Interaction Between Two Hypercomplex Fields in a Hypercomplex General Relativity Framework

Autonomous Working Note

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Abstract

We construct a minimal autonomous model describing the interaction between two internal hypercomplex fields in a connection–curvature framework. The central idea is that a hypercomplex field is represented by a non-commutative internal connection, so that the interaction between two sectors may arise in four distinct ways: through kinetic mixing of curvatures, through a crossed commutator between connections, through coupling via emergent geometry, or through locking of internal structures. Here we retain the most directly calculable minimal model: two internal connections A_μ and B_μ, their curvatures $F^{(A)}_{\mu\nu}$ and $F^{(B)}_{\mu\nu}$, a kinetic mixing term proportional to ϵ, and a direct non-commutative interaction term proportional to λ. We derive the schematic field equations as well as the complete effective energy–momentum tensor. The document is self-contained and may serve as a basis for a paper section or for a later appendix.

1 Purpose of the note

This note formalizes a simple but structurally important idea: *what happens when two hypercomplex fields coexist and interact within the same geometric sector?*

The viewpoint adopted here is the following. In a geometric reformulation of symplectic or fibered type, internal non-commutativity is not carried directly by spacetime coordinates, but by an internal structure, typically in the form of a connection on an internal fiber. Consequently, a hypercomplex field may be represented by an internal connection, and its dynamics by a non-Abelian Yang–Mills-type curvature.

Our goal is to build a minimal autonomous model, namely:

- simple enough to be calculable;
- general enough to capture the physical idea;
- clean enough to be reused in a scientific manuscript.

2 Hypotheses, intuitions, model, formal results

2.1 Hypotheses

We introduce two internal hypercomplex fields, represented by two connections

$$A_\mu, \qquad B_\mu, \tag{1}$$

defined on observable spacetime, but taking values in a non-commutative internal algebra. We denote this algebra by $\mathcal{H}$, without imposing at this stage a unique realization; one may think of a quaternionic structure, or of a larger internal algebra containing a quaternionic subsector.

Two readings are possible.

1. A_μ and B_μ live in the *same* internal sector, and may therefore interact directly through a commutator.
2. A_μ and B_μ live in *two distinct subsectors* of a larger structure; direct interaction may then be absent, reduced, or mediated by emergent geometry.

In this note, we adopt the simplest framework: the two fields are close enough for crossed terms to make invariant sense, in particular

$$[A_\mu, B_\nu] \neq 0 \tag{2}$$

is allowed in general.

2.2 Physical intuitions

The physical idea may be summarized by three images.

- **Two internal curvatures.** Each hypercomplex field curves the internal fiber. One may therefore feel the presence of the other through the total curvature.
- **Two competing transport rules.** If A_μ and B_μ do not induce the same internal transport, their incompatibility is naturally measured by a crossed commutator.
- **A geometric interference energy.** Even if the two sectors remain distinct, their energy densities may mix through a bilinear term in the curvatures.

2.3 Chosen minimal model

The chosen model is based on two internal curvatures

$$F^{(A)}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{3}$$

$$F^{(B)}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]. \tag{4}$$

To their standard kinetic terms we add:

- a **kinetic mixing** between curvatures, with coefficient ϵ;
- a **direct non-commutative interaction term**, with coefficient λ.

2.4 Formal results obtained

The main formal results are:

1. the minimal two-field action;
2. the equations of motion in schematic covariant form;
3. the complete effective energy–momentum tensor of the system;
4. a clear identification of the limiting cases ($\epsilon = 0$, $\lambda = 0$, or $[A, B] = 0$).

3 Minimal action for two hypercomplex fields

We consider the effective action

$$S[A, B, g] = \int d^4x \sqrt{-g}\, \mathcal{L}_{\text{eff}}, \tag{5}$$

with

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}\,\mathrm{Tr}\Big(F^{(A)}_{\mu\nu}F^{(A)\mu\nu}\Big) - \frac{1}{4}\,\mathrm{Tr}\Big(F^{(B)}_{\mu\nu}F^{(B)\mu\nu}\Big) + \mathcal{L}_{\text{int}}, \tag{6}$$

and

$$\mathcal{L}_{\text{int}} = -\frac{\epsilon}{2}\,\mathrm{Tr}\Big(F^{(A)}_{\mu\nu}F^{(B)\mu\nu}\Big) + \lambda\,\mathrm{Tr}([A_\mu, B_\nu][A^\mu, B^\nu])\,. \tag{7}$$

The first term in (7) describes a *kinetic mixing* between the two hypercomplex sectors. The second term measures their *direct non-commutative interaction.*

Remark 1. *The ϵ term couples the* curvatures*, whereas the λ term couples the* algebras*. This distinction is conceptually essential: the former mixes propagation, whereas the latter penalizes or favors certain non-commuting configurations.*

4 Structure of the crossed commutator

It is useful to introduce the crossed tensor

$$\mathcal{C}_{\mu\nu} := [A_\mu, B_\nu]. \tag{8}$$

The following regime plays a central role:

$$\mathcal{C}_{\mu\nu} = 0 \quad \Longleftrightarrow \quad [A_\mu, B_\nu] = 0 \text{ for all } \mu, \nu. \tag{9}$$

In that case, the direct hypercomplex interaction disappears and only the kinetic mixing, if present, survives. Conversely, whenever $\mathcal{C}_{\mu\nu} \neq 0$, the system possesses a genuine internal interaction energy.

5 Equations of motion

Variation of the action with respect to A_μ and then to B_μ yields, in schematic form, the equations

$$D^{(A)}_\mu F^{(A)\mu\nu} + \epsilon\, D^{(A)}_\mu F^{(B)\mu\nu} + \lambda\, \mathcal{J}^\nu_{AB} = 0, \tag{10}$$

$$D^{(B)}_\mu F^{(B)\mu\nu} + \epsilon\, D^{(B)}_\mu F^{(A)\mu\nu} + \lambda\, \mathcal{J}^\nu_{BA} = 0, \tag{11}$$

where $D^{(A)}_\mu$ and $D^{(B)}_\mu$ denote the covariant derivatives in the two sectors, and where $\mathcal{J}^\nu_{AB}$, $\mathcal{J}^\nu_{BA}$ are the effective currents arising from the variation of the quartic commutator term.

5.1 Form of the interaction currents

Without pushing the full detailed computation here, one may anticipate the general structure of these currents. Writing

$$\delta\mathcal{C}_{\mu\nu} = [\delta A_\mu, B_\nu] + [A_\mu, \delta B_\nu], \tag{12}$$

the variation of the term

$$\mathrm{Tr}(\mathcal{C}_{\mu\nu}\mathcal{C}^{\mu\nu}) \tag{13}$$

necessarily produces contributions of the type

$$\mathcal{J}^{\nu}_{AB} \sim [B_\mu, [A^\nu, B^\mu]], \qquad \mathcal{J}^{\nu}_{BA} \sim [A_\mu, [B^\nu, A^\mu]], \tag{14}$$

up to symmetrization and convention-dependent factors.

The main qualitative point is the following:

> equations (10)–(11) no longer describe two independent fields, but two fields that back-react on each other through both their curvatures and their non-commutation.

6 Effective energy–momentum tensor

6.1 Definition

The energy–momentum tensor of the system is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}. \tag{15}$$

It is natural to decompose it as

$$T_{\mu\nu} = T^{(A)}_{\mu\nu} + T^{(B)}_{\mu\nu} + T^{(\epsilon)}_{\mu\nu} + T^{(\lambda)}_{\mu\nu}. \tag{16}$$

6.2 Individual contributions of the two sectors

The proper contribution of the field A_μ is

$$T^{(A)}_{\mu\nu} = \mathrm{Tr}\Big(F^{(A)}_{\mu\rho}F^{(A)\rho}{}_{\nu}\Big) - \frac{1}{4}g_{\mu\nu}\,\mathrm{Tr}\Big(F^{(A)}_{\rho\sigma}F^{(A)\rho\sigma}\Big). \tag{17}$$

Similarly, for the field B_μ,

$$T^{(B)}_{\mu\nu} = \mathrm{Tr}\Big(F^{(B)}_{\mu\rho}F^{(B)\rho}{}_{\nu}\Big) - \frac{1}{4}g_{\mu\nu}\,\mathrm{Tr}\Big(F^{(B)}_{\rho\sigma}F^{(B)\rho\sigma}\Big). \tag{18}$$

These two terms are the direct analogues of the standard Yang–Mills energy–momentum tensor.

6.3 Kinetic mixing term

The kinetic mixing term yields

$$T^{(\epsilon)}_{\mu\nu} = \epsilon\left[\frac{1}{2}\,\mathrm{Tr}\Big(F^{(A)}_{\mu\rho}F^{(B)\rho}{}_{\nu} + F^{(B)}_{\mu\rho}F^{(A)\rho}{}_{\nu}\Big) - \frac{1}{2}g_{\mu\nu}\,\mathrm{Tr}\Big(F^{(A)}_{\rho\sigma}F^{(B)\rho\sigma}\Big)\right]. \tag{19}$$

This term may be read as a *common geometric correlation energy.* The two fields no longer carry only their own energy; they also store a mixed energy due to the correlation of their curvatures.

6.4 Direct non-commutative interaction term

Introducing

$$\mathcal{C}_{\mu\nu} := [A_\mu, B_\nu], \tag{20}$$

the quartic term gives rise to a tensor of the form

$$T^{(\lambda)}_{\mu\nu} = -2\lambda\, \mathrm{Tr}(\mathcal{C}_{\mu\rho}\mathcal{C}_\nu{}^\rho + \mathcal{C}_{\nu\rho}\mathcal{C}_\mu{}^\rho) + \lambda g_{\mu\nu}\, \mathrm{Tr}(\mathcal{C}_{\rho\sigma}\mathcal{C}^{\rho\sigma})\,. \tag{21}$$

When no ambiguity arises from symmetrization in μ, ν, one may write more compactly

$$T^{(\lambda)}_{\mu\nu} \simeq -4\lambda\, \mathrm{Tr}(\mathcal{C}_{\mu\rho}\mathcal{C}_\nu{}^\rho) + \lambda g_{\mu\nu}\, \mathrm{Tr}(\mathcal{C}_{\rho\sigma}\mathcal{C}^{\rho\sigma})\,. \tag{22}$$

6.5 Complete expression

Collecting the blocks (17), (18), (19) and (21), one obtains

$$\begin{aligned}
T_{\mu\nu} = {} & \mathrm{Tr}\Big(F^{(A)}_{\mu\rho}F^{(A)\rho}{}_\nu\Big) - \frac{1}{4}g_{\mu\nu}\,\mathrm{Tr}\Big(F^{(A)}_{\rho\sigma}F^{(A)\rho\sigma}\Big) \\
& + \mathrm{Tr}\Big(F^{(B)}_{\mu\rho}F^{(B)\rho}{}_\nu\Big) - \frac{1}{4}g_{\mu\nu}\,\mathrm{Tr}\Big(F^{(B)}_{\rho\sigma}F^{(B)\rho\sigma}\Big) \\
& + \epsilon\left[\frac{1}{2}\,\mathrm{Tr}\Big(F^{(A)}_{\mu\rho}F^{(B)\rho}{}_\nu + F^{(B)}_{\mu\rho}F^{(A)\rho}{}_\nu\Big) - \frac{1}{2}g_{\mu\nu}\,\mathrm{Tr}\Big(F^{(A)}_{\rho\sigma}F^{(B)\rho\sigma}\Big)\right] \\
& - 2\lambda\, \mathrm{Tr}(\mathcal{C}_{\mu\rho}\mathcal{C}_\nu{}^\rho + \mathcal{C}_{\nu\rho}\mathcal{C}_\mu{}^\rho) + \lambda g_{\mu\nu}\, \mathrm{Tr}(\mathcal{C}_{\rho\sigma}\mathcal{C}^{\rho\sigma})\,.
\end{aligned} \tag{23}$$

7 Physical interpretation of the effective tensor

Expression (23) naturally reads as the sum of three physical contents.

7.1 Two individual energies

The first two blocks are the energy and pressure densities separately carried by each hypercomplex field.

7.2 An interference energy

The ϵ term represents an interference or correlation energy between the two sectors. It does not express direct non-commutativity, but rather a co-dynamics of the curvatures.

7.3 An algebraic incompatibility energy

The λ term measures the energetic cost associated with the internal misalignment of the two connections. Depending on the sign of λ:

- if $\lambda > 0$, strongly non-commuting configurations are penalized;
- if $\lambda < 0$, the theory may favor non-trivial configurations, possibly related to domains, defects, or condensed states.

8 Limiting cases

Case 1: completely decoupled sectors

$$\epsilon = 0, \qquad \lambda = 0. \tag{24}$$

Then

$$T_{\mu\nu} = T^{(A)}_{\mu\nu} + T^{(B)}_{\mu\nu}. \tag{25}$$

One simply has two independent internal geometric fluids.

Case 2: kinetic mixing only

$$\epsilon \neq 0, \qquad \lambda = 0. \tag{26}$$

The two fields remain algebraically separate, but their curvatures mix dynamically.

Case 3: direct non-commutative interaction only

$$\epsilon = 0, \qquad \lambda \neq 0. \tag{27}$$

The new physics then comes solely from the crossed commutator sector.

Case 4: exact commutation

$$[A_\mu, B_\nu] = 0 \quad \forall \mu, \nu. \tag{28}$$

Then

$$\mathcal{C}_{\mu\nu} = 0, \qquad T^{(\lambda)}_{\mu\nu} = 0, \tag{29}$$

and the entire direct interaction disappears.

9 Link with emergent geometry

The previous model may be viewed as an effective subsector of a broader theory in which the observable metric emerges from a symplectic structure and an internal structure. In that reading, two hypercomplex fields may interact not only through the explicit terms (7), but also because they jointly contribute to emergent geometry.

Schematically, if the effective metric depends on two internal sectors,

$$g(X, Y) = \Omega\big(X, \mathcal{J}(A, B)Y\big), \tag{30}$$

then the fields A_μ and B_μ indirectly modify:

- the effective connection;
- the effective curvature;
- the cosmological or galactic dynamics;
- the observable gravitational content.

In other words, even in the absence of explicit direct interaction, two hypercomplex fields may couple because they *co-construct* the effective geometry.

10 Methodological discussion

This document deliberately imposes neither a unique hypercomplex algebra, nor a precise cosmological reduction, nor a fully fixed sign convention. This restraint is methodological.

What is fixed here is the *framework*:

1. two internal connections;
2. two individual curvatures;
3. a bilinear kinetic mixing;
4. a quartic interaction arising from the crossed commutator;
5. a calculable effective energy–momentum tensor.

The points to be fixed at a later stage are:

- the exact realization of the internal algebra;
- the trace and normalization conventions;
- the FLRW or static spherical reduction;
- the physically admissible sign of the λ term;
- the detailed covariant conservation of $T_{\mu\nu}$ once the equations of motion are imposed.

11 Conclusion

The minimal model constructed here shows that an interaction between two hypercomplex fields may be described cleanly in a connection–curvature language. Two distinct mechanisms appear immediately.

1. A **kinetic mixing** between curvatures, expressing a dynamical correlation between sectors.
2. A **direct non-commutative interaction**, measuring the algebraic misalignment of the two connections.

The resulting effective energy–momentum tensor is the most immediately exploitable output of this construction. It provides the effective gravitational source of a system with two coupled hypercomplex fields, and naturally opens the way toward cosmological applications, geometric halo-type solutions, or internal domain structures.

In this sense, the interaction between two hypercomplex fields is not merely a technical generalization; it already sketches a multi-sector physics of internal geometry.

A Short reusable version for a manuscript

For quick use in a paper, the model may be condensed as follows.

We introduce two hypercomplex connections A_μ and B_μ with curvatures

$$F^{(A)}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \qquad F^{(B)}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu].$$

The minimal two-field action is

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{4} \mathrm{Tr}\Big(F^{(A)}_{\mu\nu} F^{(A)\mu\nu}\Big) - \frac{1}{4} \mathrm{Tr}\Big(F^{(B)}_{\mu\nu} F^{(B)\mu\nu}\Big) - \frac{\epsilon}{2} \mathrm{Tr}\Big(F^{(A)}_{\mu\nu} F^{(B)\mu\nu}\Big) + \lambda \, \mathrm{Tr}([A_\mu, B_\nu][A^\mu, B^\nu])\right.$$

The effective energy–momentum tensor decomposes into individual contributions, a kinetic mixing term, and a direct non-commutative interaction term.

B Remark on sign conventions

The exact sign of the quartic block in the energy–momentum tensor depends on the following conventions:

- the metric signature;
- the definition of $T_{\mu\nu}$ via variation with respect to $g^{\mu\nu}$ or $g_{\mu\nu}$;
- the normalization of the internal trace;
- the symmetrization chosen in the contraction of the term $[A_\mu, B_\nu][A^\mu, B^\nu]$.

The general tensorial structure presented in this document nevertheless remains stable: a quadratic part in $\mathcal{C}_{\mu\rho}\mathcal{C}_\nu{}^\rho$ and a part proportional to $g_{\mu\nu} \,\mathrm{Tr}(\mathcal{C}^2)$.