predict $a_{\min}$ through (9.27) and verify compatibility with BBN and the absence of observable anomalies.

10 Quantization perspectives

The HGR framework is currently formulated as a classical geometric theory. Nevertheless, its internal structure naturally admits a well-defined quantization program. In particular, the symplectic formulation introduced in this work equips the internal sector with a closed and non-degenerate two-form ω, which defines a Poisson algebra of observables on the internal fiber.

This structure provides a natural starting point for geometric quantization or deformation quantization, in which the classical Poisson brackets associated with ω are promoted to quantum commutators. In this interpretation, the internal non-commutativity appearing in HGR is not postulated as fundamental; it emerges as the quantized limit of an underlying symplectic geometry.

At the present stage, the gravitational sector is treated classically at the background level, while quantum effects are introduced coherently through perturbations around cosmological backgrounds. This strategy places the HGR model within an effective field theory framework, in which quantum fluctuations of the metric and internal fields may be computed and confronted with cosmological observations.

A complete non-perturbative quantization of the theory, including all gravitational degrees of freedom, remains an open problem and is left for future work. However, the present formulation already defines a controlled and testable semi-quantum regime, suited to phenomenological and cosmological applications.

Appendix A Dictionary between the historical HGR formulation and the symplectic reformulation

This appendix establishes an explicit correspondence between the objects of Hypercomplex General Relativity (historical formulation) and those of the modern symplectic reformulation.

Historical HGR (hypercomplex)	**Symplectic formulation**
Hypercomplex coordinates $x^{\mu i} h_i$	Coordinates (x^μ, q^a) on $\mathcal{M} = M^4 \times \mathcal{F}$
Quaternionic basis h_i	Basis of the internal fiber $\mathcal{F} \simeq \mathbb{H}$
Non-commutativity $[h_i, h_j] \neq 0$	Curvature of the internal connection $\mathcal{A}$
Four-vector $X = x^\alpha \varphi^i h_i e_\alpha$	Section of the total tangent bundle $T\mathcal{M}$
$\partial_\mu h_i = H^{\ j}_{\mu i} h_j$	$\nabla_\mu u_i = H^{\ j}_{\mu i} u_j$ (internal connection)
$\partial_\mu \varphi^i = \Phi^{\ i}_{\mu j} \varphi^j$	Weyl connection / scale gauge ϕ_μ
Christoffel symbols $\Gamma^{\ \beta}_{\mu\alpha}$	Affine connection on M^4
Transport $\nabla_\mu X$	Covariant derivative on the fibration $\mathcal{M} \to M^4$
Commutator $[\nabla_\mu, \nabla_\nu]$	Total curvature of the fibered connection
Tensor $R^{(\Gamma)}{}_{\sigma\mu\nu}{}^{\rho}$	Effective Riemannian curvature
Tensor $F^i{}_{j\mu\nu}(\Phi)$	Weyl field $F^{(\phi)} = \mathrm{d}\phi$
Tensor $T^m_{n\mu\nu}(H)$	Internal curvature $\mathcal{F}_{\mu\nu}$ (quaternionic Yang–Mills type)
Crossed terms $C_{\mu\nu}$	Base–fiber couplings (symplectic mixing)
Fundamental metric $g_{\mu\nu}$	Emergent metric $g(X,Y) = \Omega(X, JY)$
Postulated Weyl scale gauge	Conformal compatibility $\nabla g = \phi \otimes g$
Lagrangian $R + F^2 + T^2 + \cdots$	Symplectic action $\int \Omega \wedge \Omega + \int \mathcal{F} \wedge \star\mathcal{F}$

Interpretation. This correspondence shows that the symplectic reformulation does not alter the physical content of HGR, but changes its fundamental geometric structure.

Quantization

Appendix B Canonical quantization of the internal HGR sector

B.1 Internal action. We consider the internal Yang–Mills sector:

$$S_A = \int d^4x\sqrt{-g}\,\frac{1}{4g_A^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \tag{B.1}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{B.2}$$

B.2 3+1 decomposition. The canonical variables are A_i^a and

$$E_a^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i^a} = \frac{\sqrt{h}}{g_A^2}F_a^{0i}. \tag{B.3}$$

B.3 Hamiltonian. The Hamiltonian density is written as:

$$\mathcal{H}_A = \frac{g_A^2}{2\sqrt{h}}E_a^iE_i^a + \frac{\sqrt{h}}{4g_A^2}F_{ij}^aF_a^{ij} + A_0^a\mathcal{G}_a, \tag{B.4}$$

where the Gauss constraint is

$$\mathcal{G}_a = D_iE_a^i. \tag{B.5}$$

Closure of the constraints (internal sector). We introduce the smeared constraint

$$G[\lambda] \equiv \int_\Sigma d^3x\,\lambda^a(x)\,G_a(x), \qquad G_a = D_iE^i{}_a.$$

Using the canonical brackets, one verifies that $G[\lambda]$ generates the internal gauge symmetry:

$$\delta_\lambda A_i^a = \{A_i^a, G[\lambda]\} = -D_i\lambda^a, \qquad \delta_\lambda E^i{}_a = \{E^i{}_a, G[\lambda]\} = f_{ab}{}^c\lambda^bE^i{}_c.$$

It follows that the Gauss algebra closes:

$$\{G[\lambda], G[\mu]\} = G[[\lambda,\mu]], \qquad [\lambda,\mu]^c \equiv f_{ab}{}^c\lambda^a\mu^b,$$

and that the internal Hamiltonian, built from gauge invariants (E^2, F^2), is gauge invariant:

$$\{G[\lambda], H_A\} \approx 0,$$

where $\approx$ denotes weak equality, i.e. equality on the constraint surface.

B.4 Quantization. The Poisson brackets become:

$$[A_i^a(x), E_b^j(y)] = i\hbar\delta_i^j\delta_b^a\delta^{(3)}(x-y). \tag{B.6}$$

The physical states satisfy:

$$\hat{\mathcal{G}}_a(x)\,|\Psi\rangle = 0. \tag{B.7}$$

Appendix C Construction of an internal observable X and quantum promotion

C.1 Gauge invariants and 3+1 decomposition. We consider an internal connection $A_\mu = A^a_\mu T_a$ and its curvature

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a{}_{bc} A^b_\mu A^c_\nu. \tag{C.1}$$

The usual gauge invariants are

$$\mathcal{I}_1 = \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \qquad \mathcal{I}_2 = \mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}), \tag{C.2}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

On a spatial slice Σ, we define the internal "electric" and "magnetic" fields:

$$E^a_i \equiv F^a_{0i}, \qquad B^{i\,a} \equiv \frac{1}{2}\varepsilon^{ijk}F^a_{jk}. \tag{C.3}$$

We introduce the gauge-invariant scalars

$$\omega^2 \equiv E^a_i E^i_a, \qquad J^2 \equiv B^a_i B^i_a. \tag{C.4}$$

C.2 Definition of the internal observable X. We define the HGR-compatible internal observable:

$$\boxed{X(x) \equiv \alpha\Big(E^a_i E^i_a + B^a_i B^i_a\Big)^p.} \tag{C.5}$$

This quantity is local, scalar, and invariant under the internal gauge symmetry.

C.3 Quantum promotion. In the "connection" representation, we set

$$\hat{E}^i_a(x) = -\,i\hbar\frac{\delta}{\delta A^a_i(x)}, \qquad \hat{B}^i_a(x) = \frac{1}{2}\varepsilon^{ijk}\hat{F}^a_{jk}[A](x), \tag{C.6}$$

and define the operator

$$\boxed{\hat{X}(x) = \alpha\Big(\hat{E}^a_i \hat{E}^i_a + \hat{B}^a_i \hat{B}^i_a\Big)^p.} \tag{C.7}$$

An ordering prescription, e.g. Weyl symmetrization, may be adopted in order to define $\hat{X}^2$ unambiguously.

Since $\hat{E}$ and $\hat{B}(\hat{A})$ do not commute, the definition of $\hat{X}^2$ requires an ordering prescription. By default, we adopt Weyl ordering (symmetrization), which guarantees Hermiticity and avoids bias between electric and magnetic contributions, while reproducing the classical limit as $\hbar \to 0$.

C.4 Effective Weyl threshold. We promote the triggering threshold to a state-dependent quantity:

$$\boxed{W^{\mathrm{eff}}_\star(x) \equiv \kappa_X \langle\Psi|\hat{X}^2(x)|\Psi\rangle.} \tag{C.8}$$

The triggering condition for a local bounce can then be expressed as

$$W(x) \geq W^{\mathrm{eff}}_\star(x). \tag{C.9}$$

Bounce condition (minimal formulation). In the domain-based approach, HGR corrections are locally activated when the curvature invariant W crosses an effective threshold $W_\star^{\rm eff}$. In the quantized version of the internal sector, this threshold becomes state-dependent:

$$W_\star^{\rm eff} \equiv \kappa_X \, \langle \hat{X}^2 \rangle,$$

where $\hat{X}$ is constructed from internal invariants and may be effectively saturated, thereby ensuring a bounded response.

A *bounce* then corresponds, at the effective level, to the existence of an instant t_b such that: (i) $H(t_b) = 0$ (halt of contraction), and (ii) $\dot{H}(t_b) > 0$ (return to expansion), in a region where the activation condition is satisfied, i.e. $W(t_b) \gtrsim W_\star^{\rm eff}$. In other words, the minimal physical condition for a local bounce is the activation of the HGR term with an effective sign leading to $\dot{H}(t_b) > 0$. The saturation of X ensures that $W_\star^{\rm eff}$ remains finite, preventing uncontrolled activation in the ultra-curved regime.

Effective Raychaudhuri equation and sign criterion. At the effective level (classical metric, quantized internal sector), one may write the evolution equation for H in the minimal form

$$\dot{H} = \dot{H}_{\rm GR} + \Delta_{\rm HGR}(W;\, W_\star^{\rm eff}),$$

where $\dot{H}_{\rm GR}$ collects the standard content (matter + Λ), while $\Delta_{\rm HGR}$ is a correction that activates locally when $W \gtrsim W_\star^{\rm eff}$. In a threshold/domain formulation, this activation can be modeled by an interpolation function $\mathcal{A}(W/W_\star^{\rm eff})$ which vanishes below threshold and approaches unity above threshold, so that

$$\Delta_{\rm HGR}(W;\, W_\star^{\rm eff}) \equiv \mathcal{A}\!\left(\frac{W}{W_\star^{\rm eff}}\right) \Xi_{\rm HGR},$$

where $\Xi_{\rm HGR}$ encodes the *effective sign* and amplitude of the correction.

At the bounce time t_b, one has $H(t_b) = 0$, and the dynamical condition is

$$\dot{H}(t_b) = \dot{H}_{\rm GR}(t_b) + \mathcal{A}\!\left(\frac{W(t_b)}{W_\star^{\rm eff}}\right) \Xi_{\rm HGR} \;>\; 0.$$

Thus, in an activated domain ($W(t_b) \gtrsim W_\star^{\rm eff}$), a bounce occurs if the HGR contribution locally dominates the GR focusing tendency, i.e. if $\Xi_{\rm HGR}$ is sufficiently positive. Internal saturation enters only through $W_\star^{\rm eff} = \kappa_X \langle \hat{X}^2 \rangle$ and guarantees that activation remains controlled in the ultra-curved regime.

Appendix D Ordering prescription and semi-classical approximation of $\langle \hat{X}^2 \rangle$

We recall the definition

(D.1) $$X(x) = \alpha\, Q(x)^p, \qquad Q(x) \equiv E^a_i E^i_a + B^a_i B^i_a,$$

hence

(D.2) $$X^2 = \alpha^2 Q^{2p}.$$

After quantization, $\hat{E}$ and $\hat{B}(A)$ do not commute, and an ordering prescription is required in order to define $\hat{X}^2 = \alpha^2\, \widehat{Q^{2p}}$.

D.1 Ordering prescription (minimal choice). By default, we adopt Weyl ordering (symmetrization) for $\widehat{Q^{2p}}$, which avoids any bias between electric and magnetic contributions.

D.2 Semi-classical approximation. In a semi-classical state $|\Psi\rangle$ such that

$$\langle \hat{Q} \rangle = Q_{\rm cl}, \qquad (\Delta Q)^2 \equiv \langle \hat{Q}^2 \rangle - \langle \hat{Q} \rangle^2, \qquad \Delta Q \ll Q_{\rm cl}, \tag{D.3}$$

one obtains, using a leading-order cumulant expansion,

$$\boxed{\langle \hat{X}^2 \rangle = \alpha^2 \, \langle \widehat{Q^{2p}} \rangle \;\approx\; \alpha^2 \, Q_{\rm cl}^{2p} \left[1 + \frac{(2p)(2p-1)}{2} \frac{(\Delta Q)^2}{Q_{\rm cl}^2} \right]}. \tag{D.4}$$

Thus,

$$\boxed{\langle \hat{X}^2 \rangle \approx X_{\rm cl}^2 + \Delta X^2} \tag{D.5}$$

with

$$X_{\rm cl}^2 = \alpha^2 Q_{\rm cl}^{2p}, \qquad \Delta X^2 \simeq \alpha^2 Q_{\rm cl}^{2p} \, \frac{(2p)(2p-1)}{2} \frac{(\Delta Q)^2}{Q_{\rm cl}^2}. \tag{D.6}$$

D.3 Effective threshold and statistical interpretation. The Weyl threshold becomes state-dependent:

$$W_\star^{\rm eff}(x) = \kappa_X \, \langle \hat{X}^2(x) \rangle. \tag{D.7}$$

Defining the equivalent threshold in terms of Q,

$$Q_\star \equiv \left(\frac{W}{\kappa_X \alpha^2} \right)^{\frac{1}{2p}}, \tag{D.8}$$

the triggering probability in a "domain" approach is written as

$$P_{\rm bounce} = \mathbb{P}(Q \geq Q_\star), \tag{D.9}$$

and, if $\ln Q$ is approximately Gaussian,

$$P_{\rm bounce} \simeq \frac{1}{2} \, \mathrm{erfc}\left(\frac{\ln Q_\star - \mu_Q}{\sqrt{2}\, \sigma_Q} \right). \tag{D.10}$$

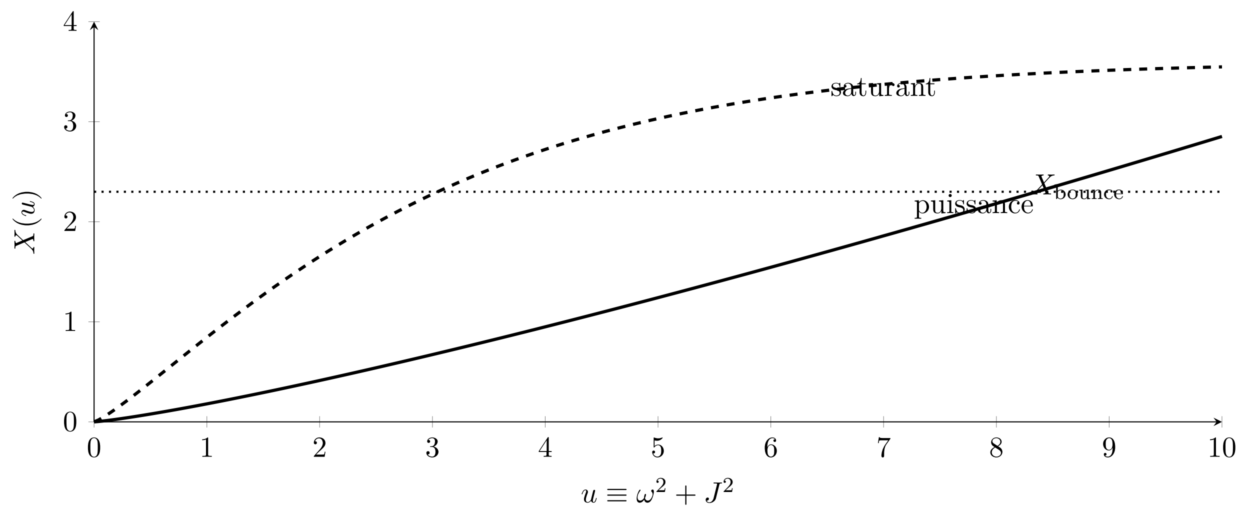


FIGURE D.1. Saturation interne (schéma). $u = \omega^2 + J^2$. La loi puissance $X = \alpha u^p$ diverge, tandis qu'une loi saturante $X = X_{\max}\left(1 - e^{-(u/\Lambda^2)^p}\right)$ tend vers $X_{\max}$. La ligne pointillée indique un seuil $X_{\rm bounce}$ (rebond).

Appendix E Dark-matter-dominated galaxies and hypercomplex halos in HGR

E.1 Observational context. Some recently observed galaxies exhibit an extremely low baryonic fraction. These objects are often referred to as:

- *dark galaxies*,
- *ultra-diffuse galaxies* (UDGs),
- dark-matter-dominated halos.

The observational evidence is essentially gravitational:

- globular clusters orbiting around a gravitational center,
- measured velocity dispersion,
- sometimes weak gravitational lensing,
- dynamical estimates of the halo mass.

These observations generally imply

$$M_{\mathrm{halo}} \gg M_{\mathrm{baryon}} \tag{E.1}$$

In some extreme systems:

$$\frac{M_{\mathrm{baryon}}}{M_{\mathrm{halo}}} \sim 10^{-3} - 10^{-4} \tag{E.2}$$

whereas the expected cosmological baryon fraction is

$$\frac{M_{\mathrm{baryon}}}{M_{\mathrm{halo}}} \approx 0.15. \tag{E.3}$$

These estimates come from dynamical analyses (velocity dispersion, globular-cluster dynamics) and not from a direct measurement of the invisible mass.

E.2 Standard interpretation in the ΛCDM model. In the standard cosmological framework, galaxy formation follows the hierarchical scenario

$$\text{dark-matter halo} \to \text{capture of baryonic gas} \to \text{star formation.} \tag{E.4}$$

In extremely diffuse galaxies, this baryonic process appears to have failed.

Three main mechanisms are usually invoked.

Violent stellar feedback. The first generations of stars can expel gas through:

- stellar winds
- supernova explosions
- intense ultraviolet radiation.

Gravitational stripping. In a galaxy-cluster environment:

$$\text{ram pressure} + \text{tidal forces} \tag{E.5}$$

can strip baryonic gas from the halo.

Suppression by reionization. During the epoch of cosmic reionization, heated gas can remain too energetic to be captured by some halos.

E.3 Conceptual limits. These extreme objects are often interpreted as an argument in favor of a real material component (dark matter), rather than a universal modification of gravity.

However, the observed variability of baryonic fractions also suggests that gravitational structure can exist independently of visible matter.

This opens the door to more fundamental geometric interpretations.

E.4 Framework of Hypercomplex General Relativity. In Hypercomplex General Relativity (HGR), effective gravity may contain a contribution from a hypercomplex field.

The effective metric may be written as

$$g_{\mu\nu} = g_{\mu\nu}^{(R)} + g_{\mu\nu}^{(\mathbb{H})} \tag{E.6}$$

where $g_{\mu\nu}^{(\mathbb{H})}$ represents the hypercomplex contribution.

The effective energy–momentum tensor then becomes

$$T_{\mu\nu}^{\text{eff}} = T_{\mu\nu}^{\text{baryon}} + T_{\mu\nu}^{\mathbb{H}}. \tag{E.7}$$

In some regimes:

$$T_{\mu\nu}^{\mathbb{H}} \gg T_{\mu\nu}^{\text{baryon}}. \tag{E.8}$$

A gravitational halo can therefore exist even in the almost complete absence of baryonic matter.

E.5 Hypothesis: hypercomplex halos. Hypothesis H1.

There may exist astrophysical domains in which the hypercomplex energy density dominates the baryonic contribution:

$$\rho_{\mathbb{H}} \gg \rho_{\text{baryon}}. \tag{E.9}$$

In that case, the observable gravitational structure would correspond to a condensate of the hypercomplex field.

E.6 Possible structure. A galaxy dominated by the hypercomplex field could follow the scheme:

(E.10)

condensation of the hypercomplex field → gravitational well → partial capture of baryons.

If baryonic capture is inefficient, the object appears almost invisible.

E.7 Failure of baryonic coupling. It is also possible that the coupling

$$\mathbb{H} \leftrightarrow T_{\mu\nu}^{\text{baryon}} \tag{E.11}$$

is weak in some domains.

The gravitational structure can then form without any significant accumulation of ordinary matter.

E.8 Topological domains. In some geometric extensions, these structures could correspond to:

- topological defects,
- gravitational solitons,
- Weyl or torsion domains.

These structures produce a metric perturbation

$$\Delta g_{\mu\nu} \tag{E.12}$$

generating effective gravity similar to that of a dark-matter halo.

E.9 Observational predictions. If this interpretation is correct, several observable signatures could appear:

(1) atypical dynamical dispersion,
(2) anomalous mass–globular-cluster relation,
(3) mass profile incompatible with a standard NFW halo,
(4) modified gravitational-lensing signatures.

E.10 Relevance for HGR. These galaxies are particularly interesting astrophysical laboratories because they minimize baryonic effects and isolate the pure gravitational structure.

They therefore constitute privileged systems for testing geometric extensions of general relativity.

E.11 Conceptual analogy. A simple analogy is to compare a galaxy to a Christmas tree:

- in the ΛCDM model, the tree corresponds to the dark-matter halo and the garlands to baryonic matter;
- in MOND, modified gravity affects the apparent brightness of the garlands;
- in HGR, the very structure of the tree could arise from a hypercomplex geometry of the gravitational field.

Appendix F Galactic baryonic flux and ejection by stellar feedback

F.1 Motivation. Star-formation processes produce feedback mechanisms (*stellar feedback*) capable of expelling a significant fraction of the baryonic gas from a galaxy.

The main mechanisms are:

- stellar winds,
- supernova explosions,
- intense ultraviolet radiation.

These mechanisms can generate galactic winds capable of transporting gas beyond the galactic disk.

The fundamental physical question is then:

How much baryonic matter can be expelled, and out to what distance?

F.2 Gravitational binding energy. For a galaxy with total mass M and characteristic radius R, the gravitational binding energy is approximately

$$E_{\text{bind}} \sim \frac{GM^2}{R}. \tag{F.1}$$

For a typical galaxy:

$$M \sim 10^{11} M_{\odot}, \quad R \sim 50 \text{ kpc}.$$

This gives the order of magnitude

$$E_{\text{bind}} \sim 10^{59} \text{ J}. \tag{F.2}$$

The energy released by a supernova is typically

$$E_{\text{SN}} \sim 10^{44} \text{ J}. \tag{F.3}$$

The number of supernovae required to expel the gas completely would therefore be

$$N_{\text{SN}} \sim \frac{E_{\text{bind}}}{E_{\text{SN}}} \sim 10^{15}. \tag{F.4}$$

This shows that complete expulsion of baryonic matter is difficult for massive galaxies.

F.3 Ejection velocity. Galactic winds typically have velocities

$$v_{\text{wind}} \sim 300 - 1000 \text{ km s}^{-1}. \tag{F.5}$$

The maximum distance reached by the gas can be estimated by comparing this velocity with the escape velocity of the halo.

For a typical galactic halo:

$$v_{\text{esc}} \sim 500 \text{ km s}^{-1}. \tag{F.6}$$

Thus, part of the expelled gas remains gravitationally bound to the halo.

F.4 Circumgalactic medium. The expelled gas generally forms a diffuse halo called the

Circumgalactic Medium (CGM)

characterized by:

$$T \sim 10^5 - 10^6 \text{ K}, \tag{F.7}$$

$$R \sim 100 - 500 \text{ kpc}, \tag{F.8}$$

$$M_{\text{gas}} \sim 10^9 - 10^{10} M_\odot. \tag{F.9}$$

This gas may then fall back toward the galaxy in a cycle called a

galactic fountain.

F.5 Ejection into the intergalactic medium. In some extreme cases (dwarf galaxies or gravitational interactions), a fraction of the gas can be expelled into the intergalactic medium.

The gas is then distributed in the

Warm-Hot Intergalactic Medium (WHIM)

characterized by:

$$T \sim 10^5 - 10^7 \text{ K}. \tag{F.10}$$

This medium could contain a large fraction of the missing baryons of the universe.

F.6 Baryonic flux. The mass flux ejected by a galactic wind may be estimated as

$$\dot{M}_{\text{out}} \sim \eta \dot{M}_\star, \tag{F.11}$$

where

- $\dot{M}_\star$ is the star-formation rate,
- η is the wind mass-loading factor.

Observations typically give

$$\eta \sim 1 - 10 \tag{F.12}$$

for strongly star-forming galaxies.

F.7 Limits of expulsion. To reach a distance of several megaparsecs, the ejection velocity would need to be

$$v \gtrsim 3000 \text{ km s}^{-1}. \tag{F.13}$$

Such velocities are generally not reached by standard galactic winds.

Thus, most of the expelled gas remains confined in the circumgalactic halo or in the immediate intergalactic neighborhood.

F.8 Possible interpretation in HGR. Within the framework of Hypercomplex General Relativity (HGR), another interpretation may be considered.

Effective gravity may contain a contribution from the hypercomplex field:

$$T_{\mu\nu}^{\text{eff}} = T_{\mu\nu}^{\text{baryon}} + T_{\mu\nu}^{H}. \tag{F.14}$$

In some regimes:

$$T_{\mu\nu}^{H} \gg T_{\mu\nu}^{\text{baryon}}. \tag{F.15}$$

In that case, a gravitational halo may exist even in the presence of a very small amount of baryonic matter.

Such structures could correspond to hypercomplex halos in which ordinary matter was never efficiently captured or was expelled during the earliest phases of galactic evolution.

F.9 Observational consequences. Such systems could appear as:

- ultra-diffuse galaxies,
- gravitational halos detected through globular-cluster dynamics,
- structures dominated by an invisible gravitational component.

These objects are therefore interesting astrophysical laboratories for testing galaxy formation models and geometric extensions of gravity.

Appendix G Hypercomplex halos and galactic rotation curves

G.1 Motivation. A major observational property of spiral galaxies is the presence of approximately flat rotation curves at large distances from the center. In the standard framework, this property is generally interpreted as the signature of a dark-matter halo.

Within the framework of Hypercomplex General Relativity (HGR), one may consider that part of this effective gravity arises from a hypercomplex geometric contribution, without necessarily postulating a new particle component.

G.2 Hypothesis of a spherical hypercomplex halo. Consider a static, spherically symmetric effective halo described by an effective hypercomplex energy density $\rho_H(r)$.

We assume that, at large distances, the density follows the asymptotic law

$$\rho_H(r) = \frac{\rho_0 r_0^2}{r^2}, \tag{G.1}$$

where ρ_0 and r_0 are characteristic constants of the halo.

This law is especially interesting because it naturally leads to a linear growth of the enclosed mass.

G.3 Interior effective mass. The effective mass contained within a radius r is

$$M_H(r) = 4\pi \int_0^r \rho_H(r')\, r'^2\, dr'. \tag{G.2}$$

Using (G.1), one obtains

$$M_H(r) = 4\pi\rho_0 r_0^2 \int_0^r dr' = 4\pi\rho_0 r_0^2\, r. \tag{G.3}$$

Thus,

$$M_H(r) \propto r. \tag{G.4}$$

G.4 Rotation curve. In the Newtonian approximation, the circular velocity satisfies

$$v_c^2(r) = \frac{GM(r)}{r}. \tag{G.5}$$

If the dominant contribution at large distances is $M_H(r)$ given by (G.3), then

$$v_c^2(r) = \frac{G}{r}\left(4\pi\rho_0 r_0^2 r\right) = 4\pi G\rho_0 r_0^2. \tag{G.6}$$

Therefore

$$v_c(r) \approx \text{constant}. \tag{G.7}$$

A flat rotation curve is thus recovered naturally.

G.5 HGR interpretation. In HGR, this effective density may be interpreted as arising from the hypercomplex sector of the energy–momentum tensor:

$$T^{\text{eff}}_{\mu\nu} = T^{\text{baryon}}_{\mu\nu} + T^{H}_{\mu\nu}. \tag{G.8}$$

In the external-halo regime, one may assume that

$$T^{H}_{\mu\nu} \gg T^{\text{baryon}}_{\mu\nu}, \tag{G.9}$$

so that the orbital dynamics is dominated by the hypercomplex contribution.

The profile $\rho_H(r) \propto r^{-2}$ may then be viewed as a stationary effective solution of the internal geometric sector.

G.6 Possible origin of the $1/r^2$ profile. Such a profile may emerge if the hypercomplex field forms a quasi-static self-gravitating halo whose energy is distributed according to a scaling law.

For example, if a scalar invariant of the internal field $X(r)$ satisfies asymptotically

$$X(r) \propto \frac{1}{r}, \tag{G.10}$$

and if the effective density is quadratic in this invariant,

$$\rho_H(r) \propto X(r)^2, \tag{G.11}$$

then one immediately obtains

$$\rho_H(r) \propto \frac{1}{r^2}. \tag{G.12}$$

This possibility is consistent with a reading in which the internal HGR sector acts as a geometric halo structure rather than as ordinary particle matter.

G.7 Minimal effective model. The total gravitational potential may be parametrized as

$$\Phi(r) = \Phi_b(r) + \Phi_H(r), \tag{G.13}$$

where Φ_b is the baryonic contribution and Φ_H the hypercomplex contribution.

The hypercomplex contribution then satisfies

$$\nabla^2 \Phi_H = 4\pi G \rho_H(r). \tag{G.14}$$

With $\rho_H(r) = \rho_0 r_0^2/r^2$, one obtains a radial field

$$\frac{d\Phi_H}{dr} = \frac{GM_H(r)}{r^2} = \frac{4\pi G \rho_0 r_0^2}{r}, \tag{G.15}$$

which, after integration, gives

$$\Phi_H(r) \sim 4\pi G \rho_0 r_0^2 \ln r + \text{const}. \tag{G.16}$$

The logarithmic potential is precisely the classical behavior associated with asymptotically constant orbital velocities.

G.8 Observational consequences. If hypercomplex halos do reproduce rotation curves, one expects to observe:

(1) asymptotically flat orbital velocities,
(2) a baryons $\to$ hypercomplex-halo transition at large radius,
(3) effective mass profiles compatible with $M(r) \propto r$,
(4) possible deviations from the standard NFW profile in intermediate regions.

G.9 Conclusion. The HGR framework therefore makes it possible, at least at the effective level, to consider a geometric explanation of galactic rotation curves.

An asymptotic hypercomplex density of the form

$$\rho_H(r) \propto \frac{1}{r^2} \tag{G.17}$$

implies

$$M_H(r) \propto r, \qquad v_c(r) \approx \text{constant}, \tag{G.18}$$

which directly reproduces the observed behavior of galactic halos.

This possibility makes hypercomplex halos natural candidates for the HGR interpretation of large-scale galactic dynamics.

G.10 Minimal definition of the RGHCli model. We consider a four-dimensional Lorentzian manifold (M, g) equipped with a spin structure.

The RGHCli extension is based on an internal algebraic structure built from the tensor product

$$\mathcal{A}_{RGHCli} = Cl(1,3) \otimes \mathbb{H},$$

where $Cl(1,3)$ denotes the Clifford algebra associated with the Lorentzian metric, and $\mathbb{H}$ denotes the quaternion algebra.

At each point $x \in M$, one associates an algebraic fiber

$$\mathcal{F}_x \simeq Cl(1,3) \otimes \mathbb{H}.$$

The extended covariant derivative is defined by

$$\mathcal{D}_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} + W_\mu + g_H A_\mu^I \tau_I,$$

where ω_μ^{ab} is the spin connection, W_μ is a Weyl-type conformal field, and A_μ^I is an internal quaternionic connection.

The total curvature is defined by

$$\mathcal{F}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu].$$

The minimal effective action is written as

$$S = \int_M d^4x\, \sqrt{-g}\left[\frac{1}{2\kappa}R - \frac{1}{4}F_{\mu\nu}^{(W)}F_{(W)}^{\mu\nu} - \frac{1}{4}F_{\mu\nu}^I F_I^{\mu\nu} + \mathcal{L}_{\text{mix}} + \mathcal{L}_m\right].$$

Appendix H Sketch of a Clifford–quaternionic hypercomplex extension (RGHCli)

H.1 Motivation. Within the framework of Hypercomplex General Relativity (HGR), it is natural to consider an extension in which the underlying geometric structure is no longer limited to a real Lorentzian metric equipped with its Levi–Civita connection, but also includes a dynamical internal hypercomplex sector. The guiding idea of this extension, denoted here by RGHCli, is that observable spacetime may only be the real projection of a richer geometric structure combining a relativistic Clifford algebra with an internal quaternionic fiber.

The purpose of this appendix is not to propose a complete theory, but to set up a minimal coherent mathematical architecture that may serve as a basis for future developments.

H.2 Extended algebraic structure. We introduce the effective algebra

$$\mathcal{A}_{\mathrm{RGHCli}} = Cl(1,3) \otimes \mathbb{H}, \tag{H.1}$$

where:

- $Cl(1,3)$ denotes the Clifford algebra of Lorentzian signature, adapted to the local structure of relativistic spacetime;
- $\mathbb{H}$ denotes the quaternion algebra, interpreted here as an internal rotation/gauge-like sector.

The $Cl(1,3)$ sector naturally carries:

- spacetime vectors;
- bivectors associated with rotations and boosts;
- local spinorial structures.

The quaternionic sector $\mathbb{H}$ carries:

- an internal triplet structure;
- an $SU(2)$-type dynamics;
- additional geometric degrees of freedom that are not directly observable.

H.3 Base manifold and internal fiber. We keep a classical four-dimensional Lorentzian manifold M, equipped with an observable real metric $g_{\mu\nu}$. At each point $x \in M$, one associates an algebraic fiber

$$\mathcal{F}_x \simeq Cl(1,3) \otimes \mathbb{H}. \tag{H.2}$$

The total structure can therefore be viewed as an extended algebraic bundle

$$\mathcal{F} \longrightarrow M. \tag{H.3}$$

From this perspective, experimentally measured quantities are not necessarily the fundamental objects themselves, but rather effective projections of the complete structure.

H.4 Observable projection principle. One of the conceptual assumptions of RGHCli is that the observable geometry corresponds to a real projection of a richer structure. Formally, one may introduce a projection map

$$\Pi_{\mathrm{obs}} : Cl(1,3) \otimes \mathbb{H} \longrightarrow \mathbb{R} \quad \text{or} \quad \Pi_{\mathrm{obs}} : Cl(1,3) \otimes \mathbb{H} \longrightarrow TM. \tag{H.4}$$

In a conservative first version of the model, the real metric $g_{\mu\nu}$ is kept as the fundamental observable object, and the hypercomplex extension is carried not by the metric itself, but by the associated connections and curvatures.

H.5 Extended total connection. We introduce an extended covariant derivation

$$\mathcal{D}_\mu = \partial_\mu + \frac{1}{4}\,\omega_\mu^{ab}\gamma_{ab} + W_\mu \mathbf{1} + g_H A_\mu^I \tau_I, \tag{H.5}$$

where:

- ω_μ^{ab} is the spin connection associated with the relativistic sector;
- γ_{ab} are the bivector generators of $Cl(1,3)$;
- W_μ is a Weyl-type field associated with a conformal/dilatational structure;
- A_μ^I is an internal quaternionic connection;
- τ_I denote the internal generators associated with the quaternionic sector;
- g_H is the coupling constant of the hypercomplex sector.

This expression naturally separates three components:

(1) the standard relativistic geometry;
(2) the Weyl-type conformal sector;
(3) the internal quaternionic geometry.

H.6 Total curvature. The curvature associated with the total connection is defined by

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = \mathcal{F}_{\mu\nu}. \tag{H.6}$$

It can be schematically decomposed as

$$\mathcal{F}_{\mu\nu} = \frac{1}{4}R_{\mu\nu}^{ab}\gamma_{ab} + F_{\mu\nu}^{(W)}\mathbf{1} + g_H F_{\mu\nu}^I \tau_I + \mathcal{F}_{\mu\nu}^{(\mathrm{mix})}, \tag{H.7}$$

where:

- $R_{\mu\nu}^{ab}$ is the curvature of the spin/gravitational sector;
- $F_{\mu\nu}^{(W)} = \partial_\mu W_\nu - \partial_\nu W_\mu$ is the curvature of the Weyl sector;
-

$$F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I + g_H \epsilon^I{}_{JK} A_\mu^J A_\nu^K \tag{H.8}$$

  is the curvature of the quaternionic sector;
- $\mathcal{F}_{\mu\nu}^{(\mathrm{mix})}$ collects the coupling terms between the different sectors.

These mixed terms constitute the specific core of the RGHCli extension.

H.7 Minimal effective action. A minimal effective action compatible with this structure may be written as

$$S = \int d^4x\,\sqrt{-g}\,\left[\frac{1}{2\kappa}R - \frac{1}{4}F_{\mu\nu}^{(W)}F_{(W)}^{\mu\nu} - \frac{1}{4}F_{\mu\nu}^I F_I^{\mu\nu} + \mathcal{L}_{\mathrm{mix}} + \mathcal{L}_m\right]. \tag{H.9}$$

The term $\mathcal{L}_{\mathrm{mix}}$ encodes the interactions between the relativistic curvature, the Weyl sector, and the internal hypercomplex sector. As an indication, one may consider contributions of the form

$$\mathcal{L}_{\mathrm{mix}} = \xi\, R\, A_\mu^I A_I^\mu + \eta\, W_\mu J_H^\mu + \zeta\, F_{\mu\nu}^{(W)} F_I^{\mu\nu} \Phi^I, \tag{H.10}$$

where ξ, η, ζ are coupling constants, while Φ^I and J_H^μ represent effective variables arising from the hypercomplex sector.

H.8 Effective field equations. Varying the action with respect to the metric schematically leads to modified Einstein equations:

$$G_{\mu\nu} = \kappa \left(T^{(m)}_{\mu\nu} + T^{(W)}_{\mu\nu} + T^{(H)}_{\mu\nu} + T^{(\mathrm{mix})}_{\mu\nu} \right). \tag{H.11}$$

The effective stress–energy tensor may therefore be defined as

$$T^{\mathrm{eff}}_{\mu\nu} = T^{(m)}_{\mu\nu} + T^{(W)}_{\mu\nu} + T^{(H)}_{\mu\nu} + T^{(\mathrm{mix})}_{\mu\nu}. \tag{H.12}$$

In this interpretation, observed gravitation is no longer due exclusively to ordinary matter, but also receives geometric contributions arising from:

- the conformal structure;
- the internal quaternionic curvature;
- coupling terms between sectors.

H.9 Physical interpretation. In such an extension, it becomes possible to interpret some effective gravitational components as arising from an internal geometric structure rather than from additional particulate matter. In particular, one may consider configurations for which

$$T^{(m)}_{\mu\nu} \ll T^{(H)}_{\mu\nu} + T^{(\mathrm{mix})}_{\mu\nu}, \tag{H.13}$$

which opens the possibility of gravitational halos with apparently low baryonic content.

In a homogeneous cosmological context, the Friedmann equations would then be modified by additional effective densities and pressures,

$$\rho_{\mathrm{eff}} = \rho_m + \rho_W + \rho_H + \rho_{\mathrm{mix}}, \qquad p_{\mathrm{eff}} = p_m + p_W + p_H + p_{\mathrm{mix}}, \tag{H.14}$$

allowing, in principle, the emergence of bounce, saturation, or effective-acceleration regimes.

H.10 Methodological position. It should be emphasized that the present appendix is only a first structural sketch. In a first stage, the most robust methodological option is to:

- keep an observable real metric $g_{\mu\nu}$;
- place the extension in the connections, curvatures, and internal sectors;
- introduce a complete hypercomplex metric only at a later stage, once the causal, variational, and energetic consistency conditions have been clarified.

H.11 Summary. The RGHCli extension proposed here rests on the idea that a mixed structure

$$Cl(1,3) \otimes \mathbb{H} \tag{H.15}$$

may provide a natural support for an extended gravitational geometry combining:

- local relativistic geometry;
- a Weyl-type conformal structure;
- an internal quaternionic dynamics;
- coupling terms capable of generating an additional effective gravitation.

This architecture thus offers a conceptual framework for exploring a generalization of HGR in which internal geometric contributions participate directly in the observable gravitational dynamics.

H.12 Minimal affine ansatz for the hypercomplex sector. Within the framework of HGR, the internal quaternionic structure may be interpreted not only as a gauge sector contributing to the effective stress–energy tensor, but also as a geometric structure capable of modifying parallel transport in observable spacetime.

In order to capture this possibility at the simplest level, we introduce here a minimal affine ansatz in which the effective spacetime connection receives an additional contribution from the internal hypercomplex sector.

In a strong version of HGR, one may postulate that the quaternionic sector contributes not only to the effective stress–energy tensor, but also to the observable affine connection.

The observable geometry is then described by an effective connection ${\Gamma^\rho}_{\mu\nu,\mathrm{eff}}$, obtained by completing the Levi–Civita connection with a correction depending on the internal hypercomplex field.

$$ {\Gamma^\rho}_{\mu\nu,\mathrm{eff}} = {\Gamma^\rho}_{\mu\nu}(g) + \Delta{\Gamma^\rho}_{\mu\nu}(W) + \Delta{\Gamma^\rho}_{\mu\nu}(H), \tag{H.16} $$

where the hypercomplex contribution is taken, at the minimal level, in the form

$$ \Delta{\Gamma^\rho}_{\mu\nu}(H) = \frac{\lambda}{2}\left(A^I_\mu {\Sigma^\rho}_{\nu I} + A^I_\nu {\Sigma^\rho}_{\mu I}\right). \tag{H.17} $$

This expression may be geometrically interpreted as the projection onto observable spacetime of a connection defined on the total fibered space $M_4 \times F$, where F denotes the internal hypercomplex fiber.

Here, A^I_μ denotes the internal quaternionic connection, ${\Sigma^\rho}_{\nu I}$ a base–fiber projection tensor, and λ a geometric coupling constant. This expression encodes the idea that the internal hypercomplex structure directly modifies parallel transport and the geodesics of observable spacetime.

Appendix I Effective Friedmann equation in the HGR–Weyl framework and structural comparison with Damour–Kogan–Papazoglou bigravity

I.1 Effective Friedmann equation in the HGR–Weyl framework. This appendix makes explicit the effective form of the Friedmann equation in the HGR–Weyl framework and clarifies in what sense its additional geometric term plays a structurally analogous role, without being identical in content, to the bimetric interaction term studied by Damour, Kogan and Papazoglou.

We consider a homogeneous and isotropic FLRW spacetime,

$$ds^2 = -dt^2 + a(t)^2\, d\Sigma_k^2, \qquad H := \frac{\dot a}{a}, \tag{I.1}$$

where $a(t)$ is the scale factor, $k \in \{-1, 0, +1\}$ is the normalized spatial curvature, and H is the Hubble parameter.

In the standard framework, the Friedmann equation reads

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}. \tag{I.2}$$

In the HGR framework enriched by an internal hypercomplex structure and by a Weyl-type contribution, we postulate that the effective cosmological dynamics receives an additional geometric correction. We therefore write

$$\boxed{H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} + \rho_{\mathrm{eff}}^{(\mathrm{HGR-W})}} \tag{I.3}$$

with

$$\boxed{\rho_{\mathrm{eff}}^{(\mathrm{HGR-W})} = \Phi(I_{\mathrm{HC}}) + \Psi(\varphi, \dot\varphi, H)} \tag{I.4}$$

where:

- I_{HC} denotes an effective invariant of the internal hypercomplex sector;
- $\varphi(t)$ denotes the homogeneous component of the Weyl field, through

$$W_\mu = (\varphi(t), 0, 0, 0). \tag{I.5}$$

At the first non-trivial order, these contributions may be parametrized as

$$\Phi(I_{\mathrm{HC}}) \simeq \alpha\, \omega_J^2 + \beta\, \dot\omega_J^2 + \gamma\, H\omega_J, \tag{I.6}$$

where ω_J represents an effective mode associated with the hypercomplex structure, and

$$\Psi(\varphi, \dot\varphi, H) \simeq A\,\varphi^2 + B\,H\varphi + C\,\dot\varphi. \tag{I.7}$$

One thus obtains the explicit effective equation

$$\boxed{H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} + \alpha\,\omega_J^2 + \beta\,\dot\omega_J^2 + \gamma\,H\omega_J + A\,\varphi^2 + B\,H\varphi + C\,\dot\varphi} \tag{I.8}$$

The corresponding acceleration equation can be written in the general form

$$\boxed{\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} + \widetilde{\Phi}(I_{\mathrm{HC}}) + \widetilde{\Psi}(\varphi, \dot\varphi, H)} \tag{I.9}$$

where the functions $\widetilde{\Phi}$ and $\widetilde{\Psi}$ depend on the details of the chosen effective coupling.

A cosmological bounce at $t = t_b$ is then characterized by the conditions

$$H(t_b) = 0, \qquad \ddot a(t_b) > 0. \tag{I.10}$$

Using (I.3), this imposes

$$0 = \frac{8\pi G}{3}\rho_b + \frac{\Lambda}{3} - \frac{k}{a_b^2} + \rho_{\mathrm{eff},b}^{(\mathrm{HGR-W})}, \tag{I.11}$$

whereas (I.9) requires

$$\widetilde{\Phi}(I_{\mathrm{HC},b}) + \widetilde{\Psi}(\varphi_b, \dot{\varphi}_b, H_b) > \frac{4\pi G}{3}(\rho_b + 3p_b) - \frac{\Lambda}{3}. \tag{I.12}$$

This form admits a clear structural analogy with the nonlinear bigravity studied by Damour, Kogan and Papazoglou: in their case, the effective cosmic acceleration arises from an interaction potential between two dynamical metrics; in the present framework, it arises from an effective geometric density associated with the hypercomplex sector and/or the Weyl field. The relevant parallel is therefore not an identification of the two theories, but a correspondence of roles in the effective cosmological equation:

$$\text{Damour–Kogan–Papazoglou bigravity} \;:\; H^2 \sim \frac{\Lambda}{3} + \rho_{\mathrm{int}}, \tag{I.13}$$

to be compared with

$$\text{HGR–Weyl framework} \;:\; H^2 \sim \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} + \rho_{\mathrm{eff}}^{(\mathrm{HGR-W})}. \tag{I.14}$$

Thus, in both cases, an additional geometric term occupies the same structural position in the Friedmann dynamics, although its microscopic origin is different:

$$\boxed{\rho_{\mathrm{DE}}^{\mathrm{eff}} \sim \begin{cases} \rho_{\text{bimetric interaction}}, & \text{in Damour–Kogan–Papazoglou,} \\ \Phi(I_{\mathrm{HC}}) + \Psi(\varphi, \dot{\varphi}, H), & \text{in the HGR–Weyl framework.} \end{cases}} \tag{I.15}$$

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