

# A Dirichlet-Form Rewriting of the Weil Explicit Formula and an Unconditional Hilbert–Pólya Candidate

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## Abstract

For  $\lambda > 1$ , let  $I_\lambda = [\lambda^{-1}, \lambda]$  and  $H_\lambda = L^2(I_\lambda, d^\times u)$ . We show that the boundary-free semilocal Weil quadratic form on  $H_\lambda$ , after an explicit scalar shift determined by the von Mangoldt function and by the digamma function at  $1/4$ , extends to a nonnegative closed symmetric Dirichlet form  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ . It decomposes canonically as the sum of a continuous pure-jump archimedean energy with explicit Lévy measure  $\nu(dr) = \pi^{-1}(1 - e^{-2r})^{-1}e^{-r/2}dr$  and a finite arithmetic part consisting of dilation-induced discrete jumps and a prime-power killing term indexed by  $n \leq \lambda^2$ . The form is irreducible, its self-adjoint generator  $L_\lambda$  has compact resolvent, and the unique  $L^2$ -normalized ground state  $\xi_\lambda$  is strictly positive almost everywhere and even under  $u \mapsto u^{-1}$ . The Fourier–Mellin transform of  $\xi_\lambda$ , and of every even-Galerkin approximant  $\xi_{\lambda,N}$ , is an entire function of exponential type whose zeros are all real, unconditionally on the Riemann Hypothesis. A zeta-regularized determinant identity moreover identifies the zero set of  $\hat{\xi}_{\lambda,N}$ , with multiplicity, with the spectrum of an explicit finite self-adjoint operator, thereby furnishing an unconditional candidate in the spirit of Hilbert–Pólya.

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## 1. Introduction

Let  $\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_p (1 - p^{-s})^{-1}$  for  $\Re s > 1$ , and let

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2) \zeta(s), \quad \Xi(z) = \xi\left(\frac{1}{2} + iz\right),$$

denote the completed zeta function and its even real entire avatar. The Riemann Hypothesis (RH) asserts that every zero of  $\Xi$  is real. The aim of this paper is twofold; both results are established unconditionally.

1.1. *A Dirichlet-form rewriting of the Weil explicit formula.* The Riemann–Weil explicit formula encodes, in a single distributional identity, the zeros of  $\zeta$ , the archimedean local factor, and the prime powers [1] and [2]. In the semilocal framework of Connes and Connes–Consani–Moscovici [3], [4], [5], [6], and [7], the semilocal Weil form  $QW_\lambda$  acts on  $H_\lambda = L^2(I_\lambda, d^\times u)$ ,  $I_\lambda = [\lambda^{-1}, \lambda]$ , and decomposes into archimedean, finite-rank boundary, and prime-power terms. After removing the bounded boundary term and adding the explicit shift

$$c_{D,\lambda} = c_\infty + c_{p,\lambda}, \quad c_\infty = \log \pi - \psi\left(\frac{1}{4}\right), \quad c_{p,\lambda} = 2 \sum_{1 < n \leq \lambda^2} \Lambda(n) n^{-1/2},$$

the resulting form  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  is proved to be a nonnegative closed symmetric Dirichlet form on  $H_\lambda$ . It splits canonically as  $\mathcal{E}_\lambda = \mathcal{E}_{\infty,\lambda} + \mathcal{E}_{p,\lambda}$ , where, in logarithmic coordinates,  $\mathcal{E}_{\infty,\lambda}$  is a continuous pure-jump Lévy–Khintchine energy with measure

$$\nu(dr) = \frac{1}{\pi} \cdot \frac{e^{-r/2}}{1 - e^{-2r}} dr, \quad r > 0,$$

and  $\mathcal{E}_{p,\lambda}$  is a finite sum of  $U_n$ -induced discrete jumps together with a prime-power killing term, both indexed by  $n \leq \lambda^2$ . This rewriting establishes a structural correspondence between the Riemann–Weil explicit formula and the Lévy–Khintchine representation of nonlocal Dirichlet forms, allowing direct transfer of probabilistic Markovian techniques into the analytic study of  $\zeta$ .

1.2. *An unconditional Hilbert–Pólya candidate.* Combining the structure of  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  with the finite-interval real-zero theorem of Connes–van Suijlekom [8], we obtain the following unconditional results.

- (i) The strict positivity of the Lévy density of  $\nu$  on  $(0, \infty)$  implies that  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  is irreducible; the associated semigroup and resolvents are positivity improving.
- (ii) The high-frequency divergence of the archimedean Fourier symbol  $\chi_\lambda(t) = \int_0^\infty (1 - \cos tr) \nu(dr)$  implies, via the Kolmogorov–Riesz criterion, that  $L_\lambda$  has compact resolvent and discrete spectrum. Perron–Frobenius theory then yields that the bottom eigenvalue  $\mu_0(\lambda)$  is simple, and the unique  $L^2$ -normalized ground state  $\xi_\lambda$  is strictly positive almost everywhere on  $I_\lambda$  and inversion-even.
- (iii) The Fourier–Mellin transform

$$\widehat{\xi}_\lambda(z) = \int_{\lambda^{-1}}^\lambda \xi_\lambda(u) u^{-iz} d^\times u$$

is an entire function of exponential type whose zeros are all real; the same conclusion holds for every even-Galerkin approximant  $\widehat{\xi}_{\lambda,N}$ . A zeta-regularized determinant identity moreover identifies the zero set of  $\widehat{\xi}_{\lambda,N}$ , with multiplicity, with the spectrum of an explicit finite self-adjoint operator  $D_{\log}^{(\lambda,N)}$ .

- (iv) Numerical experiments at  $(\lambda, N) = (10, 120)$  with mpmath precision  $\text{dps} = 50$  in the boundary-free model match each of the first twenty nontrivial zeros  $\frac{1}{2} + i\gamma_k$  of  $\zeta$  ( $1 \leq k \leq 20$ ) to the nearest positive zero of  $\widehat{\xi}_{\lambda,N}$ ; the resulting mean absolute error is  $\text{MAE} \approx 3.40 \times 10^{-1}$ .

Across the five-point geometric sweep  $\lambda^2 \in \{13, 25, 50, 100, 200\}$  documented in Section 6, the boundary-free MAE decreases overall from 0.680 at  $\lambda^2 = 13$  to 0.267 at  $\lambda^2 = 200$ , with one upward step at  $\lambda^2 = 100$  (Kendall  $\tau = -0.6$ , continuous-decrease ratio  $\text{CR} = 0.75$ ); the worst-case error  $\max_{k \leq 20} |q_k - \gamma_k|$  decreases strictly monotonically along the sweep ( $\tau = -1$ ,  $\text{CR} = 1$ ; a descriptive log-log linear fit on the five sampled points gives  $R^2 = 0.98$ , reported as a within-range trend coefficient and not extrapolated<sup>1</sup>). A complementary four-point  $N$ -convergence sweep at  $\lambda^2 = 50$  shows that the MAE is stable to relative change of order  $10^{-3}$  for  $N \in [60, 120]$ ; under the working assumption that the same Galerkin saturation holds at the other sweep points (where  $N/L \approx 25$  is fixed throughout), the within-range refinement of Table 3 is consistent with being driven by the window scale  $\lambda$  rather than by the Galerkin dimension  $N$ . The single-point monotonicity diagnostics yield  $|\beta| < 0.05$  and Spearman  $|\rho| < 0.22$  uniformly across the sweep, so the rank correlation between the residual error and the index  $k$  of the matched zero is not statistically significant at the 5% level; this contrasts with the boundary-retained variant of [7], where the per-row error distribution is highly dispersed, with  $\text{MAE}/\text{GM} > 10^{10}$  at  $\lambda^2 = 13$ , a small subset of zeros being matched to far higher precision than the rest.

The absolute precision of the boundary-free construction at moderate window scale is therefore modest, and the within-range trends documented above do not, by themselves, establish any asymptotic statement; their role is to record the absence, in the tested range, of the pathological signatures (extreme dispersion,  $N$ -instability) that would falsify the boundary-free model as a carrier for the diagonal limit problem of Section 7.

The construction is unconditional:  $D_{\log}^{(\lambda,N)}$  provides, without assuming RH and without any unverified hypothesis on the spectral form, a

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<sup>1</sup>For a sample of size  $n = 5$ , the value  $R^2 = 0.98$  merely records the relative linearity of the five sampled points in the log-log plane and does not constitute statistical evidence for a power-law decay; we report it descriptively, not inferentially.

Hilbert–Pólya-type candidate self-adjoint operator whose spectrum is real and whose lowest elements track the first twenty nontrivial zeros of  $\zeta$  within absolute error of order  $10^{-1}$  at the tested parameters. Section 6 reports the numerical experiments and the sensitivity analysis that distinguishes the two truncations, and Section 7 formulates the global limit problem whose resolution would yield RH.

1.3. *Organization.* Section 2 constructs the window Hilbert space, the compressed dilations, and the semilocal Weil form. Section 3 establishes the Dirichlet-form decomposition of the boundary-free Weil form. Section 4 proves irreducibility and positivity improvement. Section 5 establishes compact resolvent, ground-state simplicity and strict positivity, the real-zero theorems, and the determinant identity. Section 6 reports the numerical experiments. Section 7 formulates a global Fourier-side limit problem whose resolution would yield RH.

## 2. The Window Hilbert Space and the Semilocal Weil Form

2.1. *The idèle class group, the modulus homomorphism, and the scaling axis.* We begin by recalling the idèle class group of  $\mathbb{Q}$  and the modulus map on it.

DEFINITION 2.1. Since this paper concerns the Riemann zeta function over the rational field, throughout this section we take the global field to be  $K = \mathbb{Q}$ . For each finite prime  $p$ , let  $\mathbb{Q}_p$  denote the  $p$ -adic field and  $\mathbb{Z}_p$  its ring of integers; at the infinite place we write  $\mathbb{Q}_\infty = \mathbb{R}$ .

Define the adèle ring by

$$\mathbb{A}_{\mathbb{Q}} := \mathbb{R} \times \prod_p {}'(\mathbb{Q}_p, \mathbb{Z}_p),$$

where  $\prod_p {}'$  denotes the restricted direct product: if  $x = (x_p)_p \in \mathbb{A}_{\mathbb{Q}}$ , then  $x_\infty \in \mathbb{R}$ ,  $x_p \in \mathbb{Q}_p$ , and  $x_p \in \mathbb{Z}_p$  for all but finitely many primes  $p$ .

Equipped with the restricted product topology,  $\mathbb{A}_{\mathbb{Q}}$  is a locally compact topological ring. Its group of units is

$$I_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}^\times = \mathbb{R}^\times \times \prod_p {}'(\mathbb{Q}_p^\times, \mathbb{Z}_p^\times).$$

One should note that the subspace topology inherited from  $\mathbb{A}_{\mathbb{Q}}$  does not, in general, make  $I_{\mathbb{Q}}$  into a topological group, since inversion need not be continuous there. Thus  $I_{\mathbb{Q}}$  must be endowed with the restricted product topology; equivalently, one may use the topology induced by the embedding  $x \mapsto (x, x^{-1}) \in \mathbb{A}_{\mathbb{Q}} \times \mathbb{A}_{\mathbb{Q}}$ . With this topology,  $I_{\mathbb{Q}}$  is a locally compact topological group.

Let

$$\Delta : \mathbb{Q}^\times \hookrightarrow \mathbb{A}_{\mathbb{Q}}^\times, \quad \Delta(a) = (a)_v,$$

be the diagonal embedding. Its image, still denoted by  $\mathbb{Q}^\times$ , is the principal idèle subgroup. The idèle class group is then defined by

$$C_{\mathbb{Q}} := \mathbb{A}_{\mathbb{Q}}^\times / \mathbb{Q}^\times.$$

2.1.1. *The modulus homomorphism descends from  $\mathbb{A}_{\mathbb{Q}}^\times$  to  $C_{\mathbb{Q}}$ .* For each place  $v$ , choose the standard normalized absolute value  $|\cdot|_v$ : at the infinite place this is the usual absolute value, while at a finite prime  $p$  it is the  $p$ -adic absolute value normalized by  $|p|_p = p^{-1}$ . Then for every  $a \in \mathbb{Q}^\times$ , the product formula holds:

$$\prod_v |a|_v = 1.$$

For an idèle  $x = (x_v)_v \in \mathbb{A}_{\mathbb{Q}}^\times$ , define its idèlic norm by

$$\|x\|_{\mathbb{A}} := \prod_v |x_v|_v \in \mathbb{R}_{>0}^\times.$$

Since  $x_p \in \mathbb{Z}_p^\times$  for all but finitely many  $p$ , and  $|x_p|_p = 1$  whenever  $x_p \in \mathbb{Z}_p^\times$ , only finitely many factors differ from 1, so the product is well defined. By multiplicativity in each component,  $x \mapsto \|x\|_{\mathbb{A}}$  is a continuous group homomorphism

$$\|\cdot\|_{\mathbb{A}} : \mathbb{A}_{\mathbb{Q}}^\times \rightarrow \mathbb{R}_{>0}^\times.$$

On the other hand, for every  $a \in \mathbb{Q}^\times$ , the product formula gives

$$\|\Delta(a)\|_{\mathbb{A}} = \prod_v |a|_v = 1.$$

Hence  $\|\cdot\|_{\mathbb{A}}$  is identically 1 on  $\mathbb{Q}^\times$  and therefore descends uniquely to a continuous group homomorphism

$$\|\cdot\| : C_{\mathbb{Q}} \rightarrow \mathbb{R}_{>0}^\times.$$

Define

$$C_{\mathbb{Q}}^1 := \ker(\|\cdot\| : C_{\mathbb{Q}} \rightarrow \mathbb{R}_{>0}^\times).$$

Equivalently, if we write

$$I_{\mathbb{Q}}^1 := \{x \in \mathbb{A}_{\mathbb{Q}}^\times : \|x\|_{\mathbb{A}} = 1\},$$

then there is a natural isomorphism

$$C_{\mathbb{Q}}^1 \cong I_{\mathbb{Q}}^1 / \mathbb{Q}^\times.$$

For a general number field  $K$ , one defines similarly

$$\|\cdot\|_{\mathbb{A}} : I_K \rightarrow \mathbb{R}_{>0}^\times, \quad C_K^1 := \ker(\|\cdot\| : C_K \rightarrow \mathbb{R}_{>0}^\times).$$

If  $K$  is a function field, then the image of the modulus map is the discrete subgroup  $q^{\mathbb{Z}} \subset \mathbb{R}_{>0}^\times$ . In the present paper we work only with  $K = \mathbb{Q}$ , so the scaling axis is simply  $\mathbb{R}_{>0}^\times$ .

2.1.2. *The image of  $\|\cdot\|$  is all of  $\mathbb{R}_{>0}^\times$ .* For any  $u \in \mathbb{R}_{>0}^\times$ , define

$$\tilde{s}(u) := (u; 1, 1, 1, \dots) \in \mathbb{R}^\times \times \prod_p' (\mathbb{Q}_p^\times, \mathbb{Z}_p^\times) = \mathbb{A}_\mathbb{Q}^\times.$$

Since  $1 \in \mathbb{Z}_p^\times$  for every finite prime  $p$ , the element  $\tilde{s}(u)$  satisfies the restricted product condition and is therefore an idèle. Let

$$s(u) := \tilde{s}(u)\mathbb{Q}^\times \in C_\mathbb{Q}$$

denote its class in the idèle class group. By definition,

$$\|s(u)\| = \|\tilde{s}(u)\|_\mathbb{A} = |u|_\infty \cdot \prod_p |1|_p = u.$$

Thus, for every  $u > 0$ , there exists  $s(u) \in C_\mathbb{Q}$  such that  $\|s(u)\| = u$ . Hence

$$\text{Im}(\|\cdot\|) = \mathbb{R}_{>0}^\times,$$

so the modulus map is surjective in the case  $K = \mathbb{Q}$ .

Moreover, the map

$$s : \mathbb{R}_{>0}^\times \rightarrow C_\mathbb{Q}, \quad u \mapsto s(u)$$

is a continuous group homomorphism and satisfies

$$\|\cdot\| \circ s = \text{Id}_{\mathbb{R}_{>0}^\times}.$$

Indeed,  $\tilde{s}(uv) = \tilde{s}(u)\tilde{s}(v)$ , hence  $s(uv) = s(u)s(v)$ ; continuity follows from the continuity of  $u \mapsto \tilde{s}(u) \in \mathbb{A}_\mathbb{Q}^\times$  and of the quotient map  $\mathbb{A}_\mathbb{Q}^\times \rightarrow C_\mathbb{Q}$ .

Because the normalized absolute values  $|\cdot|_v$  are nonnegative, the idélic norm  $\|\cdot\|_\mathbb{A}$  inherently takes values in  $\mathbb{R}_{>0}^\times$ . This ensures that the image of the modulus map is restricted to the positive reals, aligning naturally with the fact that the continuous section  $s$  is defined solely on the positive real axis  $\mathbb{R}_{>0}^\times$ . Therefore, the scaling axis is intrinsically  $\mathbb{R}_{>0}^\times$  rather than  $\mathbb{R}^\times$ .

2.1.3. *The short exact sequence.* By definition,

$$C_\mathbb{Q}^1 = \ker(\|\cdot\| : C_\mathbb{Q} \rightarrow \mathbb{R}_{>0}^\times).$$

Since the previous subsection proved that  $\|\cdot\|$  is surjective, one obtains the short exact sequence

$$1 \rightarrow C_\mathbb{Q}^1 \rightarrow C_\mathbb{Q} \xrightarrow{\|\cdot\|} \mathbb{R}_{>0}^\times \rightarrow 1.$$

Here the left-hand arrow is the inclusion map. Exactness means precisely that

$$\text{Im}(C_\mathbb{Q}^1 \rightarrow C_\mathbb{Q}) = \ker(\|\cdot\| : C_\mathbb{Q} \rightarrow \mathbb{R}_{>0}^\times),$$

and exactness at the right end is just the surjectivity of  $\|\cdot\|$ .

Since we have already constructed a continuous group homomorphism

$$s : \mathbb{R}_{>0}^\times \rightarrow C_\mathbb{Q}$$

satisfying

$$\|\cdot\| \circ s = \text{Id}_{\mathbb{R}_{>0}^\times},$$

this short exact sequence is split in the category of topological groups. In other words,  $C_{\mathbb{Q}}$  can be viewed as a split extension of its unit-modulus part  $C_{\mathbb{Q}}^1$  by the scaling axis  $\mathbb{R}_{>0}^\times$ .

2.1.4. *Identifying the quotient  $C_{\mathbb{Q}}/C_{\mathbb{Q}}^1$  with  $\mathbb{R}_{>0}^\times$ .* Let

$$\pi : C_{\mathbb{Q}} \rightarrow C_{\mathbb{Q}}/C_{\mathbb{Q}}^1$$

be the quotient map. Since

$$C_{\mathbb{Q}}^1 = \ker(\|\cdot\| : C_{\mathbb{Q}} \rightarrow \mathbb{R}_{>0}^\times),$$

the first isomorphism theorem yields a group isomorphism

$$C_{\mathbb{Q}}/C_{\mathbb{Q}}^1 \cong \mathbb{R}_{>0}^\times.$$

More precisely, there exists a unique group homomorphism

$$\overline{\|\cdot\|} : C_{\mathbb{Q}}/C_{\mathbb{Q}}^1 \rightarrow \mathbb{R}_{>0}^\times$$

such that

$$\overline{\|\cdot\|} \circ \pi = \|\cdot\|.$$

Explicitly,

$$\overline{\|\cdot\|}(gC_{\mathbb{Q}}^1) = \|g\|, \quad g \in C_{\mathbb{Q}}.$$

This is well defined because if  $g_1C_{\mathbb{Q}}^1 = g_2C_{\mathbb{Q}}^1$ , then  $g_2^{-1}g_1 \in C_{\mathbb{Q}}^1 = \ker(\|\cdot\|)$ , hence  $\|g_1\| = \|g_2\|$ .

On the other hand, from the previously constructed section

$$s : \mathbb{R}_{>0}^\times \rightarrow C_{\mathbb{Q}}$$

with

$$\|\cdot\| \circ s = \text{Id}_{\mathbb{R}_{>0}^\times},$$

one gets, by composing with the quotient map, a continuous homomorphism

$$\bar{s} := \pi \circ s : \mathbb{R}_{>0}^\times \rightarrow C_{\mathbb{Q}}/C_{\mathbb{Q}}^1.$$

We now show that  $\overline{\|\cdot\|}$  and  $\bar{s}$  are inverse to each other. For any  $u \in \mathbb{R}_{>0}^\times$ ,

$$\overline{\|\cdot\|}(\bar{s}(u)) = \overline{\|\cdot\|}(\pi(s(u))) = \|s(u)\| = u,$$

so

$$\overline{\|\cdot\|} \circ \bar{s} = \text{Id}_{\mathbb{R}_{>0}^\times}.$$

Conversely, for any  $g \in C_{\mathbb{Q}}$ ,

$$\bar{s}(\overline{\|\cdot\|}(\pi(g))) = \bar{s}(\|g\|) = \pi(s(\|g\|)).$$

But

$$\|g s(\|g\|)^{-1}\| = \|g\| \|s(\|g\|)\|^{-1} = \|g\| \|g\|^{-1} = 1,$$



hence

$$g s(\|g\|)^{-1} \in C_{\mathbb{Q}}^1.$$

Therefore,

$$\pi(g) = \pi(s(\|g\|)) = \bar{s}(\|g\|) = \bar{s}(\overline{\|\cdot\|}(\pi(g))),$$

and thus

$$\bar{s} \circ \overline{\|\cdot\|} = \text{Id}_{C_{\mathbb{Q}}/C_{\mathbb{Q}}^1}.$$

Hence

$$\overline{\|\cdot\|} : C_{\mathbb{Q}}/C_{\mathbb{Q}}^1 \xrightarrow{\sim} \mathbb{R}_{>0}^{\times}$$

is a topological group isomorphism with inverse  $\bar{s}$ .

**2.1.5. Splitting of the short exact sequence and direct product decomposition.** The preceding discussion shows that the short exact sequence splits in the category of topological groups. We may write the corresponding topological group isomorphism explicitly. Define

$$\Phi : C_{\mathbb{Q}}^1 \times \mathbb{R}_{>0}^{\times} \rightarrow C_{\mathbb{Q}}, \quad \Phi(h, u) := h s(u).$$

Since  $C_{\mathbb{Q}}$  is abelian and  $s$  is a group homomorphism,  $\Phi$  is itself a continuous group homomorphism.

Define also

$$\Psi : C_{\mathbb{Q}} \rightarrow C_{\mathbb{Q}}^1 \times \mathbb{R}_{>0}^{\times}, \quad \Psi(g) := (g s(\|g\|)^{-1}, \|g\|).$$

This is well defined because the first component satisfies

$$\|g s(\|g\|)^{-1}\| = 1, \quad g s(\|g\|)^{-1} \in C_{\mathbb{Q}}^1.$$

Continuity follows from the continuity of  $\|\cdot\|$ ,  $s$ , multiplication, and inversion.

Now verify that  $\Phi$  and  $\Psi$  are inverse to each other. For  $(h, u) \in C_{\mathbb{Q}}^1 \times \mathbb{R}_{>0}^{\times}$ ,

$$\Psi(\Phi(h, u)) = \Psi(h s(u)) = (h s(u) s(\|h s(u)\|)^{-1}, \|h s(u)\|).$$

Since  $h \in C_{\mathbb{Q}}^1$ , one has  $\|h\| = 1$ , and therefore

$$\|h s(u)\| = \|h\| \|s(u)\| = u.$$

Thus

$$\Psi(\Phi(h, u)) = (h s(u) s(u)^{-1}, u) = (h, u).$$

Conversely, for  $g \in C_{\mathbb{Q}}$ ,

$$\Phi(\Psi(g)) = \Phi(g s(\|g\|)^{-1}, \|g\|) = g s(\|g\|)^{-1} s(\|g\|) = g.$$

Hence  $\Phi^{-1} = \Psi$ , and therefore

$$C_{\mathbb{Q}} \cong C_{\mathbb{Q}}^1 \times \mathbb{R}_{>0}^{\times}.$$

Equivalently, the idèle class group decomposes as the direct product of its unit-modulus part and the scaling axis. This decomposition depends on the continuous section

$$s(u) = (u; 1, 1, \dots) \mathbb{Q}^\times.$$

2.1.6. *Hilbert spaces on the multiplicative group.* By Fujisaki's lemma, the group  $C_{\mathbb{Q}}^1$  is compact. Let  $\mu_1$  be the Haar measure on  $C_{\mathbb{Q}}^1$ , normalized by

$$\mu_1(C_{\mathbb{Q}}^1) = 1.$$

On  $\mathbb{R}_{>0}^\times$  we take the standard Haar measure

$$d^\times u = du/u.$$

Pushing forward the product measure  $\mu_1 \otimes d^\times u$  via  $\Phi$  gives a Haar measure on  $C_{\mathbb{Q}}$ :

$$\mu_C := \Phi_*(\mu_1 \otimes d^\times u).$$

Throughout the sequel,  $L^2(C_{\mathbb{Q}})$  means  $L^2(C_{\mathbb{Q}}, \mu_C)$ .

Define

$$U_\Phi : L^2(C_{\mathbb{Q}}, \mu_C) \longrightarrow L^2(C_{\mathbb{Q}}^1 \times \mathbb{R}_{>0}^\times, \mu_1 \otimes d^\times u)$$

by

$$(U_\Phi F)(h, u) := F(\Phi(h, u)) = F(h \cdot s(u)).$$

By construction of  $\mu_C$ , the map  $U_\Phi$  is unitary.

Next consider the left translation action of  $C_{\mathbb{Q}}^1$  on  $L^2(C_{\mathbb{Q}})$ . For each  $k \in C_{\mathbb{Q}}^1$ , define

$$(L_k F)(g) := F(k^{-1}g), \quad F \in L^2(C_{\mathbb{Q}}).$$

Since  $\mu_C$  is a left Haar measure, each  $L_k$  is unitary. Moreover,

$$(U_\Phi L_k F)(h, u) = F(k^{-1}hs(u)) = (U_\Phi F)(k^{-1}h, u).$$

Hence, in the realization furnished by  $U_\Phi$ , the group  $C_{\mathbb{Q}}^1$  acts only on the first variable  $h$  and leaves the scaling variable  $u$  unchanged.

Let

$$L^2(C_{\mathbb{Q}})^{C_{\mathbb{Q}}^1} := \{F \in L^2(C_{\mathbb{Q}}) : L_k F = F \text{ for all } k \in C_{\mathbb{Q}}^1\}.$$

Under  $U_\Phi$  this invariant subspace corresponds to

$$L^2(C_{\mathbb{Q}}^1 \times \mathbb{R}_{>0}^\times)^{C_{\mathbb{Q}}^1} = \{G \in L^2(C_{\mathbb{Q}}^1 \times \mathbb{R}_{>0}^\times) : G(k^{-1}h, u) = G(h, u) \text{ a.e.}\}.$$

This invariant subspace consists precisely of functions depending only on the second variable  $u$ . Indeed, for fixed  $u$ , the left regular representation on  $L^2(C_{\mathbb{Q}}^1, \mu_1)$  has invariant vectors exactly the constant functions. Equivalently, the orthogonal projection onto the invariant subspace is given by averaging over the group:

$$(P\varphi)(h) = \int_{C_{\mathbb{Q}}^1} \varphi(k^{-1}h) d\mu_1(k) = \int_{C_{\mathbb{Q}}^1} \varphi(k) d\mu_1(k),$$

so  $\text{Ran}(P)$  is the space of constant functions. By Fubini's theorem,

$$L^2(C_{\mathbb{Q}}^1 \times \mathbb{R}_{>0}^{\times})^{C_{\mathbb{Q}}^1} = \{G(h, u) = f(u) \text{ a.e.} : f \in L^2(\mathbb{R}_{>0}^{\times}, d^{\times}u)\}.$$

Accordingly, the map

$$X : L^2(\mathbb{R}_{>0}^{\times}, d^{\times}u) \longrightarrow L^2(C_{\mathbb{Q}}^1 \times \mathbb{R}_{>0}^{\times}, \mu_1 \otimes d^{\times}u)^{C_{\mathbb{Q}}^1}, \quad (Xf)(h, u) := f(u)$$

is an isometric isomorphism. Indeed,

$$\|Xf\|^2 = \int_{\mathbb{R}_{>0}^{\times}} \int_{C_{\mathbb{Q}}^1} |f(u)|^2 d\mu_1(h) d^{\times}u = \mu_1(C_{\mathbb{Q}}^1) \int_{\mathbb{R}_{>0}^{\times}} |f(u)|^2 d^{\times}u = \|f\|^2.$$

Transporting this identification back by  $U_{\Phi}^{-1}$  yields the natural isometric isomorphism

$$L^2(C_{\mathbb{Q}})^{C_{\mathbb{Q}}^1} \cong L^2(\mathbb{R}_{>0}^{\times}, d^{\times}u).$$

We denote this Hilbert space by

$$H_{\text{tot}} := L^2(\mathbb{R}_{>0}^{\times}, d^{\times}u), \quad \mu := d^{\times}u = du/u.$$

Equivalently,

$$L^2(C_{\mathbb{Q}}/C_{\mathbb{Q}}^1) \cong L^2(C_{\mathbb{Q}})^{C_{\mathbb{Q}}^1},$$

where the measure on the quotient is the Haar measure transported back from  $d^{\times}u$  under the identification  $C_{\mathbb{Q}}/C_{\mathbb{Q}}^1 \cong \mathbb{R}_{>0}^{\times}$ . Thus the  $C_{\mathbb{Q}}^1$ -invariant  $L^2$  functions are precisely the square-integrable functions on the scaling axis.

**2.2. The Hilbert space with window cutoff.** The window-cutoff Hilbert space is defined as follows.

**PROPOSITION 2.2.** *In the semilocal framework, let  $\lambda > 1$  be the cutoff parameter. Define the compact inversion-symmetric interval*

$$I_{\lambda} = [\lambda^{-1}, \lambda] \subset \mathbb{R}_{>0}^{\times}, \quad u \in [\lambda^{-1}, \lambda] \iff u^{-1} \in [\lambda^{-1}, \lambda].$$

*Define the cutoff multiplication operator*

$$P_{\lambda}f(u) := \mathbf{1}_{I_{\lambda}}(u)f(u).$$

*Then:*

- (1)  $P_{\lambda}$  is bounded,  $\|P_{\lambda}\| = 1$ , and satisfies  $P_{\lambda}^* = P_{\lambda}$  and  $P_{\lambda}^2 = P_{\lambda}$ . Hence it is an orthogonal projection on  $H_{\text{tot}}$ , and  $\text{Ran}(P_{\lambda})$  is a closed subspace.
- (2) Let  $R_{\lambda}f = f|_{I_{\lambda}}$  be the restriction operator, and let

$$(E_{\lambda}g)(u) = \begin{cases} g(u), & u \in I_{\lambda}, \\ 0, & u \notin I_{\lambda}, \end{cases}$$

*be the zero extension. Then  $R_{\lambda}E_{\lambda} = \text{Id}$ ,  $E_{\lambda}R_{\lambda} = P_{\lambda}$ , and  $E_{\lambda}$  yields an isometric isomorphism*

$$L^2(I_{\lambda}, \mu|_{I_{\lambda}}) \cong \text{Ran}(P_{\lambda}).$$

*Proof.* Let

$$H_{\text{tot}} = L^2(\mathbb{R}_{>0}^\times, d^\times u), \quad \Sigma := \mathcal{B}(\mathbb{R}_{>0}^\times), \quad \mu = d^\times u = du/u.$$

By the usual definition of  $L^p$  spaces, elements of  $L^2(\mathbb{R}_{>0}^\times, d^\times u)$  are equivalence classes of measurable functions, where

$$f \sim g \iff \mu(\{u \in \mathbb{R}_{>0}^\times : f(u) \neq g(u)\}) = 0.$$

Since  $I_\lambda = [\lambda^{-1}, \lambda] \subset \mathbb{R}_{>0}^\times$  is closed, one has  $I_\lambda \in \Sigma$ , and its characteristic function  $\mathbf{1}_{I_\lambda}$  is measurable.

(i) *Well-definedness.* For an equivalence class  $[f] \in L^2(\mathbb{R}_{>0}^\times, d^\times u)$ , choose any representative  $f$  and define

$$P_\lambda[f] := [\varphi_\lambda f], \quad \varphi_\lambda := \mathbf{1}_{I_\lambda}.$$

If  $f \sim g$ , then with  $N_1 = \{u : f(u) \neq g(u)\}$  one has  $\mu(N_1) = 0$ , and

$$\{u : \varphi_\lambda(u)f(u) \neq \varphi_\lambda(u)g(u)\} \subset N_1.$$

Hence  $\varphi_\lambda f \sim \varphi_\lambda g$ , so  $P_\lambda$  is well defined. The same argument shows that the restriction operator  $R_\lambda$  is well defined on  $L^2(I_\lambda, \mu|_{I_\lambda})$ , and the zero extension  $E_\lambda$  is also well defined.

(ii) *Linearity, boundedness, and the operator norm.* The operator  $P_\lambda$  is clearly linear. Since  $\|\varphi_\lambda\|_\infty = 1$ , for every  $f \in L^2(\mathbb{R}_{>0}^\times, \mu)$ ,

$$\|P_\lambda f\|_2^2 = \int_{\mathbb{R}_{>0}^\times} |\varphi_\lambda(u)f(u)|^2 d\mu(u) \leq \|\varphi_\lambda\|_\infty^2 \|f\|_2^2 = \|f\|_2^2.$$

Thus  $\|P_\lambda\| \leq 1$ . On the other hand,

$$\mu(I_\lambda) = \int_{\lambda^{-1}}^{\lambda} \frac{du}{u} = 2 \log \lambda \in (0, \infty),$$

so  $\mathbf{1}_{I_\lambda} \in L^2(\mathbb{R}_{>0}^\times, \mu)$ , and  $P_\lambda \mathbf{1}_{I_\lambda} = \mathbf{1}_{I_\lambda}$ . Therefore  $\|P_\lambda\| \geq 1$ , hence  $\|P_\lambda\| = 1$ .

(iii) *Orthogonal projection.* Since pointwise  $\mathbf{1}_{I_\lambda}^2 = \mathbf{1}_{I_\lambda}$ , one has  $P_\lambda^2 = P_\lambda$ . Moreover, for  $f, g \in H_{\text{tot}}$ ,

$$\langle P_\lambda f, g \rangle = \int \mathbf{1}_{I_\lambda}(u) f(u) \overline{g(u)} d\mu(u) = \int f(u) \overline{\mathbf{1}_{I_\lambda}(u) g(u)} d\mu(u) = \langle f, P_\lambda g \rangle,$$

so  $P_\lambda^* = P_\lambda$ . Hence  $P_\lambda$  is an orthogonal projection and  $\text{Ran}(P_\lambda)$  is closed.

(iv) *Identification by restriction and zero extension.* For  $g \in L^2(I_\lambda, \mu_\lambda)$ , one has

$$\|E_\lambda g\|_{L^2(\mathbb{R}_{>0}^\times, \mu)}^2 = \int_{\mathbb{R}_{>0}^\times} |E_\lambda g(u)|^2 d\mu(u) = \int_{I_\lambda} |g(u)|^2 d\mu(u) = \|g\|_{L^2(I_\lambda, \mu_\lambda)}^2,$$

so  $E_\lambda$  is an isometric linear embedding. For  $f \in L^2(\mathbb{R}_{>0}^\times, \mu)$ ,

$$\|R_\lambda f\|_{L^2(I_\lambda, \mu_\lambda)}^2 = \int_{I_\lambda} |f(u)|^2 d\mu(u) \leq \int_{\mathbb{R}_{>0}^\times} |f(u)|^2 d\mu(u) = \|f\|_{L^2(\mathbb{R}_{>0}^\times, \mu)}^2,$$

so  $R_\lambda$  is bounded.

For every  $g \in L^2(I_\lambda, \mu_\lambda)$ , one has  $E_\lambda g = g$  a.e. on  $I_\lambda$ , hence  $R_\lambda E_\lambda = \text{Id}$ . Conversely, for every  $f \in L^2(\mathbb{R}_{>0}^\times, \mu)$ ,

$$(E_\lambda R_\lambda f)(u) = \begin{cases} f(u), & u \in I_\lambda, \\ 0, & u \notin I_\lambda, \end{cases} = \mathbf{1}_{I_\lambda}(u) f(u) = (P_\lambda f)(u)$$

a.e., so  $E_\lambda R_\lambda = P_\lambda$ .

Finally, we establish the equality of ranges in two steps. First, the identity  $E_\lambda R_\lambda = P_\lambda$  gives, for every  $F \in \text{Ran}(P_\lambda)$  with  $F = P_\lambda f$ ,

$$F = P_\lambda f = E_\lambda R_\lambda f \in \text{Ran}(E_\lambda),$$

so  $\text{Ran}(P_\lambda) \subset \text{Ran}(E_\lambda)$ . Conversely, for every  $g \in L^2(I_\lambda, \mu_\lambda)$ , the function  $E_\lambda g$  vanishes off  $I_\lambda$  by definition; hence  $\mathbf{1}_{I_\lambda}(u) (E_\lambda g)(u) = (E_\lambda g)(u)$  for every  $u \in \mathbb{R}_{>0}^\times$ , that is,  $P_\lambda E_\lambda g = E_\lambda g$ , and therefore  $E_\lambda g \in \text{Ran}(P_\lambda)$ . Combining both inclusions,

$$\text{Ran}(E_\lambda) = \text{Ran}(P_\lambda),$$

and  $E_\lambda$  furnishes the isometric isomorphism

$$E_\lambda : L^2(I_\lambda, \mu_\lambda) \xrightarrow{\sim} \text{Ran}(P_\lambda).$$

Thus we may define the Hilbert space with window cutoff by

$$H_\lambda := L^2(I_\lambda, \mu_\lambda) = L^2([\lambda^{-1}, \lambda], d^\times u),$$

and identify it with  $\text{Ran}(P_\lambda) \subset H_{\text{tot}}$ .  $\square$

LEMMA 2.3. *Let*

$$\ell := \log \lambda, \quad I_\lambda := [\lambda^{-1}, \lambda] \subset \mathbb{R}_{>0}^\times, \quad I_x := [-\ell, \ell] \subset \mathbb{R}.$$

*Write*

$$H_\lambda = L^2(I_\lambda, d^\times u), \quad d^\times u = du/u, \quad H_x := L^2(I_x, dx).$$

*Define*

$$Z : H_\lambda \rightarrow H_x, \quad (Zf)(x) := f(e^x).$$

*More precisely, if  $f$  denotes an  $L^2$ -equivalence class in  $H_\lambda$ , then  $Zf$  denotes the  $L^2$ -equivalence class of the function  $x \mapsto f(e^x)$ . Then  $Z$  is a unitary isomorphism.*

*Proof.* First we check that  $Z$  is well defined. Let  $f_1, f_2 \in H_\lambda$  satisfy  $f_1 = f_2$  almost everywhere on  $I_\lambda$ . Define

$$N_2 := \{u \in I_\lambda : f_1(u) \neq f_2(u)\}, \quad M_2 := \{x \in I_x : f_1(e^x) \neq f_2(e^x)\}.$$

Under the change of variables  $u = e^x$ , one has  $dx = d^\times u$ , hence

$$\int_{M_2} dx = \int_{N_2} d^\times u = 0.$$

Thus  $f_1(e^x) = f_2(e^x)$  almost everywhere on  $I_x$ , so  $Z$  is well defined on  $L^2$  equivalence classes. Linearity is immediate.

To prove that  $Z$  is an isometry, note that for any  $f \in H_\lambda$ , the map  $u = e^x$  gives a bijection  $I_x \rightarrow I_\lambda$  and  $dx = d^\times u$ , so

$$\|Zf\|_{H_x}^2 = \int_{-\ell}^{\ell} |f(e^x)|^2 dx = \int_{\lambda^{-1}}^{\lambda} |f(u)|^2 d^\times u = \|f\|_{H_\lambda}^2.$$

Hence  $Z$  is an isometric linear map.

Finally, to prove surjectivity, let  $g \in H_x$  and define

$$(Yg)(u) := g(\log u), \quad u \in I_\lambda.$$

The same change of variables shows that  $Y$  is well defined and isometric. Moreover, for  $f \in H_\lambda$  and  $g \in H_x$ ,

$$(YZf)(u) = f(e^{\log u}) = f(u) \quad \text{a.e. on } I_\lambda,$$

and

$$(ZYg)(x) = g(\log(e^x)) = g(x) \quad \text{a.e. on } I_x.$$

Therefore,

$$YZ = \text{Id}_{H_\lambda}, \quad ZY = \text{Id}_{H_x}.$$

Thus  $Y = Z^{-1}$ , and  $Z$  is unitary.  $\square$

**2.3. Dilations on the total space and semilocal compression.** In this subsection we construct the compressed dilation operators that will later be used in the definition of the semilocal Weil quadratic form. To avoid confusion with later summation indices over integers  $n$ , the dilation parameter will always be denoted by  $a \in \mathbb{R}_{>0}^\times$ .

Let

$$H_{\text{tot}} = L^2(\mathbb{R}_{>0}^\times, d^\times u), \quad d^\times u = du/u,$$

with inner product linear in the first variable:

$$\langle F, G \rangle_{H_{\text{tot}}} := \int_{\mathbb{R}_{>0}^\times} F(u) \overline{G(u)} d^\times u.$$

For any  $a \in \mathbb{R}_{>0}^\times$ , define the dilation operator

$$(S_a F)(u) := F(au), \quad F \in H_{\text{tot}}.$$

**LEMMA 2.4.** *For each  $a \in \mathbb{R}_{>0}^\times$ , the operator  $S_a$  is unitary on  $H_{\text{tot}}$ , and*

$$S_a S_b = S_{ab}, \quad S_a^* = S_{a^{-1}}, \quad a, b \in \mathbb{R}_{>0}^\times.$$

*Proof.* Since  $d^\times u$  is invariant under multiplicative translation, the change of variables  $v = au$  gives

$$\|S_a F\|_{H_{\text{tot}}}^2 = \int_{\mathbb{R}_{>0}^\times} |F(au)|^2 d^\times u = \int_{\mathbb{R}_{>0}^\times} |F(v)|^2 d^\times v = \|F\|_{H_{\text{tot}}}^2.$$

Thus  $S_a$  is isometric. Since  $(S_a S_{a^{-1}} F)(u) = F(a \cdot a^{-1} u) = F(u)$  and symmetrically  $(S_{a^{-1}} S_a F)(u) = F(u)$  for every  $F \in H_{\text{tot}}$  and every  $u \in \mathbb{R}_{>0}^\times$ , the operator  $S_{a^{-1}}$  is a two-sided inverse of  $S_a$ ; hence  $S_a$  is bijective, and therefore unitary.

For  $F, G \in H_{\text{tot}}$ , the substitution  $v = au$  yields

$$\langle S_a F, G \rangle_{H_{\text{tot}}} = \int_{\mathbb{R}_{>0}^\times} F(au) \overline{G(u)} d^\times u = \int_{\mathbb{R}_{>0}^\times} F(v) \overline{G(a^{-1}v)} d^\times v = \langle F, S_{a^{-1}} G \rangle_{H_{\text{tot}}}.$$

Hence  $S_a^* = S_{a^{-1}}$ . Finally,

$$(S_a S_b F)(u) = S_b F(au) = F(bau) = F((ab)u) = S_{ab} F(u),$$

so  $S_a S_b = S_{ab}$ .  $\square$

LEMMA 2.5. *The zero extension operator  $E_\lambda$  is an isometric embedding, the restriction operator  $R_\lambda$  is bounded, and*

$$E_\lambda^* = R_\lambda, \quad R_\lambda^* = E_\lambda, \quad \|E_\lambda\| = \|R_\lambda\| = 1.$$

*Proof.* For  $f \in H_\lambda$ ,

$$\|E_\lambda f\|_{H_{\text{tot}}}^2 = \int_{\mathbb{R}_{>0}^\times} |E_\lambda f(u)|^2 d^\times u = \int_{I_\lambda} |f(u)|^2 d^\times u = \|f\|_{H_\lambda}^2,$$

so  $E_\lambda$  is an isometric embedding; in particular,  $\|E_\lambda\| = 1$ .

For  $F \in H_{\text{tot}}$ ,

$$\|R_\lambda F\|_{H_\lambda}^2 = \int_{I_\lambda} |F(u)|^2 d^\times u \leq \int_{\mathbb{R}_{>0}^\times} |F(u)|^2 d^\times u = \|F\|_{H_{\text{tot}}}^2,$$

so  $R_\lambda$  is bounded and  $\|R_\lambda\| \leq 1$ .

Moreover, for  $f \in H_\lambda$  and  $F \in H_{\text{tot}}$ ,

$$\langle E_\lambda f, F \rangle_{H_{\text{tot}}} = \int_{I_\lambda} f(u) \overline{F(u)} d^\times u = \langle f, R_\lambda F \rangle_{H_\lambda},$$

so  $E_\lambda^* = R_\lambda$ , hence  $R_\lambda^* = E_\lambda$ .

Finally, since  $R_\lambda E_\lambda = \text{Id}_{H_\lambda}$ ,

$$1 = \|\text{Id}_{H_\lambda}\| = \|R_\lambda E_\lambda\| \leq \|R_\lambda\| \|E_\lambda\| = \|R_\lambda\|.$$

Combined with  $\|R_\lambda\| \leq 1$ , this implies  $\|R_\lambda\| = 1$ .  $\square$

DEFINITION 2.6. For any  $a \in \mathbb{R}_{>0}^\times$ , define the compressed dilation operator by

$$U_a := R_\lambda S_a E_\lambda : H_\lambda \rightarrow H_\lambda.$$

Under the identification  $H_\lambda \cong \text{Ran}(P_\lambda) \subset H_{\text{tot}}$  furnished by  $E_\lambda$ , the operator  $U_a$  corresponds to the compression

$$P_\lambda S_a P_\lambda|_{\text{Ran}(P_\lambda)}.$$

More precisely,

$$E_\lambda U_a = E_\lambda R_\lambda S_a E_\lambda = P_\lambda S_a E_\lambda = P_\lambda S_a P_\lambda E_\lambda.$$

PROPOSITION 2.7. *For each  $a \in \mathbb{R}_{>0}^\times$ , the operator  $U_a$  is bounded and satisfies:*

- (1)  $\|U_a\| \leq 1$ ;
- (2)  $U_a^* = U_{a^{-1}}$ ;
- (3) *for every  $f \in H_\lambda$  and any representative of its equivalence class, for almost every  $u \in I_\lambda$ ,*

$$(U_a f)(u) = \mathbf{1}_{I_\lambda}(au) f(au).$$

Consequently,

$$\text{ess supp}(U_a f) \subset I_\lambda \cap a^{-1} I_\lambda,$$

and

$$\|U_a f\|_{H_\lambda}^2 = \int_{I_\lambda \cap a^{-1} I_\lambda} |f(au)|^2 d^\times u = \int_{a(I_\lambda \cap a^{-1} I_\lambda)} |f(v)|^2 d^\times v \leq \|f\|_{H_\lambda}^2.$$

*Proof.* By submultiplicativity of the operator norm and the norm identities above,

$$\|U_a\| \leq \|R_\lambda\| \|S_a\| \|E_\lambda\| = 1 \cdot 1 \cdot 1 = 1.$$

Using  $(AB)^* = B^* A^*$  together with

$$E_\lambda^* = R_\lambda, \quad R_\lambda^* = E_\lambda, \quad S_a^* = S_{a^{-1}},$$

one obtains

$$U_a^* = (R_\lambda S_a E_\lambda)^* = E_\lambda^* S_a^* R_\lambda^* = R_\lambda S_{a^{-1}} E_\lambda = U_{a^{-1}}.$$

Now let  $f \in H_\lambda$ . Its zero extension satisfies

$$(E_\lambda f)(u) = \begin{cases} f(u), & u \in I_\lambda, \\ 0, & u \notin I_\lambda. \end{cases}$$

Hence for almost every  $u \in I_\lambda$ ,

$$(U_a f)(u) = (R_\lambda S_a E_\lambda f)(u) = (S_a E_\lambda f)(u) = (E_\lambda f)(au) = \begin{cases} f(au), & au \in I_\lambda, \\ 0, & au \notin I_\lambda, \end{cases}$$

which we abbreviate, with the convention that the symbol  $f(au)$  at points  $au \notin I_\lambda$  is irrelevant since it is multiplied by 0, as

$$(1) \quad (U_a f)(u) = \mathbf{1}_{I_\lambda}(au) f(au) \quad \text{for a.e. } u \in I_\lambda.$$

The support inclusion follows immediately. Finally, by this pointwise formula and the invariance of multiplicative Haar measure, the substitution  $v = au$  gives

$$\|U_a f\|_{H_\lambda}^2 = \int_{I_\lambda} \mathbf{1}_{I_\lambda}(au) |f(au)|^2 d^\times u = \int_{I_\lambda \cap a^{-1} I_\lambda} |f(au)|^2 d^\times u = \int_{a(I_\lambda \cap a^{-1} I_\lambda)} |f(v)|^2 d^\times v \leq \int_{I_\lambda} |f(v)|^2 d^\times v$$

□



*Remark 2.8.* The family  $(S_a)_{a \in \mathbb{R}_{>0}^\times}$  forms a unitary representation on the total space. By contrast, the compressed operators  $U_a$  generally do not form a representation; in general,

$$U_a U_b \neq U_{ab}.$$

A concrete counterexample is provided by  $\lambda = 4$ ,  $a = 2$ ,  $b = 1/2$ , and  $f := \mathbf{1}_{(2,4]} \in H_\lambda$ , where  $I_\lambda = [1/4, 4]$ . On one hand,  $S_1 = I_{H_{\text{tot}}}$  together with  $R_\lambda E_\lambda = \text{Id}_{H_\lambda}$  (Proposition 2.2) gives

$$U_{ab} = U_1 = R_\lambda S_1 E_\lambda = R_\lambda E_\lambda = \text{Id}_{H_\lambda}, \quad U_1 f = f \neq 0 \text{ a.e. on } (2, 4].$$

On the other hand, by (1) applied to  $U_{1/2}$ ,

$$(U_{1/2} f)(u) = \mathbf{1}_{I_\lambda}(u/2) f(u/2) \quad \text{for a.e. } u \in I_\lambda.$$

The factor  $\mathbf{1}_{I_\lambda}(u/2)$  vanishes whenever  $u \in [1/4, 1/2]$ , while for  $u \in [1/2, 4]$  one has  $u/2 \in [1/4, 2]$ , on which  $f = \mathbf{1}_{(2,4]}$  vanishes identically. Hence

$$U_{1/2} f = 0 \text{ a.e. on } I_\lambda, \quad U_2 U_{1/2} f = U_2 \cdot 0 = 0,$$

which contradicts  $U_1 f = f \neq 0$ . Therefore  $U_2 U_{1/2} \neq U_{2 \cdot 1/2}$ , as claimed.

In what follows we use only the boundedness, adjoint relation, and pointwise formula for  $U_a$ , and never any representation property.

**2.4. Fourier transform and the semilocal Weil quadratic form.** In this subsection we derive the semilocal Weil quadratic form rigorously from the compressed dilation operators  $U_a$  constructed above. Throughout the paper we adopt the following conventions:

- (1) the inner product is linear in the first variable and conjugate-linear in the second;
- (2) the Haar measure on the multiplicative group  $\mathbb{R}_{>0}^\times$  is  $d^\times u = du/u$ ;
- (3) for  $f \in H_\lambda$ , its Fourier transform in logarithmic coordinates is

$$\widehat{f}(z) = \int_{I_\lambda} f(u) u^{-iz} d^\times u, \quad z \in \mathbb{C},$$

where  $I_\lambda = [\lambda^{-1}, \lambda]$ , so the integral converges absolutely for every  $z \in \mathbb{C}$ .

To match the semilocal Weil quadratic form of Connes–Consani–Moscovici, let

$$\mathcal{V}_\lambda := \text{span}\{V_n : n \in \mathbb{Z}\} \subset H_\lambda,$$

where  $V_n(u) := L^{-1/2} \exp(2\pi i n \log(\lambda u)/L)$  for  $u \in I_\lambda$  and zero elsewhere, with  $L := 2 \log \lambda$ . By [5, §2] (cited in [7, Propositions 3.3 and 3.4]),  $\mathcal{V}_\lambda$  is a form core for  $QW_\lambda$ , and  $QW_\lambda$  is a lower-bounded, lower-semicontinuous, closed quadratic form. Accordingly, we first carry out all computations on the core  $\mathcal{V}_\lambda$  and then extend uniquely to  $\text{Dom}(QW_\lambda)$  by closure.

2.4.1. *Fourier transform on the multiplicative group and its logarithmic expression.* The continuous characters of the multiplicative group  $\mathbb{R}_{>0}^\times$  are

$$\chi_t(u) = u^{it} = e^{it \log u}, \quad t \in \mathbb{R},$$

so its Pontryagin dual can be identified with  $\mathbb{R}$ . Thus the Fourier transform on  $H_\lambda$  is

$$\widehat{f}(t) = \int_{I_\lambda} f(u) u^{-it} d^\times u, \quad t \in \mathbb{R}.$$

If  $\ell = \log \lambda$  and one defines in logarithmic coordinates

$$\varphi(x) := f(e^x), \quad x \in [-\ell, \ell],$$

then, with  $u = e^x$  and  $d^\times u = dx$ ,

$$\widehat{f}(t) = \int_{-\ell}^{\ell} \varphi(x) e^{-itx} dx.$$

Hence the Fourier transform on the multiplicative group is exactly the ordinary Fourier transform in logarithmic coordinates.

2.4.2. *Comparison with the interval convention  $[0, L]$  in Connes's paper.* Connes–Consani–Moscovici take

$$L = 2 \log \lambda = 2\ell$$

and use the coordinate

$$y = \log(\lambda u) = x + \ell \in [0, L]$$

to identify the interval  $[\lambda^{-1}, \lambda]$  with  $[0, L]$ . Writing

$$\varphi(x) = f(e^x), \quad x \in [-\ell, \ell], \quad \varphi_{\text{CC}}(y) := \varphi(y - \ell), \quad y \in [0, L],$$

one gets

$$\widehat{\varphi}_{\text{CC}}(t) = \int_0^L \varphi(y - \ell) e^{-ity} dy = e^{-i\ell t} \int_{-\ell}^{\ell} \varphi(x) e^{-itx} dx = e^{-i\ell t} \widehat{\varphi}(t).$$

Therefore, for real  $t$ ,

$$|\widehat{\varphi}_{\text{CC}}(t)|^2 = |\widehat{\varphi}(t)|^2,$$

and also

$$\widehat{\varphi}_{\text{CC}}(i/2) \overline{\widehat{\varphi}_{\text{CC}}(-i/2)} = \widehat{\varphi}(i/2) \overline{\widehat{\varphi}(-i/2)}.$$

Thus translating the interval  $[0, L]$  to the symmetric interval  $[-\ell, \ell]$  merely multiplies the Fourier transform by a phase factor; the archimedean term and the boundary term in the semilocal Weil formula remain numerically unchanged. See [7, Proposition 3.2 and equations (3.19)–(3.20)].

2.4.3. *Definition of the correlation function  $F_{f,g}$ .* We now introduce the correlation function used throughout the Fourier-formalism discussion.

DEFINITION 2.9. For  $f, g \in \mathcal{V}_\lambda$ , viewed as functions on  $\mathbb{R}_{>0}^\times$  by zero extension, define

$$F_{f,g}(a) := \int_{\mathbb{R}_{>0}^\times} f(au) \overline{g(u)} d^\times u, \quad a \in \mathbb{R}_{>0}^\times.$$

Since  $f$  and  $g$  are supported in  $I_\lambda$ , this can also be written as

$$F_{f,g}(a) = \int_{I_\lambda \cap a^{-1}I_\lambda} f(au) \overline{g(u)} d^\times u.$$

This definition is consistent with the compressed dilation operators from Subsection 2.3: for every  $a > 0$ ,

$$F_{f,g}(a) = \langle U_a f, g \rangle_{H_\lambda}.$$

Indeed, from

$$U_a f(u) = \mathbf{1}_{I_\lambda}(au) f(au) \quad \text{a.e. on } I_\lambda,$$

one gets

$$\langle U_a f, g \rangle_{H_\lambda} = \int_{I_\lambda} \mathbf{1}_{I_\lambda}(au) f(au) \overline{g(u)} d^\times u = \int_{I_\lambda \cap a^{-1}I_\lambda} f(au) \overline{g(u)} d^\times u.$$

DEFINITION 2.10 (WEIL CLASS). Following the convention of [5, §2] and [7, §3], the *Weil class*  $\mathcal{W}(\mathbb{R}_{>0}^\times)$  consists of those complex-valued functions  $F$  on  $\mathbb{R}_{>0}^\times$  satisfying the following two conditions:

- (1) *Smoothness.*  $F$  has a continuous derivative on  $\mathbb{R}_{>0}^\times$  except at finitely many points, at which both  $F$  and  $F'$  may have at most jump discontinuities of the first kind; at every such exceptional point, the value of  $F$  (resp. of  $F'$ ) is defined as the half-sum of its right and left limits.
- (2) *Decay.*  $F$  decays sufficiently rapidly as  $u \rightarrow 0^+$  and as  $u \rightarrow \infty$  to ensure the absolute convergence of the archimedean integral  $\int_{\mathbb{R}} \widehat{F}(t) w(t) dt$  and of the prime sum  $\sum_p W_p(F)$  entering the Riemann–Weil functional  $\Psi(F)$  (defined later in this subsection, in §2.4.5 below; cf. [2]).

In particular, every continuous, piecewise smooth function with compact support in  $\mathbb{R}_{>0}^\times$  belongs to  $\mathcal{W}(\mathbb{R}_{>0}^\times)$ , since condition (2) is then trivially satisfied.

LEMMA 2.11. *For any  $f, g \in \mathcal{V}_\lambda$ , the function  $F_{f,g}$  is continuous, piecewise  $C^\infty$ , and satisfies*

$$\text{supp}(F_{f,g}) \subset [\lambda^{-2}, \lambda^2].$$

*In particular,  $F_{f,g} \in \mathcal{W}(\mathbb{R}_{>0}^\times)$ , where  $\mathcal{W}(\mathbb{R}_{>0}^\times)$  is the Weil class of Definition 2.10.*

*Proof.* Write

$$\varphi(x) := f(e^x), \quad \psi(x) := g(e^x), \quad x \in I_x = [-\ell, \ell],$$

and extend both by zero to all of  $\mathbb{R}$ . If  $a = e^y$ , then

$$F_{f,g}(e^y) = \int_{\mathbb{R}} \varphi(x+y) \overline{\psi(x)} dx.$$

This is the ordinary correlation function on the additive group, hence it is continuous and piecewise smooth. Since both  $\varphi$  and  $\psi$  are supported in  $[-\ell, \ell]$ , the integral can be nonzero only if

$$[-\ell, \ell] \cap [-\ell - y, \ell - y] \neq \emptyset,$$

that is, only if  $|y| \leq 2\ell$ . Equivalently,

$$e^y \in [e^{-2\ell}, e^{2\ell}] = [\lambda^{-2}, \lambda^2].$$

This proves the support inclusion. Since  $F_{f,g}$  is compactly supported, continuous, and piecewise smooth on  $\mathbb{R}_{>0}^\times$ , both conditions of Definition 2.10 are satisfied; hence  $F_{f,g} \in \mathcal{W}(\mathbb{R}_{>0}^\times)$ .  $\square$

2.4.4. *Fourier transform of the correlation function.* The Fourier transform of the correlation function is computed in the following proposition.

PROPOSITION 2.12. *For any  $f, g \in \mathcal{V}_\lambda$  and the corresponding function  $F_{f,g}$ , one has, for every  $z \in \mathbb{C}$ ,*

$$\widehat{F}_{f,g}(z) = \widehat{f}(z) \overline{\widehat{g}(\bar{z})}.$$

*In particular, for real  $t$ ,*

$$\widehat{F}_{f,g}(t) = \widehat{f}(t) \overline{\widehat{g}(t)},$$

*and*

$$\widehat{F}_{f,g}(i/2) = \widehat{f}(i/2) \overline{\widehat{g}(-i/2)}, \quad \widehat{F}_{f,g}(-i/2) = \widehat{f}(-i/2) \overline{\widehat{g}(i/2)}.$$

*Proof.* Since  $f$  and  $g$  are supported on the finite-measure interval  $I_\lambda$ , they belong to  $L^1(I_\lambda, d^\times u)$ . Since  $F_{f,g}$  is supported in  $[\lambda^{-2}, \lambda^2]$ , for each fixed  $z \in \mathbb{C}$  the function  $a \mapsto a^{-iz}$  is bounded on this compact interval. Hence all of the following integrals converge absolutely and Fubini's theorem applies:

$$\widehat{F}_{f,g}(z) = \int_{\mathbb{R}_{>0}^\times} F_{f,g}(a) a^{-iz} d^\times a = \int_{\mathbb{R}_{>0}^\times} \left( \int_{\mathbb{R}_{>0}^\times} f(au) \overline{g(u)} d^\times u \right) a^{-iz} d^\times a.$$

Interchanging the order of integration gives

$$\widehat{F}_{f,g}(z) = \int_{\mathbb{R}_{>0}^\times} \overline{g(u)} \left( \int_{\mathbb{R}_{>0}^\times} f(au) a^{-iz} d^\times a \right) d^\times u.$$

For fixed  $u > 0$ , set  $v = au$ . Since  $d^\times v = d^\times a$  and

$$a^{-iz} = (v/u)^{-iz} = v^{-iz} u^{iz},$$

one gets

$$\int_{\mathbb{R}_{>0}^\times} f(au) a^{-iz} d^\times a = \int_{\mathbb{R}_{>0}^\times} f(v)(v/u)^{-iz} d^\times v = u^{iz} \int_{\mathbb{R}_{>0}^\times} f(v) v^{-iz} d^\times v = u^{iz} \widehat{f}(z).$$

Substituting back yields

$$\widehat{F}_{f,g}(z) = \widehat{f}(z) \int_{\mathbb{R}_{>0}^\times} \overline{g(u)} u^{iz} d^\times u.$$

But by definition,

$$\widehat{g}(\bar{z}) = \int_{\mathbb{R}_{>0}^\times} g(u) u^{-i\bar{z}} d^\times u,$$

so taking complex conjugates gives

$$\overline{\widehat{g}(\bar{z})} = \int_{\mathbb{R}_{>0}^\times} \overline{g(u)} u^{iz} d^\times u.$$

Therefore,

$$\widehat{F}_{f,g}(z) = \widehat{f}(z) \overline{\widehat{g}(\bar{z})}.$$

The remaining formulas follow by specializing to  $z = t$  and  $z = \pm i/2$ .  $\square$

2.4.5. *Matching with the explicit formula.* In the Fourier-version explicit formula of Connes–Consani–Moscovici, [7, (3.10)], one defines

$$\Psi(F) := W_{0,2}(F) - W_{\mathbb{R}}(F) - \sum_p W_p(F),$$

where

$$W_{0,2}(F) = \widehat{F}(i/2) + \widehat{F}(-i/2), \quad W_p(F) = (\log p) \sum_{m \geq 1} p^{-m/2} (F(p^m) + F(p^{-m})),$$

and

$$W_{\mathbb{R}}(F) = -W_{\infty}(F), \quad W_{\infty}(F) = \int_{\mathbb{R}} \widehat{F}(t) \frac{2\theta'(t)}{2\pi} dt.$$

Here

$$\theta(t) = -\frac{t}{2} \log \pi + \Im \log \Gamma\left(\frac{1}{4} + \frac{it}{2}\right).$$

See [7, (3.7)–(3.11)].

Because our convention is linearity in the first variable, we define the semilocal Weil sesquilinear form by

$$QW_{\lambda}(f, g) := \Psi(F_{f,g}), \quad f, g \in \mathcal{V}_{\lambda}.$$

Thus  $QW_{\lambda}$  is linear in the first variable and conjugate-linear in the second, in accordance with the conventions of this paper.

2.4.6. *Exact computation of the three components.* Fix  $f, g \in \mathcal{V}_\lambda$  and abbreviate  $F_{f,g}$  by  $F$ .

(i) *The archimedean term.* By Proposition 2.12,

$$\widehat{F}(t) = \widehat{f}(t) \overline{\widehat{g}(t)}, \quad t \in \mathbb{R}.$$

Hence

$$-W_{\mathbb{R}}(F) = W_{\infty}(F) = \int_{\mathbb{R}} \widehat{f}(t) \overline{\widehat{g}(t)} \frac{2\theta'(t)}{2\pi} dt.$$

Writing

$$w(t) := \frac{2\theta'(t)}{2\pi} = \frac{\theta'(t)}{\pi},$$

the archimedean contribution becomes

$$Q_{\infty,\lambda}(f, g) := \int_{\mathbb{R}} \widehat{f}(t) \overline{\widehat{g}(t)} w(t) dt.$$

(ii) *The boundary term.* Since

$$W_{0,2}(F) = \widehat{F}(i/2) + \widehat{F}(-i/2),$$

Proposition 2.12 gives

$$\widehat{F}(i/2) = \widehat{f}(i/2) \overline{\widehat{g}(-i/2)}, \quad \widehat{F}(-i/2) = \widehat{f}(-i/2) \overline{\widehat{g}(i/2)}.$$

Therefore,

$$B_{\lambda}(f, g) := W_{0,2}(F) = \widehat{f}(i/2) \overline{\widehat{g}(-i/2)} + \widehat{f}(-i/2) \overline{\widehat{g}(i/2)}.$$

Moreover,

$$\widehat{f}(i/2) = \int_{I_{\lambda}} f(u) u^{1/2} d^{\times} u, \quad \widehat{f}(-i/2) = \int_{I_{\lambda}} f(u) u^{-1/2} d^{\times} u.$$

Since  $u^{\pm 1/2} \in L^{\infty}(I_{\lambda})$  and  $\mu(I_{\lambda}) = 2 \log \lambda < \infty$ , these are bounded linear functionals on  $H_{\lambda}$ . Consequently  $B_{\lambda}$  is a bounded perturbation of rank at most two.

(iii) *The prime term.* By the support statement above,

$$\text{supp}(F) \subset [\lambda^{-2}, \lambda^2].$$

Hence for every prime  $p$ , in

$$W_p(F) = (\log p) \sum_{m \geq 1} p^{-m/2} (F(p^m) + F(p^{-m}))$$

only the terms with  $p^m \leq \lambda^2$  can be nonzero. Thus the total sum is in fact finite. Rewriting it in terms of the von Mangoldt function  $\Lambda(n)$  yields

$$\sum_p W_p(F) = \sum_{1 < n \leq \lambda^2} \Lambda(n) n^{-1/2} (F(n) + F(n^{-1})).$$

On the other hand, from  $F_{f,g}(a) = \langle U_a f, g \rangle$  and  $U_a^* = U_{a^{-1}}$ , one has for integers  $n \geq 2$ ,

$$F(n) = \langle U_n f, g \rangle, \quad F(n^{-1}) = \langle U_{n^{-1}} f, g \rangle = \langle U_n^* f, g \rangle.$$

Therefore, defining

$$T(n) := n^{-1/2}(U_n + U_n^*),$$

one sees that  $T(n)$  is a bounded self-adjoint operator on  $H_\lambda$  and

$$\langle T(n)f, g \rangle = n^{-1/2}(\langle U_n f, g \rangle + \langle U_n^* f, g \rangle) = n^{-1/2}(F(n) + F(n^{-1})).$$

Hence the prime contribution is

$$Q_{p,\lambda}(f, g) := - \sum_p W_p(F) = - \sum_{1 < n \leq \lambda^2} \Lambda(n) \langle T(n)f, g \rangle_{H_\lambda}.$$

**2.4.7. Explicit formula for the semilocal Weil sesquilinear form.** Putting everything together, for all  $f, g \in \mathcal{V}_\lambda$  one has

$$QW_\lambda(f, g) = \int_{\mathbb{R}} \widehat{f}(t) \overline{\widehat{g}(t)} w(t) dt + \widehat{f}(i/2) \overline{\widehat{g}(-i/2)} + \widehat{f}(-i/2) \overline{\widehat{g}(i/2)} - \sum_{1 < n \leq \lambda^2} \Lambda(n) \langle T(n)f, g \rangle_{H_\lambda},$$

where

$$w(t) = \frac{2\theta'(t)}{2\pi}, \quad T(n) = n^{-1/2}(U_n + U_n^*).$$

This is exactly the semilocal formula [7, (3.19)–(3.20)], written in the present convention that the inner product is linear in the first variable. The same paper also proves that  $QW_\lambda$  is lower bounded and lower semicontinuous, and that  $\mathcal{V}_\lambda$  is a core for it.

**2.4.8. Extension from the core to the full domain.** Since  $QW_\lambda$  is a closed quadratic form and  $\mathcal{V}_\lambda$  is a core, the formula above first holds on  $\mathcal{V}_\lambda$  and then uniquely determines the closure. In other words, one defines and proves the formula first for  $f, g \in \mathcal{V}_\lambda$  and then extends it, by closure, to a sesquilinear form on  $\text{Dom}(QW_\lambda)$ . From this point of view, the boundary term  $B_\lambda$  and the prime contribution  $Q_{p,\lambda}(f, g)$  are bounded perturbations; the archimedean integral term is the leading component governing closedness.

*Remark.* From this point onward, the genuine object of interest is no longer  $QW_\lambda$  itself, but the form obtained after removing the boundary term:

$$Q_{D,\lambda}(f, g) := QW_\lambda(f, g) - B_\lambda(f, g) = Q_{\infty,\lambda}(f, g) + Q_{p,\lambda}(f, g).$$

The reason is that  $B_\lambda$  is only a bounded finite-rank perturbation. In Section 3 we shall introduce an explicit shift

$$\mathcal{E}_\lambda(f, f) = Q_{D,\lambda}(f, f) + c_{D,\lambda} \|f\|_{H_\lambda}^2, \quad c_{D,\lambda} = c_\infty + c_{p,\lambda}, \quad \mathcal{F}_\lambda = \text{Dom}(Q_{D,\lambda}),$$

where  $c_\infty$  is the archimedean renormalization constant and  $c_{p,\lambda}$  is the truncated prime renormalization constant. It is this explicitly shifted form  $\mathcal{E}_\lambda$  that carries the Markovian structure to be identified later. One will prove that  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$

is a nonnegative, lower-semicontinuous, symmetric, and irreducible nonlocal Dirichlet form. Consequently, the resolvent of the associated self-adjoint operator  $L_\lambda$  is positivity improving; hence the spectrum of  $L_\lambda$  is discrete, its lowest eigenvalue is simple, and the corresponding ground state  $\xi_\lambda$  is a positive even function. This is precisely the starting point for the real-zero argument developed in the subsequent sections.

### 3. Dirichlet-Form Structure of the Weil Quadratic Form

In this section we prove that, after removing the boundary term  $B_\lambda$ , the semilocal Weil quadratic form becomes, after an explicit constant shift, a non-negative, closed, symmetric Dirichlet form on  $H_\lambda$ . Its structure has two essential components: the archimedean part gives a continuous pure-jump contribution, while the truncated prime part produces finitely many arithmetic discrete jump terms together with a killing term.

3.1. *Main theorem.* We adopt the following convention throughout this section.

*Remark 3.1.* A *complex normal contraction* is a function  $\Phi : \mathbb{C} \rightarrow \mathbb{C}$  such that

$$\Phi(0) = 0, \quad |\Phi(z) - \Phi(w)| \leq |z - w| \quad \text{for all } z, w \in \mathbb{C}.$$

A closed, densely defined, symmetric, lower-semibounded sesquilinear form  $(\mathcal{Q}, \text{Dom}(\mathcal{Q}))$  on a complex Hilbert space  $H$  is called a *symmetric Dirichlet form* if  $\mathcal{Q} \geq 0$  and if, for every complex normal contraction  $\Phi$  and every  $f \in \text{Dom}(\mathcal{Q})$ , one has  $\Phi \circ f \in \text{Dom}(\mathcal{Q})$  and  $\mathcal{Q}(\Phi \circ f, \Phi \circ f) \leq \mathcal{Q}(f, f)$ . This is the complex-valued formulation of the Beurling–Deny II condition used in [14, Sec. I.4, Thm. 4.4]; via the standard real/imaginary decomposition of the eigenfunction it reduces to the real-valued condition of Fukushima–Oshima–Takeda [13, Thm. 1.4.1], and is equivalent to the sub-Markovianity of the associated  $C_0$ -semigroup  $T_t = e^{-tL}$  on  $H$ .

We now state the main structural result of this section.

**THEOREM 3.2.** *For each fixed  $\lambda > 1$ , let  $a_\infty$  be the constant defined in (19) and set*

$$(2) \quad c_\infty := -2\pi a_\infty, \quad c_{p,\lambda} := 2 \sum_{1 < n \leq \lambda^2} \Lambda(n) n^{-1/2}, \quad c_{D,\lambda} := c_\infty + c_{p,\lambda}.$$

*Define*

$$(3) \quad \mathcal{E}_\lambda(f, f) := Q_{D,\lambda}(f, f) + c_{D,\lambda} \|f\|_{H_\lambda}^2, \quad \mathcal{F}_\lambda := \text{Dom}(Q_{D,\lambda}) \subset H_\lambda, \quad f \in \mathcal{F}_\lambda.$$

*Then the following statements hold.*



- (1)  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  is a nonnegative, closed, symmetric Dirichlet form on  $H_\lambda$  in the sense of Remark 3.1, with form core  $\mathcal{V}_\lambda$  defined in Subsection 2.4.
- (2) The archimedean part  $\mathcal{E}_{\infty, \lambda}(f, f) := Q_{\infty, \lambda}(f, f) + c_\infty \|f\|_{H_\lambda}^2$  admits the Lévy–Khinchine pure-jump representation

$$(4) \quad \mathcal{E}_{\infty, \lambda}(f, f) = \pi \int_0^\infty \|g(\cdot + r) - g\|_{L^2(\mathbb{R})}^2 \nu(dr), \quad g := E_x Z f \in L^2(\mathbb{R}),$$

where  $Z : H_\lambda \xrightarrow{\sim} L^2(I_x, dx)$  is the logarithmic unitary of Lemma 2.3,  $E_x : L^2(I_x, dx) \hookrightarrow L^2(\mathbb{R}, dx)$  is the zero extension, and

$$(5) \quad \nu(dr) := \frac{1}{\pi} \frac{e^{-r/2}}{1 - e^{-2r}} dr, \quad r > 0,$$

is a Lévy measure on  $(0, \infty)$ . The archimedean shift admits the explicit form

$$(6) \quad c_\infty = \log \pi - \psi\left(\frac{1}{4}\right) = \gamma + \frac{\pi}{2} + 3 \log 2 + \log \pi > 0.$$

- (3) The prime part  $\mathcal{E}_{p, \lambda}(f, f) := Q_{p, \lambda}(f, f) + c_{p, \lambda} \|f\|_{H_\lambda}^2$  is everywhere defined and bounded on  $H_\lambda$ , and admits the discrete jump+killing decomposition

$$(7) \quad \mathcal{E}_{p, \lambda}(f, f) = \sum_{1 < n \leq \lambda^2} \Lambda(n) n^{-1/2} \|f - U_n f\|_{H_\lambda}^2 + \sum_{1 < n \leq \lambda^2} \Lambda(n) n^{-1/2} (\|f\|_{H_\lambda}^2 - \|U_n f\|_{H_\lambda}^2),$$

where  $U_n$  is the compressed dilation operator from Definition 2.6 and  $\Lambda$  is the von Mangoldt function.

- (4) The shifted form decomposes additively:

$$(8) \quad \mathcal{E}_\lambda(f, f) = \mathcal{E}_{\infty, \lambda}(f, f) + \mathcal{E}_{p, \lambda}(f, f), \quad f \in \mathcal{F}_\lambda,$$

with both summands individually nonnegative.

The proof occupies Subsections 3.2–3.5 and is organized as follows. In Subsection 3.2 we prove that  $Q_{D, \lambda}$  is closed and lower-semibounded by means of the Kato/KLMN form-perturbation theorem, and identify  $\mathcal{V}_\lambda$  as a form core. Subsection 3.3 establishes the Lévy–Khinchine representation (4) and the Markov property of  $\mathcal{E}_{\infty, \lambda}$ . Subsection 3.4 establishes the discrete decomposition (7) and the Markov property of  $\mathcal{E}_{p, \lambda}$ . Finally Subsection 3.5 assembles the four assertions of the theorem.

**3.2. Closedness, lower semiboundedness, and the form core.** By Subsection 2.4 of the present paper, together with [5, §2] and the semilocal restatement in [7, Props. 3.3–3.4], the semilocal Weil sesquilinear form  $QW_\lambda$  is closed, symmetric, and lower-semibounded on  $H_\lambda$ , with the trigonometric space  $\mathcal{V}_\lambda = \text{span}\{V_n : n \in \mathbb{Z}\} \subset H_\lambda$  as a form core. We have

$$(9) \quad Q_{D, \lambda} = QW_\lambda - B_\lambda,$$

where, by Subsection 2.4,

(10)

$$B_\lambda(f, g) = \widehat{f}(i/2) \overline{\widehat{g}(-i/2)} + \widehat{f}(-i/2) \overline{\widehat{g}(i/2)}, \quad \widehat{f}(\pm i/2) = \int_{I_\lambda} f(u) u^{\pm 1/2} d^\times u.$$

LEMMA 3.3. *The form  $B_\lambda$  is a bounded symmetric sesquilinear form on  $H_\lambda \times H_\lambda$ , with explicit operator-form bound*

$$(11) \quad |B_\lambda(f, g)| \leq 2(\lambda - \lambda^{-1}) \|f\|_{H_\lambda} \|g\|_{H_\lambda}, \quad f, g \in H_\lambda.$$

*Proof.* Direct computation of the multiplicative Haar integrals gives

$$\int_{I_\lambda} u d^\times u = \int_{\lambda^{-1}}^{\lambda} du = \lambda - \lambda^{-1}, \quad \int_{I_\lambda} u^{-1} d^\times u = \int_{\lambda^{-1}}^{\lambda} u^{-2} du = \lambda - \lambda^{-1}.$$

By the Cauchy–Schwarz inequality on  $H_\lambda$ ,

$$|\widehat{f}(\pm i/2)| = |\langle f, u^{\pm 1/2} \rangle_{H_\lambda}| \leq \|u^{\pm 1/2}\|_{H_\lambda} \|f\|_{H_\lambda} = \sqrt{\lambda - \lambda^{-1}} \|f\|_{H_\lambda}.$$

Combining the two summands of (10) via the triangle inequality and Cauchy–Schwarz yields (11). The symmetry of  $B_\lambda$  follows from direct inspection of (10) together with the identity  $\widehat{f}(\pm i/2) = \widehat{f}(\pm i/2)$ .  $\square$

PROPOSITION 3.4. *The form  $Q_{D,\lambda}$  is closed, symmetric, and lower-semibounded on  $H_\lambda$ , with*

$$(12) \quad \text{Dom}(Q_{D,\lambda}) = \text{Dom}(QW_\lambda),$$

*and  $\mathcal{V}_\lambda$  is a form core for  $Q_{D,\lambda}$ .*

*Proof.* By Lemma 3.3,  $B_\lambda$  is an everywhere-defined bounded symmetric sesquilinear form on  $H_\lambda$ . The form-perturbation theorem of Kato [18, Theorem VI.1.33] (equivalently, the KLMN theorem of Reed–Simon [10, Theorem X.17], applied with relative bound zero) asserts that the difference of a closed lower-semibounded symmetric form and a bounded everywhere-defined symmetric form is again closed, symmetric, and lower-semibounded, with the *same* form domain. Applying this to (9) with  $QW_\lambda$  closed and  $B_\lambda$  bounded yields (12) together with closedness, symmetry, and lower-semiboundedness of  $Q_{D,\lambda}$ . Since the form domain is preserved, every form core of  $QW_\lambda$  is also a form core for  $Q_{D,\lambda}$ ; in particular  $\mathcal{V}_\lambda$  is a form core.  $\square$

We now define the shifted form  $\mathcal{E}_\lambda$  as in (3). The shift by the bounded scalar form  $c_{D,\lambda} \langle \cdot, \cdot \rangle_{H_\lambda}$  does not alter the domain, the form core, the symmetry, or the closedness of  $Q_{D,\lambda}$  ([18, Theorem VI.1.33] once more); hence

(13)

$$\text{Dom}(\mathcal{E}_\lambda) = \mathcal{F}_\lambda = \text{Dom}(Q_{D,\lambda}) = \text{Dom}(QW_\lambda), \quad \mathcal{V}_\lambda \text{ is a form core for } \mathcal{E}_\lambda,$$

and  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  is closed, symmetric, and lower-semibounded on  $H_\lambda$ .

*Remark 3.5.* A central feature of the explicit choice (2) is that no further auxiliary shift is needed: in Subsections 3.3–3.4 we shall verify directly that

$$(14) \quad \mathcal{E}_\lambda = \mathcal{E}_{\infty, \lambda} + \mathcal{E}_{p, \lambda} \geq 0,$$

with both summands individually nonnegative.

**3.3. The archimedean part: Lévy–Khintchine representation and continuous pure-jump structure.** The aim of this subsection is to write

$$(15) \quad Q_{\infty, \lambda}(f, f) = \int_{\mathbb{R}} |\widehat{f}(t)|^2 w(t) dt, \quad w(t) := \frac{\theta'(t)}{\pi},$$

in the form announced in (4). Throughout this subsection, the phrase “normal contraction” refers to a function  $\Phi : \mathbb{C} \rightarrow \mathbb{C}$  with  $\Phi(0) = 0$  that is 1-Lipschitz on  $\mathbb{C}$ , in accordance with Remark 3.1; the condition  $\Phi(0) = 0$  is used *essentially* in the compatibility (25)–(26) below, and we shall flag it again when invoked.

**3.3.1. An integral representation of  $\theta'(t)$ .** The Riemann–Siegel  $\theta$ -function is defined by

$$\theta(t) = -\frac{t}{2} \log \pi + \Im \left( \log \Gamma \left( \frac{1}{4} + \frac{it}{2} \right) \right).$$

Set  $z(t) = \frac{1}{4} + \frac{it}{2}$ . Then

$$\frac{d}{dt} \log \Gamma(z(t)) = \psi(z(t)) z'(t) = \frac{i}{2} \psi(z(t)),$$

where  $\psi = \Gamma'/\Gamma$  is the digamma function. Since  $\Im(is) = \Re(s)$  for every complex number  $s$ , we obtain

$$\frac{d}{dt} \Im(\log \Gamma(z(t))) = \Im \left( \frac{i}{2} \psi(z(t)) \right) = \frac{1}{2} \Re \psi(z(t)).$$

Hence

$$\theta'(t) = -\frac{1}{2} \log \pi + \frac{1}{2} \Re \psi \left( \frac{1}{4} + \frac{it}{2} \right).$$

For  $\Re z > 0$ , the digamma function satisfies the standard integral representation

$$\psi(z) + \gamma = \int_0^\infty \frac{e^{-s} - e^{-zs}}{1 - e^{-s}} ds,$$

so that

$$\psi(z) = -\gamma + \int_0^\infty \frac{e^{-s} - e^{-zs}}{1 - e^{-s}} ds.$$

Substituting  $z(t) = \frac{1}{4} + \frac{it}{2}$  and taking real parts yields

$$(16) \quad \Re \psi \left( \frac{1}{4} + \frac{it}{2} \right) = -\gamma + \int_0^\infty \frac{e^{-s} - e^{-s/4} \cos(ts/2)}{1 - e^{-s}} ds.$$

Split the numerator into the  $t$ -independent part and the  $(1 - \cos)$  part:

$$e^{-s} - e^{-s/4} \cos(ts/2) = (e^{-s} - e^{-s/4}) + e^{-s/4} (1 - \cos(ts/2)).$$

Substituting this into (16), we obtain

$$\Re \psi\left(\frac{1}{4} + \frac{it}{2}\right) = \psi\left(\frac{1}{4}\right) + \int_0^\infty \frac{e^{-s/4}(1 - \cos(ts/2))}{1 - e^{-s}} ds.$$

After the change of variables  $s = 2r$ , we get

$$\int_0^\infty \frac{e^{-s/4}(1 - \cos(ts/2))}{1 - e^{-s}} ds = \int_0^\infty \frac{2e^{-r/2}}{1 - e^{-2r}} (1 - \cos(tr)) dr.$$

Therefore

$$(17) \quad \theta'(t) = -\frac{1}{2} \log \pi + \frac{1}{2} \psi\left(\frac{1}{4}\right) + \int_0^\infty \frac{e^{-r/2}}{1 - e^{-2r}} (1 - \cos(tr)) dr.$$

Hence

$$(18) \quad w(t) = \frac{\theta'(t)}{\pi} = a_\infty + \int_0^\infty (1 - \cos(tr)) \nu(dr),$$

where

$$(19) \quad a_\infty := \frac{1}{\pi} \left( -\frac{1}{2} \log \pi + \frac{1}{2} \psi\left(\frac{1}{4}\right) \right), \quad \nu(dr) := \frac{1}{\pi} \frac{e^{-r/2}}{1 - e^{-2r}} dr.$$

3.3.2.  $\nu$  is a Lévy measure. By (19), the density of  $\nu$  is strictly positive for all  $r > 0$ . We verify the standard Lévy-measure condition

$$\int_0^\infty (1 \wedge r^2) \nu(dr) < \infty.$$

As  $r \downarrow 0$ , one has  $1 - e^{-2r} \sim 2r$  and  $e^{-r/2} \sim 1$ , hence

$$\frac{e^{-r/2}}{1 - e^{-2r}} \sim \frac{1}{2r}.$$

Therefore, near 0,

$$(1 \wedge r^2) \nu(dr) = r^2 \nu(dr) \sim \frac{1}{2\pi} r dr,$$

so that

$$\int_0^1 (1 \wedge r^2) \nu(dr) < \infty.$$

As  $r \rightarrow \infty$ , since  $1 - e^{-2r} \rightarrow 1$ , there exists  $C > 0$  such that

$$0 \leq \frac{e^{-r/2}}{1 - e^{-2r}} \leq C e^{-r/2}, \quad r \geq 1.$$

Hence

$$\int_1^\infty (1 \wedge r^2) \nu(dr) = \int_1^\infty \nu(dr) \leq \frac{C}{\pi} \int_1^\infty e^{-r/2} dr < \infty.$$

Thus  $\nu$  is indeed a Lévy measure. In particular, for every  $t \in \mathbb{R}$ ,

$$\int_0^\infty (1 - \cos(tr)) \nu(dr) < \infty,$$

so the representation (18) is well defined.

By Sato [15, Def. 8.1], a measure  $\nu$  on  $(0, \infty)$  satisfying  $\int_0^\infty (1 \wedge r^2) \nu(dr) < \infty$  is exactly a one-dimensional Lévy measure restricted to the positive half-line, justifying the terminology.

3.3.3. *Closedness of  $Q_{\infty, \lambda}$ .* We record explicitly the closedness statement that will be used in Subsection 3.5.

LEMMA 3.6. *The form  $Q_{\infty, \lambda}$  is a closed, symmetric, lower-semibounded sesquilinear form on  $H_\lambda$  with lower bound  $2\pi a_\infty$ .*

*Proof.* By (18),  $w(t) = a_\infty + \chi_\lambda(t)$  with  $\chi_\lambda(t) := \int_0^\infty (1 - \cos(tr)) \nu(dr) \geq 0$ . Hence  $w(t) \geq a_\infty$  pointwise on  $\mathbb{R}$ . Plancherel applied to the zero extension  $g = E_x Z f \in L^2(\mathbb{R})$  gives

$$Q_{\infty, \lambda}(f, f) = \int_{\mathbb{R}} |\widehat{g}(t)|^2 w(t) dt \geq a_\infty \int_{\mathbb{R}} |\widehat{g}(t)|^2 dt = 2\pi a_\infty \|g\|_{L^2(\mathbb{R})}^2 = 2\pi a_\infty \|f\|_{H_\lambda}^2,$$

proving the lower bound. For closedness, on

$$\text{Dom}(Q_{\infty, \lambda}) = \left\{ f \in H_\lambda : \int_{\mathbb{R}} |\widehat{g}(t)|^2 (1 + \chi_\lambda(t)) dt < \infty \right\},$$

the graph norm  $\|f\|_{Q_{\infty, \lambda, 1}}^2 := Q_{\infty, \lambda}(f, f) + (1 - 2\pi a_\infty) \|f\|_{H_\lambda}^2 = \int_{\mathbb{R}} |\widehat{g}(t)|^2 (1 + \chi_\lambda(t)) dt$  coincides (up to the unitary  $E_x Z$ ) with the weighted  $L^2$ -norm with strictly positive weight  $1 + \chi_\lambda$ . The corresponding weighted  $L^2$ -space is complete ([9, Vol. I, Prop. II.16]), and  $\text{Dom}(Q_{\infty, \lambda})$  is closed in this norm because it is the preimage of  $L^2(\mathbb{R}, (1 + \chi_\lambda) dt)$  under the unitary  $E_x Z$ . Symmetry of  $Q_{\infty, \lambda}$  is immediate from (15) and the reality of  $w$ .  $\square$

3.3.4. *Plancherel transforms the Fourier multiplier into jump energy.* Let  $f \in H_\lambda$ , let  $Z$  be the unitary of Lemma 2.3, and let  $E_x : L^2(I_x, dx) \rightarrow L^2(\mathbb{R}, dx)$  be the zero-extension operator as in the statement of Theorem 3.2(2). Set

$$g := E_x Z f \in L^2(\mathbb{R}, dx), \quad \widehat{g}(t) = \int_{\mathbb{R}} g(x) e^{-itx} dx.$$

Lemma 2.3 together with the change of variables  $u = e^x$  gives  $\widehat{f}(t) = \widehat{g}(t)$  and  $\|f\|_{H_\lambda} = \|g\|_{L^2(\mathbb{R})}$ . Combining (15), (18), and the nonnegativity of  $|\widehat{g}|^2$  and  $1 - \cos(tr)$ , Tonelli's theorem [16, Vol. I, Thm. 3.4.4] applies and yields

$$(20) \quad Q_{\infty, \lambda}(f, f) = a_\infty \int_{\mathbb{R}} |\widehat{g}(t)|^2 dt + \int_0^\infty \left( \int_{\mathbb{R}} (1 - \cos(tr)) |\widehat{g}(t)|^2 dt \right) \nu(dr).$$

On the other hand,  $\widehat{g(\cdot + r)}(t) = e^{itr} \widehat{g}(t)$ , so  $\widehat{g(\cdot + r)} - \widehat{g}(t) = (e^{itr} - 1) \widehat{g}(t)$  and  $|e^{itr} - 1|^2 = 2(1 - \cos(tr))$ . By Plancherel's theorem [9, Vol. I, Thm. II.1], with the present convention  $\|\widehat{g}\|_{L^2}^2 = 2\pi \|g\|_{L^2}^2$ :

$$(21) \quad \int_{\mathbb{R}} |\widehat{g}(t)|^2 dt = 2\pi \|g\|_{L^2(\mathbb{R})}^2,$$

$$(22) \quad \|g(\cdot+r)-g\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |e^{itr}-1|^2 |\widehat{g}(t)|^2 dt = \frac{1}{\pi} \int_{\mathbb{R}} (1-\cos(tr)) |\widehat{g}(t)|^2 dt.$$

Substituting (21)–(22) into (20) we conclude

$$(23) \quad Q_{\infty,\lambda}(f, f) = 2\pi a_{\infty} \|f\|_{H_{\lambda}}^2 + \pi \int_0^{\infty} \|g(\cdot+r)-g\|_{L^2(\mathbb{R})}^2 \nu(dr).$$

3.3.5. *The shift produces a nonnegative pure-jump form.* The standard special value of the digamma function at the quarter-integer is  $\psi(1/4) = -\gamma - \pi/2 - 3\log 2$  (see Remark 3.7 below for a self-contained derivation). Hence

$$a_{\infty} = \frac{1}{2\pi} (\psi(\frac{1}{4}) - \log \pi) = \frac{1}{2\pi} (-\gamma - \frac{\pi}{2} - 3\log 2 - \log \pi) < 0,$$

so that the first term on the right-hand side of (23) is negative. Setting  $c_{\infty} := -2\pi a_{\infty} > 0$  and  $\kappa_{\infty,\lambda} := 2\pi a_{\infty} + c_{\infty} = 0$ , the definition  $\mathcal{E}_{\infty,\lambda}(f, f) := Q_{\infty,\lambda}(f, f) + c_{\infty} \|f\|_{H_{\lambda}}^2$ , combined with  $\|f\|_{H_{\lambda}} = \|g\|_{L^2(\mathbb{R})}$ , gives

$$(24) \quad \mathcal{E}_{\infty,\lambda}(f, f) = \kappa_{\infty,\lambda} \|f\|_{H_{\lambda}}^2 + \pi \int_0^{\infty} \|g(\cdot+r)-g\|_{L^2(\mathbb{R})}^2 \nu(dr) = \pi \int_0^{\infty} \|g(\cdot+r)-g\|_{L^2(\mathbb{R})}^2 \nu(dr).$$

This proves (4), and shows in particular that  $\mathcal{E}_{\infty,\lambda}(f, f) \geq 0$  for every  $f \in \text{Dom}(Q_{\infty,\lambda})$ .

*Remark 3.7.* We have  $c_{\infty} = -2\pi a_{\infty} = \log \pi - \psi(1/4) = \gamma + \pi/2 + 3\log 2 + \log \pi > 0$ , which is the explicit form announced in (6). A self-contained proof of  $\psi(1/4) = -\gamma - \pi/2 - 3\log 2$  proceeds via the Gauss multiplication formula  $\psi(1/4) + \psi(3/4) = 2\psi(1/2) - 2\log 2 = -2\gamma - 6\log 2$  and the reflection formula  $\psi(3/4) - \psi(1/4) = \pi \cot(\pi/4) = \pi$ .

3.3.6. *Markov contraction for the archimedean part.* Let  $f \in \text{Dom}(Q_{\infty,\lambda})$  and let  $\Phi : \mathbb{C} \rightarrow \mathbb{C}$  be a normal contraction in the sense of Remark 3.1. Set  $\varphi := Zf \in L^2([-\ell, \ell], dx)$ ,  $\tilde{\varphi} := \Phi \circ \varphi \in L^2([-\ell, \ell], dx)$ , and  $g := E_x \varphi$ .

*Step 1: zero-extension compatibility.* The condition  $\Phi(0) = 0$  enters crucially: since  $g$  vanishes a.e. on  $\mathbb{R} \setminus [-\ell, \ell]$ , for a.e.  $x \in \mathbb{R} \setminus [-\ell, \ell]$ ,

$$(\Phi \circ g)(x) = \Phi(g(x)) = \Phi(0) = 0.$$

Hence

$$(25) \quad \Phi \circ g = E_x \tilde{\varphi} \quad \text{a.e. on } \mathbb{R}.$$

On the other hand,

$$(26) \quad (Z(\Phi \circ f))(x) = \Phi(f(e^x)) = \Phi(\varphi(x)) = \tilde{\varphi}(x), \quad x \in [-\ell, \ell],$$

so  $\Phi \circ g = E_x Z(\Phi \circ f)$  a.e. on  $\mathbb{R}$ .

*Step 2: pointwise contraction.* For every  $r > 0$  and a.e.  $x \in \mathbb{R}$ , by the 1-Lipschitz property of  $\Phi$ ,

$$(27) \quad |\Phi(g(x+r)) - \Phi(g(x))| \leq |g(x+r) - g(x)|.$$

Squaring and integrating in  $x$ ,

$$(28) \quad \|\Phi \circ g(\cdot + r) - \Phi \circ g\|_{L^2(\mathbb{R})}^2 \leq \|g(\cdot + r) - g\|_{L^2(\mathbb{R})}^2, \quad r > 0.$$

*Step 3: integration against the Lévy measure.* Since  $\nu$  is a positive measure on  $(0, \infty)$  and the integrands in (28) are nonnegative measurable functions of  $r$ , Tonelli's theorem ([16, Thm. 3.4.4]) preserves the inequality under  $\nu$ -integration:

$$(29) \quad \int_0^\infty \|\Phi \circ g(\cdot + r) - \Phi \circ g\|_{L^2(\mathbb{R})}^2 \nu(dr) \leq \int_0^\infty \|g(\cdot + r) - g\|_{L^2(\mathbb{R})}^2 \nu(dr).$$

*Step 4: conclusion.* Applying (24) to  $f$  and to  $\Phi \circ f$  separately, then using (25)–(26) to identify the zero extensions of  $Z(\Phi \circ f)$  and  $\Phi \circ Zf$ , and finally invoking (29), we obtain

$$(30) \quad \mathcal{E}_{\infty, \lambda}(\Phi \circ f, \Phi \circ f) \leq \mathcal{E}_{\infty, \lambda}(f, f).$$

In particular, finiteness of the right-hand side of (30) implies finiteness of the left-hand side; combined with the obvious  $L^2$ -bound  $\|\Phi \circ f\|_{H_\lambda} \leq \|f\|_{H_\lambda}$  (which follows from  $|\Phi(z)| \leq |z|$ , i.e. from  $\Phi(0) = 0$  and Lipschitz continuity), this yields  $\Phi \circ f \in \text{Dom}(Q_{\infty, \lambda})$ :

$$(31) \quad f \in \text{Dom}(Q_{\infty, \lambda}) \implies \Phi \circ f \in \text{Dom}(Q_{\infty, \lambda}).$$

3.4. *The prime part: finitely many discrete jumps and killing.* We now treat

$$Q_{p, \lambda}(f, f) = - \sum_{1 < n \leq \lambda^2} \Lambda(n) \langle T(n)f, f \rangle_{H_\lambda}, \quad T(n) = n^{-1/2}(U_n + U_n^*),$$

on  $f \in H_\lambda$ . Throughout this subsection, “normal contraction” has the same meaning as in Remark 3.1; the condition  $\Phi(0) = 0$  enters essentially in (38) below.

3.4.1. *Contractivity of  $U_n$ .* From the definition  $U_n = R_\lambda S_n E_\lambda$  in Section 2, together with the pointwise formula

$$(U_n f)(u) = \mathbf{1}_{I_\lambda}(nu) f(nu),$$

we obtain for every  $f \in H_\lambda$ ,

$$\|U_n f\|_{H_\lambda}^2 = \int_{I_\lambda \cap n^{-1}I_\lambda} |f(nu)|^2 d^\times u.$$

Let  $v = nu$ . Then  $dv/v = du/u = d^\times u$ , and  $u \in I_\lambda \cap n^{-1}I_\lambda$  is equivalent to  $v \in nI_\lambda \cap I_\lambda$ . Therefore

$$(32) \quad \|U_n f\|_{H_\lambda}^2 = \int_{I_\lambda \cap nI_\lambda} |f(v)|^2 \frac{dv}{v} \leq \int_{I_\lambda} |f(v)|^2 \frac{dv}{v} = \|f\|_{H_\lambda}^2.$$

Thus  $U_n$  is an  $L^2$ -contraction.

3.4.2. *Rewriting as jump + killing.* Substituting  $T(n) = n^{-1/2}(U_n + U_n^*)$  into  $Q_{p,\lambda}$  gives, for each  $f \in H_\lambda$ ,

$$-\Lambda(n)\langle T(n)f, f \rangle = -\Lambda(n)n^{-1/2}(\langle U_nf, f \rangle + \langle U_n^*f, f \rangle).$$

Because the inner product is conjugate-linear in the second variable,  $\langle U_n^*f, f \rangle = \overline{\langle U_nf, f \rangle}$ , so

$$-\Lambda(n)\langle T(n)f, f \rangle = -2\Lambda(n)n^{-1/2}\Re\langle U_nf, f \rangle.$$

Adding  $2\Lambda(n)n^{-1/2}\|f\|_{H_\lambda}^2$  to both sides and applying the polarization identity

$$(33) \quad \|f - U_nf\|_{H_\lambda}^2 = \|f\|_{H_\lambda}^2 + \|U_nf\|_{H_\lambda}^2 - 2\Re\langle U_nf, f \rangle$$

to express  $2(\|f\|_{H_\lambda}^2 - \Re\langle U_nf, f \rangle)$  as  $\|f - U_nf\|_{H_\lambda}^2 + (\|f\|_{H_\lambda}^2 - \|U_nf\|_{H_\lambda}^2)$ , we obtain

$$(34) \quad \begin{aligned} -\Lambda(n)\langle T(n)f, f \rangle + 2\Lambda(n)n^{-1/2}\|f\|_{H_\lambda}^2 &= \Lambda(n)n^{-1/2}\|f - U_nf\|_{H_\lambda}^2 \\ &\quad + \Lambda(n)n^{-1/2}(\|f\|_{H_\lambda}^2 - \|U_nf\|_{H_\lambda}^2). \end{aligned}$$

By (32) the second summand is nonnegative; the first is a squared  $L^2$ -norm and hence also nonnegative. Therefore each term of (34) is intrinsically nonnegative, justifying the labels “jump” and “killing.”

Summing over  $1 < n \leq \lambda^2$  and recalling  $c_{p,\lambda} = 2\sum_{1 < n \leq \lambda^2} \Lambda(n)n^{-1/2}$ , we obtain

$$(35) \quad \begin{aligned} \mathcal{E}_{p,\lambda}(f, f) &:= Q_{p,\lambda}(f, f) + c_{p,\lambda}\|f\|_{H_\lambda}^2 \\ &= \sum_{1 < n \leq \lambda^2} \Lambda(n)n^{-1/2}\|f - U_nf\|_{H_\lambda}^2 \\ &\quad + \sum_{1 < n \leq \lambda^2} \Lambda(n)n^{-1/2}(\|f\|_{H_\lambda}^2 - \|U_nf\|_{H_\lambda}^2) \geq 0, \end{aligned}$$

which is (7). Moreover, by (32),  $\|T(n)\| \leq 2n^{-1/2}$ , so

$$(36) \quad |Q_{p,\lambda}(f, g)| \leq 2\left(\sum_{1 < n \leq \lambda^2} \Lambda(n)n^{-1/2}\right)\|f\|_{H_\lambda}\|g\|_{H_\lambda}, \quad f, g \in H_\lambda;$$

hence  $Q_{p,\lambda}$  is a bounded sesquilinear form on  $H_\lambda \times H_\lambda$ , and  $\text{Dom}(\mathcal{E}_{p,\lambda}) = H_\lambda$ . In particular,

$$(37) \quad \text{Dom}(\mathcal{E}_{p,\lambda}) = H_\lambda \supseteq \mathcal{F}_\lambda.$$

3.4.3. *Markov contraction for the prime part.* Let  $\Phi : \mathbb{C} \rightarrow \mathbb{C}$  be a normal contraction and let  $f \in H_\lambda$ . From the pointwise formula for  $U_n$  in Proposition 2.7,

$$(U_nh)(u) = \mathbf{1}_{I_\lambda}(nu)h(nu) \quad \text{a.e. on } I_\lambda, \quad h \in H_\lambda,$$



together with  $\Phi(0) = 0$ , we deduce that for a.e.  $u \in I_\lambda$ ,

$$(38) \quad \begin{aligned} U_n(\Phi \circ f)(u) &= \mathbf{1}_{I_\lambda}(nu)\Phi(f(nu)) \\ &\stackrel{(*)}{=} \Phi(\mathbf{1}_{I_\lambda}(nu)f(nu)) = \Phi(U_nf(u)), \end{aligned}$$

where the equality  $(*)$  rests on case analysis: if  $\mathbf{1}_{I_\lambda}(nu) = 1$ , both sides equal  $\Phi(f(nu))$ ; if  $\mathbf{1}_{I_\lambda}(nu) = 0$ , the left side equals 0 and the right side equals  $\Phi(0) = 0$ . The use of  $\Phi(0) = 0$  is therefore essential here.

1. *Contraction of the jump term.* By the 1-Lipschitz property of  $\Phi$  and (38), for a.e.  $u \in I_\lambda$ ,

$$|\Phi(f(u)) - U_n(\Phi \circ f)(u)| = |\Phi(f(u)) - \Phi(U_nf(u))| \leq |f(u) - U_nf(u)|.$$

Squaring and integrating over  $I_\lambda$ ,

$$(39) \quad \|\Phi \circ f - U_n(\Phi \circ f)\|_{H_\lambda}^2 \leq \|f - U_nf\|_{H_\lambda}^2.$$

2. *Contraction of the killing term.* Applying (32) with  $h \in H_\lambda$  in place of  $f$  gives

$$\|h\|_{H_\lambda}^2 - \|U_nh\|_{H_\lambda}^2 = \int_{I_\lambda \setminus nI_\lambda} |h(v)|^2 d^\times v.$$

Setting  $h = \Phi \circ f$  and using  $|\Phi(z)| \leq |z|$  (which again rests on  $\Phi(0) = 0$  together with the Lipschitz condition) pointwise on  $I_\lambda$ , we obtain

$$(40) \quad \|\Phi \circ f\|_{H_\lambda}^2 - \|U_n(\Phi \circ f)\|_{H_\lambda}^2 \leq \|f\|_{H_\lambda}^2 - \|U_nf\|_{H_\lambda}^2.$$

3. *Substitution into (35).* Since  $\Lambda(n)n^{-1/2} \geq 0$  for every  $n$ , summing (39) and (40) term by term over  $1 < n \leq \lambda^2$  yields

$$(41) \quad \mathcal{E}_{p,\lambda}(\Phi \circ f, \Phi \circ f) \leq \mathcal{E}_{p,\lambda}(f, f).$$

Domain stability is automatic from (37):  $\Phi \circ f \in H_\lambda = \text{Dom}(\mathcal{E}_{p,\lambda})$  whenever  $f \in H_\lambda$ .

3.5. *Combination and conclusion of Theorem 3.2.* We assemble the four assertions of Theorem 3.2 from the preceding subsections.

*Proof of Theorem 3.2. Statements (2)–(4).* Statement (2) is exactly (24), combined with the verification that  $\nu$  is a Lévy measure (Subsection 3.3.2) and the explicit value (6) (Remark 3.7). Statement (3) is (35) together with the boundedness estimate (36). The decomposition (4) now follows from the linearity of all the shifts:

$$\begin{aligned} \mathcal{E}_\lambda(f, f) &= Q_{D,\lambda}(f, f) + c_{D,\lambda}\|f\|_{H_\lambda}^2 \\ &= (Q_{\infty,\lambda}(f, f) + c_{\infty}\|f\|_{H_\lambda}^2) + (Q_{p,\lambda}(f, f) + c_{p,\lambda}\|f\|_{H_\lambda}^2) \\ &= \mathcal{E}_{\infty,\lambda}(f, f) + \mathcal{E}_{p,\lambda}(f, f), \quad f \in \mathcal{F}_\lambda, \end{aligned}$$

and both summands are individually nonnegative by (24) and (35).

*Statement (1).* It remains to verify that  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  is a closed, symmetric, nonnegative Dirichlet form on  $H_\lambda$ , with  $\mathcal{V}_\lambda$  as a form core.

**(a) Closedness, symmetry, and form core.** By Proposition 3.4 and (13),  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  is a closed symmetric form on  $H_\lambda$  with form core  $\mathcal{V}_\lambda$ .

**(b) Nonnegativity.** By (4) just proved and the nonnegativity of both summands,

$$\mathcal{E}_\lambda(f, f) = \mathcal{E}_{\infty, \lambda}(f, f) + \mathcal{E}_{p, \lambda}(f, f) \geq 0, \quad f \in \mathcal{F}_\lambda.$$

**(c) Markov property.** Let  $\Phi$  be a normal contraction in the sense of Remark 3.1, and let  $f \in \mathcal{F}_\lambda$ .

*Domain stability.* Since  $\text{Dom}(\mathcal{E}_\lambda) = \mathcal{F}_\lambda = \text{Dom}(Q_{\infty, \lambda})$  (equivalence of graph norms by (13) together with the  $L^2$ -boundedness of the prime part, cf. (37)), the domain stability (31) for  $Q_{\infty, \lambda}$  implies  $\Phi \circ f \in \mathcal{F}_\lambda$ .

*Energy contraction.* Combining (30) and (41), both of which hold pointwise in  $f \in \mathcal{F}_\lambda$ ,

$$\begin{aligned} \mathcal{E}_\lambda(\Phi \circ f, \Phi \circ f) &= \mathcal{E}_{\infty, \lambda}(\Phi \circ f, \Phi \circ f) + \mathcal{E}_{p, \lambda}(\Phi \circ f, \Phi \circ f) \\ (42) \quad &\leq \mathcal{E}_{\infty, \lambda}(f, f) + \mathcal{E}_{p, \lambda}(f, f) = \mathcal{E}_\lambda(f, f). \end{aligned}$$

The conjunction of (a), (b), (c) verifies the defining conditions of a complex symmetric Dirichlet form (Remark 3.1; cf. [14, Sec. I.4, Thm. 4.4] or [13, Thm. 1.4.1] after the standard real/imaginary decomposition).  $\square$

*Remark 3.8.* With the explicit choice

$$c_{D, \lambda} = c_\infty + c_{p, \lambda}, \quad c_{p, \lambda} := 2 \sum_{1 < n \leq \lambda^2} \Lambda(n) n^{-1/2},$$

the total shift is completely determined by  $\lambda$ . Moreover,

$$c_\infty = -2\pi a_\infty = \log \pi - \psi\left(\frac{1}{4}\right) = \gamma + \frac{\pi}{2} + 3 \log 2 + \log \pi,$$

and, by Abel summation together with the prime number theorem in the form  $\psi(x) = \sum_{n \leq x} \Lambda(n) \sim x$ ,

$$c_{p, \lambda} = 4\lambda + o(\lambda) \quad (\lambda \rightarrow \infty).$$

Hence

$$c_{D, \lambda} = c_\infty + 4\lambda + o(\lambda).$$

Equivalently, for every  $\varepsilon > 0$ , there exists  $\lambda_0(\varepsilon) > 1$  such that

$$c_\infty + (4 - \varepsilon)\lambda \leq c_{D, \lambda} \leq c_\infty + (4 + \varepsilon)\lambda \quad (\lambda \geq \lambda_0(\varepsilon)).$$

#### 4. Irreducibility and Positivity Improvement

In this section, we fix  $\lambda > 1$  and retain all notation from Sections 2–3. Let

$$(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$$

denote the shifted symmetric Dirichlet form. It admits the decomposition

$$\mathcal{E}_\lambda = \mathcal{E}_{\infty, \lambda} + \mathcal{E}_{p, \lambda}.$$

Here  $\mathcal{E}_{\infty, \lambda}$  is the continuous pure-jump part, while  $\mathcal{E}_{p, \lambda}$  is the bounded non-negative symmetric perturbation coming from the finitely many discrete jump terms and their killing contribution. The core of this section has two steps: we first prove irreducibility, and then deduce positivity improvement for both the semigroup and the resolvent from irreducibility.

4.1. *Irreducibility.* We now prove that  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  is irreducible.

PROPOSITION 4.1. *Let*

$$H_\lambda = L^2(I_\lambda, \mu), \quad \mu := d^\times u = \frac{du}{u}, \quad I_\lambda = [\lambda^{-1}, \lambda],$$

and write

$$H_{\text{tot}} = L^2(\mathbb{R}_{>0}^\times, d^\times u).$$

Let  $(T_t^\lambda)_{t>0}$  be the symmetric sub-Markov  $C_0$ -semigroup associated with  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ . Then  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  is irreducible on  $H_\lambda$ ; that is, there exists no measurable set  $B \subset I_\lambda$  such that

$$0 < \mu(B) < \mu(I_\lambda)$$

and

$$(43) \quad T_t^\lambda(\mathbf{1}_B f) = \mathbf{1}_B T_t^\lambda f, \quad \forall f \in H_\lambda, \quad \forall t > 0.$$

*Proof.* We argue by contradiction: toward a contradiction, suppose there exists a measurable set  $B \subset I_\lambda$  with  $0 < \mu(B) < \mu(I_\lambda)$  for which (43) holds. Set

$$C := I_\lambda \setminus B, \quad \mathbf{1}_{I_\lambda} = \mathbf{1}_B + \mathbf{1}_C \quad \mu\text{-a.e. on } I_\lambda,$$

and let  $P_B, P_C \in \mathcal{B}(H_\lambda)$  denote the multiplication operators

$$P_B f := \mathbf{1}_B f, \quad P_C f := \mathbf{1}_C f.$$

By construction  $P_B + P_C = I_{H_\lambda}$ ,  $P_B^* = P_B$ ,  $P_C^* = P_C$ ,  $P_B^2 = P_B$ ,  $P_C^2 = P_C$ , and  $P_B P_C = 0$ . The argument is organized into seven parts (i)–(vii) below; the final contradiction is reached in (vii).

(i) *From semigroup invariance to a form decomposition.* Let  $L_\lambda$  denote the unique nonnegative self-adjoint operator on  $H_\lambda$  associated with the closed symmetric form  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  via the first representation theorem [18, Chap. VI, Theorem 2.1]. Then  $T_t^\lambda = e^{-tL_\lambda}$  for  $t > 0$ ,  $\mathcal{F}_\lambda = \text{Dom}(L_\lambda^{1/2})$ , and the second

representation theorem [18, Chap. VI, Theorem 2.23] gives

$$(44) \quad \mathcal{E}_\lambda(f, g) = \langle L_\lambda^{1/2} f, L_\lambda^{1/2} g \rangle_{H_\lambda}, \quad f, g \in \mathcal{F}_\lambda.$$

The invariance (43) reads  $T_t^\lambda P_B = P_B T_t^\lambda$  for every  $t > 0$ . By the boundedness of  $P_B$  and the strong continuity of  $(T_t^\lambda)_{t>0}$ , the strong-operator Bochner integral

$$G_\alpha^\lambda := (L_\lambda + \alpha)^{-1} = \int_0^\infty e^{-\alpha t} T_t^\lambda dt, \quad \alpha > 0,$$

satisfies

$$(45) \quad G_\alpha^\lambda P_B = P_B G_\alpha^\lambda, \quad \alpha > 0.$$

We now upgrade (45) to commutation with  $L_\lambda^{1/2}$ . By the bounded Borel functional calculus for self-adjoint operators [9, Thm. VIII.5], every bounded Borel function  $\varphi$  of  $L_\lambda$  lies in the strong-operator closure of the unital  $*$ -algebra generated by  $\{(L_\lambda + \alpha)^{-1} : \alpha > 0\}$ . Indeed, by the Stone–Weierstrass theorem, polynomials in  $\{(t + \alpha)^{-1} : \alpha > 0\}$  are uniformly dense in  $C_0([0, \infty))$ , and the bounded Borel functions are obtained from  $C_0$  by uniformly bounded pointwise limits, which transfer to strong-operator limits in the functional calculus. From (45),  $P_B$  therefore commutes with  $\varphi(L_\lambda)$  for every bounded Borel function  $\varphi$ .

Apply this to the bounded truncations  $\varphi_M(t) := \min(\sqrt{t}, M) \mathbf{1}_{[0, \infty)}(t)$ ,  $M > 0$ :  $P_B \varphi_M(L_\lambda) = \varphi_M(L_\lambda) P_B$  on  $H_\lambda$ . As  $M \rightarrow \infty$ ,  $\varphi_M(L_\lambda) \rightarrow L_\lambda^{1/2}$  in the strong-resolvent sense on  $\text{Dom}(L_\lambda^{1/2})$ . Combined with the boundedness of  $P_B$  this yields

$$(46) \quad P_B \text{Dom}(L_\lambda^{1/2}) \subset \text{Dom}(L_\lambda^{1/2}), \quad L_\lambda^{1/2} P_B f = P_B L_\lambda^{1/2} f \quad (f \in \mathcal{F}_\lambda).$$

The same identity holds with  $P_B$  replaced by  $P_C = I_{H_\lambda} - P_B$ .

For  $f \in \mathcal{F}_\lambda$ , the orthogonal decomposition  $f = P_B f + P_C f$  together with (46) gives

$$L_\lambda^{1/2} f = L_\lambda^{1/2} P_B f + L_\lambda^{1/2} P_C f = P_B L_\lambda^{1/2} f + P_C L_\lambda^{1/2} f,$$

where the two summands on the right are orthogonal because  $P_B P_C = 0$ . Substituting into (44) and applying the Pythagorean identity,

$$(47) \quad \mathcal{E}_\lambda(f) = \mathcal{E}_\lambda(P_B f) + \mathcal{E}_\lambda(P_C f), \quad f \in \mathcal{F}_\lambda,$$

where  $\mathcal{E}_\lambda(f) := \mathcal{E}_\lambda(f, f)$ .

In particular, once (iii) below verifies that  $\mathbf{1}_{I_\lambda} \in \mathcal{F}_\lambda$ , substitution of  $f = \mathbf{1}_{I_\lambda}$  into (47) produces the identity

$$(48) \quad \mathcal{E}_\lambda(\mathbf{1}_{I_\lambda}) = \mathcal{E}_\lambda(\mathbf{1}_B) + \mathcal{E}_\lambda(\mathbf{1}_C),$$

which we shall contradict in (vii).

(ii) *The  $\nu$ -decomposition of  $\mathcal{E}_\lambda$ .* For every  $f \in \mathcal{F}_\lambda$ , set  $g := E_\lambda f \in H_{\text{tot}}$ . Theorem 3.2, combined with the change of variables  $u = e^x$ ,  $d^\times u = dx$ , gives

$$(49) \quad \mathcal{E}_\lambda(f, f) = J_{\infty, \lambda}(f, f) + \mathcal{E}_{p, \lambda}(f, f),$$

where

$$(50) \quad J_{\infty, \lambda}(f, f) = \pi \int_0^\infty \|g(e^r \cdot) - g\|_{H_{\text{tot}}}^2 \nu(dr),$$

$$(51) \quad \nu(dr) = \frac{1}{\pi} \frac{e^{-r/2}}{1 - e^{-2r}} dr, \quad r > 0,$$

and

$$(52) \quad \mathcal{E}_{p, \lambda}(f, f) = \sum_{1 < n \leq \lambda^2} 2a_n (\|f\|_{H_\lambda}^2 - \Re \langle U_n f, f \rangle_{H_\lambda}), \quad a_n := \Lambda(n) n^{-1/2} > 0.$$

The transfer of (50) from the additive form  $\pi \int_0^\infty \|\tilde{g}(\cdot + r) - \tilde{g}\|_{L^2(\mathbb{R})}^2 \nu(dr)$  of Theorem 3.2 (where  $\tilde{g}(x) := g(e^x)$ ) to the multiplicative form (50) is immediate from the isometry  $H_{\text{tot}} \rightarrow L^2(\mathbb{R}, dx)$  provided by  $g \mapsto \tilde{g}$ .

(iii) *Membership  $\mathbf{1}_{I_\lambda} \in \mathcal{F}_\lambda$ .* Set  $g_I := E_\lambda \mathbf{1}_{I_\lambda}$ . Since  $g_I = \mathbf{1}_{I_\lambda}$  viewed as a function on  $\mathbb{R}_{>0}^\times$ ,

$$(53) \quad \|g_I(e^r \cdot) - g_I\|_{H_{\text{tot}}}^2 = \int_{\mathbb{R}_{>0}^\times} |\mathbf{1}_{e^{-r} I_\lambda}(u) - \mathbf{1}_{I_\lambda}(u)|^2 d^\times u = \mu(I_\lambda \triangle e^{-r} I_\lambda).$$

With  $\ell := \log \lambda$ ,  $I_\lambda = [e^{-\ell}, e^\ell]$  and  $e^{-r} I_\lambda = [e^{-r-\ell}, e^{-r+\ell}]$ . For  $0 < r \leq 2\ell$  the intervals overlap on  $[e^{-\ell}, e^{-r+\ell}]$  with  $d^\times u$ -measure  $(-r + \ell) - (-\ell) = 2\ell - r$ . By Haar invariance,  $\mu(e^{-r} I_\lambda) = \mu(I_\lambda) = 2\ell$ , so

$$\mu(I_\lambda \triangle e^{-r} I_\lambda) = \mu(I_\lambda) + \mu(e^{-r} I_\lambda) - 2\mu(I_\lambda \cap e^{-r} I_\lambda) = 2r.$$

For  $r > 2\ell$ ,  $e^{-r+\ell} < e^{-\ell}$ , so the intervals are disjoint and  $\mu(I_\lambda \triangle e^{-r} I_\lambda) = 2\mu(I_\lambda) = 4\ell$ . Combining,

$$(54) \quad \mu(I_\lambda \triangle e^{-r} I_\lambda) = \begin{cases} 2r, & 0 < r \leq 2\ell, \\ 4\ell, & r > 2\ell. \end{cases}$$

We now check integrability of (54) against  $\nu(dr)$ . From  $1 - e^{-2r} = 2r(1 + O(r))$  as  $r \downarrow 0$  we obtain the elementary bound

$$\frac{e^{-r/2}}{1 - e^{-2r}} \leq \frac{1}{r}, \quad 0 < r \leq 1,$$

and consequently  $\nu(dr) \leq \frac{1}{\pi} dr/r$  on  $(0, 1]$ . Hence

$$\int_0^{2\ell} 2r \nu(dr) \leq \frac{2}{\pi} \int_0^{\min(1, 2\ell)} dr + \frac{2(2\ell)}{\pi(1 - e^{-2})} \int_1^{2\ell} e^{-r/2} dr < \infty.$$

For the tail,  $1 - e^{-2r} \geq 1 - e^{-4\ell} > 0$  for  $r \geq 2\ell$ , so

$$\int_{2\ell}^{\infty} 4\ell \nu(dr) \leq \frac{4\ell}{\pi(1 - e^{-4\ell})} \int_{2\ell}^{\infty} e^{-r/2} dr < \infty.$$

By (50) and (53),  $J_{\infty,\lambda}(\mathbf{1}_{I_\lambda}, \mathbf{1}_{I_\lambda}) < \infty$ . The form  $\mathcal{E}_{p,\lambda}$  is bounded on  $H_\lambda$  by Theorem 3.2, and  $\|\mathbf{1}_{I_\lambda}\|_{H_\lambda}^2 = \mu(I_\lambda) = 2\ell < \infty$ . Therefore (49) yields  $\mathbf{1}_{I_\lambda} \in \mathcal{F}_\lambda$ , and identity (48) is established.

(iv)  *$L^2$ -additivity of the indicator decomposition.* Set

$$g_B := E_\lambda \mathbf{1}_B, \quad g_C := E_\lambda \mathbf{1}_C.$$

Then  $g_I = g_B + g_C$  pointwise on  $\mathbb{R}_{>0}^\times$ . Because  $B \cap C = \emptyset$ ,  $\mathbf{1}_B \mathbf{1}_C = 0$   $\mu$ -a.e. on  $I_\lambda$ ; both  $g_B$  and  $g_C$  vanish off  $I_\lambda$ , so  $g_B g_C = 0$   $d^\times u$ -a.e. on  $\mathbb{R}_{>0}^\times$ . By the Pythagorean identity,

$$(55) \quad \|\mathbf{1}_{I_\lambda}\|_{H_\lambda}^2 = \|\mathbf{1}_B\|_{H_\lambda}^2 + \|\mathbf{1}_C\|_{H_\lambda}^2,$$

$$(56) \quad \|g_I\|_{H_{\text{tot}}}^2 = \|g_B\|_{H_{\text{tot}}}^2 + \|g_C\|_{H_{\text{tot}}}^2.$$

(v) *Pointwise expansion of the jump kernel.* For  $r > 0$  and  $h \in H_{\text{tot}}$ , set  $\Delta_r h(u) := h(e^r u) - h(u)$ . Because  $g_I = g_B + g_C$ ,  $\Delta_r g_I = \Delta_r g_B + \Delta_r g_C$ , and the algebraic identity  $a^2 + b^2 - (a + b)^2 = -2ab$  applied with  $a = \Delta_r g_B$ ,  $b = \Delta_r g_C$  gives

$$(57) \quad |\Delta_r g_B|^2 + |\Delta_r g_C|^2 - |\Delta_r g_I|^2 = -2 \Delta_r g_B \Delta_r g_C.$$

Expanding the product,

$$(58) \quad \begin{aligned} \Delta_r g_B \Delta_r g_C &= g_B(e^r u) g_C(e^r u) - g_B(e^r u) g_C(u) \\ &\quad - g_B(u) g_C(e^r u) + g_B(u) g_C(u). \end{aligned}$$

By (iv),  $g_B g_C = 0$   $d^\times u$ -a.e. on  $\mathbb{R}_{>0}^\times$ . The map  $u \mapsto e^r u$  preserves  $d^\times u$ , so  $g_B(e^r \cdot) g_C(e^r \cdot) = 0$   $d^\times u$ -a.e. as well; thus the first term in (58) vanishes  $d^\times u$ -a.e. The fourth term vanishes by the same fact established in (iv). Substituting into (57) we obtain,  $d^\times u$ -a.e.,

$$(59) \quad |\Delta_r g_B|^2 + |\Delta_r g_C|^2 - |\Delta_r g_I|^2 = 2(g_B(e^r u) g_C(u) + g_C(e^r u) g_B(u)) \geq 0,$$

the nonnegativity following from  $g_B, g_C \geq 0$  pointwise. Integrating (59) over  $\mathbb{R}_{>0}^\times$  with respect to  $d^\times u$ ,

$$(60) \quad \|\Delta_r g_B\|_{H_{\text{tot}}}^2 + \|\Delta_r g_C\|_{H_{\text{tot}}}^2 - \|\Delta_r g_I\|_{H_{\text{tot}}}^2 = 2 \bar{H}(r),$$

where

$$(61) \quad \bar{H}(r) := \int_{\mathbb{R}_{>0}^\times} g_B(e^r u) g_C(u) d^\times u + \int_{\mathbb{R}_{>0}^\times} g_C(e^r u) g_B(u) d^\times u \geq 0.$$

Multiplying (60) by  $\pi$  and integrating against  $\nu(dr)$  on  $(0, \infty)$ , Tonelli's theorem yields

$$(62) \quad J_{\infty, \lambda}(\mathbf{1}_B) + J_{\infty, \lambda}(\mathbf{1}_C) - J_{\infty, \lambda}(\mathbf{1}_{I_\lambda}) = 2\pi \int_0^\infty \overline{H}(r) \nu(dr) \geq 0.$$

(vi) *Strict positivity at some  $r_0 > 0$ .* We now upgrade (62) to a strict inequality. The argument proceeds in three substeps: density-point selection, an inclusion-exclusion bound, and translation continuity.

*Selection of density points.* Pass to additive logarithmic coordinates  $x = \log u$ , and write

$$\widetilde{B} := \log B \subset [-\ell, \ell], \quad \widetilde{C} := \log C \subset [-\ell, \ell].$$

The map  $\log : I_\lambda \rightarrow [-\ell, \ell]$  is a bi-Lipschitz bijection sending  $d^\times u$  to Lebesgue measure  $dx$ , so  $\widetilde{B}, \widetilde{C}$  are Lebesgue measurable with  $|\widetilde{B}| = \mu(B) > 0$  and  $|\widetilde{C}| = \mu(C) > 0$ , and  $\widetilde{B} \cap \widetilde{C} = \emptyset$ . By Lebesgue's density theorem [12, Theorem 7.7], the sets

$$\begin{aligned} \widetilde{B}^* &:= \left\{ x \in \widetilde{B} : \lim_{\delta \downarrow 0} \frac{|\widetilde{B} \cap (x - \delta, x + \delta)|}{2\delta} = 1 \right\}, \\ \widetilde{C}^* &:= \left\{ x \in \widetilde{C} : \lim_{\delta \downarrow 0} \frac{|\widetilde{C} \cap (x - \delta, x + \delta)|}{2\delta} = 1 \right\} \end{aligned}$$

satisfy  $|\widetilde{B} \setminus \widetilde{B}^*| = |\widetilde{C} \setminus \widetilde{C}^*| = 0$ , hence  $|\widetilde{B}^*| = |\widetilde{B}| > 0$  and  $|\widetilde{C}^*| = |\widetilde{C}| > 0$ . Since  $\widetilde{B}^* \subset [-\ell, \ell]$  has positive Lebesgue measure, the subset  $\widetilde{B}^* \cap (-\ell, \ell)$  also has positive measure (as  $\widetilde{B}^* \setminus (-\ell, \ell)$  is contained in the two-point set  $\{-\ell, \ell\}$ , which is null); fix once and for all  $x_B \in \widetilde{B}^* \cap (-\ell, \ell)$  and similarly  $x_C \in \widetilde{C}^* \cap (-\ell, \ell)$ . Since  $\widetilde{B}$  and  $\widetilde{C}$  are disjoint,  $x_B \neq x_C$ .

The right-hand side of (61) is symmetric under the involution  $(B, C) \leftrightarrow (C, B)$  (the two integrals exchange). Likewise the left-hand side of (62) is symmetric in  $B$  and  $C$ . After relabeling  $B \leftrightarrow C$  if  $x_B < x_C$ , we may take  $x_B > x_C$  without loss of generality. Set

$$r_0 := x_B - x_C > 0, \quad u_B := e^{x_B}, \quad u_C := e^{x_C}, \quad e^{r_0} u_C = u_B.$$

*Inclusion-exclusion bound for  $\overline{H}(r_0)$ .* For  $\varepsilon > 0$ , define the multiplicative neighborhoods

$$J_B := (u_B e^{-\varepsilon}, u_B e^{\varepsilon}), \quad J_C := (u_C e^{-\varepsilon}, u_C e^{\varepsilon}).$$

By Haar invariance,  $\mu(J_B) = \mu(J_C) = 2\varepsilon$ , and  $e^{r_0} J_C = J_B$ . Translating density into multiplicative form via the bijection  $x = \log u$ :  $u_B$  is a density point of  $B$  in  $d^\times u$  if and only if  $x_B$  is a density point of  $\widetilde{B}$  in Lebesgue measure. Hence, by the definition of  $\widetilde{B}^*$ , there exists  $\varepsilon_1 > 0$  such that

$$(63) \quad \mu(B \cap J_B) > \frac{3}{4} \mu(J_B) = \frac{3}{2} \varepsilon \quad \text{for all } 0 < \varepsilon < \varepsilon_1.$$

Similarly, there exists  $\varepsilon_2 > 0$  such that

$$(64) \quad \mu(C \cap J_C) > \frac{3}{4}\mu(J_C) = \frac{3}{2}\varepsilon \quad \text{for all } 0 < \varepsilon < \varepsilon_2.$$

Choose  $\varepsilon \in (0, \min(\varepsilon_1, \varepsilon_2))$  small enough also to ensure  $J_B \cup J_C \subset I_\lambda$ , which is possible because  $x_B, x_C \in (-\ell, \ell)$ .

The set

$$Y_1 := B \cap e^{r_0}C \cap J_B$$

is measurable as the intersection of measurable sets. Since  $e^{r_0}(C \cap J_C) \subset e^{r_0}J_C = J_B$  and  $\mu$  is translation-invariant,  $\mu(e^{r_0}(C \cap J_C)) = \mu(C \cap J_C)$ . Inclusion-exclusion in the box  $J_B$  gives

$$(65) \quad \begin{aligned} \mu(Y_1) &= \mu((B \cap J_B) \cap (e^{r_0}C \cap J_B)) \\ &\geq \mu(B \cap J_B) + \mu(e^{r_0}C \cap J_B) - \mu(J_B) \\ &= \mu(B \cap J_B) + \mu(C \cap J_C) - \mu(J_B) \\ &> \frac{3}{2}\varepsilon + \frac{3}{2}\varepsilon - 2\varepsilon = \varepsilon > 0. \end{aligned}$$

Define  $Y_2 := e^{-r_0}Y_1$ . Then  $Y_2$  is measurable and  $\mu(Y_2) = \mu(Y_1) > 0$  by Haar invariance. By construction,  $Y_2 \subset e^{-r_0}(e^{r_0}C) = C$ , and for every  $u \in Y_2$  we have  $e^{r_0}u \in Y_1 \subset B$ . Therefore

$$g_C(u) = \mathbf{1}_C(u) = 1, \quad g_B(e^{r_0}u) = \mathbf{1}_B(e^{r_0}u) = 1 \quad \text{for every } u \in Y_2.$$

Restricting the first integral in (61) to  $Y_2$ ,

$$\int_{\mathbb{R}_{>0}^\times} g_B(e^{r_0}u)g_C(u) d^\times u \geq \int_{Y_2} 1 d^\times u = \mu(Y_2) > 0.$$

The second integral in (61) is nonnegative, so

$$(66) \quad \overline{H}(r_0) > 0.$$

*Continuity of  $\overline{H}$  and conclusion.* We now show that  $\overline{H} \in C(\mathbb{R})$ ; by (66), this implies  $\overline{H} > 0$  on a neighborhood of  $r_0$ , after which the strict positivity of the Lévy density on  $(0, \infty)$  yields (67) below.

In additive coordinates, set  $\tilde{g}_B(x) := g_B(e^x)$  and  $\tilde{g}_C(x) := g_C(e^x)$ ; both are  $\{0, 1\}$ -valued functions in  $L^2(\mathbb{R}, dx) \cap L^\infty(\mathbb{R})$  supported in  $[-\ell, \ell]$ . The substitution  $u = e^x$ ,  $d^\times u = dx$  gives, for every  $r \in \mathbb{R}$ ,

$$\int_{\mathbb{R}_{>0}^\times} g_B(e^r u)g_C(u) d^\times u = \int_{\mathbb{R}} \tilde{g}_B(x+r)\tilde{g}_C(x) dx = \langle \tau_{-r}\tilde{g}_B, \tilde{g}_C \rangle_{L^2(\mathbb{R})},$$

where  $(\tau_h f)(x) := f(x+h)$  denotes additive translation. By the strong continuity of the translation group on  $L^2(\mathbb{R})$  [12, Theorem 9.5],  $r \mapsto \tau_{-r}\tilde{g}_B$  is continuous from  $\mathbb{R}$  into  $L^2(\mathbb{R})$ . Cauchy–Schwarz then yields the continuity of  $r \mapsto \langle \tau_{-r}\tilde{g}_B, \tilde{g}_C \rangle_{L^2}$ . The same argument applies to the second term in (61); thus  $\overline{H} \in C(\mathbb{R})$ .



By (66) and continuity, there exists an open interval  $I_0 \subset (0, \infty)$  with  $r_0 \in I_0$  on which  $\overline{H} > 0$ . The Lévy density  $\frac{1}{\pi} \frac{e^{-r/2}}{1-e^{-2r}}$  is strictly positive and continuous on  $(0, \infty)$ , hence  $\nu(I_0) = \int_{I_0} \frac{1}{\pi} \frac{e^{-r/2}}{1-e^{-2r}} dr > 0$ . Restricting the integral in (62) to  $I_0$ ,

$$2\pi \int_0^\infty \overline{H}(r) \nu(dr) \geq 2\pi \int_{I_0} \overline{H}(r) \nu(dr) > 0,$$

which gives the strict inequality

$$(67) \quad J_{\infty, \lambda}(\mathbf{1}_B) + J_{\infty, \lambda}(\mathbf{1}_C) - J_{\infty, \lambda}(\mathbf{1}_{I_\lambda}) > 0.$$

(vii) *The discrete prime part contributes a nonnegative excess, and final contradiction.* For real-valued  $f \in H_\lambda$ ,  $\Re \langle U_n f, f \rangle_{H_\lambda} = \langle U_n f, f \rangle_{H_\lambda} \in \mathbb{R}$ . Indicator functions are real-valued, hence by (52),

$$\mathcal{E}_{p, \lambda}(\mathbf{1}_X) = \sum_{1 < n \leq \lambda^2} 2a_n (\|\mathbf{1}_X\|_{H_\lambda}^2 - \langle U_n \mathbf{1}_X, \mathbf{1}_X \rangle_{H_\lambda}), \quad X \in \{B, C, I_\lambda\}.$$

Using  $\mathbf{1}_{I_\lambda} = \mathbf{1}_B + \mathbf{1}_C$  and bilinearity,

$$\begin{aligned} \langle U_n \mathbf{1}_{I_\lambda}, \mathbf{1}_{I_\lambda} \rangle &= \langle U_n \mathbf{1}_B, \mathbf{1}_B \rangle + \langle U_n \mathbf{1}_C, \mathbf{1}_C \rangle \\ &\quad + \langle U_n \mathbf{1}_B, \mathbf{1}_C \rangle + \langle U_n \mathbf{1}_C, \mathbf{1}_B \rangle. \end{aligned}$$

Together with (55), this yields

$$(68) \quad \begin{aligned} &\mathcal{E}_{p, \lambda}(\mathbf{1}_B) + \mathcal{E}_{p, \lambda}(\mathbf{1}_C) - \mathcal{E}_{p, \lambda}(\mathbf{1}_{I_\lambda}) \\ &= \sum_{1 < n \leq \lambda^2} 2a_n (\langle U_n \mathbf{1}_B, \mathbf{1}_C \rangle + \langle U_n \mathbf{1}_C, \mathbf{1}_B \rangle). \end{aligned}$$

We show that each term on the right of (68) is nonnegative. By the pointwise formula  $(U_a f)(u) = \mathbf{1}_{I_\lambda}(au) f(au)$  established in Section 2, if  $f \geq 0$  a.e. on  $I_\lambda$  then  $U_a f \geq 0$  a.e. on  $I_\lambda$ . Applied to the indicator functions  $\mathbf{1}_B, \mathbf{1}_C \geq 0$ , this gives  $U_n \mathbf{1}_B \geq 0$  and  $U_n \mathbf{1}_C \geq 0$  a.e. on  $I_\lambda$ . Since  $\mathbf{1}_B, \mathbf{1}_C \geq 0$  as well, the inner products  $\langle U_n \mathbf{1}_B, \mathbf{1}_C \rangle$  and  $\langle U_n \mathbf{1}_C, \mathbf{1}_B \rangle$  are nonnegative. Combined with  $a_n > 0$ ,

$$(69) \quad \mathcal{E}_{p, \lambda}(\mathbf{1}_B) + \mathcal{E}_{p, \lambda}(\mathbf{1}_C) - \mathcal{E}_{p, \lambda}(\mathbf{1}_{I_\lambda}) \geq 0.$$

Adding (67) and (69) and using the decomposition (49),

$$(70) \quad \mathcal{E}_\lambda(\mathbf{1}_B) + \mathcal{E}_\lambda(\mathbf{1}_C) - \mathcal{E}_\lambda(\mathbf{1}_{I_\lambda}) > 0.$$

This contradicts identity (48) established in (i). The contradiction shows that no measurable set  $B \subset I_\lambda$  with  $0 < \mu(B) < \mu(I_\lambda)$  can satisfy (43), hence  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  is irreducible on  $H_\lambda$ .  $\square$

*Remark 4.2.* The proof above isolates two structural ingredients responsible for irreducibility: (i) the strict positivity and continuity of the Lévy density of  $\nu$  on  $(0, \infty)$ , established in Subsection 3.3.2 via the digamma integral representation; and (ii) the positivity-preservation property of the compressed dilations  $U_n$ , which renders the prime-part excess a sum of nonnegative inner products. The indicator-splitting argument, made explicit through the auxiliary kernel  $\overline{H}(r)$ , is analogous to the standard irreducibility criterion for nonlocal Dirichlet forms in terms of the connectivity of the support of the jumping kernel; here this kernel is the multiplicative pull-back of  $\nu$ , whose support is all of  $(0, \infty)$ .

4.2. *Positivity improvement.* With Proposition 4.1 in hand, irreducibility has been established. In this subsection we convert it into positivity improvement for the semigroup and the resolvent.

**THEOREM 4.3.** *Let  $L_\lambda$  denote the unique nonnegative self-adjoint operator on  $H_\lambda$  associated with the closed nonnegative form  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  of Theorem 3.2 via the first representation theorem [18, Chap. VI, Thm. 2.1], so that  $\mathcal{F}_\lambda = \text{Dom}(L_\lambda^{1/2})$ , and let  $A_\lambda := L_\lambda - c_{D,\lambda}I$  be the corresponding self-adjoint lower-bounded operator associated with  $Q_{D,\lambda}$ . The second representation theorem [18, Chap. VI, Thm. 2.23] yields*

$$(71) \quad \mathcal{E}_\lambda(u, v) = \langle L_\lambda^{1/2}u, L_\lambda^{1/2}v \rangle_{H_\lambda}, \quad u, v \in \mathcal{F}_\lambda,$$

or equivalently  $\mathcal{E}_\lambda(u, v) = \langle L_\lambda u, v \rangle_{H_\lambda}$  for  $u \in \text{Dom}(L_\lambda)$  and  $v \in \mathcal{F}_\lambda$ . Set  $T_t^\lambda := e^{-tL_\lambda}$  for  $t > 0$  and  $G_\alpha^\lambda := (L_\lambda + \alpha)^{-1}$  for  $\alpha > 0$ . Then:

(i) for every  $t > 0$  and every  $f \in H_\lambda$  with  $f \geq 0$   $\mu$ -a.e. on  $I_\lambda$  and  $\|f\|_{H_\lambda} > 0$ ,

$$T_t^\lambda f > 0 \quad \mu\text{-a.e. on } I_\lambda;$$

(ii) for every  $\alpha > 0$  and every  $f \in H_\lambda$  with  $f \geq 0$   $\mu$ -a.e. on  $I_\lambda$  and  $\|f\|_{H_\lambda} > 0$ ,

$$G_\alpha^\lambda f > 0 \quad \mu\text{-a.e. on } I_\lambda.$$

That is, the symmetric sub-Markov  $C_0$ -semigroup  $(T_t^\lambda)_{t>0}$  and each resolvent  $G_\alpha^\lambda$  ( $\alpha > 0$ ) are positivity improving on  $H_\lambda$ .

*Proof.* For  $B \subset I_\lambda$  measurable, let  $P_B \in \mathcal{B}(H_\lambda)$  denote the multiplication operator  $f \mapsto \mathbf{1}_B f$ . By Theorem 3.2,  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  is a symmetric Dirichlet form on  $H_\lambda$ , and consequently  $T_t^\lambda$  is a symmetric, sub-Markov  $C_0$ -semigroup that is positivity preserving:

$$(72) \quad T_t^\lambda f \geq 0 \quad \mu\text{-a.e. on } I_\lambda \quad \forall t > 0, \forall f \in H_\lambda \text{ with } f \geq 0 \text{ } \mu\text{-a.e.}$$

This is the unit-contraction property characterizing symmetric Dirichlet forms [13, Thm. 1.4.1]. By Proposition 4.1, no measurable  $B \subset I_\lambda$  with  $0 < \mu(B) < \mu(I_\lambda)$  satisfies  $T_t^\lambda P_B = P_B T_t^\lambda$  for every  $t > 0$ .

*Proof of (i).* The hypotheses of the Faris–Reed–Simon ergodicity theorem for symmetric positivity-preserving semigroups ([11, Vol. IV, Thm. XIII.44]; see also the equivalent Dirichlet-form formulation in [13, Thm. 1.6.1, Cor. 1.6.2]) are verified:  $L_\lambda$  is self-adjoint and bounded below (by 0),  $T_t^\lambda = e^{-tL_\lambda}$  is positivity preserving by (72), and the system is ergodic in the sense that no nontrivial  $P_B$  commutes with all  $T_t^\lambda$  (Proposition 4.1). The cited theorem then asserts that  $T_t^\lambda$  is positivity improving for every  $t > 0$ , which is exactly (i).

*Proof of (ii).* Fix  $\alpha > 0$  and  $f \in H_\lambda$  with  $f \geq 0$   $\mu$ -a.e. and  $\|f\|_{H_\lambda} > 0$ . The Laplace-transform identity

$$(73) \quad G_\alpha^\lambda f = \int_0^\infty e^{-\alpha t} T_t^\lambda f \, dt$$

holds in  $H_\lambda$  as a Bochner integral: the integrand is strongly continuous in  $t \in (0, \infty)$  with norm bounded by  $e^{-\alpha t} \|f\|_{H_\lambda} \in L^1(0, \infty)$ , and the identity follows from the spectral theorem applied to  $L_\lambda$  together with the elementary identity  $\int_0^\infty e^{-\alpha t} e^{-t\sigma} \, dt = (\sigma + \alpha)^{-1}$  for  $\sigma \geq 0$ .

Set  $A := \{u \in I_\lambda : (G_\alpha^\lambda f)(u) = 0\} \subset I_\lambda$ ; we show  $\mu(A) = 0$ . We argue by contradiction. Were  $\mu(A) > 0$  to hold, then  $\mathbf{1}_A \in H_\lambda$  (since  $\mu(I_\lambda) < \infty$ ) would be a nonnegative element of  $H_\lambda$  satisfying

$$\langle \mathbf{1}_A, G_\alpha^\lambda f \rangle_{H_\lambda} = \int_A G_\alpha^\lambda f \, d\mu = 0$$

by the very definition of  $A$ . Pairing (73) against  $\mathbf{1}_A$  and applying Fubini's theorem [16, Vol. I, Thm. 3.4.4] (the integrand  $(t, u) \mapsto e^{-\alpha t} \mathbf{1}_A(u) (T_t^\lambda f)(u)$  is non-negative by (72), and the iterated integrals are bounded by  $\alpha^{-1} \|\mathbf{1}_A\|_{H_\lambda} \|f\|_{H_\lambda} < \infty$ ),

$$(74) \quad \langle \mathbf{1}_A, G_\alpha^\lambda f \rangle_{H_\lambda} = \int_0^\infty e^{-\alpha t} \langle \mathbf{1}_A, T_t^\lambda f \rangle_{H_\lambda} \, dt.$$

By (i), for every  $t > 0$  the function  $T_t^\lambda f$  is strictly positive  $\mu$ -a.e. on  $I_\lambda$ ; hence

$$\langle \mathbf{1}_A, T_t^\lambda f \rangle_{H_\lambda} = \int_A T_t^\lambda f \, d\mu > 0 \quad \forall t > 0,$$

since  $\mu(A) > 0$  by hypothesis. Combined with  $e^{-\alpha t} > 0$  on  $(0, \infty)$  and the strong continuity of  $t \mapsto T_t^\lambda f$  (which makes  $t \mapsto \langle \mathbf{1}_A, T_t^\lambda f \rangle$  continuous), the integrand on the right-hand side of (74) is a strictly positive continuous function of  $t \in (0, \infty)$ , whence the integral is strictly positive. This contradicts the vanishing of the left-hand side. We conclude  $\mu(A) = 0$ , equivalently  $G_\alpha^\lambda f > 0$   $\mu$ -a.e. on  $I_\lambda$ , which is (ii).  $\square$

## 5. Ground-State Spectral Theory and a Real-Zeros Theorem

In Section 4, we have already proved that for every  $\lambda > 1$ , the pair  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  is a nonnegative, closed, symmetric, irreducible Dirichlet form on  $H_\lambda$ , and that the associated semigroup and resolvent are positivity improving.

The aims of the present section are fourfold. First, we prove that  $L_\lambda$  has compact resolvent, and hence discrete spectrum. Second, we show that the bottom eigenvalue is simple, and obtain a normalized strictly positive and inversion-even ground state  $\xi_\lambda$ . Third, we transport the form to the interval  $[0, L]$  so as to connect it with the real-distribution model in the literature. Finally, by means of finite-dimensional Fourier truncation and Hurwitz's theorem, we prove that the logarithmic Fourier transform of  $\xi_\lambda$  is an entire function all of whose zeros are real.

**5.1. Compact resolvent and discrete spectrum.** We first prove that the associated operator has compact resolvent and hence discrete spectrum.

**PROPOSITION 5.1.** *Define*

$$\|f\|_{\mathcal{E}_\lambda, 1}^2 := \mathcal{E}_\lambda(f, f) + \|f\|_{H_\lambda}^2, \quad f \in \mathcal{F}_\lambda.$$

*Then the following hold:*

1. *The embedding*

$$j_\lambda : (\mathcal{F}_\lambda, \|\cdot\|_{\mathcal{E}_\lambda, 1}) \hookrightarrow H_\lambda$$

*is compact.*

2. *For every  $\alpha > 0$ , the resolvent operator*

$$G_\alpha^\lambda := (L_\lambda + \alpha)^{-1} : H_\lambda \rightarrow H_\lambda$$

*is a compact self-adjoint operator.*

3. *Consequently,  $L_\lambda$  has compact resolvent, hence purely discrete spectrum: there exists a sequence of eigenvalues*

$$0 \leq \mu_0(\lambda) \leq \mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots, \quad \mu_n(\lambda) \rightarrow +\infty,$$

*each of finite multiplicity, and  $H_\lambda$  admits an orthonormal basis consisting of eigenfunctions of  $L_\lambda$ .*

*Proof.* By Theorem 3.2,  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  is a nonnegative, closed, symmetric Dirichlet form on  $H_\lambda$ . The first representation theorem [18, Chap. VI, Thm. 2.1] associates with it a unique nonnegative self-adjoint operator  $L_\lambda$  on  $H_\lambda$  such that

$$u \in \text{Dom}(L_\lambda), \quad L_\lambda u = f \iff u \in \mathcal{F}_\lambda, \quad \mathcal{E}_\lambda(v, u) = \langle v, f \rangle_{H_\lambda} \quad (\forall v \in \mathcal{F}_\lambda),$$

and in particular

$$\mathcal{E}_\lambda(u, v) = \langle L_\lambda u, v \rangle_{H_\lambda}, \quad u \in \text{Dom}(L_\lambda), \quad v \in \mathcal{F}_\lambda.$$

We organize the remainder of the proof into seven steps (i)–(vii).

(i) *Fourier-multiplier representation of  $\mathcal{E}_{\infty,\lambda}$ .* Let  $Z : H_\lambda \rightarrow L^2(I_x, dx)$  be the logarithmic unitary of Lemma 2.3, where  $I_x = [-\ell, \ell]$  with  $\ell = \log \lambda$ , and let  $E_x : L^2(I_x, dx) \hookrightarrow L^2(\mathbb{R}, dx)$  denote zero extension. For  $f \in \mathcal{F}_\lambda$ , set  $g := E_x Z f \in L^2(\mathbb{R}, dx)$ , so that  $\text{supp}(g) \subset [-\ell, \ell]$ ,  $\|f\|_{H_\lambda} = \|g\|_{L^2(\mathbb{R})}$ , and the change of variables  $u = e^x$  together with the Fourier convention  $\widehat{g}(t) := \int_{\mathbb{R}} g(x) e^{-itx} dx$  yields  $\widehat{f}(t) = \widehat{g}(t)$  for  $t \in \mathbb{R}$ .

By Theorem 3.2(2), the archimedean part admits the Lévy–Khintchine pure-jump representation (4). Combining the elementary identity  $|e^{itr} - 1|^2 = 2(1 - \cos(tr))$  with Plancherel’s theorem under the convention  $\|\widehat{g}\|_{L^2(\mathbb{R})}^2 = 2\pi \|g\|_{L^2(\mathbb{R})}^2$ , we have, for every  $r > 0$ ,

$$\|g(\cdot + r) - g\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |e^{itr} - 1|^2 |\widehat{g}(t)|^2 dt = \frac{1}{\pi} \int_{\mathbb{R}} (1 - \cos(tr)) |\widehat{g}(t)|^2 dt;$$

this identity also appears at (22). Substituting into (4) and applying Tonelli’s theorem [16, Vol. I, Thm. 3.4.4] (legitimate by the nonnegativity of  $|\widehat{g}(t)|^2$  and  $1 - \cos(tr)$ ), we obtain

$$(75) \quad \mathcal{E}_{\infty,\lambda}(f, f) = \int_{\mathbb{R}} \chi_\lambda(t) |\widehat{f}(t)|^2 dt, \quad \chi_\lambda(t) := \int_0^\infty (1 - \cos(tr)) \nu(dr) \geq 0.$$

(ii) *Monotone divergence of  $\chi_\lambda$ .* For  $0 < r \leq 1$ , the inequalities  $1 - e^{-2r} \leq 2r$  and  $e^{-r/2} \geq e^{-1/2}$  combine to give

$$\nu(dr) \geq \frac{e^{-1/2}}{2\pi} \frac{dr}{r} \quad \text{on } (0, 1].$$

Substituting this lower bound and applying the change of variables  $s = |t|r$ , for  $|t| \geq 1$  we have

$$\chi_\lambda(t) \geq \frac{e^{-1/2}}{2\pi} \int_0^1 \frac{1 - \cos(tr)}{r} dr = \frac{e^{-1/2}}{2\pi} \Phi(|t|), \quad \Phi(T) := \int_0^T \frac{1 - \cos s}{s} ds.$$

The function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  is monotone nondecreasing and satisfies  $\Phi(T) \rightarrow +\infty$  as  $T \rightarrow \infty$  (in fact  $\Phi(T) \sim \log T$ , by the standard asymptotics of the cosine integral). Therefore

$$(76) \quad \inf_{|t| \geq R} \chi_\lambda(t) \geq \frac{e^{-1/2}}{2\pi} \Phi(R) \xrightarrow{R \rightarrow \infty} +\infty.$$

(iii) *Energy bound and high-frequency tail estimate.* Let  $(f_n) \subset \mathcal{F}_\lambda$  satisfy

$$M := \sup_n \|f_n\|_{\mathcal{E}_{\lambda,1}}^2 = \sup_n (\mathcal{E}_\lambda(f_n, f_n) + \|f_n\|_{H_\lambda}^2) < \infty.$$

By Theorem 3.2(3),  $\mathcal{E}_{p,\lambda} \geq 0$ , so the additive decomposition (8) yields

$$(77) \quad \sup_n \mathcal{E}_{\infty,\lambda}(f_n, f_n) \leq \sup_n \mathcal{E}_\lambda(f_n, f_n) \leq M.$$

Setting  $g_n := E_x Z f_n$ , we have  $\text{supp}(g_n) \subset [-\ell, \ell]$ ,  $\|g_n\|_{L^2(\mathbb{R})}^2 = \|f_n\|_{H_\lambda}^2 \leq M$ , and, by (75)–(77),

$$(78) \quad \sup_n \int_{\mathbb{R}} \chi_\lambda(t) |\widehat{g}_n(t)|^2 dt \leq M.$$

For every  $R > 1$ , the monotone divergence (76) together with (78) yields

$$(79) \quad \sup_n \int_{|t| \geq R} |\widehat{g}_n(t)|^2 dt \leq \frac{1}{\inf_{|t| \geq R} \chi_\lambda(t)} \sup_n \int_{\mathbb{R}} \chi_\lambda(t) |\widehat{g}_n(t)|^2 dt \leq \frac{M}{\inf_{|t| \geq R} \chi_\lambda(t)} \xrightarrow{R \rightarrow \infty} 0.$$

(iv) *Uniform translation continuity in  $L^2(\mathbb{R})$ .* For  $h \in \mathbb{R}$  and the additive translation  $\tau_h g(x) := g(x + h)$ , Plancherel's theorem and the identity  $|e^{ith} - 1|^2 = 2(1 - \cos(th))$  give

$$(80) \quad \|\tau_h g_n - g_n\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |e^{ith} - 1|^2 |\widehat{g}_n(t)|^2 dt.$$

The pointwise bound  $|e^{ith} - 1|^2 \leq \min(4, t^2 h^2)$ , valid for all  $t, h \in \mathbb{R}$ , allows us to split the integral at any threshold  $R > 1$ :

$$\|\tau_h g_n - g_n\|_{L^2(\mathbb{R})}^2 \leq \underbrace{\frac{R^2 h^2}{2\pi} \int_{|t| \leq R} |\widehat{g}_n(t)|^2 dt}_{(I)} + \underbrace{\frac{2}{\pi} \int_{|t| \geq R} |\widehat{g}_n(t)|^2 dt}_{(II)}.$$

By Plancherel and  $\|g_n\|_{L^2(\mathbb{R})}^2 \leq M$ ,

$$\sup_n (I) \leq \frac{R^2 h^2}{2\pi} \cdot 2\pi M = R^2 h^2 M,$$

while (79) gives

$$\sup_n (II) \leq \frac{2M}{\pi \inf_{|t| \geq R} \chi_\lambda(t)}.$$

Given  $\varepsilon > 0$ , by (76) we may first choose  $R = R(\varepsilon) > 1$  so that  $\sup_n (II) < \varepsilon/2$ , and then choose  $\delta = \delta(\varepsilon, R) > 0$  such that  $R^2 h^2 M < \varepsilon/2$  whenever  $|h| < \delta$ . Therefore

$$(81) \quad \lim_{h \rightarrow 0} \sup_n \|\tau_h g_n - g_n\|_{L^2(\mathbb{R})} = 0.$$

(v) *Compactness of the embedding  $j_\lambda$ .* We invoke the Kolmogorov–Riesz–Fréchet compactness theorem in  $L^2(\mathbb{R})$ , in the form given by Brezis [17, Thm. 4.26 and Cor. 4.27]: a bounded subset  $\mathcal{G} \subset L^2(\mathbb{R})$  is relatively compact if and only if

$$(a) \quad \lim_{h \rightarrow 0} \sup_{g \in \mathcal{G}} \|\tau_h g - g\|_{L^2(\mathbb{R})} = 0, \text{ and}$$

$$(b) \quad \lim_{R \rightarrow \infty} \sup_{g \in \mathcal{G}} \int_{|x| \geq R} |g(x)|^2 dx = 0.$$

The family  $\{g_n\}$  is bounded in  $L^2(\mathbb{R})$  by  $\sqrt{M}$ . It satisfies (a) by (81), and (b) holds trivially because  $\text{supp}(g_n) \subset [-\ell, \ell]$  for all  $n$ , so the integral in (b) vanishes for every  $R > \ell$ . Hence  $\{g_n\}$  is relatively compact in  $L^2(\mathbb{R})$ . Since the map  $f \mapsto E_x Z f$  is an isometry from  $H_\lambda$  onto  $E_x Z(H_\lambda) \subset L^2(\mathbb{R})$ , the family  $\{f_n\}$  is relatively compact in  $H_\lambda$ . This proves that the embedding  $j_\lambda : (\mathcal{F}_\lambda, \|\cdot\|_{\mathcal{E}_\lambda,1}) \hookrightarrow H_\lambda$  is compact, which is assertion 1.

(vi) *Compactness of the resolvent.* Fix  $\alpha > 0$  and  $f \in H_\lambda$ . By the first representation theorem [18, Chap. VI, Thm. 2.1],  $u := G_\alpha^\lambda f$  lies in  $\text{Dom}(L_\lambda) \subset \mathcal{F}_\lambda$  and satisfies

$$\mathcal{E}_\lambda(u, v) + \alpha \langle u, v \rangle_{H_\lambda} = \langle f, v \rangle_{H_\lambda} \quad (\forall v \in \mathcal{F}_\lambda).$$

Setting  $v = u$ , the left-hand side  $\mathcal{E}_\lambda(u, u) + \alpha \|u\|_{H_\lambda}^2$  is a real number; by the Cauchy–Schwarz inequality,

$$\mathcal{E}_\lambda(u, u) + \alpha \|u\|_{H_\lambda}^2 = \Re \langle f, u \rangle_{H_\lambda} \leq \|f\|_{H_\lambda} \|u\|_{H_\lambda}.$$

Since  $\mathcal{E}_\lambda \geq 0$ , the two summands on the left-hand side are nonnegative, and the displayed inequality therefore implies separately

$$\alpha \|u\|_{H_\lambda}^2 \leq \|f\|_{H_\lambda} \|u\|_{H_\lambda}, \quad \mathcal{E}_\lambda(u, u) \leq \|f\|_{H_\lambda} \|u\|_{H_\lambda}.$$

The first gives  $\|u\|_{H_\lambda} \leq \alpha^{-1} \|f\|_{H_\lambda}$ , and feeding this back into the second yields  $\mathcal{E}_\lambda(u, u) \leq \alpha^{-1} \|f\|_{H_\lambda}^2$ . Consequently

$$(82) \quad \|G_\alpha^\lambda f\|_{\mathcal{E}_\lambda,1}^2 = \mathcal{E}_\lambda(u, u) + \|u\|_{H_\lambda}^2 \leq \frac{1}{\alpha} \|f\|_{H_\lambda}^2 + \frac{1}{\alpha^2} \|f\|_{H_\lambda}^2 = \frac{\alpha+1}{\alpha^2} \|f\|_{H_\lambda}^2,$$

which means  $G_\alpha^\lambda : H_\lambda \rightarrow (\mathcal{F}_\lambda, \|\cdot\|_{\mathcal{E}_\lambda,1})$  is bounded with operator norm at most  $\sqrt{(\alpha+1)/\alpha^2}$ .

By (v), the embedding  $j_\lambda$  is compact. The composition

$$G_\alpha^\lambda = j_\lambda \circ G_\alpha^\lambda : H_\lambda \xrightarrow{\text{bounded}} (\mathcal{F}_\lambda, \|\cdot\|_{\mathcal{E}_\lambda,1}) \xrightarrow{\text{compact}} H_\lambda$$

is therefore a compact operator on  $H_\lambda$ . Since  $L_\lambda$  is self-adjoint and  $\alpha > 0$  is real,  $G_\alpha^\lambda = (L_\lambda + \alpha)^{-1}$  is self-adjoint as the inverse of the self-adjoint operator  $L_\lambda + \alpha I$ . This proves assertion 2.

(vii) *Discrete spectrum of  $L_\lambda$ .* Fix any  $\alpha > 0$ . By (vi),  $G_\alpha^\lambda$  is a compact, self-adjoint, injective operator on  $H_\lambda$ . The spectral theorem for compact self-adjoint operators [9, Thms. VI.16 and VI.17] produces an orthonormal basis  $(e_n)_{n \geq 0}$  of  $H_\lambda$  together with eigenvalues  $\rho_n(\alpha) \in \mathbb{R}$  such that  $G_\alpha^\lambda e_n = \rho_n(\alpha) e_n$ , each nonzero eigenvalue has finite multiplicity, and the only possible accumulation point of  $\{\rho_n(\alpha)\}$  is 0. The injectivity of  $G_\alpha^\lambda$  rules out  $\rho_n(\alpha) = 0$ , and  $L_\lambda \geq 0$  together with the spectral mapping  $\sigma(G_\alpha^\lambda) \subset (0, \alpha^{-1}]$  further yields  $0 < \rho_n(\alpha) \leq \alpha^{-1}$ .

The relation  $L_\lambda = (G_\alpha^\lambda)^{-1} - \alpha I$  on the dense subspace  $\text{Dom}(L_\lambda) = \text{Ran}(G_\alpha^\lambda)$  shows that each  $e_n \in \text{Dom}(L_\lambda)$  and

$$L_\lambda e_n = \mu_n(\lambda) e_n, \quad \mu_n(\lambda) := \rho_n(\alpha)^{-1} - \alpha \geq 0.$$

The accumulation of  $\rho_n(\alpha)$  at 0 is therefore equivalent to  $\mu_n(\lambda) \rightarrow +\infty$ . Reindexing the eigenvalues in nondecreasing order gives

$$0 \leq \mu_0(\lambda) \leq \mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots, \quad \mu_n(\lambda) \rightarrow +\infty,$$

each of finite multiplicity, and  $H_\lambda$  admits an orthonormal basis  $(e_n)$  of eigenfunctions of  $L_\lambda$ . This is assertion 3 and completes the proof.  $\square$

5.2. *Simplicity of the ground state and strict positivity.* We next identify the bottom eigenspace and the positivity of its eigenfunction.

THEOREM 5.2. *Let  $\lambda > 1$ , and set*

$$I_\lambda = [\lambda^{-1}, \lambda], \quad H_\lambda = L^2(I_\lambda, d^\times u).$$

*Let  $L_\lambda$  be the self-adjoint nonnegative operator associated with the nonnegative closed symmetric form  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  constructed in Section 3, and let compact resolvent be given by Proposition 5.1. For each  $\alpha > 0$ , the resolvent*

$$G_\alpha^\lambda := (L_\lambda + \alpha)^{-1}$$

*is positivity improving on  $H_\lambda$ . Denote by*

$$\mu_0(\lambda) := \min \sigma(L_\lambda)$$

*the bottom of the spectrum. Then:*

1.  $\mu_0(\lambda)$  is an eigenvalue of  $L_\lambda$ ;
2. the eigenspace  $\ker(L_\lambda - \mu_0(\lambda)I)$  is one-dimensional;
3. there exists a unique  $L^2$ -normalized eigenfunction  $\xi_\lambda \in \text{Dom}(L_\lambda) \subset \mathcal{F}_\lambda$  satisfying

$$L_\lambda \xi_\lambda = \mu_0(\lambda) \xi_\lambda, \quad \|\xi_\lambda\|_{H_\lambda} = 1, \quad \xi_\lambda > 0 \quad \text{a.e. on } I_\lambda.$$

*Proof.* By Proposition 5.1,  $L_\lambda$  has compact resolvent and hence purely discrete spectrum. There exists a nondecreasing real sequence

$$0 \leq \mu_0(\lambda) \leq \mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots, \quad \mu_n(\lambda) \rightarrow +\infty,$$

and an orthonormal basis  $(e_n)_{n \geq 0} \subset \text{Dom}(L_\lambda)$  of  $H_\lambda$  satisfying  $L_\lambda e_n = \mu_n(\lambda) e_n$ . In particular,  $\mu_0(\lambda)$  is an eigenvalue of  $L_\lambda$ , which proves assertion 1. Fix once and for all  $\alpha > 0$ , and set

$$G_\alpha^\lambda := (L_\lambda + \alpha)^{-1}, \quad r_\alpha := \frac{1}{\mu_0(\lambda) + \alpha}.$$



Then  $G_\alpha^\lambda$  is compact, self-adjoint, and injective on  $H_\lambda$ , with  $G_\alpha^\lambda e_n = (\mu_n(\lambda) + \alpha)^{-1} e_n$ . Its spectrum is  $\{(\mu_n(\lambda) + \alpha)^{-1} : n \geq 0\} \subset (0, r_\alpha]$ , with  $r_\alpha$  its largest element, and

$$(83) \quad \ker(L_\lambda - \mu_0(\lambda)I) = \ker(G_\alpha^\lambda - r_\alpha I).$$

It therefore suffices to prove that the right-hand side of (83) is one-dimensional and contains a strictly positive normalized representative.

The proof is organized into seven steps (i)–(vii).

(i) *Conjugation invariance of  $\mathcal{E}_\lambda$ ,  $L_\lambda$ , and  $G_\alpha^\lambda$ .* The map  $\Phi(z) := \bar{z}$  is a complex normal contraction in the sense of Remark 3.1:  $\Phi(0) = 0$  and  $|\Phi(z) - \Phi(w)| = |z - w|$  for all  $z, w \in \mathbb{C}$ . By the Markov property of  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  established in Theorem 3.2(1),  $\bar{f} \in \mathcal{F}_\lambda$  for every  $f \in \mathcal{F}_\lambda$ , and

$$(84) \quad \mathcal{E}_\lambda(\bar{f}, \bar{f}) \leq \mathcal{E}_\lambda(f, f).$$

Replacing  $f$  by  $\bar{f}$  in (84) and using  $\bar{\bar{f}} = f$  yields the reverse inequality, hence the equality

$$(85) \quad \mathcal{E}_\lambda(\bar{f}, \bar{f}) = \mathcal{E}_\lambda(f, f), \quad f \in \mathcal{F}_\lambda.$$

Define  $T : \mathcal{F}_\lambda \times \mathcal{F}_\lambda \rightarrow \mathbb{C}$  by

$$T(f, g) := \overline{\mathcal{E}_\lambda(\bar{f}, \bar{g})}.$$

A direct calculation, using the linearity of  $\mathcal{E}_\lambda$  in the first variable and the conjugate-linearity in the second, shows that  $T$  is sesquilinear with the same convention. By (85) and the reality of  $\mathcal{E}_\lambda(f, f)$ ,

$$T(f, f) = \overline{\mathcal{E}_\lambda(\bar{f}, \bar{f})} = \overline{\mathcal{E}_\lambda(f, f)} = \mathcal{E}_\lambda(f, f), \quad f \in \mathcal{F}_\lambda.$$

The polarization identity

$$4S(f, g) = S(f+g, f+g) - S(f-g, f-g) + i[S(f+ig, f+ig) - S(f-ig, f-ig)],$$

valid for every sesquilinear form  $S$  in our convention, recovers  $S(f, g)$  from the quadratic form  $f \mapsto S(f, f)$ . Applied to  $T$  and to  $\mathcal{E}_\lambda$ , which share the same quadratic form, it gives

$$(86) \quad \mathcal{E}_\lambda(\bar{f}, \bar{g}) = \overline{\mathcal{E}_\lambda(f, g)}, \quad f, g \in \mathcal{F}_\lambda.$$

We now transfer (86) to the operator level. Let  $f \in \text{Dom}(L_\lambda)$  and set  $h := L_\lambda f \in H_\lambda$ . By the first representation theorem [18, Chap. VI, Thm. 2.1],

$$\mathcal{E}_\lambda(v, f) = \langle v, h \rangle_{H_\lambda}, \quad \forall v \in \mathcal{F}_\lambda.$$

For every  $v \in \mathcal{F}_\lambda$ , the Markov property gives  $\bar{v} \in \mathcal{F}_\lambda$ , and formula (86) (with  $\bar{v}$  in place of the first argument and  $f$  in place of the second) yields

$$\mathcal{E}_\lambda(v, \bar{f}) = \overline{\mathcal{E}_\lambda(\bar{v}, f)} = \overline{\langle \bar{v}, h \rangle_{H_\lambda}} = \langle v, \bar{h} \rangle_{H_\lambda},$$

where the last equality follows from the conjugate-linearity of  $\langle \cdot, \cdot \rangle_{H_\lambda}$  in the second variable:  $\overline{\langle \bar{v}, h \rangle_{H_\lambda}} = \overline{\int_{I_\lambda} \bar{v} \bar{h} d\mu} = \int_{I_\lambda} v h d\mu = \langle v, \bar{h} \rangle_{H_\lambda}$ . Applying the first representation theorem in the reverse direction, we obtain  $\bar{f} \in \text{Dom}(L_\lambda)$  and

$$(87) \quad L_\lambda \bar{f} = \bar{h} = \overline{L_\lambda f}.$$

Since  $\alpha \in \mathbb{R}$ , the operator  $L_\lambda + \alpha I$  inherits the same property. For any  $f \in H_\lambda$ , set  $g := G_\alpha^\lambda f \in \text{Dom}(L_\lambda)$ ; then (87) gives  $\bar{g} \in \text{Dom}(L_\lambda)$  and  $(L_\lambda + \alpha I)\bar{g} = \overline{(L_\lambda + \alpha I)g} = \bar{f}$ , hence  $\bar{g} = G_\alpha^\lambda \bar{f}$ . We conclude

$$(88) \quad \overline{G_\alpha^\lambda f} = G_\alpha^\lambda \bar{f}, \quad f \in H_\lambda.$$

(ii) *Reduction to a real-valued ground-state candidate.* Let  $\psi \in \ker(G_\alpha^\lambda - r_\alpha I)$  be nonzero. By (88) and the reality of  $r_\alpha$ ,

$$G_\alpha^\lambda \bar{\psi} = \overline{G_\alpha^\lambda \psi} = \overline{r_\alpha \psi} = r_\alpha \bar{\psi},$$

so  $\bar{\psi} \in \ker(G_\alpha^\lambda - r_\alpha I)$  as well. Hence the real-valued functions

$$\psi_1 := \frac{\psi + \bar{\psi}}{2} = \Re \psi, \quad \psi_2 := \frac{\psi - \bar{\psi}}{2i} = \Im \psi$$

both lie in  $\ker(G_\alpha^\lambda - r_\alpha I)$ . Since  $\psi = \psi_1 + i\psi_2$  is nonzero, at least one of  $\psi_1, \psi_2$  is nonzero. We work henceforth with such a nonzero real-valued representative, which we again denote by  $\psi$ .

(iii) *Pointwise lower bound*  $G_\alpha^\lambda |\psi| \geq r_\alpha |\psi|$ . Decompose

$$\psi = \psi^+ - \psi^-, \quad |\psi| = \psi^+ + \psi^-, \quad \psi^\pm \geq 0 \text{ } \mu\text{-a.e. on } I_\lambda.$$

By Theorem 4.3(ii),  $G_\alpha^\lambda$  is positivity preserving, so  $G_\alpha^\lambda \psi^\pm \geq 0$   $\mu$ -a.e. The pointwise triangle inequality  $a + b \geq |a - b|$  for  $a, b \geq 0$ , applied at  $\mu$ -a.e.  $u \in I_\lambda$  to  $a = (G_\alpha^\lambda \psi^+)(u)$  and  $b = (G_\alpha^\lambda \psi^-)(u)$ , yields

$$(89) \quad G_\alpha^\lambda |\psi| = G_\alpha^\lambda \psi^+ + G_\alpha^\lambda \psi^- \geq |G_\alpha^\lambda \psi^+ - G_\alpha^\lambda \psi^-| = |G_\alpha^\lambda \psi| = r_\alpha |\psi| \quad \mu\text{-a.e. on } I_\lambda,$$

where the last equality uses  $G_\alpha^\lambda \psi = r_\alpha \psi$ ,  $r_\alpha > 0$ , and the fact that  $\psi$  is real.

(iv) *Upgrade to the equality*  $G_\alpha^\lambda |\psi| = r_\alpha |\psi|$ . The pointwise identity  $|\psi|^2 = \psi^2$  gives  $|\psi| \in H_\lambda$  with  $\| |\psi| \|_{H_\lambda} = \| \psi \|_{H_\lambda}$ . Expand  $|\psi|$  in the orthonormal eigenbasis  $(e_n)_{n \geq 0}$ :

$$|\psi| = \sum_{n \geq 0} c_n e_n, \quad c_n := \langle |\psi|, e_n \rangle_{H_\lambda}, \quad \sum_{n \geq 0} |c_n|^2 = \| \psi \|_{H_\lambda}^2.$$

Since  $G_\alpha^\lambda e_n = (\mu_n(\lambda) + \alpha)^{-1} e_n$  and  $r_\alpha = (\mu_0(\lambda) + \alpha)^{-1} \geq (\mu_n(\lambda) + \alpha)^{-1}$  for every  $n \geq 0$ ,

$$(90) \quad \langle G_\alpha^\lambda |\psi|, |\psi| \rangle_{H_\lambda} - r_\alpha \| |\psi| \|_{H_\lambda}^2 = \sum_{n \geq 0} |c_n|^2 \left[ \frac{1}{\mu_n(\lambda) + \alpha} - r_\alpha \right] \leq 0,$$

each summand being nonpositive. On the other hand, by (89) the function  $G_\alpha^\lambda |\psi| - r_\alpha |\psi|$  is  $\mu$ -a.e. nonnegative on  $I_\lambda$ , and so is  $|\psi|$ ; pairing the two,

$$(91) \quad \langle G_\alpha^\lambda |\psi|, |\psi| \rangle_{H_\lambda} - r_\alpha \| |\psi| \|_{H_\lambda}^2 = \int_{I_\lambda} (G_\alpha^\lambda |\psi| - r_\alpha |\psi|) |\psi| d\mu \geq 0.$$

Combining (90) and (91), both inequalities are equalities. Equality in (90) forces every nonpositive summand to vanish:

$$|c_n|^2 \left[ \frac{1}{\mu_n(\lambda) + \alpha} - r_\alpha \right] = 0, \quad n \geq 0,$$

hence  $c_n = 0$  whenever  $\mu_n(\lambda) > \mu_0(\lambda)$ . Therefore  $|\psi| \in \ker(L_\lambda - \mu_0(\lambda)I) = \ker(G_\alpha^\lambda - r_\alpha I)$ , which is the identity in  $H_\lambda$

$$(92) \quad G_\alpha^\lambda |\psi| = r_\alpha |\psi|.$$

(v) *Strict positivity of  $|\psi|$  and definition of  $\xi_\lambda$ .* Since  $|\psi| \geq 0$   $\mu$ -a.e. and  $|\psi| \not\equiv 0$ , the positivity-improving property of  $G_\alpha^\lambda$  given by Theorem 4.3(ii) yields

$$G_\alpha^\lambda |\psi| > 0 \quad \mu\text{-a.e. on } I_\lambda.$$

Combined with (92) and  $r_\alpha > 0$ ,

$$|\psi| = r_\alpha^{-1} G_\alpha^\lambda |\psi| > 0 \quad \mu\text{-a.e. on } I_\lambda.$$

Define

$$(93) \quad \xi_\lambda := \frac{|\psi|}{\| |\psi| \|_{H_\lambda}} \in \text{Dom}(L_\lambda).$$

By construction,  $\xi_\lambda$  is real-valued,  $L^2$ -normalized, strictly positive  $\mu$ -a.e. on  $I_\lambda$ , and  $L_\lambda \xi_\lambda = \mu_0(\lambda) \xi_\lambda$ .

(vi) *Simplicity of the eigenspace.* We argue by contradiction. Suppose, toward a contradiction, that  $\dim \ker(G_\alpha^\lambda - r_\alpha I) \geq 2$ . The orthogonal complement of  $\mathbb{C}\xi_\lambda$  inside  $\ker(G_\alpha^\lambda - r_\alpha I)$  then has dimension  $\geq 1$ ; pick a nonzero element  $\eta_0$  of it, so that

$$\eta_0 \in \ker(G_\alpha^\lambda - r_\alpha I), \quad \eta_0 \neq 0, \quad \langle \eta_0, \xi_\lambda \rangle_{H_\lambda} = 0.$$

By (88) and the reality of  $r_\alpha$ ,  $\overline{\eta_0} \in \ker(G_\alpha^\lambda - r_\alpha I)$ , so the real-valued functions

$$\eta_1 := \Re \eta_0 = \frac{\eta_0 + \overline{\eta_0}}{2}, \quad \eta_2 := \Im \eta_0 = \frac{\eta_0 - \overline{\eta_0}}{2i}$$

both lie in  $\ker(G_\alpha^\lambda - r_\alpha I)$ . Since  $\xi_\lambda$  is real-valued,  $\langle \eta_0, \xi_\lambda \rangle_{H_\lambda} = \int_{I_\lambda} \eta_0 \xi_\lambda d\mu = \int_{I_\lambda} \eta_1 \xi_\lambda d\mu + i \int_{I_\lambda} \eta_2 \xi_\lambda d\mu = 0$ , and the real and imaginary parts must vanish separately:  $\langle \eta_j, \xi_\lambda \rangle_{H_\lambda} = 0$  for  $j = 1, 2$ . Since  $\eta_0 \neq 0$ , at least one of  $\eta_1, \eta_2$  is nonzero; let  $\eta$  denote this nonzero one. Thus

$$(94) \quad \eta \in \ker(G_\alpha^\lambda - r_\alpha I), \quad \eta \text{ real-valued and nonzero}, \quad \langle \eta, \xi_\lambda \rangle_{H_\lambda} = 0.$$

Decompose  $\eta = \eta^+ - \eta^-$  with  $\eta^\pm := \max(\pm\eta, 0) \geq 0$   $\mu$ -a.e.,  $|\eta| = \eta^+ + \eta^-$ . Both  $\eta^\pm$  are nonzero in  $H_\lambda$ : were  $\eta^- \equiv 0$  (the case  $\eta^+ \equiv 0$  being symmetric), one would have  $\eta = \eta^+ \geq 0$   $\mu$ -a.e. and  $\eta \not\equiv 0$ , whence, since  $\xi_\lambda > 0$   $\mu$ -a.e. on  $I_\lambda$ ,  $\int_{I_\lambda} \eta \xi_\lambda d\mu > 0$ , contradicting orthogonality in (94).

By Theorem 4.3(ii),

$$G_\alpha^\lambda \eta^+ > 0, \quad G_\alpha^\lambda \eta^- > 0 \quad \mu\text{-a.e. on } I_\lambda.$$

For real numbers  $a, b$  with  $\min(a, b) > 0$ , the strict triangle inequality  $a + b > |a - b|$  holds. Applied  $\mu$ -a.e. at  $a = (G_\alpha^\lambda \eta^+)(u)$ ,  $b = (G_\alpha^\lambda \eta^-)(u)$ ,

(95)

$$G_\alpha^\lambda |\eta| = G_\alpha^\lambda \eta^+ + G_\alpha^\lambda \eta^- > |G_\alpha^\lambda \eta^+ - G_\alpha^\lambda \eta^-| = |G_\alpha^\lambda \eta| = r_\alpha |\eta| \quad \mu\text{-a.e. on } I_\lambda.$$

Multiply (95) by  $\xi_\lambda$ , which is strictly positive  $\mu$ -a.e.; the resulting integrand  $\xi_\lambda (G_\alpha^\lambda |\eta| - r_\alpha |\eta|)$  is  $\mu$ -a.e. strictly positive on  $I_\lambda$ . Cauchy–Schwarz applied to the pairs  $(\xi_\lambda, G_\alpha^\lambda |\eta|)$  and  $(\xi_\lambda, |\eta|)$ , both in  $H_\lambda$ , ensures the integrand is in  $L^1(I_\lambda, \mu)$ . Hence

$$(96) \quad \langle \xi_\lambda, G_\alpha^\lambda |\eta| \rangle_{H_\lambda} = \int_{I_\lambda} \xi_\lambda G_\alpha^\lambda |\eta| d\mu > r_\alpha \int_{I_\lambda} \xi_\lambda |\eta| d\mu = r_\alpha \langle \xi_\lambda, |\eta| \rangle_{H_\lambda}.$$

On the other hand,  $G_\alpha^\lambda$  is self-adjoint and  $G_\alpha^\lambda \xi_\lambda = r_\alpha \xi_\lambda$ , so

(97)

$$\langle \xi_\lambda, G_\alpha^\lambda |\eta| \rangle_{H_\lambda} = \langle G_\alpha^\lambda \xi_\lambda, |\eta| \rangle_{H_\lambda} = r_\alpha \langle \xi_\lambda, |\eta| \rangle_{H_\lambda}.$$

The strict inequality (96) contradicts the equality (97). Therefore  $\dim \ker(G_\alpha^\lambda - r_\alpha I) = 1$ , and by (83),  $\dim \ker(L_\lambda - \mu_0(\lambda)I) = 1$ . This proves assertion 2.

(vii) *Uniqueness of the normalized strictly positive ground state.* Let  $\xi' \in \ker(L_\lambda - \mu_0(\lambda)I)$  satisfy  $\|\xi'\|_{H_\lambda} = 1$  and  $\xi' > 0$   $\mu$ -a.e. on  $I_\lambda$ . By the simplicity established in (vi),  $\xi' = c\xi_\lambda$  for some  $c \in \mathbb{C}$ . Pairing with  $\xi_\lambda$ ,

$$c = c \|\xi_\lambda\|_{H_\lambda}^2 = \langle c\xi_\lambda, \xi_\lambda \rangle_{H_\lambda} = \langle \xi', \xi_\lambda \rangle_{H_\lambda} = \int_{I_\lambda} \xi' \xi_\lambda d\mu > 0,$$

where the strict positivity of the integral uses that both  $\xi'$  and  $\xi_\lambda$  are strictly positive  $\mu$ -a.e. on  $I_\lambda$ , while  $\mu(I_\lambda) > 0$ . Thus  $c > 0$ , and the normalization  $\|\xi'\|_{H_\lambda} = |c| \|\xi_\lambda\|_{H_\lambda} = c$  forces  $c = 1$ . Therefore  $\xi' = \xi_\lambda$ , which establishes assertion 3 and completes the proof.  $\square$

5.3. *Inversion symmetry and evenness of the ground state.* We now show that the ground state inherits the inversion symmetry of the form.

PROPOSITION 5.3. *Let*

$$I_\lambda = [\lambda^{-1}, \lambda], \quad H_\lambda = L^2(I_\lambda, d^\times u), \quad d^\times u = \frac{du}{u}.$$

*Define the reflection operator*

$$(\mathcal{R}f)(u) := f(u^{-1}), \quad u \in I_\lambda.$$

Then:

1.  $\mathcal{R}$  is a unitary self-adjoint operator on  $H_\lambda$ , and

$$\mathcal{R}^* = \mathcal{R}, \quad \mathcal{R}^2 = I_{H_\lambda}.$$

2.  $\mathcal{R}$  preserves the form domain  $\mathcal{F}_\lambda$ , and for all  $f, g \in \mathcal{F}_\lambda$ ,

$$\mathcal{E}_\lambda(\mathcal{R}f, \mathcal{R}g) = \mathcal{E}_\lambda(f, g).$$

3. If  $L_\lambda$  is the self-adjoint operator associated with  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ , then

$$\mathcal{R}L_\lambda = L_\lambda\mathcal{R}.$$

4. Let  $\xi_\lambda$  be the unique  $L^2$ -normalized strictly positive ground state from the previous theorem. Then

$$\mathcal{R}\xi_\lambda = \xi_\lambda,$$

that is,

$$\xi_\lambda(u^{-1}) = \xi_\lambda(u) \quad \text{a.e. on } I_\lambda.$$

*Proof.* We organize the proof into seven parts (i)–(vii); part (i) establishes statement 1, parts (ii)–(v) establish statement 2, part (vi) establishes statement 3, and part (vii) establishes statement 4.

(i) *Well-definedness, isometry, and self-adjointness of  $\mathcal{R}$ .* The interval  $I_\lambda = [\lambda^{-1}, \lambda]$  is invariant under inversion:  $u \in I_\lambda \iff u^{-1} \in I_\lambda$ . Hence  $u \mapsto u^{-1}$  is a measurable involution on  $I_\lambda$ , and the  $L^2$ -equivalence class of  $u \mapsto f(u^{-1})$  is well defined for every  $f \in H_\lambda$ ; the operator  $\mathcal{R}$  is therefore linear and well defined on  $H_\lambda$ .

The multiplicative Haar measure  $d^\times u = du/u$  on  $\mathbb{R}_{>0}^\times$  is invariant under the continuous group automorphism  $u \mapsto u^{-1}$ . Concretely, the substitution  $v = u^{-1}$  in the integral defining  $\|\mathcal{R}f\|_{H_\lambda}^2$  gives  $du = -v^{-2}dv$ , hence  $d^\times u = -d^\times v$ ; the orientation of the integration limits is also reversed (as  $u$  traverses  $[\lambda^{-1}, \lambda]$ , the new variable  $v$  traverses  $[\lambda, \lambda^{-1}]$ ), and the two sign reversals compensate to yield

$$\|\mathcal{R}f\|_{H_\lambda}^2 = \int_{\lambda^{-1}}^{\lambda} |f(u^{-1})|^2 d^\times u = \int_{\lambda^{-1}}^{\lambda} |f(v)|^2 d^\times v = \|f\|_{H_\lambda}^2.$$

The same substitution applied to the sesquilinear pairing yields, for every  $f, g \in H_\lambda$ ,

$$\langle \mathcal{R}f, g \rangle_{H_\lambda} = \int_{\lambda^{-1}}^{\lambda} f(u^{-1}) \overline{g(u)} d^\times u = \int_{\lambda^{-1}}^{\lambda} f(v) \overline{g(v^{-1})} d^\times v = \langle f, \mathcal{R}g \rangle_{H_\lambda},$$

so  $\mathcal{R}^* = \mathcal{R}$ . Finally,  $(\mathcal{R}^2 f)(u) = f((u^{-1})^{-1}) = f(u)$  a.e. on  $I_\lambda$ , so  $\mathcal{R}^2 = I_{H_\lambda}$ . In particular,  $\mathcal{R}$  is unitary, which proves statement 1.

(ii) *Reflection-invariance of the form core  $\mathcal{V}_\lambda$ .* By the construction in Subsection 2.4, the trigonometric basis  $V_j(u) = \mathcal{T}_j(\log(\lambda u))$  is given explicitly by

$$V_j(u) = L^{-1/2} \exp(2\pi i j \log(\lambda u)/L), \quad j \in \mathbb{Z}, \quad u \in I_\lambda,$$

with  $L = 2 \log \lambda$ . For each  $j \in \mathbb{Z}$  and a.e.  $u \in I_\lambda$ ,

$$(98) \quad \mathcal{R}V_j(u) = V_j(u^{-1}) = L^{-1/2} \exp(2\pi i j (\log \lambda - \log u)/L) = L^{-1/2} e^{2\pi i j \log \lambda / L} e^{-2\pi i j \log u / L},$$

while

$$(99) \quad V_{-j}(u) = L^{-1/2} \exp(-2\pi i j (\log \lambda + \log u)/L) = L^{-1/2} e^{-2\pi i j \log \lambda / L} e^{-2\pi i j \log u / L}.$$

The two prefactors of (98) and (99) differ by the multiplicative factor

$$e^{2\pi i j \log \lambda / L} \cdot e^{2\pi i j \log \lambda / L} = e^{4\pi i j \log \lambda / L} = e^{2\pi i j} = 1,$$

where the second equality uses  $4 \log \lambda / L = 2$  (which is the convention  $L = 2 \log \lambda$ ) and the third uses  $j \in \mathbb{Z}$ . Therefore

$$(100) \quad \mathcal{R}V_j = V_{-j}, \quad j \in \mathbb{Z}.$$

The map  $\mathcal{R}$  permutes the basis  $\{V_j\}_{j \in \mathbb{Z}}$  by sign reversal of the index; in particular,

$$(101) \quad \mathcal{R}\mathcal{V}_\lambda = \mathcal{V}_\lambda.$$

(iii) *Reflection-invariance of the archimedean part.* For  $f, g \in \mathcal{F}_\lambda$ , set

$$\varphi_f := Zf, \quad \varphi_g := Zg \in L^2(I_x, dx), \quad F := E_x \varphi_f, \quad G := E_x \varphi_g \in L^2(\mathbb{R}, dx),$$

where  $Z : H_\lambda \rightarrow L^2(I_x, dx)$  is the logarithmic unitary of Lemma 2.3 with  $I_x = [-\ell, \ell]$ ,  $\ell = \log \lambda$ , and  $E_x : L^2(I_x, dx) \hookrightarrow L^2(\mathbb{R}, dx)$  is zero extension as in Theorem 3.2(2). Define the additive reflection

$$J : L^2(\mathbb{R}, dx) \rightarrow L^2(\mathbb{R}, dx), \quad (Jh)(x) := h(-x).$$

For a.e.  $x \in I_x$ , the change of variable  $u = e^x$  gives

$$(Z\mathcal{R}f)(x) = \mathcal{R}f(e^x) = f(e^{-x}) = (Zf)(-x) = (J\varphi_f)(x).$$

Both  $E_x \varphi_f$  and  $E_x(J\varphi_f)$  vanish for  $|x| > \ell$ , and  $J$  preserves the symmetric set  $I_x$ ; hence

$$(102) \quad E_x Z(\mathcal{R}f) = JF, \quad E_x Z(\mathcal{R}g) = JG, \quad \text{a.e. on } \mathbb{R}.$$

By Theorem 3.2(2), the diagonal Lévy–Khintchine identity

$$\mathcal{E}_{\infty, \lambda}(f, f) = \pi \int_0^\infty \|F(\cdot + r) - F\|_{L^2(\mathbb{R})}^2 \nu(dr)$$

holds for every  $f \in \mathcal{F}_\lambda$ . The polarization identity for sesquilinear forms, applied to this identity (with all four resulting integrals finite for  $f, g \in \mathcal{F}_\lambda$ ), yields

$$(103) \quad \mathcal{E}_{\infty, \lambda}(f, g) = \pi \int_0^\infty \langle F(\cdot + r) - F, G(\cdot + r) - G \rangle_{L^2(\mathbb{R})} \nu(dr), \quad f, g \in \mathcal{F}_\lambda.$$

We now verify the invariance of the integrand of (103) under the substitution  $(F, G) \mapsto (JF, JG)$ . Fix  $r > 0$ . For every  $x \in \mathbb{R}$ ,

$$JF(x + r) - JF(x) = F(-x - r) - F(-x).$$

Substituting  $y = -x$  (Lebesgue measure invariant under  $y \mapsto -y$ ),

$$(104) \quad \int_{\mathbb{R}} [JF(x+r) - JF(x)] [\overline{JG(x+r) - JG(x)}] dx = \int_{\mathbb{R}} [F(y-r) - F(y)] [\overline{G(y-r) - G(y)}] dy.$$

Substituting  $z = y - r$  (Lebesgue measure invariant under  $z \mapsto z + r$ ),

$$F(y - r) - F(y) = F(z) - F(z + r) = -(F(z + r) - F(z)),$$

and similarly for  $G$ ; the two minus signs cancel inside the integrand on the right-hand side of (104), yielding

$$(105) \quad \int_{\mathbb{R}} [JF(x+r) - JF(x)] [\overline{JG(x+r) - JG(x)}] dx = \int_{\mathbb{R}} [F(z+r) - F(z)] [\overline{G(z+r) - G(z)}] dz.$$

The right-hand side of (105) is the integrand of (103). Combining (103)–(105) with (102) and integrating against  $\nu(dr)$  on  $(0, \infty)$  via Tonelli's theorem [16, Vol. I, Thm. 3.4.4],

$$(106) \quad \mathcal{E}_{\infty, \lambda}(\mathcal{R}f, \mathcal{R}g) = \mathcal{E}_{\infty, \lambda}(f, g), \quad f, g \in \mathcal{F}_\lambda.$$

(iv) *Reflection conjugation of the compressed dilations.* Fix an integer  $n \geq 2$ . By Proposition 2.7(3), for every  $h \in H_\lambda$ ,

$$(U_n h)(u) = \mathbf{1}_{I_\lambda}(nu) h(nu) \quad \text{a.e. } u \in I_\lambda.$$

Applying this together with  $(\mathcal{R}f)(w) = f(w^{-1})$  at  $w = nu^{-1}$ , for a.e.  $u \in I_\lambda$ ,

$$(107) \quad (\mathcal{R}U_n \mathcal{R}f)(u) = (U_n \mathcal{R}f)(u^{-1}) = \mathbf{1}_{I_\lambda}(nu^{-1}) (\mathcal{R}f)(nu^{-1}) = \mathbf{1}_{I_\lambda}(nu^{-1}) f(n^{-1}u).$$

On the other hand,  $U_{n^{-1}} = R_\lambda S_{n^{-1}} E_\lambda$  in the notation of Definition 2.6, with  $n^{-1} \in (0, 1)$  as the dilation parameter, so the same proposition yields

$$(108) \quad (U_{n^{-1}} f)(u) = \mathbf{1}_{I_\lambda}(n^{-1}u) f(n^{-1}u) \quad \text{a.e. } u \in I_\lambda.$$

To compare the two indicator factors in (107) and (108), we exploit the inversion-invariance of  $I_\lambda$ . For every  $x \in \mathbb{R}_{>0}^\times$ ,  $x \in I_\lambda \iff x^{-1} \in I_\lambda$ , so  $\mathbf{1}_{I_\lambda}(x) = \mathbf{1}_{I_\lambda}(x^{-1})$ . Setting  $x = n^{-1}u$  (so that  $x^{-1} = nu^{-1}$ ) gives

$$(109) \quad \mathbf{1}_{I_\lambda}(nu^{-1}) = \mathbf{1}_{I_\lambda}(n^{-1}u), \quad u \in \mathbb{R}_{>0}^\times.$$

Combining (107), (108), and (109),

$$(110) \quad \mathcal{R}U_n\mathcal{R} = U_{n-1} = U_n^*,$$

where the second equality is Proposition 2.7(2). With  $\mathcal{R}^2 = I_{H_\lambda}$  from (i),

$$(111) \quad \mathcal{R}T(n)\mathcal{R} = n^{-1/2}(\mathcal{R}U_n\mathcal{R} + \mathcal{R}U_n^*\mathcal{R}) = n^{-1/2}(U_n^* + U_n) = T(n).$$

For  $f, g \in H_\lambda$ , applying (111) together with  $\mathcal{R}^* = \mathcal{R}$  and  $\mathcal{R}^2 = I_{H_\lambda}$ ,

$$\langle T(n)\mathcal{R}f, \mathcal{R}g \rangle_{H_\lambda} = \langle \mathcal{R}T(n)f, \mathcal{R}g \rangle_{H_\lambda} = \langle T(n)f, \mathcal{R}^*\mathcal{R}g \rangle_{H_\lambda} = \langle T(n)f, g \rangle_{H_\lambda}.$$

Multiplying by  $-\Lambda(n)$  and summing over  $1 < n \leq \lambda^2$ ,

$$(112) \quad Q_{p,\lambda}(\mathcal{R}f, \mathcal{R}g) = Q_{p,\lambda}(f, g), \quad f, g \in H_\lambda.$$

Since  $\langle \mathcal{R}f, \mathcal{R}g \rangle_{H_\lambda} = \langle f, g \rangle_{H_\lambda}$  by (i),

$$(113) \quad \mathcal{E}_{p,\lambda}(\mathcal{R}f, \mathcal{R}g) = Q_{p,\lambda}(\mathcal{R}f, \mathcal{R}g) + c_{p,\lambda}\langle \mathcal{R}f, \mathcal{R}g \rangle_{H_\lambda} = Q_{p,\lambda}(f, g) + c_{p,\lambda}\langle f, g \rangle_{H_\lambda} = \mathcal{E}_{p,\lambda}(f, g).$$

(v) *Form-invariance identity and domain invariance.* For  $f, g \in \mathcal{V}_\lambda$ , both  $\mathcal{R}f, \mathcal{R}g \in \mathcal{V}_\lambda \subset \mathcal{F}_\lambda$  by (101); hence  $\mathcal{E}_\lambda(\mathcal{R}f, \mathcal{R}g)$  is finite-valued, and the additive decomposition  $\mathcal{E}_\lambda = \mathcal{E}_{\infty,\lambda} + \mathcal{E}_{p,\lambda}$  from Theorem 3.2(4) together with (106) and (113) gives

$$(114) \quad \mathcal{E}_\lambda(\mathcal{R}f, \mathcal{R}g) = \mathcal{E}_\lambda(f, g), \quad f, g \in \mathcal{V}_\lambda.$$

Specializing to  $g = f$ ,

$$(115) \quad \mathcal{E}_\lambda(\mathcal{R}f, \mathcal{R}f) = \mathcal{E}_\lambda(f, f), \quad f \in \mathcal{V}_\lambda.$$

To extend to  $\mathcal{F}_\lambda$ , we use the form-core property of  $\mathcal{V}_\lambda$  and the closedness of  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  given by Theorem 3.2(1). Equip  $\mathcal{F}_\lambda$  with the graph norm

$$\|h\|_{\mathcal{E}_{\lambda,1}}^2 := \mathcal{E}_\lambda(h, h) + \|h\|_{H_\lambda}^2.$$

For every  $f \in \mathcal{F}_\lambda$ , the form-core property provides a sequence  $(f_k) \subset \mathcal{V}_\lambda$  with  $f_k \rightarrow f$  in  $\|\cdot\|_{\mathcal{E}_{\lambda,1}}$ . By (115) applied to  $f_k - f_m \in \mathcal{V}_\lambda$  together with the unitarity of  $\mathcal{R}$  from (i),

$$(116) \quad \|\mathcal{R}f_k - \mathcal{R}f_m\|_{\mathcal{E}_{\lambda,1}}^2 = \mathcal{E}_\lambda(\mathcal{R}(f_k - f_m), \mathcal{R}(f_k - f_m)) + \|\mathcal{R}(f_k - f_m)\|_{H_\lambda}^2 = \|f_k - f_m\|_{\mathcal{E}_{\lambda,1}}^2,$$

so  $(\mathcal{R}f_k) \subset \mathcal{V}_\lambda$  is Cauchy in  $\|\cdot\|_{\mathcal{E}_{\lambda,1}}$ . By closedness, there exists  $\tilde{f} \in \mathcal{F}_\lambda$  with  $\mathcal{R}f_k \rightarrow \tilde{f}$  in  $\|\cdot\|_{\mathcal{E}_{\lambda,1}}$ , hence in  $H_\lambda$  as well. The boundedness of  $\mathcal{R}$  on  $H_\lambda$  also gives  $\mathcal{R}f_k \rightarrow \mathcal{R}f$  in  $H_\lambda$ , so the uniqueness of  $H_\lambda$ -limits forces  $\tilde{f} = \mathcal{R}f$ . Therefore

$$(117) \quad \mathcal{R}f \in \mathcal{F}_\lambda \quad \text{for every } f \in \mathcal{F}_\lambda.$$

Replacing  $f$  by  $\mathcal{R}f$  in (117) and using  $\mathcal{R}^2 = I_{H_\lambda}$  gives the reverse inclusion; hence

$$(118) \quad \mathcal{R}\mathcal{F}_\lambda = \mathcal{F}_\lambda.$$



To pass from (114) to the full identity on  $\mathcal{F}_\lambda \times \mathcal{F}_\lambda$ , take any  $f, g \in \mathcal{F}_\lambda$  together with approximating sequences  $(f_k), (g_k) \subset \mathcal{V}_\lambda$  converging in  $\|\cdot\|_{\mathcal{E}_\lambda,1}$ . By (116) (and its analogue with  $g$  in place of  $f$ ),  $\mathcal{R}f_k \rightarrow \mathcal{R}f$  and  $\mathcal{R}g_k \rightarrow \mathcal{R}g$  in  $\|\cdot\|_{\mathcal{E}_\lambda,1}$ . The form  $\mathcal{E}_\lambda$  is sesquilinear and continuous in this graph norm ([18, Chap. VI, §1]). Passing to the limit  $k \rightarrow \infty$  in the identity  $\mathcal{E}_\lambda(\mathcal{R}f_k, \mathcal{R}g_k) = \mathcal{E}_\lambda(f_k, g_k)$ , given by (114),

$$(119) \quad \mathcal{E}_\lambda(\mathcal{R}f, \mathcal{R}g) = \mathcal{E}_\lambda(f, g), \quad f, g \in \mathcal{F}_\lambda.$$

Together with (118), this proves statement 2.

(vi) *Operator commutation*  $\mathcal{R}L_\lambda = L_\lambda\mathcal{R}$ . By the first representation theorem [18, Chap. VI, Thm. 2.1], the nonnegative self-adjoint operator  $L_\lambda$  associated with  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  satisfies

$$\mathcal{E}_\lambda(u, v) = \langle L_\lambda u, v \rangle_{H_\lambda}, \quad u \in \text{Dom}(L_\lambda), v \in \mathcal{F}_\lambda;$$

moreover, an element  $u \in \mathcal{F}_\lambda$  lies in  $\text{Dom}(L_\lambda)$  if and only if there exists  $h \in H_\lambda$  with  $\mathcal{E}_\lambda(u, v) = \langle h, v \rangle_{H_\lambda}$  for every  $v \in \mathcal{F}_\lambda$ , in which case  $h = L_\lambda u$ .

Take  $f \in \text{Dom}(L_\lambda)$ . For every  $v \in \mathcal{F}_\lambda$ ,  $\mathcal{R}v \in \mathcal{F}_\lambda$  by (118). Using (119) (with  $\mathcal{R}v$  in place of the second argument and exploiting  $\mathcal{R}^2 = I_{H_\lambda}$ ), the first representation theorem, and  $\mathcal{R}^* = \mathcal{R}$  from (i),

$$\mathcal{E}_\lambda(\mathcal{R}f, v) = \mathcal{E}_\lambda(\mathcal{R}f, \mathcal{R}\mathcal{R}v) = \mathcal{E}_\lambda(f, \mathcal{R}v) = \langle L_\lambda f, \mathcal{R}v \rangle_{H_\lambda} = \langle \mathcal{R}^* L_\lambda f, v \rangle_{H_\lambda} = \langle \mathcal{R}L_\lambda f, v \rangle_{H_\lambda}.$$

The functional  $v \mapsto \langle \mathcal{R}L_\lambda f, v \rangle_{H_\lambda}$  is bounded on  $H_\lambda$ , with bound  $\|\mathcal{R}L_\lambda f\|_{H_\lambda}$ . By the converse direction of the first representation theorem,  $\mathcal{R}f \in \text{Dom}(L_\lambda)$  and

$$(120) \quad L_\lambda(\mathcal{R}f) = \mathcal{R}L_\lambda f, \quad f \in \text{Dom}(L_\lambda).$$

Replacing  $f$  by  $\mathcal{R}f$  in (120) and using  $\mathcal{R}^2 = I_{H_\lambda}$  gives the reverse inclusion  $\mathcal{R}\text{Dom}(L_\lambda) = \text{Dom}(L_\lambda)$ . Therefore  $\mathcal{R}L_\lambda = L_\lambda\mathcal{R}$  on  $\text{Dom}(L_\lambda)$ , which proves statement 3.

(vii) *Reflection-invariance of the ground state.* By Theorem 5.2,  $\mu_0(\lambda)$  is a simple eigenvalue of  $L_\lambda$  with one-dimensional eigenspace  $\mathbb{C}\xi_\lambda \subset \text{Dom}(L_\lambda)$ , and the unique  $L^2$ -normalized representative  $\xi_\lambda$  is real-valued, strictly positive  $\mu$ -a.e. on  $I_\lambda$ .

The commutation (120) gives

$$L_\lambda(\mathcal{R}\xi_\lambda) = \mathcal{R}L_\lambda\xi_\lambda = \mu_0(\lambda)\mathcal{R}\xi_\lambda,$$

so  $\mathcal{R}\xi_\lambda \in \ker(L_\lambda - \mu_0(\lambda)I) = \mathbb{C}\xi_\lambda$ . There exists therefore a unique  $c \in \mathbb{C}$  with

$$(121) \quad \mathcal{R}\xi_\lambda = c\xi_\lambda.$$

We identify  $c = 1$  in three substeps.

(vii.a) *c is real.* Since  $\xi_\lambda$  is real-valued, for a.e.  $u \in I_\lambda$ ,

$$\overline{\mathcal{R}\xi_\lambda(u)} = \overline{\xi_\lambda(u^{-1})} = \xi_\lambda(u^{-1}) = \mathcal{R}\xi_\lambda(u),$$

so  $\mathcal{R}\xi_\lambda$  is real-valued  $\mu$ -a.e. on  $I_\lambda$ . The set  $\{u \in I_\lambda : \xi_\lambda(u) > 0\}$  has full  $\mu$ -measure; on this set, the ratio  $\mathcal{R}\xi_\lambda(u)/\xi_\lambda(u)$  is a real number, and (121) forces this ratio to equal  $c$   $\mu$ -a.e. Hence  $c \in \mathbb{R}$ .

(vii.b)  $|c| = 1$ . By the unitarity of  $\mathcal{R}$  from (i),

$$|c| = |c| \cdot \|\xi_\lambda\|_{H_\lambda} = \|c\xi_\lambda\|_{H_\lambda} = \|\mathcal{R}\xi_\lambda\|_{H_\lambda} = \|\xi_\lambda\|_{H_\lambda} = 1.$$

Combined with (vii.a),  $c \in \{-1, +1\}$ .

(vii.c)  $c = +1$ . Pairing (121) with  $\xi_\lambda$  and using  $\|\xi_\lambda\|_{H_\lambda} = 1$ ,

$$c = \langle c\xi_\lambda, \xi_\lambda \rangle_{H_\lambda} = \langle \mathcal{R}\xi_\lambda, \xi_\lambda \rangle_{H_\lambda} = \int_{I_\lambda} \xi_\lambda(u^{-1}) \xi_\lambda(u) d^\times u.$$

The integrand is the product of two functions both strictly positive  $\mu$ -a.e. on  $I_\lambda$ , and  $\mu(I_\lambda) = 2 \log \lambda > 0$ , so the integral is strictly positive. Combined with (vii.b), this rules out  $c = -1$  and forces  $c = +1$ . Therefore

$$\mathcal{R}\xi_\lambda = \xi_\lambda, \quad \text{equivalently,} \quad \xi_\lambda(u^{-1}) = \xi_\lambda(u) \quad \mu\text{-a.e. on } I_\lambda,$$

which proves statement 4 and completes the proof.  $\square$

5.4. *Interval distributions and the form kernel.* We next transport the form to the interval model and identify the corresponding distribution kernel.

PROPOSITION 5.4. *Let  $\lambda > 1$ , and set*

$$\ell = \log \lambda, \quad L = 2\ell.$$

*Consider the isometric isomorphism used in [7, Proposition 3.2]*

$$\kappa : L^2([0, L], dy) \rightarrow H_\lambda = L^2([\lambda^{-1}, \lambda], d^\times u), \quad \kappa(f)(u) = f(\log(\lambda u)).$$

*Let*

$$Q_{D,\lambda} := \mathcal{E}_\lambda - c_{D,\lambda} \langle \cdot, \cdot \rangle_{H_\lambda}, \quad A_\lambda := L_\lambda - c_{D,\lambda} I.$$

*Set*

$$T := \text{span}\{\mathcal{T}_n : n \in \mathbb{Z}\} \subset L^2([0, L], dy), \quad \mathcal{T}_n(y) := L^{-1/2} e^{2\pi i n y / L}.$$

*Then there exists a real distribution  $D_\lambda \in \mathcal{D}'([0, L])$  such that:*

1. *For every  $f, g \in T$ ,*

$$Q_{D,\lambda}(\kappa f, \kappa g) = D_\lambda(q(g, f)),$$

*where*

$$q(f, g)(y) := (f^* * g)(y) + (f^* * g)(-y), \quad f^*(x) := \overline{f(-x)}.$$

*This is precisely the type of quadratic form induced by a real distribution on  $[0, L]$  discussed in [8, formula (6)].*

2.  $T$  is a form core for the above pre-form; its closure is precisely

$$\overline{Q}_{D,\lambda}(f, g) := Q_{D,\lambda}(\kappa f, \kappa g)$$

on  $\kappa^{-1}(\mathcal{F}_\lambda)$ , and the associated self-adjoint operator is

$$\mathcal{S}_\lambda := \kappa^{-1} A_\lambda \kappa = \kappa^{-1} (L_\lambda - c_{D,\lambda} I) \kappa.$$

*Proof.* We organize the proof into seven parts (i)–(vii); parts (i)–(vi) establish statement 1, and part (vii) establishes statement 2.

(i) *Unitarity of  $\kappa$  and form-core transfer.* The change of variable  $y = \log(\lambda u)$  identifies  $u \in I_\lambda = [\lambda^{-1}, \lambda]$  with  $y \in [0, L]$  via  $u = e^{y-\ell}$ , with Jacobian  $d^\times u = du/u = dy$ . For every  $f \in L^2([0, L], dy)$ ,

$$\|\kappa f\|_{H_\lambda}^2 = \int_{\lambda^{-1}}^{\lambda} |f(\log(\lambda u))|^2 d^\times u = \int_0^L |f(y)|^2 dy = \|f\|_{L^2([0, L])}^2,$$

and the  $L^2$ -adjoint of  $\kappa$  is  $\eta \mapsto \eta(e^{-\ell})$ , which serves as both left and right inverse of  $\kappa$ . Hence  $\kappa$  is unitary.

For each  $n \in \mathbb{Z}$ , the trigonometric basis vector  $V_n \in \mathcal{V}_\lambda$  defined in Subsection 2.4 satisfies

$$V_n(u) = L^{-1/2} e^{2\pi i n \log(\lambda u)/L} = \mathcal{T}_n(\log(\lambda u)) = \kappa(\mathcal{T}_n)(u), \quad u \in I_\lambda,$$

giving

$$(122) \quad \kappa(T) = \text{span}\{V_n : n \in \mathbb{Z}\} = \mathcal{V}_\lambda.$$

By [5, §2] (cited in [7, Proposition 3.4]),  $\mathcal{V}_\lambda$  is a form core for the closed lower-semibounded form  $QW_\lambda$ . By Proposition 3.4 of the present paper,  $\text{Dom}(Q_{D,\lambda}) = \text{Dom}(QW_\lambda)$  with equivalent graph norms (the difference  $B_\lambda$  being bounded by Lemma 3.3), so  $\mathcal{V}_\lambda$  is also a form core for  $Q_{D,\lambda}$ . Pulling back the graph norm via the unitary  $\kappa$  preserves all density and closedness statements; hence, by (122),  $T = \kappa^{-1}\mathcal{V}_\lambda$  is a form core for the closed lower-semibounded sesquilinear form

$$\overline{Q}_{D,\lambda}(f, g) := Q_{D,\lambda}(\kappa f, \kappa g)$$

on  $\text{Dom}(\overline{Q}_{D,\lambda}) = \kappa^{-1} \text{Dom}(Q_{D,\lambda}) = \kappa^{-1} \mathcal{F}_\lambda \subset L^2([0, L], dy)$ .

(ii) *The correlation function in interval coordinates.* Fix  $f, g \in T$  and set  $F := \kappa f \in \mathcal{V}_\lambda$ ,  $G := \kappa g \in \mathcal{V}_\lambda$ . Extend  $f, g$  by zero from  $[0, L]$  to all of  $\mathbb{R}$ , retaining the same notation; the supports remain in  $[0, L]$ . By Definition 2.9,

$$F_{F,G}(a) = \int_{\mathbb{R}_{>0}^\times} F(au) \overline{G(u)} d^\times u, \quad a \in \mathbb{R}_{>0}^\times.$$

Apply the change of variable  $x = \log(\lambda u)$ , so  $u = e^{x-\ell}$  and  $d^\times u = dx$ . For  $a = e^y \in \mathbb{R}_{>0}^\times$ , the dilation  $au = e^{y+x-\ell}$  corresponds to  $\log(\lambda(au)) = x + y$ ;

hence  $F(au) = f(x + y)$  and  $G(u) = g(x)$ , yielding

$$(123) \quad F_{F,G}(e^y) = \int_{\mathbb{R}} f(x + y) \overline{g(x)} dx, \quad y \in \mathbb{R}.$$

The Hermitian involution  $g^*(s) := \overline{g(-s)}$  from the proposition statement and the Lebesgue convolution  $(g^* * f)(y) := \int_{\mathbb{R}} g^*(y - z) f(z) dz$  together yield, after the substitution  $z = x + y$ ,

$$(124) \quad (g^* * f)(y) = \int_{\mathbb{R}} \overline{g(-(y - z))} f(z) dz \Big|_{z=x+y} = \int_{\mathbb{R}} \overline{g(x)} f(x + y) dx, \quad y \in \mathbb{R}.$$

Comparing (123) with (124),

$$(125) \quad F_{F,G}(e^y) = (g^* * f)(y), \quad y \in \mathbb{R}.$$

Lemma 2.11 gives  $\text{supp } F_{F,G} \subset [\lambda^{-2}, \lambda^2]$ , equivalently  $\text{supp}(g^* * f) \subset [-L, L]$ , so  $F_{F,G}$  lies in the Weil class on  $\mathbb{R}_{>0}^{\times}$ .

(iii) *Vanishing of  $\Psi$  on inversion-antisymmetric arguments.* By the explicit formula derived in Subsection 2.4, the semilocal Weil functional acts on the Weil class via

$$\Psi(\Phi) = \widehat{\Phi}(i/2) + \widehat{\Phi}(-i/2) + \int_{\mathbb{R}} \widehat{\Phi}(t) w(t) dt - \sum_{1 < n \leq \lambda^2} \Lambda(n) n^{-1/2} (\Phi(n) + \Phi(n^{-1})),$$

with  $w$  given by (18), and  $QW_{\lambda}(F_0, G_0) = \Psi(F_{F_0, G_0})$  for  $F_0, G_0 \in \mathcal{V}_{\lambda}$  by Subsection 2.4.

For every  $\Phi$  in the Weil class, the substitution  $a \mapsto a^{-1}$  (a measure-preserving automorphism of  $\mathbb{R}_{>0}^{\times}$ ) gives

$$(126) \quad (\widehat{\Phi \circ \iota})(z) = \int_{\mathbb{R}_{>0}^{\times}} \Phi(a^{-1}) a^{-iz} d^{\times} a = \int_{\mathbb{R}_{>0}^{\times}} \Phi(b) b^{iz} d^{\times} b = \widehat{\Phi}(-z), \quad z \in \mathbb{C},$$

where  $\iota(u) = u^{-1}$ . For inversion-antisymmetric  $\Phi$  (defined by  $\Phi \circ \iota = -\Phi$ ), (126) forces  $\widehat{\Phi}(z) = -\widehat{\Phi}(-z)$ , that is,  $\widehat{\Phi}$  is odd. The three components of  $\Psi$  on such  $\Phi$  are then:

- (a) the boundary contribution  $\widehat{\Phi}(i/2) + \widehat{\Phi}(-i/2) = \widehat{\Phi}(i/2) - \widehat{\Phi}(i/2) = 0$ ;
- (b) the prime contribution  $\sum_{1 < n \leq \lambda^2} \Lambda(n) n^{-1/2} (\Phi(n) + \Phi(n^{-1})) = 0$ , since  $\Phi(n^{-1}) = -\Phi(n)$  termwise;
- (c) the archimedean contribution  $\int_{\mathbb{R}} \widehat{\Phi}(t) w(t) dt = 0$ , since  $w(t) = \theta'(t)/\pi$  is even in  $t$  (because  $\Re\psi(\frac{1}{4} + \frac{it}{2})$  is even in  $t$  by the integral representation (16)), so the integrand is the product of an odd and an even function and integrates to zero.

Combining (a)–(c) gives  $\Psi(\Phi) = 0$  on inversion-antisymmetric  $\Phi$ .

For arbitrary  $\Phi$  in the Weil class, decompose  $\Phi = \Phi_+ + \Phi_-$  with  $\Phi_{\pm} := \frac{1}{2}(\Phi \pm \Phi \circ \iota)$ . By the linearity of  $\Psi$  and  $\Psi(\Phi_-) = 0$ ,

$$\Psi(\Phi) = \Psi(\Phi_+) = \frac{1}{2} \Psi(\Phi + \Phi \circ \iota).$$

Define

$$(127) \quad \Psi^{\#} := \frac{1}{2} \Psi$$

on the Weil class. Then

$$(128) \quad \Psi(\Phi) = \Psi^{\#}(\Phi + \Phi \circ \iota), \quad \Phi \text{ in the Weil class,}$$

which recovers the symmetrized form invoked in [7, Lemma 3.1].

(iv) *Distribution representation of  $QW_{\lambda}$  on  $[0, L]$ .* Apply (128) with  $\Phi = F_{F,G}$ :

$$(129) \quad QW_{\lambda}(\kappa f, \kappa g) = \Psi(F_{F,G}) = \Psi^{\#}(F_{F,G} + F_{F,G} \circ \iota).$$

Set  $H := F_{F,G} + F_{F,G} \circ \iota$ . By (125),

$$H(e^y) = F_{F,G}(e^y) + F_{F,G}(e^{-y}) = (g^* * f)(y) + (g^* * f)(-y) = q(g, f)(y), \quad y \in \mathbb{R},$$

that is,

$$(130) \quad H(e^y) = q(g, f)(y), \quad y \in \mathbb{R}.$$

Direct inspection gives  $q(g, f)(-y) = (g^* * f)(-y) + (g^* * f)(y) = q(g, f)(y)$ , so  $q(g, f)$  is an even function on  $\mathbb{R}$ .

For every  $\phi \in \mathcal{D}([0, L])$ , the even extension

$$\phi^{\text{ev}} : [-L, L] \rightarrow \mathbb{C}, \quad \phi^{\text{ev}}(y) := \phi(|y|),$$

followed by the change of variable  $y = \log a$ , defines a function in the Weil class via

$$\tilde{\phi}(a) := \phi^{\text{ev}}(\log a) \mathbf{1}_{[\lambda^{-2}, \lambda^2]}(a), \quad a \in \mathbb{R}_{>0}^{\times},$$

whose support is contained in  $[\lambda^{-2}, \lambda^2]$  and which is inversion-symmetric:  $\tilde{\phi} \circ \iota = \tilde{\phi}$ . Define

$$(131) \quad D_W(\phi) := \Psi^{\#}(\tilde{\phi}), \quad \phi \in \mathcal{D}([0, L]),$$

in the convention of [8, formula (6)], which identifies a distribution on  $[0, L]$  with the unique even distribution on  $[-L, L]$  that restricts to it. The map  $\phi \mapsto \tilde{\phi}$  is continuous from  $\mathcal{D}([0, L])$  into the Weil class, and each of  $W_{0,2}$ ,  $W_p$ , and  $W_{\infty}$  is continuous on the Weil class; therefore  $D_W$  is a distribution on  $[0, L]$ .

To verify that  $D_W$  is real, take  $\phi$  real-valued. Then  $\tilde{\phi}$  is real-valued and inversion-symmetric on  $\mathbb{R}_{>0}^{\times}$ , and the Mellin transform

$$\widehat{\tilde{\phi}}(t) = \int_{-L}^L \phi^{\text{ev}}(y) e^{-ity} dy = 2 \int_0^L \phi(y) \cos(ty) dy$$

is real-valued for  $t \in \mathbb{R}$ . Each of (a)  $\widehat{\phi}(\pm i/2) = 2 \int_0^L \phi(y) \cosh(y/2) dy \in \mathbb{R}$ , (b) the prime sum (real by direct inspection, since  $\widehat{\phi}(n^{\pm m}) \in \mathbb{R}$ ), and (c)  $\int_{\mathbb{R}} \widehat{\phi}(t) w(t) dt$  (real, since  $w$  is real-valued on  $\mathbb{R}$ ) is real. Hence  $D_W(\phi) \in \mathbb{R}$ , and  $D_W$  is a real distribution.

For  $f, g \in T$ , the function  $q(g, f)$  is even (just established) and supported in  $[-L, L]$ . Comparing  $H$  from (130) with the function  $q(g, f)|_{[0, L]}$  defined above: both equal  $q(g, f)(\log a)$  on  $[1, \lambda^2]$  and  $q(g, f)(-\log a) = q(g, f)(\log a^{-1})$  on  $[\lambda^{-2}, 1]$  (by evenness of  $q(g, f)$ ), and both vanish outside  $[\lambda^{-2}, \lambda^2]$ . Hence  $H = q(g, f)|_{[0, L]}$ , and (129)–(131) together give

$$(132) \quad QW_\lambda(\kappa f, \kappa g) = D_W(q(g, f)), \quad f, g \in T,$$

where  $q(g, f)$  on the right denotes its restriction to  $[0, L]$ .

(v) *The boundary kernel as a smooth distribution.* The boundary form  $B_\lambda$  from Lemma 3.3 is, by (10),

$$B_\lambda(F, G) = \widehat{F}(i/2) \widehat{G}(-i/2) + \widehat{F}(-i/2) \widehat{G}(i/2), \quad F, G \in H_\lambda.$$

For  $F = \kappa f$  and  $u = e^{y-\ell}$ ,  $d^\times u = dy$ ,  $u^{\pm 1/2} = e^{\pm(y-\ell)/2}$ ,

$$(133) \quad (\widehat{\kappa f})(\pm i/2) = \int_{\lambda^{-1}}^\lambda f(\log(\lambda u)) u^{\pm 1/2} d^\times u = \int_0^L f(y) e^{\pm(y-\ell)/2} dy.$$

Substituting (133) and using the reality of the exponentials,

$$\begin{aligned} B_\lambda(\kappa f, \kappa g) &= \int_0^L \int_0^L f(x) \overline{g(y)} \left[ e^{(x-\ell)/2 - (y-\ell)/2} + e^{-(x-\ell)/2 + (y-\ell)/2} \right] dx dy \\ &= \int_0^L \int_0^L f(x) \overline{g(y)} \cdot 2 \cosh\left(\frac{x-y}{2}\right) dx dy. \end{aligned}$$

For each  $y \in [0, L]$ , the substitution  $z := x - y \in [-y, L - y]$  in the inner integral, together with the zero extension of  $f$  outside  $[0, L]$ , gives

$$\int_0^L f(x) \cdot 2 \cosh\left(\frac{x-y}{2}\right) dx = \int_{\mathbb{R}} 2 \cosh(z/2) f(y+z) dz,$$

the integrand vanishing outside  $z \in [-L, L]$ . Fubini's theorem then yields

$$B_\lambda(\kappa f, \kappa g) = \int_{-L}^L 2 \cosh(z/2) \left[ \int_{\mathbb{R}} f(y+z) \overline{g(y)} dy \right] dz = \int_{-L}^L 2 \cosh(z/2) (g^* f)(z) dz,$$

where the inner integral equals  $(g^* f)(z)$  by (124).

The function  $z \mapsto 2 \cosh(z/2)$  is even on  $[-L, L]$ . Splitting the integral at  $z = 0$  and substituting  $z \mapsto -z$  on  $[-L, 0]$ ,

$$B_\lambda(\kappa f, \kappa g) = \int_0^L 2 \cosh(z/2) \left[ (g^* f)(z) + (g^* f)(-z) \right] dz = \int_0^L 2 \cosh(z/2) q(g, f)(z) dz.$$

Define the real distribution

$$(134) \quad D_B(\phi) := \int_0^L 2 \cosh(z/2) \phi(z) dz, \quad \phi \in \mathcal{D}([0, L]),$$

given by the smooth even density  $2 \cosh(z/2)$ . Then

$$(135) \quad B_\lambda(\kappa f, \kappa g) = D_B(q(g, f)), \quad f, g \in T.$$

(vi) *Construction of  $D_\lambda$ .* Combining (132) and (135) via  $Q_{D,\lambda} = QW_\lambda - B_\lambda$ ,

$$Q_{D,\lambda}(\kappa f, \kappa g) = QW_\lambda(\kappa f, \kappa g) - B_\lambda(\kappa f, \kappa g) = D_W(q(g, f)) - D_B(q(g, f)).$$

Define

$$(136) \quad D_\lambda := D_W - D_B \in \mathcal{D}'([0, L]).$$

Both  $D_W$  and  $D_B$  are real distributions on  $[0, L]$ , by (iv) and (v); therefore  $D_\lambda$  is a real distribution as well, and

$$Q_{D,\lambda}(\kappa f, \kappa g) = D_\lambda(q(g, f)), \quad f, g \in T.$$

This identifies the form  $(f, g) \mapsto Q_{D,\lambda}(\kappa f, \kappa g)$  on  $T \times T$  with the quadratic form induced by a real distribution on  $[0, L]$  in the framework of [8, formula (6)], completing statement 1.

(vii) *Form core and the associated self-adjoint operator.* By (i),  $T$  is a form core for the closed lower-semibounded sesquilinear form  $\overline{Q}_{D,\lambda}$  on  $\text{Dom}(\overline{Q}_{D,\lambda}) = \kappa^{-1}\mathcal{F}_\lambda \subset L^2([0, L], dy)$ , with closure

$$\overline{Q}_{D,\lambda}(f, g) = Q_{D,\lambda}(\kappa f, \kappa g), \quad f, g \in \text{Dom}(\overline{Q}_{D,\lambda}).$$

The first representation theorem [18, Chap. VI, Theorem 2.1] associates with  $\overline{Q}_{D,\lambda}$  a unique self-adjoint operator  $\mathcal{S}_\lambda$  on  $L^2([0, L], dy)$ .

In  $H_\lambda$ , the operator associated with  $\mathcal{E}_\lambda$  via the same representation theorem is  $L_\lambda$ , by definition. Since  $Q_{D,\lambda} = \mathcal{E}_\lambda - c_{D,\lambda}\langle \cdot, \cdot \rangle_{H_\lambda}$  differs from  $\mathcal{E}_\lambda$  by the bounded scalar form  $c_{D,\lambda}\langle \cdot, \cdot \rangle_{H_\lambda}$ , the first representation theorem applied to both forms (which share the form domain  $\mathcal{F}_\lambda$ ) gives

$$A_\lambda = L_\lambda - c_{D,\lambda}I$$

as the operator associated with  $Q_{D,\lambda}$  on  $H_\lambda$ .

For any  $f \in \kappa^{-1}\text{Dom}(A_\lambda)$  and  $g \in \kappa^{-1}\mathcal{F}_\lambda$ , the unitarity  $\kappa^* = \kappa^{-1}$  from (i) yields

$$\overline{Q}_{D,\lambda}(f, g) = Q_{D,\lambda}(\kappa f, \kappa g) = \langle A_\lambda(\kappa f), \kappa g \rangle_{H_\lambda} = \langle \kappa^{-1}A_\lambda \kappa f, g \rangle_{L^2([0, L])}.$$

The functional  $g \mapsto \langle \kappa^{-1}A_\lambda \kappa f, g \rangle_{L^2([0, L])}$  is bounded on  $L^2([0, L], dy)$ , with bound  $\|\kappa^{-1}A_\lambda \kappa f\|_{L^2([0, L])} = \|A_\lambda \kappa f\|_{H_\lambda}$ . By the converse direction of the first representation theorem applied to  $\overline{Q}_{D,\lambda}$ ,  $f \in \text{Dom}(\mathcal{S}_\lambda)$  and  $\mathcal{S}_\lambda f = \kappa^{-1}A_\lambda \kappa f$ .

The reverse inclusion follows by the same argument with  $\kappa$  and  $\kappa^{-1}$  exchanged. Therefore

$$\mathcal{S}_\lambda = \kappa^{-1} A_\lambda \kappa = \kappa^{-1} (L_\lambda - c_{D,\lambda} I) \kappa,$$

which is precisely the form of lower-bounded essentially self-adjoint operator analyzed in [8, Theorem 6.1]. This proves statement 2 and completes the proof.  $\square$

LEMMA 5.5 (CLOSED FORM AND ADMISSIBLE TEST-FUNCTION CLASS FOR  $D_\lambda$ ). *Let  $\lambda > 1$ , set  $L := 2 \log \lambda$ , and let  $D_\lambda \in \mathcal{D}'([0, L])$  be the real distribution constructed in Proposition 5.4. Then  $D_\lambda$  admits the closed-form expression*

$$(137) \quad D_\lambda(\phi) = \int_{\mathbb{R}} \widehat{\phi}_c(t) w(t) dt - \sum_{1 < n \leq \lambda^2} \Lambda(n) n^{-1/2} \phi(\log n),$$

in which the smooth  $\cosh(y/2)$ -densities present in  $D_W$  and  $D_B$  have cancelled exactly. Here

$$\widehat{\phi}_c(t) := \int_0^L \phi(y) \cos(ty) dy,$$

$w(t) = \theta'(t)/\pi$  is the archimedean weight of (18), and  $\Lambda$  is the von Mangoldt function.

Let

$$(138) \quad \mathcal{A}_L := \{\phi \in C([0, L]) : \phi \text{ is piecewise } C^2 \text{ on } [0, L] \text{ and } \phi(L) = 0\}.$$

Then for every  $\phi \in \mathcal{A}_L$ , the right-hand side of (137) is absolutely convergent. Moreover, on  $C^\infty([0, L]) \cap \mathcal{A}_L$ , the right-hand side of (137) agrees with the distributional pairing  $D_\lambda(\phi)$  of Proposition 5.4. Consequently, (137) unambiguously defines  $D_\lambda$  as a real linear functional on  $\mathcal{A}_L$ , extending the distributional pairing.

*Proof. Closed form.* For  $\phi \in C^\infty([0, L])$ , the inversion-symmetric extension  $\widetilde{\phi}(a) = \phi^{\text{ev}}(\log a) \mathbf{1}_{[\lambda^{-2}, \lambda^2]}(a)$  introduced in the proof of Proposition 5.4 (iv) has Mellin transform

$$\widehat{\widetilde{\phi}}(t) = \int_{-L}^L \phi^{\text{ev}}(y) e^{-ity} dy = 2\widehat{\phi}_c(t), \quad \widehat{\widetilde{\phi}}(\pm i/2) = 2 \int_0^L \phi(y) \cosh(y/2) dy.$$

The prime contribution evaluates as  $\widetilde{\phi}(p^m) = \widetilde{\phi}(p^{-m}) = \phi(m \log p)$  for  $p^m \leq \lambda^2$ , equivalently  $2 \sum_{1 < n \leq \lambda^2} \Lambda(n) n^{-1/2} \phi(\log n)$  in the von Mangoldt rewriting of §2.4. Inserting these into  $\Psi^\# = \frac{1}{2} \Psi$  in (131),

$$D_W(\phi) = 2 \int_0^L \phi(y) \cosh(y/2) dy + \int_{\mathbb{R}} \widehat{\phi}_c(t) w(t) dt - \sum_{1 < n \leq \lambda^2} \Lambda(n) n^{-1/2} \phi(\log n).$$

Subtracting  $D_B(\phi) = 2 \int_0^L \phi(y) \cosh(y/2) dy$  of (134) cancels the smooth term and yields (137).



*Absolute convergence on  $\mathcal{A}_L$ .* Let  $\phi \in \mathcal{A}_L$ , with  $\phi'$  possibly jumping at finitely many points  $0 = y_0 < y_1 < \cdots < y_K = L$ . Two integrations by parts on  $\widehat{\phi}_c$ , applied subinterval by subinterval and collecting boundary contributions, give for  $|t| \geq 1$

$$(139) \quad \widehat{\phi}_c(t) = \frac{\phi'(L^-) \cos(tL) - \phi'(0^+)}{t^2} + \frac{1}{t^2} \sum_{k=1}^{K-1} [\phi'(y_k^+) - \phi'(y_k^-)] \cos(ty_k) - \frac{1}{t^2} \int_0^L \phi''(y) \cos(ty) dy,$$

where the boundary contribution at  $y = L$  in the first integration by parts vanishes by  $\phi(L) = 0$ , and the boundary contribution at  $y = 0$  vanishes by  $\sin 0 = 0$ . Hence  $\widehat{\phi}_c(t) = O(t^{-2})$  as  $|t| \rightarrow \infty$ , with implicit constant depending only on  $\|\phi'\|_{BV([0,L])}$  and  $\|\phi''\|_{L^1([0,L])}$ .

For the archimedean weight, the inequality  $1 - \cos(tr) \leq \min(2, t^2 r^2 / 2)$  together with the Lévy-measure verification of §3.3.2 (which gives both the bound  $\nu(dr) \leq C_0 dr/r$  on  $(0, 1]$  and the exponential tail of  $\nu$  on  $[1, \infty)$ , for some absolute constant  $C_0 > 0$ ) yields

$$|w(t)| \leq |a_\infty| + \int_0^\infty \min(2, t^2 r^2 / 2) \nu(dr) \leq C(1 + \log^+ |t|), \quad t \in \mathbb{R},$$

for an absolute constant  $C > 0$  depending only on  $\nu$ . Combined with (139), the integrand  $\widehat{\phi}_c(t) w(t)$  is  $O(\log |t| / t^2)$  as  $|t| \rightarrow \infty$  and bounded on bounded sets, hence absolutely integrable on  $\mathbb{R}$ .

The prime sum in (137) is finite, with each summand bounded by  $\Lambda(n) n^{-1/2} \|\phi\|_{C([0,L])}$ . The sole potential boundary contribution  $\log n = L$  (which occurs only when  $\lambda^2$  is itself a prime power) carries  $\phi(\log n) = \phi(L) = 0$  by the definition of  $\mathcal{A}_L$ , so this endpoint introduces no ambiguity.

*Compatibility with the distributional pairing.* For  $\phi \in C^\infty([0, L]) \cap \mathcal{A}_L$ , the inversion-symmetric extension  $\widetilde{\phi}$  is continuous on  $\mathbb{R}_{>0}^\times$  since  $\widetilde{\phi}(\lambda^{\pm 2}) = \phi^{\text{ev}}(\pm L) = \phi(L) = 0$ ; hence  $\widetilde{\phi}$  lies in the Weil class of Lemma 2.11 and the pairing  $\Psi^\#(\widetilde{\phi})$  is given by the absolutely convergent expansion above. Term by term this expansion coincides with the construction of  $D_W - D_B$  in the proof of Proposition 5.4, so the right-hand side of (137) agrees with  $D_\lambda(\phi)$  as defined there. The right-hand side of (137) is real and linear on  $\mathcal{A}_L \supset C^\infty([0, L]) \cap \mathcal{A}_L$ , so it provides the announced extension.  $\square$

*Remark 5.6.* Three classes of test functions arising in §5.5–§5.6 all lie in  $\mathcal{A}_L$ :

- (a)  $q(g, f)|_{[0,L]}$  for  $f, g \in T$ , since  $q(g, f)$  is a continuous piecewise-smooth function on  $[-L, L]$  with  $q(g, f)(\pm L) = 0$  by the support-additivity of Lebesgue convolutions of functions supported in  $[0, L]$ ;
- (b)  $\sin(\alpha_k y) = \sin(2\pi k y / L)$  for  $k \in \mathbb{Z}$ , since  $\sin(\alpha_k L) = \sin(2\pi k) = 0$ ;
- (c)  $(L - y) \cos(\alpha_n y)$  for  $n \in \mathbb{Z}$ , since the factor  $(L - y)$  vanishes at  $y = L$ .

In particular, every distributional pairing of  $D_\lambda$  used in Theorem 5.7 and in §5.6 falls within the scope of Lemma 5.5.

5.5. *Real zeros of the Fourier transform of the ground state.* We now combine the spectral structure established in Sections 3–5 with the interval kernel of Proposition 5.4, and invoke the real-zeros theorem of [8, Theorem 6.1], to conclude that the Mellin transform of the ground state has only real zeros.

THEOREM 5.7. *Let  $\lambda > 1$ , set  $\ell := \log \lambda$  and  $L := 2\ell$ , and let  $I_\lambda := [\lambda^{-1}, \lambda]$ . Let  $L_\lambda$  be the nonnegative self-adjoint operator on  $H_\lambda = L^2(I_\lambda, d^\times u)$  associated with the Dirichlet form  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  of Theorem 3.2 via the first representation theorem [18, Chap. VI, Thm. 2.1], let  $\mu_0(\lambda) := \min \sigma(L_\lambda)$ , and let  $\xi_\lambda$  denote the  $L^2$ -normalized ground state furnished by Theorem 5.2 and Proposition 5.3, satisfying*

$$L_\lambda \xi_\lambda = \mu_0(\lambda) \xi_\lambda, \quad \|\xi_\lambda\|_{H_\lambda} = 1, \quad \xi_\lambda > 0 \text{ } \mu\text{-a.e. on } I_\lambda, \quad \xi_\lambda(u^{-1}) = \xi_\lambda(u) \text{ } \mu\text{-a.e. on } I_\lambda.$$

*The Mellin transform*

$$\widehat{\xi}_\lambda(z) := \int_{\lambda^{-1}}^{\lambda} \xi_\lambda(u) u^{-iz} d^\times u, \quad z \in \mathbb{C},$$

*is an entire function whose zeros all lie on the real axis.*

*Proof.* The argument transports the ground state through two unitary isomorphisms onto the symmetric interval  $I_L := [-L/2, L/2]$ , identifies the resulting sesquilinear form  $\widetilde{Q}_\lambda$  on the trigonometric polynomial space with the Schwartz-kernel quadratic form induced by an explicit real even distribution  $\widetilde{\mathcal{D}} \in \mathcal{D}'([-L, L])$ , applies [8, Theorem 6.1] to obtain real zeros of the Fourier transform of the resulting eigenfunction, and finally transfers the conclusion back to  $\widehat{\xi}_\lambda$  by an algebraic identification of Fourier transforms.

The proof is organized into nine parts (i)–(ix). Throughout,  $\ell := \log \lambda$ ,  $L := 2\ell$ ,  $I_L := [-L/2, L/2]$ , and the Lebesgue measure on  $I_L$  is denoted by  $dx$ . The convention for Fourier transforms on intervals or on  $\mathbb{R}$  is  $\widehat{g}(z) := \int g(x) e^{-izx} dx$ , in agreement with [8, §6].

(i) *The Mellin transform  $\widehat{\xi}_\lambda$  is entire.* The interval  $I_\lambda$  has finite multiplicative Haar measure

$$\mu(I_\lambda) = \int_{\lambda^{-1}}^{\lambda} d^\times u = \int_{\lambda^{-1}}^{\lambda} \frac{du}{u} = 2 \log \lambda = 2\ell < \infty.$$

Since  $\xi_\lambda \in H_\lambda$ , the Cauchy–Schwarz inequality applied to the constant function  $\mathbf{1}_{I_\lambda} \in H_\lambda$  gives

$$\|\xi_\lambda\|_{L^1(I_\lambda, d^\times u)} \leq \sqrt{\mu(I_\lambda)} \|\xi_\lambda\|_{H_\lambda} = \sqrt{2\ell} < \infty,$$

so  $\xi_\lambda \in L^1(I_\lambda, d^\times u)$ .

Fix any compact set  $K \subset \mathbb{C}$  and set  $M_K := \sup_{z \in K} |\Im z| < \infty$ . For every  $u \in I_\lambda$  and every  $z \in K$ , the bound  $|\log u| \leq \ell$  gives

$$(140) \quad |u^{-iz}| = e^{(\Im z) \log u} \leq e^{|\Im z| |\log u|} \leq e^{M_K \ell}.$$

Consequently, for every nonnegative integer  $m$ ,

$$|\xi_\lambda(u) (-i \log u)^m u^{-iz}| \leq \ell^m e^{M_K \ell} |\xi_\lambda(u)| \in L^1(I_\lambda, d^\times u), \quad u \in I_\lambda, \quad z \in K.$$

Lebesgue's dominated convergence theorem [12, Thm. 1.34], applied to the difference quotient in the  $z$ -variable, permits differentiation under the integral sign to all orders, with

$$(141) \quad \frac{d^m}{dz^m} \widehat{\xi}_\lambda(z) = \int_{\lambda^{-1}}^\lambda \xi_\lambda(u) (-i \log u)^m u^{-iz} d^\times u, \quad z \in K, \quad m \geq 0.$$

Since  $K \subset \mathbb{C}$  was an arbitrary compact subset,  $\widehat{\xi}_\lambda$  is holomorphic on  $\mathbb{C}$ , hence entire.

(ii) *Transfer to  $L^2([0, L])$  via the unitary  $\kappa$ .* By Proposition 5.4 (2), the map

$$\kappa : L^2([0, L], dy) \xrightarrow{\sim} H_\lambda, \quad (\kappa f)(u) = f(\log(\lambda u)), \quad (\kappa^{-1} \xi)(y) = \xi(e^{y-\ell}),$$

is a unitary isomorphism. The closed lower-semibounded sesquilinear form

$$\overline{Q}_{D,\lambda}(f, g) := Q_{D,\lambda}(\kappa f, \kappa g), \quad f, g \in \text{Dom}(\overline{Q}_{D,\lambda}) = \kappa^{-1} \mathcal{F}_\lambda,$$

admits the trigonometric span

$$T := \text{span}\{\mathcal{T}_n : n \in \mathbb{Z}\} \subset L^2([0, L], dy), \quad \mathcal{T}_n(y) := L^{-1/2} e^{2\pi i n y / L},$$

as a form core, and the associated self-adjoint operator on  $L^2([0, L], dy)$  is

$$(142) \quad \mathcal{S}_\lambda = \kappa^{-1} A_\lambda \kappa = \kappa^{-1} (L_\lambda - c_{D,\lambda} I) \kappa, \quad A_\lambda := L_\lambda - c_{D,\lambda} I.$$

Set  $\eta_\lambda := \kappa^{-1} \xi_\lambda \in L^2([0, L], dy)$ . Unitarity of  $\kappa$  gives  $\|\eta_\lambda\|_{L^2([0, L])} = 1$ . Applying  $\kappa^{-1}$  to both sides of  $L_\lambda \xi_\lambda = \mu_0(\lambda) \xi_\lambda$  and using  $\kappa \eta_\lambda = \xi_\lambda$  together with (142),

$$\mathcal{S}_\lambda \eta_\lambda = \kappa^{-1} A_\lambda \xi_\lambda = \kappa^{-1} ((\mu_0(\lambda) - c_{D,\lambda}) \xi_\lambda) = (\mu_0(\lambda) - c_{D,\lambda}) \eta_\lambda.$$

Set

$$(143) \quad \lambda_0 := \mu_0(\lambda) - c_{D,\lambda} \in \mathbb{R},$$

so that  $\mathcal{S}_\lambda \eta_\lambda = \lambda_0 \eta_\lambda$  in  $L^2([0, L], dy)$  and  $\|\eta_\lambda\|_{L^2([0, L])} = 1$ .

(iii)  $\lambda_0$  is the simple isolated bottom of  $\sigma(\mathcal{S}_\lambda)$ . By Proposition 5.1,  $L_\lambda$  has compact resolvent and purely discrete spectrum

$$0 \leq \mu_0(\lambda) \leq \mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots, \quad \mu_n(\lambda) \rightarrow +\infty,$$

each eigenvalue of finite multiplicity. By Theorem 5.2,  $\mu_0(\lambda)$  is a simple eigenvalue of  $L_\lambda$ . The unitary equivalence  $\mathcal{S}_\lambda = \kappa^{-1}(L_\lambda - c_{D,\lambda}I)\kappa$  preserves both the spectrum and the dimensions of all eigenspaces, so

$$\sigma(\mathcal{S}_\lambda) = \sigma(L_\lambda) - c_{D,\lambda} = \{\mu_n(\lambda) - c_{D,\lambda} : n \geq 0\},$$

each  $\mu_n(\lambda) - c_{D,\lambda}$  inheriting the multiplicity of  $\mu_n(\lambda)$ . Therefore  $\lambda_0 = \min \sigma(\mathcal{S}_\lambda)$  is a simple eigenvalue of  $\mathcal{S}_\lambda$  with one-dimensional eigenspace  $\mathbb{C}\eta_\lambda$ . The spectral gap

$$(144) \quad \delta := \mu_1(\lambda) - \mu_0(\lambda) > 0$$

is strictly positive (because  $\mu_0(\lambda)$  is simple and the spectrum is discrete), giving the isolation

$$(145) \quad \sigma(\mathcal{S}_\lambda) \setminus \{\lambda_0\} \subset [\lambda_0 + \delta, \infty).$$

(iv) *Symmetry of  $\eta_\lambda$  under  $y \mapsto L - y$ .* By Proposition 5.3,  $\xi_\lambda(u^{-1}) = \xi_\lambda(u)$  for  $\mu$ -a.e.  $u \in I_\lambda$ . Using  $L = 2\ell$  (so that  $L - \ell = \ell$  and consequently  $L - y - \ell = \ell - y$ ) and the algebraic identity  $e^{\ell-y} = (e^{y-\ell})^{-1}$ , for  $\mu$ -a.e.  $y \in [0, L]$ ,

$$\eta_\lambda(L - y) = \xi_\lambda(e^{(L-y)-\ell}) = \xi_\lambda(e^{\ell-y}) = \xi_\lambda((e^{y-\ell})^{-1}) = \xi_\lambda(e^{y-\ell}) = \eta_\lambda(y).$$

The first and last equalities are the definition of  $\eta_\lambda$  through  $\kappa^{-1}$ , the second uses  $L - y - \ell = \ell - y$ , the third uses the algebraic inversion identity above, and the fourth applies the inversion symmetry of  $\xi_\lambda$  to the point  $u = e^{y-\ell} \in I_\lambda$ . Therefore  $\eta_\lambda$  is  $\mu$ -a.e. invariant under the reflection  $y \mapsto L - y$ .

(v) *Translation to  $L^2(I_L)$  via the unitary  $\tau$ .* The translation operator

$$\tau : L^2([0, L], dy) \xrightarrow{\sim} L^2(I_L, dx), \quad (\tau f)(x) := f(x + L/2), \quad (\tau^{-1}g)(y) := g(y - L/2),$$

preserves Lebesgue measure on  $\mathbb{R}$  and is therefore unitary. Define

$$(146) \quad \tilde{\eta}_\lambda := \tau \eta_\lambda \in L^2(I_L), \quad \tilde{\mathcal{S}}_\lambda := \tau \mathcal{S}_\lambda \tau^{-1}, \quad \tilde{Q}_\lambda(\varphi, \psi) := \overline{Q}_{D,\lambda}(\tau^{-1}\varphi, \tau^{-1}\psi).$$

Since the unitary  $\tau$  preserves both the form-graph topology and the  $L^2$ -norm,  $\tilde{Q}_\lambda$  is a closed lower-semibounded sesquilinear form on  $L^2(I_L)$ , with form domain

$$(147) \quad \text{Dom}(\tilde{Q}_\lambda) = \tau \text{Dom}(\overline{Q}_{D,\lambda}) = \tau \kappa^{-1} \mathcal{F}_\lambda.$$

By the first representation theorem [18, Chap. VI, Thm. 2.1],  $\tilde{\mathcal{S}}_\lambda$  is the unique self-adjoint operator on  $L^2(I_L)$  associated with  $\tilde{Q}_\lambda$ .

Unitary equivalence preserves the spectrum and eigenspaces of  $\mathcal{S}_\lambda$ , so

$$(148) \quad \sigma(\tilde{\mathcal{S}}_\lambda) = \sigma(\mathcal{S}_\lambda), \quad \tilde{\mathcal{S}}_\lambda \tilde{\eta}_\lambda = \lambda_0 \tilde{\eta}_\lambda, \quad \|\tilde{\eta}_\lambda\|_{L^2(I_L)} = 1.$$

Combining (148) with (145),  $\lambda_0 = \min \sigma(\tilde{\mathcal{S}}_\lambda)$  is a simple isolated eigenvalue of  $\tilde{\mathcal{S}}_\lambda$  with one-dimensional eigenspace  $\mathbb{C}\tilde{\eta}_\lambda$ , and the gap is the same  $\delta > 0$  as in (144).

(vi) *Evenness of  $\tilde{\eta}_\lambda$  under  $x \mapsto -x$ .* For  $\mu$ -a.e.  $x \in I_L$ , the algebraic identity

$$(149) \quad -x + L/2 = L - (x + L/2)$$

in  $\mathbb{R}$  (verified by  $L - (x + L/2) = L/2 - x = -x + L/2$ ) combined with (iv) yields

$$\tilde{\eta}_\lambda(-x) = \eta_\lambda(-x + L/2) = \eta_\lambda(L - (x + L/2)) = \eta_\lambda(x + L/2) = \tilde{\eta}_\lambda(x).$$

The first and fourth equalities apply the definition of  $\tau$  in (146), the second uses (149), and the third invokes (iv) at the point  $z := x + L/2 \in [0, L]$ . Hence  $\tilde{\eta}_\lambda$  is  $\mu$ -a.e. even on  $I_L$  under  $x \mapsto -x$ .

(vii) *Construction of the even real distribution  $\tilde{\mathcal{D}} \in \mathcal{D}'([-L, L])$  and the Schwartz-kernel formula on the trigonometric core.*

By Proposition 5.4, there exists a real distribution  $D_\lambda \in \mathcal{D}'([0, L])$  such that

$$(150) \quad \overline{Q}_{D,\lambda}(f, g) = D_\lambda(q(g, f)|_{[0, L]}), \quad f, g \in T,$$

where, for  $f, g$  supported in  $[0, L]$  (after zero extension to  $\mathbb{R}$ ),

$$(151) \quad q(g, f)(y) := (g^* * f)(y) + (g^* * f)(-y), \quad y \in \mathbb{R}, \quad g^*(x) := \overline{g(-x)},$$

with  $*$  denoting Lebesgue convolution on  $\mathbb{R}$ . Here  $\mathcal{D}'([0, L])$  denotes the space of continuous linear functionals on  $C^\infty([0, L])$  in the standard  $\mathcal{D}$ -topology of uniform convergence of all derivatives, and analogously for  $\mathcal{D}'([-L, L])$ . By Lemma 5.5,  $D_\lambda$  admits the explicit closed form (137), which is absolutely convergent on the admissible class  $\mathcal{A}_L$  of (138); this provides an unambiguous real-valued pairing  $D_\lambda(\phi)$  for every  $\phi \in \mathcal{A}_L$ . Each distributional pairing of  $D_\lambda$  used below is verified to fall within  $\mathcal{A}_L$  via Remark 5.6; no further extension argument is required.

(vii.a) *Definition of  $\tilde{\mathcal{D}}$  and verification of reality and evenness.* For each  $\phi \in C^\infty([-L, L])$ , the function  $y \mapsto \phi(y) + \phi(-y)$  on  $[-L, L]$  is  $C^\infty$ , hence its restriction to  $[0, L]$  belongs to  $C^\infty([0, L])$ . Define

$$(152) \quad \tilde{\mathcal{D}}(\phi) := D_\lambda\left((\phi(y) + \phi(-y))|_{y \in [0, L]}\right), \quad \phi \in C^\infty([-L, L]).$$

The linear map

$$\Sigma : C^\infty([-L, L]) \rightarrow C^\infty([0, L]), \quad \Sigma\phi := (\phi(y) + \phi(-y))|_{y \in [0, L]},$$

is continuous in the  $\mathcal{D}$ -topology of uniform convergence of all derivatives, by continuity of the reflection  $\phi \mapsto \phi(-\cdot)$  on  $C^\infty([-L, L])$ , of pointwise addition, and of the restriction  $C^\infty([-L, L]) \rightarrow C^\infty([0, L])$ . Composition with the continuous functional  $D_\lambda \in \mathcal{D}'([0, L])$  yields  $\tilde{\mathcal{D}} = D_\lambda \circ \Sigma \in \mathcal{D}'([-L, L])$ .

*Reality.* For real-valued  $\phi \in C^\infty([-L, L])$ ,  $\Sigma\phi$  is real-valued, and the reality of  $D_\lambda$  established in Proposition 5.4 gives  $\tilde{\mathcal{D}}(\phi) \in \mathbb{R}$ .

*Evenness.* For every  $\phi \in C^\infty([-L, L])$ , the substitution  $y \leftrightarrow -y$  inside  $\Sigma$  gives  $\Sigma(\phi(-\cdot)) = \Sigma\phi$ , hence

$$\widetilde{\mathcal{D}}(\phi(-\cdot)) = D_\lambda(\Sigma(\phi(-\cdot))) = D_\lambda(\Sigma\phi) = \widetilde{\mathcal{D}}(\phi).$$

For the use below, we record the analogous extension class for  $\widetilde{\mathcal{D}}$ . Define

$$\widetilde{\mathcal{A}}_L := \{\phi \in C([-L, L]) : \phi \text{ is piecewise } C^2 \text{ on } [-L, L] \text{ and } \phi(\pm L) = 0\}.$$

For every  $\phi \in \widetilde{\mathcal{A}}_L$ , the symmetrization  $\Sigma\phi = (\phi(y) + \phi(-y))|_{[0, L]}$  is continuous and piecewise  $C^2$  on  $[0, L]$  with  $(\Sigma\phi)(L) = \phi(L) + \phi(-L) = 0$ , hence  $\Sigma\phi \in \mathcal{A}_L$ . By Lemma 5.5,  $D_\lambda(\Sigma\phi)$  is unambiguously defined via the closed form (137), and the formula  $\widetilde{\mathcal{D}}(\phi) := D_\lambda(\Sigma\phi)$  extends (152) to all of  $\widetilde{\mathcal{A}}_L$  as a real linear functional.

(vii.b) *Schwartz-kernel formula on  $\widetilde{T} \times \widetilde{T}$ .* Set  $\widetilde{\mathcal{T}}_n(x) := L^{-1/2}e^{2\pi i n x/L}$  for  $x \in I_L$  and  $n \in \mathbb{Z}$ . For each  $n \in \mathbb{Z}$ ,

$$(153) \quad (\tau\mathcal{T}_n)(x) = \mathcal{T}_n(x+L/2) = L^{-1/2}e^{2\pi i n(x+L/2)/L} = L^{-1/2}e^{i\pi n}e^{2\pi i n x/L} = (-1)^n \widetilde{\mathcal{T}}_n(x).$$

In particular,

$$(154) \quad \widetilde{T} := \tau T = \text{span}\{\widetilde{\mathcal{T}}_n : n \in \mathbb{Z}\} \subset L^2(I_L), \quad \tau^{-1}\widetilde{\mathcal{T}}_n = (-1)^n \mathcal{T}_n.$$

Since  $T$  is a form core for  $\overline{Q}_{D,\lambda}$  (by Proposition 5.4(2)) and  $\tau$  is unitary,  $\widetilde{T}$  is a form core for  $\widetilde{Q}_\lambda$  on  $\text{Dom}(\widetilde{Q}_\lambda)$ .

For  $\varphi, \psi \in \widetilde{T}$ , set  $f := \tau^{-1}\varphi \in T$  and  $g := \tau^{-1}\psi \in T$ . By the definition of  $\widetilde{Q}_\lambda$  in (146) and (150),

$$(155) \quad \widetilde{Q}_\lambda(\varphi, \psi) = \overline{Q}_{D,\lambda}(f, g) = D_\lambda(q(g, f)|_{[0, L]}).$$

Both  $f$  and  $g$  are supported in  $[0, L]$ , equivalently  $f(y) = \varphi(y - L/2)$  and  $g(y) = \psi(y - L/2)$  on  $\mathbb{R}$  (with  $\varphi, \psi$  extended by zero off  $I_L$ ). For  $x \in \mathbb{R}$ ,

$$(156) \quad g^*(x) = \overline{g(-x)} = \overline{\psi(-x - L/2)} = \psi^*(x + L/2),$$

where  $\psi^*(x) := \overline{\psi(-x)}$ . Substituting (156) into the definition  $(g^* * f)(y) := \int_{\mathbb{R}} g^*(y - z)f(z) dz$  and applying the change of variable  $w := z - L/2$  (so that  $dw = dz$ ,  $y - z + L/2 = y - w$ , and  $\varphi(z - L/2) = \varphi(w)$ ),

$$(157) \quad \begin{aligned} (g^* * f)(y) &= \int_{\mathbb{R}} g^*(y - z)f(z) dz = \int_{\mathbb{R}} \psi^*(y - z + L/2) \varphi(z - L/2) dz \\ &= \int_{\mathbb{R}} \psi^*(y - w) \varphi(w) dw = (\psi^* * \varphi)(y), \quad y \in \mathbb{R}. \end{aligned}$$

Substituting (157) into the definition of  $q(g, f)$  in (151) yields

$$(158) \quad q(g, f)(y) = (\psi^* * \varphi)(y) + (\psi^* * \varphi)(-y), \quad y \in \mathbb{R}.$$

Both  $\psi^*$  and  $\varphi$  have support in  $[-L/2, L/2]$  (after zero extension), so  $\psi^* * \varphi$  has support in  $[-L, L]$  by additivity of supports under Lebesgue convolution. Furthermore, since  $\varphi$  and  $\psi^*$  are bounded and compactly supported on  $\mathbb{R}$ , the convolution  $\psi^* * \varphi$  is continuous on  $\mathbb{R}$  and piecewise-smooth on  $\mathbb{R} \setminus \{-L, 0, L\}$  (with potential breaks in higher derivatives where the integration window changes endpoints), hence in particular compactly supported piecewise-smooth on  $[-L, L]$ .

By definition (152) of  $\widetilde{\mathcal{D}}$ , applied to the compactly supported piecewise-smooth function  $\phi := \psi^* * \varphi$  on  $[-L, L]$ ,

$$(159) \quad \widetilde{\mathcal{D}}((\psi^* * \varphi)|_{[-L, L]}) = D_\lambda \left( ((\psi^* * \varphi)(y) + (\psi^* * \varphi)(-y))|_{y \in [0, L]} \right) = D_\lambda(q(g, f)|_{[0, L]}),$$

where the second equality uses (158). The applicability of  $\widetilde{\mathcal{D}}$  to  $\psi^* * \varphi$ , and of  $D_\lambda$  to  $q(g, f)|_{[0, L]}$ , is justified by Lemma 5.5 together with Remark 5.6(a): the convolution  $\psi^* * \varphi$  lies in  $\widetilde{\mathcal{A}}_L$  since it is continuous on  $\mathbb{R}$ , piecewise smooth on  $\mathbb{R} \setminus \{-L, 0, L\}$ , and vanishes at  $\pm L$  by the support-additivity of Lebesgue convolutions of functions supported in  $[-L/2, L/2]$ ; consequently  $q(g, f)|_{[0, L]} = \Sigma(\psi^* * \varphi) \in \mathcal{A}_L$ .

Combining (155) with (159),

$$(160) \quad \boxed{\widetilde{Q}_\lambda(\varphi, \psi) = \widetilde{\mathcal{D}}((\psi^* * \varphi)|_{[-L, L]}), \quad \varphi, \psi \in \widetilde{T}.}$$

This is precisely the Schwartz-kernel quadratic form induced by the real even distribution  $\widetilde{\mathcal{D}}$  on  $[-L, L]$  via the kernel  $\widetilde{\mathcal{D}}(x - y)$ , i.e. the form defined on trigonometric polynomials by [8, formula (6)].

(viii) *Application of [8, Theorem 6.1].* The conditions of [8, Theorem 6.1] for the form  $\widetilde{Q}_\lambda$  on  $L^2(I_L)$  are now verified item by item, in the order in which they appear in the cited theorem.

- (a)  $L > 0$ : the strict inequality  $\lambda > 1$  gives  $L = 2 \log \lambda > 0$ .
- (b)  $\widetilde{\mathcal{D}} \in \mathcal{D}'([-L, L])$  is a real even distribution: established in (vii.a).
- (c) On the trigonometric polynomial space  $\widetilde{T} \subset L^2(I_L)$ , the sesquilinear form  $\widetilde{Q}_\lambda$  is given by the Schwartz-kernel formula [8, formula (6)] with kernel  $\widetilde{\mathcal{D}}(x - y)$ : this is (160) of (vii.b).
- (d) The form  $\widetilde{Q}_\lambda$  is closed, symmetric, and lower-semibounded on  $L^2(I_L)$ , with form core  $\widetilde{T}$  (cf. (v) and (vii.b)). Consequently, the restriction of the operator associated with  $\widetilde{Q}_\lambda$  to  $\widetilde{T}$  is essentially self-adjoint, and the closure of this restriction equals  $\widetilde{\mathcal{S}}_\lambda$ , the self-adjoint operator associated with  $\widetilde{Q}_\lambda$  via the first representation theorem [18, Chap. VI, Thm. 2.1]: this is exactly the “lower-bounded essentially self-adjoint operator” condition of [8, Theorem 6.1].

- (e)  $\lambda_0 = \min \sigma(\tilde{\mathcal{S}}_\lambda)$  is a simple isolated eigenvalue: the simplicity is established in (iii)–(v), and the isolation is (145) combined with (148).
- (f) The corresponding  $L^2$ -normalized eigenfunction  $\tilde{\eta}_\lambda$  is even on  $I_L$ : established in (vi).

By [8, Theorem 6.1], the Fourier transform

$$(161) \quad \widehat{\tilde{\eta}}_\lambda(z) := \int_{-L/2}^{L/2} \tilde{\eta}_\lambda(x) e^{-izx} dx, \quad z \in \mathbb{C},$$

is an entire function whose zeros all lie on the real axis.

(ix) *Transfer of the zero-set conclusion to  $\widehat{\xi}_\lambda$ .* We compute  $\widehat{\tilde{\eta}}_\lambda$  in terms of an auxiliary entire function defined on  $[0, L]$ , then express  $\widehat{\xi}_\lambda$  in terms of the same auxiliary function, and finally compare.

*Step (ix.1).* In (161), apply the change of variable  $y := x + L/2$  (so  $x = y - L/2$ ,  $dx = dy$ , and  $x \in [-L/2, L/2]$  corresponds to  $y \in [0, L]$ ). Using the identity  $\tilde{\eta}_\lambda(x) = (\tau\eta_\lambda)(x) = \eta_\lambda(x + L/2) = \eta_\lambda(y)$  from the definition of  $\tau$ , and  $e^{-izx} = e^{-iz(y-L/2)} = e^{izL/2} e^{-izy}$ , one obtains

$$(162) \quad \widehat{\tilde{\eta}}_\lambda(z) = \int_0^L \eta_\lambda(y) e^{izL/2} e^{-izy} dy = e^{izL/2} F_\lambda(z),$$

where

$$(163) \quad F_\lambda(z) := \int_0^L \eta_\lambda(y) e^{-izy} dy, \quad z \in \mathbb{C}.$$

The factor  $e^{izL/2}$  is nowhere zero on  $\mathbb{C}$ , so the entire functions  $\widehat{\tilde{\eta}}_\lambda$  and  $F_\lambda$  have identical zero sets, including multiplicities.

*Step (ix.2).* For  $\widehat{\xi}_\lambda$ , the relation  $\xi_\lambda(u) = (\kappa\eta_\lambda)(u) = \eta_\lambda(\log(\lambda u))$  from the definition of  $\kappa$ , together with the change of variable  $y := \log(\lambda u)$  (so  $u = e^{y-\ell}$ ,  $d^\times u = du/u = dy$ , and  $u \in [\lambda^{-1}, \lambda]$  corresponds to  $y \in [0, L]$ ), gives

$$u^{-iz} = e^{-iz(y-\ell)} = e^{iz\ell} e^{-izy},$$

and therefore

$$(164) \quad \widehat{\xi}_\lambda(z) = \int_{\lambda^{-1}}^\lambda \xi_\lambda(u) u^{-iz} d^\times u = \int_0^L \eta_\lambda(y) e^{iz\ell} e^{-izy} dy = e^{iz\ell} F_\lambda(z).$$

*Step (ix.3).* Since  $L = 2\ell$ , the exponentials in (162) and (164) agree:  $e^{izL/2} = e^{iz\ell}$ . Therefore, for every  $z \in \mathbb{C}$ ,

$$(165) \quad \widehat{\xi}_\lambda(z) = e^{iz\ell} F_\lambda(z) = e^{izL/2} F_\lambda(z) = \widehat{\tilde{\eta}}_\lambda(z).$$

The zero set of  $\widehat{\xi}_\lambda$  therefore coincides, with multiplicities, with that of  $\widehat{\tilde{\eta}}_\lambda$ , which by (viii) lies entirely on the real axis. This completes the proof.  $\square$



Summary of this section. For each fixed  $\lambda > 1$ , the following statements all hold unconditionally (i.e. without invoking the conjectural reality of the nontrivial zeros of  $\zeta$ ):

1.  $L_\lambda$  has compact resolvent and purely discrete spectrum (Proposition 5.1);
2.  $\mu_0(\lambda)$  is simple, and there exists a normalized, strictly positive, inversion-even ground state  $\xi_\lambda$  (Theorem 5.2 and Proposition 5.3);
3.  $\widehat{\xi}_\lambda$  is an entire function whose zeros are all real (Theorem 5.7).

This is precisely the “real-zeros proxy” that, in the Galerkin determinant identity of Subsection 5.6 below, links the ground state to the spectrum of an explicit finite self-adjoint operator.

5.6. *The zeta-regularized determinant of the Galerkin ground state.* We now attach to the finite Galerkin ground state a regularized determinant. This is the determinant-theoretic counterpart, in the present boundary-free Weil–Dirichlet model, of the construction used for the perturbed scaling operator in [7, Section 5]. Since our convention is that inner products are linear in the first variable, all rank-one operators below are written explicitly.

Let

$$L = 2 \log \lambda, \quad \ell = \log \lambda,$$

and keep the notation introduced in Subsection 5.5. Thus

$$\mathcal{T}_j(y) = L^{-1/2} e^{2\pi i j y / L}, \quad V_j(u) = \mathcal{T}_j(\log(\lambda u)),$$

and

$$E_N^\lambda := \text{span}\{V_j : |j| \leq N\} \subset H_\lambda.$$

The periodic logarithmic scaling operator is

$$D_{\log}^{(\lambda)} = -iu \frac{d}{du},$$

defined initially on the trigonometric span  $\bigcup_{N \geq 0} E_N^\lambda$  and closed in  $H_\lambda$ . A direct computation gives, for each  $j \in \mathbb{Z}$  and  $u \in I_\lambda$ ,

$$\frac{dV_j}{du}(u) = L^{-1/2} \frac{2\pi i j / L}{u} e^{2\pi i j \log(\lambda u) / L} = \frac{2\pi i j}{Lu} V_j(u),$$

hence

$$D_{\log}^{(\lambda)} V_j = -iu \cdot \frac{2\pi i j}{Lu} V_j = \alpha_j V_j, \quad \alpha_j := \frac{2\pi j}{L}.$$

The boundary values  $V_j(\lambda^{-1}) = L^{-1/2} = V_j(\lambda)$  confirm that each  $V_j$  satisfies the periodic boundary condition on  $I_\lambda$ . Since  $\{V_j\}_{j \in \mathbb{Z}}$  is an orthonormal basis of  $H_\lambda$  with real eigenvalues  $\{\alpha_j\}$ , the closure of  $D_{\log}^{(\lambda)}$  is self-adjoint on the dense domain  $\text{Dom}(D_{\log}^{(\lambda)}) = \{f = \sum_j f_j V_j \in H_\lambda : \sum_j |\alpha_j|^2 |f_j|^2 < \infty\}$ , which is the domain used in Theorem 5.16(ii). Let

$$Q_{D,\lambda}^{(N)} := Q_{D,\lambda}|_{E_N^\lambda \times E_N^\lambda}$$

be the Galerkin restriction of the boundary-free form. Let  $A_{\lambda,N}$  be the self-adjoint matrix representing this form in  $E_N^\lambda$ , namely

$$Q_{D,\lambda}^{(N)}(f, g) = \langle A_{\lambda,N} f, g \rangle_{H_\lambda}, \quad f, g \in E_N^\lambda.$$

We single out, for each  $\lambda > 1$ , the set of Galerkin dimensions for which the lowest Ritz problem on  $E_N^\lambda$  is non-degenerate in the inversion-even sector:

(166)

$$\mathfrak{N}_\lambda := \left\{ N \in \mathbb{Z}_{\geq 1} : \min \sigma(A_{\lambda,N}) \text{ is simple and admits an inversion-even eigenvector} \right\}.$$

For  $N \in \mathfrak{N}_\lambda$  we denote the lowest eigenvalue by

$$\varepsilon_{\lambda,N}^D := \min \sigma(A_{\lambda,N})$$

and choose a corresponding nonzero inversion-even eigenvector

$$\xi_{\lambda,N} = \sum_{|j| \leq N} \xi_j V_j \in E_N^\lambda, \quad A_{\lambda,N} \xi_{\lambda,N} = \varepsilon_{\lambda,N}^D \xi_{\lambda,N}, \quad \mathcal{R} \xi_{\lambda,N} = \xi_{\lambda,N},$$

where  $\mathcal{R}$  is the inversion of Proposition 5.3. Equivalently,

$$T_{\lambda,N} := A_{\lambda,N} - \varepsilon_{\lambda,N}^D I$$

is nonnegative on  $E_N^\lambda$  and

$$\ker T_{\lambda,N} = \mathbb{C} \xi_{\lambda,N}.$$

We also use  $T_{\lambda,N}$  for the associated Hermitian sesquilinear form

$$T_{\lambda,N}(f, g) = \langle T_{\lambda,N} f, g \rangle_{H_\lambda}.$$

*Remark.* The set  $\mathfrak{N}_\lambda$  contains every sufficiently large positive integer. Indeed, by Theorem 5.2 together with Proposition 5.3, the lowest spectral value of  $A_\lambda = L_\lambda - c_{D,\lambda} I$  is a simple isolated eigenvalue with an inversion-even eigenfunction. The Galerkin restriction  $A_{\lambda,N} \rightarrow A_\lambda$  is convergent in the strong-resolvent sense as  $N \rightarrow \infty$  since  $\bigcup_N E_N^\lambda$  is a form core for  $Q_{D,\lambda}$ . Standard Galerkin perturbation theory for a simple isolated eigenvalue (cf. [11, Vol. IV, Thm. XIII.83] together with the Davis–Kahan  $\sin \Theta$  inequality) then yields the existence of  $N_0(\lambda) < \infty$  such that  $\{N : N \geq N_0(\lambda)\} \subset \mathfrak{N}_\lambda$ . The growth rate of  $N_0(\lambda)$  plays no role in the arguments below; only membership  $N \in \mathfrak{N}_\lambda$  is used.

Define the Dirichlet-kernel vector

$$\Delta_N := L^{-1/2} \sum_{|j| \leq N} V_j \in E_N^\lambda$$

and the linear functional

$$\delta_N(f) := \langle f, \Delta_N \rangle_{H_\lambda}, \quad f \in E_N^\lambda.$$

Thus if  $f = \sum_{|j| \leq N} f_j V_j$ , then

$$\delta_N(f) = L^{-1/2} \sum_{|j| \leq N} f_j.$$

This is the normalization functional used in [7, Theorem 1.1], translated to the present convention that the inner product is linear in the first variable.

We collect six lemmas which carry, individually, the algebraic and spectral content needed to prove the regularized-determinant identity. The synthesis is the main result Theorem 5.16 below. Throughout this subsection,  $D_N := D_{\log}^{(\lambda)}|_{E_N^\lambda}$ , and the rank-one operator  $a \otimes b^*$  is defined by  $(a \otimes b^*)f := \langle f, b \rangle_{H_\lambda} a$ , with adjoint  $(a \otimes b^*)^* = b \otimes a^*$ .

**LEMMA 5.8 (REFLECTION ANTICOMMUTATION OF  $D_N$ ).** *On  $E_N^\lambda$  one has  $\mathcal{R}D_N = -D_N\mathcal{R}$ . Equivalently,  $D_N$  maps the inversion-even subspace of  $E_N^\lambda$  into the inversion-odd subspace and vice versa.*

*Proof.* For each  $|j| \leq N$ , the action  $\mathcal{R}V_j = V_{-j}$  established in Proposition 5.3, part (ii) of its proof, combined with  $D_NV_j = \alpha_j V_j$  and  $\alpha_{-j} = -\alpha_j$ , gives

$$\mathcal{R}D_NV_j = \mathcal{R}(\alpha_j V_j) = \alpha_j V_{-j} = -\alpha_{-j} V_{-j} = -D_NV_{-j} = -D_N\mathcal{R}V_j.$$

The displayed identity holds on the basis  $\{V_j\}_{|j| \leq N}$  of  $E_N^\lambda$  and therefore on all of  $E_N^\lambda$  by linearity.  $\square$

**LEMMA 5.9 (HILBERT-FORM MATRIX STRUCTURE OF  $A_{\lambda,N}$ ).** *For  $k \in \mathbb{Z}$ , set*

$$(167) \quad s_k := D_\lambda(\sin(\alpha_k y)).$$

*Then  $s_k \in \mathbb{R}$  and  $s_{-k} = -s_k$  (in particular  $s_0 = 0$ ). Moreover, on the orthonormal basis  $\{V_j\}_{|j| \leq N}$  of  $E_N^\lambda$ , the matrix elements of  $A_{\lambda,N}$  are real and given by*

$$(168) \quad \langle A_{\lambda,N} V_n, V_m \rangle_{H_\lambda} = \frac{s_n - s_m}{\pi(m - n)}, \quad m \neq n, \quad |m|, |n| \leq N,$$

and

$$(169) \quad \langle A_{\lambda,N} V_n, V_n \rangle_{H_\lambda} = \frac{2}{L} D_\lambda((L - y) \cos(\alpha_n y)), \quad |n| \leq N.$$

*In particular,  $A_{\lambda,N}$  is real symmetric in the basis  $\{V_j\}_{|j| \leq N}$ .*

*Proof.* Reality of  $s_k$  follows from reality of the distribution  $D_\lambda$  (Proposition 5.4); the relation  $s_{-k} = -s_k$  follows from  $\sin(\alpha_{-k} y) = -\sin(\alpha_k y)$  together with linearity of  $D_\lambda$ .

The pairings  $D_\lambda(\sin(\alpha_k y))$  and  $D_\lambda((L - y) \cos(\alpha_n y))$  appearing in (167) and (169) are absolutely defined by Lemma 5.5, since both test functions lie in  $\mathcal{A}_L$  (cf. Remark 5.6 (b)–(c)).

By Proposition 5.4, for every  $f, g$  in the trigonometric span  $T := \text{span}\{\mathcal{T}_n : n \in \mathbb{Z}\}$ ,

$$Q_{D,\lambda}(\kappa f, \kappa g) = D_\lambda(q(g, f)), \quad q(g, f)(y) = (g^* * f)(y) + (g^* * f)(-y),$$

with  $g^*(x) := \overline{g(-x)}$  and  $*$  denoting Lebesgue convolution on  $\mathbb{R}$  after zero extension off  $[0, L]$ .

(i) *Off-diagonal entries.* Take  $m \neq n$ . The function  $\mathcal{T}_n$ , extended by zero off  $[0, L]$ , has support in  $[0, L]$ , while  $\mathcal{T}_m^*$  satisfies  $\mathcal{T}_m^*(x) = L^{-1/2}e^{2\pi i m x/L}$  for  $x \in [-L, 0]$  and zero elsewhere. For  $y \in [0, L]$ , the convolution support condition  $\{z : y - z \in [-L, 0], z \in [0, L]\}$  reduces to  $z \in [y, L]$ , giving

$$(\mathcal{T}_m^* * \mathcal{T}_n)(y) = \int_y^L L^{-1} e^{2\pi i m(y-z)/L} e^{2\pi i n z/L} dz = \frac{e^{2\pi i m y/L} - e^{2\pi i n y/L}}{2\pi i(n - m)},$$

using  $\int_y^L e^{2\pi i(n-m)z/L} dz = L(1 - e^{2\pi i(n-m)y/L})/(2\pi i(n - m))$  together with  $e^{2\pi i(n-m)y/L} = 1$  for  $n - m \in \mathbb{Z}$ . The same computation for  $y \in [-L, 0]$ , with integration range  $[0, y + L]$ , produces the analogous formula with  $m$  and  $n$  exchanged. Therefore

$$(170) \quad q(\mathcal{T}_m, \mathcal{T}_n)(y) = (\mathcal{T}_m^* * \mathcal{T}_n)(y) + (\mathcal{T}_m^* * \mathcal{T}_n)(-y) = \frac{\sin(\alpha_m y) - \sin(\alpha_n y)}{\pi(n - m)}, \quad y \in [-L, L],$$

where the second equality uses  $e^{i\theta} - e^{-i\theta} = 2i \sin \theta$ . Combining (170) with the kernel formula and the identification  $\langle A_{\lambda,N} V_n, V_m \rangle_{H_\lambda} = Q_{D,\lambda}(V_n, V_m)$  yields (168).

(ii) *Diagonal entries.* For  $y \in [0, L]$ ,

$$(\mathcal{T}_n^* * \mathcal{T}_n)(y) = \int_y^L L^{-1} e^{2\pi i n(y-z)/L} e^{2\pi i n z/L} dz = L^{-1} e^{2\pi i n y/L} (L - y),$$

and similarly  $(\mathcal{T}_n^* * \mathcal{T}_n)(-y) = L^{-1} e^{-2\pi i n y/L} (L - y)$  for  $y \in [0, L]$ . Adding,

$$(171) \quad q(\mathcal{T}_n, \mathcal{T}_n)(y) = \frac{2(L - y) \cos(\alpha_n y)}{L}, \quad y \in [0, L].$$

Since  $D_\lambda$  is real,  $\langle A_{\lambda,N} V_n, V_n \rangle_{H_\lambda} = Q_{D,\lambda}(V_n, V_n) = D_\lambda(q(\mathcal{T}_n, \mathcal{T}_n))$  yields (169).

(iii) *Real symmetry.* The right-hand side of (168) is real and symmetric in  $(m, n)$ :  $(s_n - s_m)/\pi(m - n) = (s_m - s_n)/\pi(n - m)$ . The diagonal entries (169) are real. The matrix  $A_{\lambda,N}$  in the basis  $\{V_j\}$  is therefore real and symmetric. The same property transfers to  $T_{\lambda,N} = A_{\lambda,N} - \varepsilon_{\lambda,N}^D I$ .  $\square$

LEMMA 5.10 (RANK-TWO COMMUTATOR IDENTITY). *Define*

$$(172) \quad \beta_{\lambda,N} := -\frac{2}{L^{1/2}} \sum_{|k| \leq N} s_k V_k \in E_N^\lambda.$$

*Then  $\beta_{\lambda,N}$  is inversion-odd:*

$$(173) \quad \mathcal{R}\beta_{\lambda,N} = -\beta_{\lambda,N},$$

and on  $E_N^\lambda$  one has the operator identity

$$(174) \quad [D_N, T_{\lambda, N}] = \beta_{\lambda, N} \otimes \Delta_N^* - \Delta_N \otimes \beta_{\lambda, N}^*.$$

*Proof.* The reflection identity (173) follows from  $\mathcal{R}V_k = V_{-k}$  and  $s_{-k} = -s_k$ :

$$\mathcal{R}\beta_{\lambda, N} = -\frac{2}{L^{1/2}} \sum_{|k| \leq N} s_k V_{-k} = -\frac{2}{L^{1/2}} \sum_{|j| \leq N} s_{-j} V_j = \frac{2}{L^{1/2}} \sum_{|j| \leq N} s_j V_j = -\beta_{\lambda, N}.$$

For (174) we compare matrix entries. The eigenvalue relation  $D_N V_j = \alpha_j V_j$  with  $\alpha_j \in \mathbb{R}$  gives  $D_N^* = D_N$  on the orthonormal basis  $\{V_j\}$  and

$$(175) \quad ([D_N, T_{\lambda, N}])_{mn} = \langle [D_N, T_{\lambda, N}] V_n, V_m \rangle_{H_\lambda} = (\alpha_m - \alpha_n)(T_{\lambda, N})_{mn}.$$

For  $m \neq n$ , Lemma 5.9 gives  $(T_{\lambda, N})_{mn} = (A_{\lambda, N})_{mn} = (s_n - s_m)/\pi(m - n)$ , so  $(\alpha_m - \alpha_n)(T_{\lambda, N})_{mn} = 2(s_n - s_m)/L$ . For  $m = n$ , the prefactor  $\alpha_m - \alpha_n$  vanishes, giving  $([D_N, T_{\lambda, N}])_{nn} = 0 = 2(s_n - s_n)/L$ . Therefore

$$(176) \quad ([D_N, T_{\lambda, N}])_{mn} = \frac{2(s_n - s_m)}{L}, \quad |m|, |n| \leq N.$$

For arbitrary  $a = \sum_j a_j V_j$  and  $b = \sum_j b_j V_j$  in  $E_N^\lambda$ , the matrix elements of the rank-one operator  $a \otimes b^*$  in the basis  $\{V_j\}$  are

$$(a \otimes b^*)_{mn} = \langle (a \otimes b^*) V_n, V_m \rangle = \langle V_n, b \rangle \langle a, V_m \rangle = \overline{b_n} a_m.$$

Applied to  $a = \beta_{\lambda, N}$ ,  $b = \Delta_N = L^{-1/2} \sum_k V_k$  (so  $b_k = L^{-1/2} \in \mathbb{R}$ ),

$$(\beta_{\lambda, N} \otimes \Delta_N^*)_{mn} = L^{-1/2} \cdot \frac{-2s_m}{L^{1/2}} = -\frac{2s_m}{L},$$

and similarly, using  $s_n \in \mathbb{R}$ ,

$$(\Delta_N \otimes \beta_{\lambda, N}^*)_{mn} = \frac{\overline{-2s_n}}{L^{1/2}} \cdot L^{-1/2} = -\frac{2s_n}{L}.$$

Subtracting,

$$(\beta_{\lambda, N} \otimes \Delta_N^* - \Delta_N \otimes \beta_{\lambda, N}^*)_{mn} = \frac{-2s_m - (-2s_n)}{L} = \frac{2(s_n - s_m)}{L} = ([D_N, T_{\lambda, N}])_{mn},$$

the second equality being (176). Since both sides agree on every matrix element with respect to an orthonormal basis of the finite-dimensional space  $E_N^\lambda$ , the operator identity (174) follows.  $\square$

LEMMA 5.11 (CONNES NORMALIZATION). *For every  $N \in \mathfrak{N}_\lambda$  and every nonzero inversion-even  $\xi_{\lambda, N} \in \ker T_{\lambda, N}$ ,*

$$\delta_N(\xi_{\lambda, N}) \neq 0.$$

*Proof.* Apply the operator identity (174) to  $\xi_{\lambda,N}$ . The left-hand side is

$$[D_N, T_{\lambda,N}]\xi_{\lambda,N} = D_N T_{\lambda,N} \xi_{\lambda,N} - T_{\lambda,N} D_N \xi_{\lambda,N} = -T_{\lambda,N} D_N \xi_{\lambda,N},$$

where the second equality uses  $T_{\lambda,N} \xi_{\lambda,N} = 0$ . The right-hand side is

$$(\beta_{\lambda,N} \otimes \Delta_N^* - \Delta_N \otimes \beta_{\lambda,N}^*) \xi_{\lambda,N} = \langle \xi_{\lambda,N}, \Delta_N \rangle \beta_{\lambda,N} - \langle \xi_{\lambda,N}, \beta_{\lambda,N} \rangle \Delta_N = \delta_N(\xi_{\lambda,N}) \beta_{\lambda,N} - \langle \xi_{\lambda,N}, \beta_{\lambda,N} \rangle \Delta_N.$$

The vector  $\xi_{\lambda,N}$  is inversion-even and  $\beta_{\lambda,N}$  is inversion-odd (Lemma 5.10); together with the unitarity of  $\mathcal{R}$  and  $\mathcal{R}^2 = I$ ,

$$\langle \xi_{\lambda,N}, \beta_{\lambda,N} \rangle = \langle \mathcal{R} \xi_{\lambda,N}, \mathcal{R} \beta_{\lambda,N} \rangle = \langle \xi_{\lambda,N}, -\beta_{\lambda,N} \rangle = -\langle \xi_{\lambda,N}, \beta_{\lambda,N} \rangle,$$

hence  $\langle \xi_{\lambda,N}, \beta_{\lambda,N} \rangle = 0$ . Combining,

$$(177) \quad T_{\lambda,N} D_N \xi_{\lambda,N} = -\delta_N(\xi_{\lambda,N}) \beta_{\lambda,N}.$$

Suppose, towards a contradiction, that  $\delta_N(\xi_{\lambda,N}) = 0$ . Then  $T_{\lambda,N} D_N \xi_{\lambda,N} = 0$  by (177), that is,  $D_N \xi_{\lambda,N} \in \ker T_{\lambda,N} = \mathbb{C} \xi_{\lambda,N}$ . By Lemma 5.8 applied to the inversion-even vector  $\xi_{\lambda,N}$ ,

$$\mathcal{R}(D_N \xi_{\lambda,N}) = -D_N \mathcal{R} \xi_{\lambda,N} = -D_N \xi_{\lambda,N},$$

so  $D_N \xi_{\lambda,N}$  is inversion-odd. Combined with the inclusion  $D_N \xi_{\lambda,N} \in \mathbb{C} \xi_{\lambda,N}$  in an inversion-even subspace, this forces  $D_N \xi_{\lambda,N} = 0$ . Since  $D_N V_j = \alpha_j V_j$  with  $\alpha_j = 0 \Leftrightarrow j = 0$ , the relation  $D_N \xi_{\lambda,N} = 0$  forces  $\xi_{\lambda,N} = cV_0$  for some  $c \in \mathbb{C}$ . Nonzero  $\xi_{\lambda,N}$  gives  $c \neq 0$ , but then

$$\delta_N(\xi_{\lambda,N}) = \delta_N(cV_0) = L^{-1/2}c \neq 0,$$

contradicting  $\delta_N(\xi_{\lambda,N}) = 0$ . Therefore  $\delta_N(\xi_{\lambda,N}) \neq 0$ .  $\square$

*Remark.* By Lemma 5.11, the rescaling  $\xi_{\lambda,N} \mapsto \xi_{\lambda,N}/\delta_N(\xi_{\lambda,N})$  is permitted. Throughout the remainder of this subsection we adopt the *Connes normalization*

$$(178) \quad \delta_N(\xi_{\lambda,N}) = 1.$$

Under this normalization, identity (177) simplifies to

$$(179) \quad T_{\lambda,N} D_N \xi_{\lambda,N} = -\beta_{\lambda,N}.$$

We now turn to the rank-one perturbation operator  $R_{\lambda,N} \in \text{End}(E_N^\lambda)$  defined by

$$(180) \quad R_{\lambda,N} f := D_N f - \delta_N(f) D_N \xi_{\lambda,N} = D_N f - ((D_N \xi_{\lambda,N}) \otimes \Delta_N^*) f.$$

LEMMA 5.12 (SELF-ADJOINTNESS OF THE INDUCED QUOTIENT OPERATOR). *The operator  $R_{\lambda,N}$  vanishes on  $\mathbb{C} \xi_{\lambda,N} = \ker T_{\lambda,N}$  and induces a well-defined operator*

$$\bar{R}_{\lambda,N} : E_N^\lambda / \mathbb{C} \xi_{\lambda,N} \longrightarrow E_N^\lambda / \mathbb{C} \xi_{\lambda,N}, \quad \bar{R}_{\lambda,N}[f] := [R_{\lambda,N} f].$$

The bilinear form  $([f], [g])_{T_{\lambda,N}} := \langle T_{\lambda,N}f, g \rangle_{H_\lambda}$  is a positive-definite Hilbert space inner product on  $E_N^\lambda / \mathbb{C}\xi_{\lambda,N}$ , and  $\bar{R}_{\lambda,N}$  is self-adjoint with respect to it.

*Proof.* By the Connes normalization (178),

$$R_{\lambda,N}\xi_{\lambda,N} = D_N\xi_{\lambda,N} - \delta_N(\xi_{\lambda,N})D_N\xi_{\lambda,N} = D_N\xi_{\lambda,N} - D_N\xi_{\lambda,N} = 0,$$

so  $R_{\lambda,N}$  vanishes on  $\mathbb{C}\xi_{\lambda,N}$  and the induced map  $\bar{R}_{\lambda,N}$  on the quotient is well defined. The bilinear form  $([f], [g])_{T_{\lambda,N}} := \langle T_{\lambda,N}f, g \rangle$  is independent of the representatives  $f, g$ : the map  $f \mapsto T_{\lambda,N}f$  vanishes on  $\mathbb{C}\xi_{\lambda,N} = \ker T_{\lambda,N}$ , while  $\langle T_{\lambda,N}f, g \rangle = 0$  for  $g \in \mathbb{C}\xi_{\lambda,N}$  by self-adjointness of  $T_{\lambda,N}$ . Positive-definiteness on the quotient follows from  $T_{\lambda,N} \geq 0$  together with  $\ker T_{\lambda,N} = \mathbb{C}\xi_{\lambda,N}$ .

We verify the operator-level intertwining

$$(181) \quad T_{\lambda,N}R_{\lambda,N} = R_{\lambda,N}^*T_{\lambda,N} \quad \text{on } E_N^\lambda.$$

Abbreviate  $\xi := \xi_{\lambda,N}$ ,  $\Delta := \Delta_N$ ,  $\beta := \beta_{\lambda,N}$ ,  $T := T_{\lambda,N}$ .

(i) *Computation of  $TR_{\lambda,N}$ .* From (180),

$$TR_{\lambda,N} = TD_N - T((D_N\xi) \otimes \Delta^*) = TD_N - (TD_N\xi) \otimes \Delta^* = TD_N + \beta \otimes \Delta^*,$$

where the third equality uses  $TD_N\xi = -\beta$  from (179). Substituting  $TD_N = D_NT - [D_N, T] = D_NT - \beta \otimes \Delta^* + \Delta \otimes \beta^*$  from (174) of Lemma 5.10,

$$(182) \quad TR_{\lambda,N} = D_NT - \beta \otimes \Delta^* + \Delta \otimes \beta^* + \beta \otimes \Delta^* = D_NT + \Delta \otimes \beta^*.$$

(ii) *Computation of  $R_{\lambda,N}^*T$ .* The adjoint of the rank-one operator  $a \otimes b^*$  is  $b \otimes a^*$ , and  $D_N^* = D_N$  since  $\alpha_j \in \mathbb{R}$ . Hence

$$R_{\lambda,N}^* = D_N - \Delta \otimes (D_N\xi)^*, \quad R_{\lambda,N}^*T = D_NT - (\Delta \otimes (D_N\xi)^*)T.$$

For  $f \in E_N^\lambda$ ,

$$(\Delta \otimes (D_N\xi)^*)Tf = \langle Tf, D_N\xi \rangle \Delta = \langle f, TD_N\xi \rangle \Delta = \langle f, -\beta \rangle \Delta = -\langle f, \beta \rangle \Delta,$$

where the second equality uses  $T^* = T$  and the third uses (179). Therefore  $(\Delta \otimes (D_N\xi)^*)T = -\Delta \otimes \beta^*$ , and

$$(183) \quad R_{\lambda,N}^*T = D_NT + \Delta \otimes \beta^*.$$

Comparing (182) and (183) gives (181).

(iii) *Self-adjointness on the quotient.* For all  $f, g \in E_N^\lambda$ , by (181),

$$(\bar{R}_{\lambda,N}[f], [g])_{T_{\lambda,N}} = \langle TR_{\lambda,N}f, g \rangle = \langle R_{\lambda,N}^*Tf, g \rangle = \langle Tf, R_{\lambda,N}g \rangle = ([f], \bar{R}_{\lambda,N}[g])_{T_{\lambda,N}}.$$

The induced operator  $\bar{R}_{\lambda,N}$  on the finite-dimensional quotient is therefore symmetric in the inner product  $(\cdot, \cdot)_{T_{\lambda,N}}$ , and on a finite-dimensional Hilbert space symmetry implies self-adjointness.  $\square$

LEMMA 5.13 (MELLIN TRANSFORM OF  $\xi_{\lambda,N}$ ). *The function*

$$\widehat{\xi}_{\lambda,N}(z) := \int_{\lambda^{-1}}^{\lambda} \xi_{\lambda,N}(u) u^{-iz} d^\times u, \quad z \in \mathbb{C},$$

*is entire, and admits the closed-form expression*

$$(184) \quad \widehat{\xi}_{\lambda,N}(z) = 2L^{-1/2} \sin\left(\frac{Lz}{2}\right) \sum_{|j| \leq N} \frac{\xi_j}{z - \alpha_j}, \quad z \in \mathbb{C},$$

*where the right-hand side is interpreted by removable-singularity at each  $z = \alpha_j$ ,  $|j| \leq N$ .*

*Proof.* Entirety of  $\widehat{\xi}_{\lambda,N}$  follows from the dominated-convergence argument given in part (i) of the proof of Theorem 5.7, applied to the bounded  $L^1$ -density  $\xi_{\lambda,N}$  on the compact interval  $I_\lambda$ .

For each  $|j| \leq N$  and  $z \in \mathbb{C}$ , the change of variables  $y := \log(\lambda u)$ , with  $u = e^{y-\ell}$ ,  $d^\times u = dy$ , and  $u^{-iz} = e^{izL/2} e^{-izy}$ , gives

$$\int_{\lambda^{-1}}^{\lambda} V_j(u) u^{-iz} d^\times u = L^{-1/2} e^{izL/2} \int_0^L e^{-i(z-\alpha_j)y} dy.$$

For  $z \neq \alpha_j$ , the elementary integral  $\int_0^L e^{-i(z-\alpha_j)y} dy = (1 - e^{-i(z-\alpha_j)L})/i(z - \alpha_j)$ , combined with  $e^{-i\alpha_j L} = e^{-2\pi i j} = 1$ , yields

$$(185) \quad \int_{\lambda^{-1}}^{\lambda} V_j(u) u^{-iz} d^\times u = L^{-1/2} e^{izL/2} \frac{1 - e^{-izL}}{i(z - \alpha_j)} = \frac{2L^{-1/2} \sin(Lz/2)}{z - \alpha_j}.$$

Both sides are entire functions of  $z$  (the left-hand side by the dominated-convergence argument cited above; the right-hand side because  $2L^{-1/2} \sin(Lz/2)$  has a simple zero at each  $z = \alpha_j = 2\pi j/L$  that cancels the pole of  $1/(z - \alpha_j)$ ). The identity (185) therefore extends from  $z \notin \{\alpha_j\}$  to all  $z \in \mathbb{C}$  by continuity. Linearity in  $j$  produces (184).  $\square$

LEMMA 5.14 (FINITE-DIMENSIONAL DETERMINANT RATIO). *For every  $z \in \mathbb{C}$ ,*

$$(186) \quad \det(\overline{R}_{\lambda,N} - zI) = L^{-1/2} \det(D_N - zI) \sum_{|j| \leq N} \frac{\xi_j}{\alpha_j - z},$$

*where the right-hand side, after multiplying through by  $\prod_{|j| \leq N} (\alpha_j - z)$ , is a polynomial in  $z$ , and the identity is interpreted in the polynomial sense at  $z = \alpha_j$ .*

*Proof.* Fix  $z \in \mathbb{C} \setminus \sigma(D_N) = \mathbb{C} \setminus \{\alpha_j : |j| \leq N\}$ . The operator  $D_N - zI : E_N^\lambda \rightarrow E_N^\lambda$  is invertible, with  $(D_N - zI)^{-1} V_j = (\alpha_j - z)^{-1} V_j$ . The factorization

$$R_{\lambda,N} - zI = (D_N - zI) [I - (D_N - zI)^{-1} (D_N \xi) \otimes \Delta^*]$$



(with  $\xi = \xi_{\lambda,N}$ ,  $\Delta = \Delta_N$ ) gives, on taking finite-dimensional determinants on  $E_N^\lambda$ ,

$$\det(R_{\lambda,N} - zI) = \det(D_N - zI) \det(I - (D_N - zI)^{-1}(D_N \xi) \otimes \Delta^*).$$

Sylvester's rank-one determinant identity asserts  $\det(I + a \otimes b^*) = 1 + \langle a, b \rangle$ . We verify this in our convention: the rank-one operator  $a \otimes b^*$  has, in any orthonormal basis  $\{e_k\}$ , matrix entries  $(a \otimes b^*)_{kl} = \langle a, e_k \rangle \overline{\langle b, e_l \rangle}$ , hence trace  $\sum_k \langle a, e_k \rangle \overline{\langle b, e_k \rangle} = \langle a, b \rangle$ . Its only nonzero eigenvalue equals its trace  $\langle a, b \rangle$ , so  $\det(I + a \otimes b^*) = 1 + \langle a, b \rangle$ . Applied to  $a = -(D_N - zI)^{-1}D_N \xi$  and  $b = \Delta$ ,

$$\det(I - (D_N - zI)^{-1}(D_N \xi) \otimes \Delta^*) = 1 - \langle (D_N - zI)^{-1}D_N \xi, \Delta \rangle.$$

Using  $D_N = (D_N - zI) + zI$ ,  $(D_N - zI)^{-1}D_N \xi = \xi + z(D_N - zI)^{-1}\xi$ , so by the Connes normalization (178),

$$\langle (D_N - zI)^{-1}D_N \xi, \Delta \rangle = \langle \xi, \Delta \rangle + z \langle (D_N - zI)^{-1}\xi, \Delta \rangle = 1 + z \langle (D_N - zI)^{-1}\xi, \Delta \rangle.$$

Substituting,

$$(187) \quad \det(R_{\lambda,N} - zI) = -z \det(D_N - zI) \langle (D_N - zI)^{-1}\xi, \Delta \rangle.$$

By Lemma 5.12,  $R_{\lambda,N}\xi = 0$ , so 0 is an eigenvalue of  $R_{\lambda,N}$  with eigenvector  $\xi$ . Choosing any direct-sum complement  $W$  of  $\mathbb{C}\xi$  in  $E_N^\lambda$ , the matrix of  $R_{\lambda,N} - zI$  in the corresponding block decomposition is upper-triangular with  $(1, 1)$ -entry  $-z$  and  $(2, 2)$ -block  $\overline{R}_{\lambda,N} - zI$  (read on  $W \cong E_N^\lambda/\mathbb{C}\xi$ ). Therefore

$$\det(R_{\lambda,N} - zI) = (-z) \det(\overline{R}_{\lambda,N} - zI),$$

the right-hand-side determinant being computed using any basis of the quotient. For  $z \neq 0$ , dividing (187) by  $-z$  gives

$$\det(\overline{R}_{\lambda,N} - zI) = \det(D_N - zI) \langle (D_N - zI)^{-1}\xi, \Delta \rangle.$$

Both sides are polynomials in  $z$  (after clearing the apparent simple poles of the right-hand side at  $z = \alpha_j$  via the factor  $\det(D_N - zI) = \prod_{|j| \leq N} (\alpha_j - z)$ ), so the identity extends to all  $z \in \mathbb{C}$  by polynomial continuation:

$$(188) \quad \det(\overline{R}_{\lambda,N} - zI) = \det(D_N - zI) \langle (D_N - zI)^{-1}\xi, \Delta \rangle.$$

Computing the inner product explicitly,  $(D_N - zI)^{-1}\xi = \sum_j \frac{\xi_j}{\alpha_j - z} V_j$  and  $\Delta = L^{-1/2} \sum_k V_k$  give

$$\langle (D_N - zI)^{-1}\xi, \Delta \rangle = L^{-1/2} \sum_{|j| \leq N} \frac{\xi_j}{\alpha_j - z}.$$

Substituting in (188) yields (186).  $\square$

LEMMA 5.15 (ZETA-REGULARIZED DETERMINANT OF  $D_{\log}^{(\lambda, N)}$ ). *Let  $\mathcal{K}_{\lambda, N}$  and  $D_{\log}^{(\lambda, N)} = \overline{R}_{\lambda, N} \oplus D_{\log}^{(\lambda)}|_{(E_N^\lambda)^\perp}$  be defined as in Theorem 5.16 (ii) below. With the spectral cut convention  $(-1)^{-w} := e^{-i\pi w}$ , the zeta-regularized determinant of  $D_{\log}^{(\lambda, N)} - z$  on  $\mathcal{K}_{\lambda, N}$  satisfies, for all  $z \in \mathbb{C}$ ,*

$$(189) \quad \det_{\text{reg}}(D_{\log}^{(\lambda, N)} - z) = \det(\overline{R}_{\lambda, N} - zI) \cdot \frac{\det_{\text{reg}}(D_{\log}^{(\lambda)} - z)}{\det(D_N - zI)}.$$

Moreover,

$$(190) \quad \det_{\text{reg}}(D_{\log}^{(\lambda)} - z) = 1 - e^{-iLz}.$$

*Proof.* The spectrum of  $D_{\log}^{(\lambda)}$  is the bilateral arithmetic progression  $\{\alpha_j : j \in \mathbb{Z}\} \subset \mathbb{R}$ , with each eigenvalue simple and the orthonormal eigenbasis  $\{V_j\}_{j \in \mathbb{Z}}$  of  $H_\lambda$ . The orthogonal decomposition  $H_\lambda = E_N^\lambda \oplus (E_N^\lambda)^\perp$  reduces  $D_{\log}^{(\lambda)}$ , and

$$D_{\log}^{(\lambda)} = D_N \oplus D_{\log}^{(\lambda)}|_{(E_N^\lambda)^\perp}$$

splits as a finite-dimensional summand  $D_N$  on  $E_N^\lambda$  (with spectrum  $\{\alpha_j : |j| \leq N\}$ ) and an unbounded self-adjoint summand on  $(E_N^\lambda)^\perp$  (with spectrum  $\{\alpha_j : |j| > N\}$ ). By the multiplicativity of zeta-regularized determinants under orthogonal direct sums of self-adjoint operators with discrete spectra (eigenvalues counted with multiplicity; in which finite-dimensional summands contribute via the ordinary determinant; cf. [7, Lemma 5.8 and the surrounding discussion]),

$$(191) \quad \det_{\text{reg}}(D_{\log}^{(\lambda)} - z) = \det(D_N - zI) \cdot \det_{\text{reg}}\left(\bigoplus_{|j| > N} (\alpha_j - z)\right).$$

The same multiplicativity applied to  $D_{\log}^{(\lambda, N)} = \overline{R}_{\lambda, N} \oplus D_{\log}^{(\lambda)}|_{(E_N^\lambda)^\perp}$ , with  $\overline{R}_{\lambda, N}$  acting on the  $2N$ -dimensional Hilbert space  $E_N^\lambda/\mathbb{C}\xi$  (the underlying vector space of the first summand of  $\mathcal{K}_{\lambda, N}$ ), gives

$$(192) \quad \det_{\text{reg}}(D_{\log}^{(\lambda, N)} - z) = \det(\overline{R}_{\lambda, N} - zI) \cdot \det_{\text{reg}}\left(\bigoplus_{|j| > N} (\alpha_j - z)\right).$$

Dividing (192) by (191) cancels the regularized tail factor and yields (189).

Identity (190) is [7, Lemma 5.8]: the spectral  $\zeta$ -function  $\zeta_{D_{\log}^{(\lambda)} - z}(w) = \sum_{j \in \mathbb{Z}} (\alpha_j - z)^{-w}$  admits a meromorphic continuation to  $w = 0$ , regular at  $w = 0$ , and  $\det_{\text{reg}} = \exp(-\zeta'_{D_{\log}^{(\lambda)} - z}(0)) = 1 - e^{-iLz}$  under the spectral cut convention  $(-1)^{-w} := e^{-i\pi w}$  (see [7, Eqs. (5.32)–(5.40)]).  $\square$

THEOREM 5.16. *Let  $\lambda > 1$ , let  $N \in \mathfrak{N}_\lambda$ , and let  $\xi_{\lambda,N}$  be a corresponding inversion-even lowest-eigenvector of  $A_{\lambda,N}$ , normalized by the Connes condition (178). Define, on  $E_N^\lambda$ ,*

$$R_{\lambda,N}f := D_{\log}^{(\lambda)}f - \delta_N(f)D_{\log}^{(\lambda)}\xi_{\lambda,N}.$$

*Then the following statements hold.*

- (i)  $R_{\lambda,N}\xi_{\lambda,N} = 0$ . Hence  $R_{\lambda,N}$  induces an operator

$$\bar{R}_{\lambda,N} : E_N^\lambda / \mathbb{C}\xi_{\lambda,N} \longrightarrow E_N^\lambda / \mathbb{C}\xi_{\lambda,N},$$

*which is self-adjoint with respect to the inner product  $([f], [g])_{T_{\lambda,N}} := \langle T_{\lambda,N}f, g \rangle_{H_\lambda}$ .*

- (ii) *Let*

$$\mathcal{K}_{\lambda,N} := (E_N^\lambda / \mathbb{C}\xi_{\lambda,N}, (\cdot, \cdot)_{T_{\lambda,N}}) \oplus (E_N^\lambda)^\perp,$$

*the orthogonal direct sum of the finite-dimensional Hilbert space of (i) and the closed subspace  $(E_N^\lambda)^\perp \subset H_\lambda$  with the restricted  $H_\lambda$ -inner product, and define*

$$D_{\log}^{(\lambda,N)} := \bar{R}_{\lambda,N} \oplus D_{\log}^{(\lambda)}|_{(E_N^\lambda)^\perp}.$$

*Then  $D_{\log}^{(\lambda,N)}$  is self-adjoint on  $\mathcal{K}_{\lambda,N}$ .*

- (iii) *With the spectral cut convention  $(-1)^{-w} := e^{-i\pi w}$ , the zeta-regularized determinant satisfies*

$$(193) \quad \boxed{\det_{\text{reg}}(D_{\log}^{(\lambda,N)} - z) = -i\lambda^{-iz}\widehat{\xi}_{\lambda,N}(z), \quad z \in \mathbb{C},}$$

*where*

$$\widehat{\xi}_{\lambda,N}(z) := \int_{\lambda^{-1}}^{\lambda} \xi_{\lambda,N}(u) u^{-iz} d^\times u.$$

*Moreover,*

$$(194) \quad \boxed{\widehat{\xi}_{\lambda,N}(z) = 2L^{-1/2} \sin\left(\frac{Lz}{2}\right) \sum_{|j| \leq N} \frac{\xi_j}{z - \alpha_j}}$$

*with removable singularities at the points  $z = \alpha_j$ ,  $|j| \leq N$ .*

- (iv) *The zeros of  $\widehat{\xi}_{\lambda,N}$  coincide, with multiplicities, with the spectrum of  $D_{\log}^{(\lambda,N)}$ . In particular all zeros of  $\widehat{\xi}_{\lambda,N}$  are real.*

*Proof.* (i) The first claim  $R_{\lambda,N}\xi_{\lambda,N} = 0$  is the opening computation in the proof of Lemma 5.12 and uses only the Connes normalization (178). The induced quotient operator  $\bar{R}_{\lambda,N}$  is self-adjoint with respect to  $(\cdot, \cdot)_{T_{\lambda,N}}$  by Lemma 5.12.

(ii) The space  $\mathcal{K}_{\lambda,N}$  is a Hilbert space: the first summand carries the positive-definite inner product  $(\cdot, \cdot)_{T_{\lambda,N}}$  established in Lemma 5.12, and the second is a closed subspace of  $H_\lambda$ . By Lemma 5.12,  $\bar{R}_{\lambda,N}$  on the (finite-dimensional) first summand is self-adjoint and bounded. The unbounded operator  $D_{\log}^{(\lambda)}$

on  $H_\lambda$ , defined on the dense domain  $\text{Dom}(D_{\log}^{(\lambda)}) = \{f = \sum_j f_j V_j \in H_\lambda : \sum_j |\alpha_j|^2 |f_j|^2 < \infty\}$ , is self-adjoint, with real spectrum  $\{\alpha_j : j \in \mathbb{Z}\}$  and orthonormal eigenbasis  $\{V_j\}_{j \in \mathbb{Z}}$ . The orthogonal projection  $P_N : H_\lambda \rightarrow E_N^\lambda$  is diagonal in this eigenbasis and therefore commutes with  $D_{\log}^{(\lambda)}$  on its domain, so the orthogonal decomposition  $H_\lambda = E_N^\lambda \oplus (E_N^\lambda)^\perp$  reduces  $D_{\log}^{(\lambda)}$ . By standard reduction theory for self-adjoint operators (cf. [9, Vol. I, §VIII.3]),  $D_{\log}^{(\lambda)}|_{(E_N^\lambda)^\perp}$  is self-adjoint on  $\text{Dom}(D_{\log}^{(\lambda)}) \cap (E_N^\lambda)^\perp$ . The orthogonal direct sum  $D_{\log}^{(\lambda, N)} = \overline{R}_{\lambda, N} \oplus D_{\log}^{(\lambda)}|_{(E_N^\lambda)^\perp}$  on  $\text{Dom}(\overline{R}_{\lambda, N}) \oplus (\text{Dom}(D_{\log}^{(\lambda)}) \cap (E_N^\lambda)^\perp)$  of a bounded self-adjoint operator and an unbounded self-adjoint operator is therefore self-adjoint on  $\mathcal{K}_{\lambda, N}$ , by the standard adjoint-of-direct-sum identity  $(A \oplus B)^* = A^* \oplus B^*$  on  $\text{Dom}(A^*) \oplus \text{Dom}(B^*)$  (cf. [9, Vol. I, §VIII.1]). We emphasize that the first summand of  $\mathcal{K}_{\lambda, N}$  carries the  $T_{\lambda, N}$ -inner product  $(\cdot, \cdot)_{T_{\lambda, N}}$  rather than the restriction of  $\langle \cdot, \cdot \rangle_{H_\lambda}$ , so  $\mathcal{K}_{\lambda, N}$  is not a closed subspace of  $H_\lambda$ ; the spectrum of  $\overline{R}_{\lambda, N}$  is, however, intrinsic and independent of this choice.

(iii) The Mellin formula (194) is Lemma 5.13.

We derive (193) by combining Lemmas 5.14 and 5.15. By Lemma 5.15,

$$\det_{\text{reg}}(D_{\log}^{(\lambda, N)} - z) = \det(\overline{R}_{\lambda, N} - zI) \cdot \frac{\det_{\text{reg}}(D_{\log}^{(\lambda)} - z)}{\det(D_N - zI)}.$$

Substituting (186) from Lemma 5.14, the factor  $\det(D_N - zI)$  cancels:

$$\det_{\text{reg}}(D_{\log}^{(\lambda, N)} - z) = L^{-1/2} \sum_{|j| \leq N} \frac{\xi_j}{\alpha_j - z} \cdot \det_{\text{reg}}(D_{\log}^{(\lambda)} - z).$$

Combining with (190) from Lemma 5.15, and using the trigonometric identity  $1 - e^{-iLz} = 2ie^{-iLz/2} \sin(Lz/2)$  together with  $1/(\alpha_j - z) = -1/(z - \alpha_j)$  and  $L/2 = \ell = \log \lambda$  (so  $e^{-iLz/2} = \lambda^{-iz}$ ),

$$\det_{\text{reg}}(D_{\log}^{(\lambda, N)} - z) = -2iL^{-1/2} \lambda^{-iz} \sin(Lz/2) \sum_{|j| \leq N} \frac{\xi_j}{z - \alpha_j}.$$

Comparing the right-hand side with the formula  $\widehat{\xi}_{\lambda, N}(z) = 2L^{-1/2} \sin(Lz/2) \sum_{|j| \leq N} \xi_j / (z - \alpha_j)$  of Lemma 5.13 produces (193).

(iv) The factor  $-i\lambda^{-iz} = -ie^{-iz\ell}$  is an entire function with no zeros on  $\mathbb{C}$ . By (193),

$$\det_{\text{reg}}(D_{\log}^{(\lambda, N)} - z) = 0 \iff \widehat{\xi}_{\lambda, N}(z) = 0,$$

with multiplicities equal on both sides. The zeros of the zeta-regularized determinant of a self-adjoint operator with discrete spectrum coincide, with multiplicity, with the spectrum of that operator: this is the standard spectral interpretation of the  $\zeta$ -regularized determinant (cf. [7, §5.1], applied to the

present setting where each contributing eigenvalue is real). By (ii),  $D_{\log}^{(\lambda, N)}$  is self-adjoint on  $\mathcal{K}_{\lambda, N}$  with real spectrum  $\sigma(\overline{R}_{\lambda, N}) \sqcup \{\alpha_j : |j| > N\}$ , where  $\sqcup$  denotes the disjoint union of multisets, i.e. the union of the two spectra with eigenvalue multiplicities accumulated whenever an eigenvalue of  $\overline{R}_{\lambda, N}$  happens to coincide with some  $\alpha_j$ ,  $|j| > N$ . Combining the two displayed equivalences, the zeros of  $\widehat{\xi}_{\lambda, N}$  coincide, with multiplicity, with this real-valued multiset.  $\square$

## 6. Numerical Results

In this section we report numerical experiments organized in three layers:

- (i) a detailed single-point case study at  $(\lambda, N) = (10, 120)$ , comparing the boundary-free model  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  of Theorem 3.2 (Subsection 6.1) with the boundary-retained model of [7] (Subsection 6.2) under a common arithmetic precision  $\text{dps} = 50$  (mpmath decimal-digit precision);
- (ii) a five-point geometric parameter sweep  $\lambda^2 \in \{13, 25, 50, 100, 200\}$  in the boundary-free model, accompanied by a four-point  $N$ -convergence sweep at  $\lambda^2 = 50$ , and a three-point boundary-retained sweep at  $\lambda^2 \in \{13, 25, 50\}$  (Subsection 6.3);
- (iii) a summary of the structural and trend patterns extracted from the data within the tested range (Subsection 6.4).

For  $(\lambda, N) = (10, 120)$ ,

$$I_\lambda = [\lambda^{-1}, \lambda] = [10^{-1}, 10], \quad L = 2 \log \lambda = 2 \log 10, \quad \lambda^2 = 100,$$

so the von Mangoldt sum  $\sum_{1 < n \leq \lambda^2} \Lambda(n) n^{-1/2} \langle T(n) \cdot, \cdot \rangle$  of (7) contains the contributions from all 25 primes  $p \leq 100$  together with the higher prime powers

$$4, 8, 9, 16, 25, 27, 32, 49, 64, 81.$$

The choice  $N = 120$  matches the row  $\lambda^2 = 100$  of the boundary-free main sweep in Subsection 6.3 below; by the  $N$ -convergence diagnostic at  $\lambda^2 = 50$  in Table 5 (cf. Subsection 6.3), the boundary-free MAE has stabilized to relative change of order  $10^{-3}$  over  $N \in [60, 120]$ . Under the working assumption that an analogous Galerkin saturation holds at the present parameter pair  $(\lambda^2, N) = (100, 120)$ , where the same  $N/L \approx 25$  ratio is maintained, further enlargement of  $N$  at fixed  $\lambda$  is not expected to be informative. All computations are carried out in the finite-dimensional space

$$E_N = \text{span}\{V_n : -N \leq n \leq N\} \subset H_\lambda,$$

where, for  $u \in I_\lambda$ ,

$$V_n(u) = L^{-1/2} \exp\left(\frac{2\pi n i}{L} \log(\lambda u)\right), \quad V_n(u) = 0 \text{ for } u \notin I_\lambda.$$

Motivated by the inversion symmetry of the ground state established in Proposition 5.3, all reported computations are performed in the inversion-even subspace of  $E_N$ .

6.1. *The case with the boundary term removed.* In the boundary-free model, the closed nonnegative form  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  is obtained from

$$Q_{D,\lambda} = Q_{W\lambda} - B_\lambda, \quad B_\lambda(f, g) = \widehat{f}(i/2) \overline{\widehat{g}(-i/2)} + \widehat{f}(-i/2) \overline{\widehat{g}(i/2)},$$

by adding the explicit scalar shift  $c_{D,\lambda} \langle \cdot, \cdot \rangle_{H_\lambda}$  of (2). Equivalently, at the operator level this amounts to replacing  $L_\lambda$  by  $A_\lambda = L_\lambda - c_{D,\lambda} I$ . A scalar shift translates every eigenvalue by the same constant and preserves every eigenspace; consequently,  $Q_{D,\lambda}$  and its shifted nonnegative counterpart  $\mathcal{E}_\lambda$  admit identical normalized ground-state vectors in  $E_N$ , up to an overall sign. Throughout this subsection we write

$$\xi_{\lambda,N}^D(u) = \sum_{n=-N}^N \xi_n^D V_n(u),$$

denoting by  $\xi_{\lambda,N}^D$  the lowest Ritz vector of the restriction  $Q_{D,\lambda}|_{E_N \times E_N}$  (this is the vector denoted  $\xi_{\lambda,N}$  in Subsection 5.6). The lowest Ritz vector of  $(\mathcal{E}_\lambda)|_{E_N \times E_N}$  is represented by the same coefficient sequence  $(\xi_n^D)_{|n| \leq N}$ , again up to sign. In particular, the Fourier transform  $\widehat{\xi}_{\lambda,N}^D(t)$  and its zero set are unchanged by the addition of the constant term  $c_{D,\lambda} \langle \cdot, \cdot \rangle$ ; only the reported Ritz value is affected, through an overall translation by the constant  $c_{D,\lambda}$ .

For the basis function  $V_n$ , the Mellin/Fourier transform defined in Subsection 2.4 is given explicitly by

$$\widehat{V}_n(t) = \int_{\lambda^{-1}}^{\lambda} V_n(u) u^{-it} d^\times u = \sqrt{L} (-1)^n \operatorname{sinc}\left(n - \frac{tL}{2\pi}\right), \quad \operatorname{sinc}(x) := \frac{\sin(\pi x)}{\pi x},$$

so that

$$\widehat{\xi}_{\lambda,N}^D(t) = \sqrt{L} \sum_{n=-N}^N \xi_n^D (-1)^n \operatorname{sinc}\left(n - \frac{tL}{2\pi}\right).$$

The matrix  $A_{\lambda,N}$  representing  $Q_{D,\lambda}|_{E_N \times E_N}$  is assembled from its archimedean and arithmetic blocks, in the convention of Subsection 2.4, by

$$Q_{\infty,\lambda}(V_m, V_n) = \int_{\mathbb{R}} \widehat{V}_m(t) \overline{\widehat{V}_n(t)} w(t) dt, \quad Q_{p,\lambda}(V_m, V_n) = - \sum_{1 < r \leq \lambda^2} \Lambda(r) \langle T(r) V_m, V_n \rangle_{H_\lambda},$$

where  $T(r) = r^{-1/2}(U_r + U_r^*)$  and the archimedean weight  $w(t)$  is the function appearing in (18), namely

$$w(t) = \frac{\theta'(t)}{\pi} = \frac{1}{2\pi} \Re \psi\left(\frac{1}{4} + \frac{it}{2}\right) - \frac{\log \pi}{2\pi}.$$

The constant logarithmic term used in the numerical assembly is therefore  $-(\log \pi)/(2\pi)$ . This is forced by the definition

$$\theta(t) = -\frac{t}{2} \log \pi + \Im \log \Gamma\left(\frac{1}{4} + \frac{it}{2}\right), \quad \theta'(t) = -\frac{1}{2} \log \pi + \frac{1}{2} \Re \psi\left(\frac{1}{4} + \frac{it}{2}\right),$$

and is consistent with the global relation  $w(t) = 2\theta'(t)/(2\pi)$  used in (18). Replacing the constant by the unnormalized  $-\log \pi$  would amount to adding a scalar multiple of the identity on the archimedean block, which leaves the Ritz vectors and their Fourier-transform zeros unaffected but shifts the reported Ritz values and is inconsistent with the normalization of  $Q_{\infty, \lambda}$ .

For  $\lambda = 10$  the arithmetic block sums over all prime powers  $r \leq 100$ ; the integrals  $\int_0^L \sin(2\pi ny/L) \rho(y) dy$ ,  $\int_0^L y \cos(2\pi ny/L) \rho(y) dy$ ,  $\int_0^L (\cos(2\pi ny/L) - 1) \rho(y) dy$  with  $\rho(y) = e^{y/2}/(e^y - e^{-y})$ , in terms of which the archimedean matrix entries are expressed (cf. [7, Prop. 4.2]), are evaluated in closed form via the Gauss hypergeometric function  ${}_2F_1$ , the Lerch transcendent  $\Phi$ , and the digamma/trigamma functions; the implementation uses mpmath with arithmetic precision  $\text{dps} = 50$  (approximately 50 decimal digits) throughout. For the present parameter pair  $(\lambda, N) = (10, 120)$ , the lowest Ritz value of  $A_{\lambda, N}$  is

$$\varepsilon_{10,120}^D := \min \sigma(A_{\lambda, N})|_{(\lambda, N)=(10, 120)} \approx -14.467972.$$

The first ten positive zeros of  $\widehat{\xi}_{10,120}^D$  are

$$\begin{aligned} &1.226261, 2.666293, 4.055103, 5.432630, 6.805953, \\ &8.177498, 9.548572, 10.920557, 12.297203, 13.727588. \end{aligned}$$

These low-lying zeros are governed primarily by the window scale and exhibit quasi-lattice spacing of order  $\pi/\log 10$ .

To quantify the discrepancy with the Riemann zeros, we match the first 20 nontrivial zeros  $\frac{1}{2} + i\gamma_k$ ,  $1 \leq k \leq 20$ , with the nearest positive zeros of  $\widehat{\xi}_{10,120}^D$ . The Python source code generating the data below is contained in `Numerical Experiments.zip`.

$k$	$\gamma_k$	index $m$ of nearest positive zero	$q_m^D$	$ q_m^D - \gamma_k $
1	14.135	10	13.727588	0.4071
2	21.022	15	20.554445	0.4676
3	25.011	18	24.636507	0.3744
4	30.425	22	30.097432	0.3274
5	32.935	24	32.788130	0.1469
6	37.586	28	38.134186	0.5480
7	40.919	30	40.935902	0.0172
8	43.327	32	43.604003	0.2769
9	48.005	35	47.836049	0.1691
10	49.774	36	49.203602	0.5702
11	52.970	39	53.171597	0.2013
12	56.446	41	56.042804	0.4034
13	59.347	43	58.757974	0.5891
14	60.832	45	61.304115	0.4723
15	65.113	48	65.421598	0.3091
16	67.080	49	66.897692	0.1821
17	69.546	51	69.582517	0.0361
18	72.067	53	72.278147	0.2110
19	75.705	55	75.146905	0.5578
20	77.145	57	77.686120	0.5413

TABLE 1. *Boundary-free model at  $(\lambda, N) = (10, 120)$ ,  $\text{dps} = 50$ : the first 20 Riemann zeros  $\gamma_k$  matched to the nearest positive zeros  $q_m^D$  of  $\widehat{\xi}_{10,120}^D$ .*

These data show that the boundary-free model already provides a real-zeros proxy in the sense of Theorem 5.16 (iv): every positive zero of  $\widehat{\xi}_{10,120}^D$  listed above lies on the real axis and equals an eigenvalue of the explicit self-adjoint operator  $D_{\log}^{(\lambda, N)}$  constructed in Subsection 5.6. Numerically, however, the fit of the first twenty matched zeros to  $\gamma_1, \dots, \gamma_{20}$  at  $(\lambda, N) = (10, 120)$  is moderate, with  $\text{MAE} = 0.3404$ ,  $\text{max} = 0.5891$ , and  $\text{min} = 0.0172$ .

6.2. *The case with the boundary term retained.* In this subsection we use the truncated semilocal Weil quadratic form  $Q_{W\lambda}^N$  of [7], in which the bounded finite-rank boundary term  $B_\lambda$  of (10) is retained:

$$Q_{W\lambda}^N(V_m, V_n) = Q_{D,\lambda}(V_m, V_n) + B_\lambda(V_m, V_n).$$

We denote by  $A_{\lambda,N}^W$  the self-adjoint matrix representing  $Q_{W\lambda}^N$  on  $E_N$ , and we set

$$\varepsilon_{\lambda,N}^W := \min \sigma(A_{\lambda,N}^W).$$

Writing the lowest Ritz vector as  $\xi_{\lambda,N}^W(u) = \sum_{n=-N}^N \xi_n^W V_n(u)$ , its Fourier transform takes the same explicit form as in Subsection 6.1,

$$\widehat{\xi}_{\lambda,N}^W(t) = \sqrt{L} \sum_{n=-N}^N \xi_n^W (-1)^n \text{sinc}\left(n - \frac{tL}{2\pi}\right),$$



or equivalently, in the form used in [7, Section 5],

$$\widehat{\xi}_{\lambda,N}^W(z) = 2L^{-1/2} \sin\left(\frac{zL}{2}\right) \sum_{n=-N}^N \frac{\xi_n^W}{z - 2\pi n/L}.$$

If, at a given  $(\lambda, N)$ , the lowest eigenvalue of  $A_{\lambda,N}^W$  is simple and its eigenvector is even, then the real-zero conclusion of [8, Theorem 6.1] applies and yields that all zeros of  $\widehat{\xi}_{\lambda,N}^W$  are real and coincide, with multiplicity, with the spectrum of the self-adjoint operator  $D_{\log}^{(\lambda,N)}$  constructed in [7, Section 5]. The simplicity-and-evenness condition required by [8, Theorem 6.1] is, however, not established analytically in the boundary-retained model; in the boundary-free model it is established unconditionally by Theorems 5.2 and 5.16.

For the present example  $(\lambda, N) = (10, 120)$ , the boundary block  $B_\lambda(V_m, V_n)$  is computed in closed form (cf. (10) together with the elementary integrals  $\int_{\lambda^{-1}}^\lambda u^{\pm 1/2} d^\times u = 2 \sinh(\log \lambda/2) \dots$ ), and is added entrywise to the boundary-free matrix  $A_{\lambda,N}$  of Subsection 6.1; the same arithmetic precision  $\text{dps} = 50$  is used as in Subsection 6.1. The lowest Ritz value is

$$\varepsilon_{10,120}^W \approx -7.19 \times 10^{-49},$$

which lies within the present numerical precision of zero. The first twenty positive zeros of  $\widehat{\xi}_{10,120}^W$  are

0.751754, 2.287123, 3.787932, 5.386452, 6.916741, 8.464402, 9.924876, 11.607619,  
12.812306, 14.134725, 16.045708, 17.543527, 19.093386, 21.022040, 21.755531,  
23.864579, 25.010858, 26.651223, 28.242386, 30.424876.

Below the first nontrivial zero 14.134725 of  $\zeta$  there appear nine additional positive zeros; from the tenth positive zero onward,  $\widehat{\xi}_{10,120}^W$  reproduces the low-lying zeros of  $\zeta$  with striking accuracy. Matching the first twenty zeros of  $\zeta$  with the nearest positive zeros of  $\widehat{\xi}_{10,120}^W$  yields the table below. The Python source code generating the data is contained in `Numerical Experiments.zip`.

$k$	$\gamma_k$	index $m$ of nearest positive zero	$q_m^W$	$ q_m^W - \gamma_k $
1	14.135	10	14.134725	$1.402 \times 10^{-45}$
2	21.022	14	21.022040	$4.737 \times 10^{-46}$
3	25.011	17	25.010858	$5.126 \times 10^{-46}$
4	30.425	20	30.424876	$1.039 \times 10^{-45}$
5	32.935	22	32.935062	$5.157 \times 10^{-46}$
6	37.586	25	37.586178	$3.974 \times 10^{-45}$
7	40.919	27	40.918719	$5.132 \times 10^{-48}$
8	43.327	29	43.327073	$1.716 \times 10^{-45}$
9	48.005	32	48.005151	$5.459 \times 10^{-45}$
10	49.774	33	49.773832	$1.246 \times 10^{-45}$
11	52.970	35	52.970321	$1.027 \times 10^{-46}$
12	56.446	37	56.446248	$3.230 \times 10^{-45}$
13	59.347	39	59.347044	$9.969 \times 10^{-46}$
14	60.832	40	60.831779	$6.346 \times 10^{-45}$
15	65.113	43	65.112544	$6.294 \times 10^{-45}$
16	67.080	44	67.079811	$1.902 \times 10^{-45}$
17	69.546	46	69.546402	$1.182 \times 10^{-46}$
18	72.067	48	72.067158	$4.969 \times 10^{-46}$
19	75.705	50	75.704691	$2.455 \times 10^{-45}$
20	77.145	51	77.144840	$1.421 \times 10^{-46}$

TABLE 2. *Boundary-retained model at  $(\lambda, N) = (10, 120)$ ,  $\text{dps} = 50$  (same parameters as Table 1): the first 20 Riemann zeros  $\gamma_k$  matched to the nearest positive zeros  $q_m^W$  of  $\widehat{\xi}_{10,120}^W$ .*

At the present arithmetic precision (mpmath  $\text{dps} = 50$ , i.e. approximately 50 decimal digits), the boundary-retained model thus reproduces the first twenty zeros of  $\zeta$  to an absolute error of order  $10^{-45}$ , in striking contrast to the order  $10^{-1}$  error of the boundary-free model at the very same parameters  $(\lambda, N, \text{dps})$ . The mean absolute error is  $\text{MAE} = 1.92 \times 10^{-45}$ , with  $\text{max} = 6.35 \times 10^{-45}$  and  $\text{min} = 5.13 \times 10^{-48}$ .

6.3. *Sensitivity to truncation parameters.* We complement the moderate-scale comparison of Subsections 6.1–6.2 with a parameter sweep designed to separate the effect of the window scale  $\lambda$  from that of the Galerkin dimension  $N$ , and to characterize the error distribution beyond the single mean-absolute-error statistic.

*Statistics reported.* For each  $(\lambda, N)$  we record, on the multiset of absolute errors  $e_k := |q_k - \gamma_k|$ ,  $k = 1, \dots, 20$  (matched as in Subsection 6.1), the seven statistics  $\text{MAE} = \frac{1}{20} \sum e_k$ ,  $\text{MedAE} = \text{med}(e_k)$ ,  $\text{GM} = \exp(\frac{1}{20} \sum \log e_k)$ ,  $\text{RMSE} = \sqrt{\frac{1}{20} \sum e_k^2}$ ,  $\text{Max} = \max e_k$ ,  $\text{MaxRE} = \max(e_k/\gamma_k)$ ,  $\text{MedRE} = \text{med}(e_k/\gamma_k)$ , the per-row index slope  $\beta$  defined by the least-squares fit  $\log e_k =$

$a + \beta k$ , the Spearman rank correlation  $\rho(e_k, k)$ , and the continuous-decrease ratio  $\text{MR} = \#\{k : e_{k+1} > e_k\}/19$ .

*Boundary-free main sweep.* Table 3 reports the seven distributional statistics at the five points  $\lambda^2 \in \{13, 25, 50, 100, 200\}$  with  $N$  chosen so that  $N/L \approx 25$  in every row. The row  $\lambda^2 = 100$ ,  $N = 120$  coincides with the single-point case study of Subsection 6.1 (the lower precision  $\text{dps} = 30$  used here suffices because the boundary-free MAE at  $\lambda^2 = 50$ ,  $N = 100$  agrees to four significant digits between  $\text{dps} = 30$  in Table 5 and  $\text{dps} = 120$  in Table 8, suggesting that arithmetic precision beyond moderate  $\text{dps}$  does not perturb the boundary-free MAE at this scale).

$\lambda^2$	$N$	dps	MAE	MedAE	GM	RMSE	Max	MaxRE	MedRE
13	60	25	0.680	0.718	0.578	0.736	1.128	0.044	0.014
25	80	25	0.437	0.359	0.363	0.505	0.934	0.024	0.009
50	100	30	0.265	0.159	0.148	0.358	0.717	0.024	0.003
100	120	30	0.340	0.351	0.263	0.383	0.589	0.029	0.007
200	140	35	0.267	0.282	0.193	0.314	0.530	0.013	0.006

TABLE 3. *Boundary-free model: error statistics for the first 20 matched zeros across the geometric sweep.*

A least-squares regression of  $\log y$  on  $\log \lambda$  over the five rows yields the trend slopes  $\sigma$ , coefficients of determination  $R^2$ , Kendall rank correlations  $\tau(\lambda, y)$ , and continuous-decrease ratios CR recorded in Table 4.

metric	$\sigma$	$R^2$	$\tau$	CR
MAE	−0.616	0.715	−0.6	0.75
MedAE	−0.546	0.299	−0.6	0.75
GM	−0.730	0.548	−0.6	0.75
RMSE	−0.575	0.846	−0.8	0.75
Max	−0.575	0.981	−1.0	1.00
MaxRE	−0.661	0.659	−0.4	0.50
MedRE	−0.572	0.319	−0.6	0.75

TABLE 4. *Boundary-free model: trend analysis  $\log y \sim \sigma \log \lambda$  over the five-point sweep.*

The Max statistic shows the strongest signal: a strict monotone decrease ( $\tau = -1$ ,  $\text{CR} = 1$ ) with  $R^2 = 0.98$ .<sup>2</sup> The MAE decreases overall from 0.680 at  $\lambda^2 = 13$  to 0.267 at  $\lambda^2 = 200$  but is not monotone, with one upward step at  $\lambda^2 = 100$ . The cause of this single non-monotone step within the five-point sample is not analytically explained; we record it without further interpretation.

<sup>2</sup>The cautionary remark on the introduction concerning  $R^2 = 0.98$  on the five-point log-log fit applies, in fact, to every  $R^2$  value listed in Table 4: each is a descriptive linearity coefficient on the same five sampled points and does not, by itself, constitute inferential evidence for an asymptotic decay law.

Both quantities admit log-log linear regression slopes of order  $\sigma \approx -0.6$  on the tested range. We report these slopes as descriptive trend coefficients over the five-point sample; the sample size precludes a precise determination of any asymptotic decay law, and we do not extrapolate either slope beyond the tested range.

*N-convergence at fixed window.* To confirm that the refinement of Table 3 is driven by  $\lambda$  and not by an unsaturated Galerkin dimension, we fix  $\lambda^2 = 50$  and sweep  $N \in \{60, 80, 100, 120\}$  at fixed  $\text{dps} = 30$ .

$N$	dps	MAE	GM	Max	$\beta$	relative change in MAE
60	30	0.2658	0.1514	0.7188	-0.0398	—
80	30	0.2655	0.1489	0.7180	-0.0422	$-1.0 \times 10^{-3}$
100	30	0.2653	0.1478	0.7170	-0.0431	$-6.2 \times 10^{-4}$
120	30	0.2651	0.1469	0.7161	-0.0439	$-5.4 \times 10^{-4}$

TABLE 5. *Boundary-free model: N-convergence diagnostic at  $\lambda^2 = 50$ . The MAE has stabilized to relative change of order  $10^{-3}$  over  $N \in [60, 120]$ .*

The MAE sequence is stable to four significant digits across  $N \in [60, 120]$ , with successive relative change of order  $10^{-3}$  at every step. Since the Galerkin saturation is verified analytically only at  $\lambda^2 = 50$ , we adopt the working assumption that the same saturation holds at the other sweep points of Table 3, where  $N/L \approx 25$  is held fixed. Under this assumption, the trend recorded in Table 4 is consistent with refinement driven by the window scale  $\lambda$  rather than by the Galerkin dimension  $N$ .

*Single-point monotonicity diagnostics.* The hypothesis that the residual error  $e_k$  grows systematically with the index  $k$  of the matched zero is tested in Table 6.

$\lambda^2$	$N$	$\beta$	$R_k^2$	Spearman $\rho$	MR
13	60	+0.0012	$8.5 \times 10^{-5}$	+0.165	0.579
25	80	+0.0247	0.050	+0.214	0.632
50	100	-0.0431	0.039	-0.150	0.474
100	120	-0.0023	$2.2 \times 10^{-4}$	+0.060	0.421
200	140	-0.0154	0.009	-0.108	0.474

TABLE 6. *Boundary-free model: single-point monotonicity statistics for  $e_k$  versus  $k$ .*

Across all five rows,  $|\beta| < 0.05$ ,  $R_k^2 < 0.06$ ,  $|\rho| < 0.22$ , and the monotone ratio MR stays close to the random-baseline value 0.5. The rank correlation between the residual error and the zero-index  $k$  is therefore not statistically significant at the 5% level: across the tested range  $k \in \{1, \dots, 20\}$ , errors of comparable magnitude are observed at every index, and no high-frequency divergence is detected within the window.

*Boundary-retained comparison.* The same statistics applied to the boundary-retained model  $\widehat{\xi}_{\lambda,N}^W$  on the sweep  $\lambda^2 \in \{13, 25, 50\}$  yield Table 7.

$\lambda^2$	$N$	dps	$\varepsilon_{\lambda,N}^W$	MAE	GM	Max	MAE/GM
13	60	60	$+8.6 \times 10^{-55}$	$2.3 \times 10^{-22}$	$1.8 \times 10^{-33}$	$3.8 \times 10^{-21}$	$1.3 \times 10^{11}$
25	80	65	$-6.8 \times 10^{-49}$	$3.4 \times 10^{-42}$	$6.9 \times 10^{-47}$	$5.5 \times 10^{-41}$	$4.9 \times 10^4$
50	100	70	$-6.1 \times 10^{-64}$	$7.8 \times 10^{-64}$	$2.4 \times 10^{-64}$	$4.3 \times 10^{-63}$	3.21

TABLE 7. *Boundary-retained model: error statistics on the sweep  $\lambda^2 \in \{13, 25, 50\}$ .*

The very large MAE/GM ratios in the first two rows (exceeding  $10^4$  and  $10^{11}$  respectively) are the quantitative signature of a highly dispersed, outlier-driven error distribution: a small subset of zeros is matched to far higher precision than the rest, so that the arithmetic mean is dominated by the worst tail. This dispersion contracts at  $\lambda^2 = 50$ , where the spread between the geometric mean ( $2.4 \times 10^{-64}$ ) and the maximum ( $4.3 \times 10^{-63}$ ) is less than two orders of magnitude. In the boundary-free model, by contrast, the analogous MAE/GM ratios remain in the interval  $[1.18, 1.79]$  across all five rows of Table 3, consistent with a low-dispersion error distribution in which the arithmetic and geometric means are of comparable magnitude.

*N-instability of the boundary-retained model at fixed window.* The complementary diagnostic – varying  $N$  at fixed  $\lambda^2 = 50$  and fixed dps = 120 in both models – is reported in Table 8.

Model	$N$	dps	MAE
boundary-free	100	120	$2.65 \times 10^{-1}$
boundary-free	400	120	$2.65 \times 10^{-1}$
boundary-retained	100	120	$1.61 \times 10^{-67}$
boundary-retained	400	120	$1.20 \times 10^{-1}$

TABLE 8. *Behaviour of the MAE at fixed window scale  $\lambda^2 = 50$  and fixed arithmetic precision dps = 120. The boundary-free MAE is unchanged to three significant digits between  $N = 100$  and  $N = 400$ ; the boundary-retained MAE varies by approximately 66 orders of magnitude between the same two values of  $N$ .*

The four entries of Table 8 are evaluated at the same window scale  $\lambda^2 = 50$  and the same arithmetic precision dps = 120, so any recorded difference within the table is attributable to either the model or the Galerkin dimension  $N$  alone. Three observations follow directly from the table.

- (1) *Boundary-free N-stability at dps = 120.* The boundary-free MAE differs between  $N = 100$  and  $N = 400$  by less than 0.2% in relative terms. At  $(\lambda^2, N) = (50, 100)$  the boundary-free entry  $2.65 \times 10^{-1}$  also agrees, to four significant digits, with the dps = 30 entry of Table 5 at the same

- $(\lambda^2, N)$ , so the upgrade in arithmetic precision does not by itself perturb the boundary-free MAE at this window scale.
- (2) *Boundary-retained  $N$ -instability at  $\text{dps} = 120$ .* The boundary-retained MAE varies by approximately 66 orders of magnitude between  $N = 100$  and  $N = 400$  at the same  $(\lambda^2, \text{dps})$ . This is the direction opposite to that of the boundary-free  $N$ -convergence sweep of Table 5.
  - (3) *Convergence of the two models at  $N = 400$ .* At  $(\lambda^2, N, \text{dps}) = (50, 400, 120)$ , the boundary-retained MAE  $1.20 \times 10^{-1}$  and the boundary-free MAE  $2.65 \times 10^{-1}$  differ by less than a factor of 2.21, that is, by less than half an order of magnitude. The boundary-retained precision advantage documented at  $(\lambda, N, \text{dps}) = (10, 120, 50)$  in Subsection 6.2 is therefore parameter-dependent, not a uniform feature of the boundary-retained truncation.

We do not extrapolate any of these trends beyond the four data points displayed; we record only that, within the four-point comparison of Table 8, the boundary-free MAE is empirically insensitive to the Galerkin dimension while the boundary-retained MAE is not, and that the small- $N$  precision gap between the two models is absent at  $N = 400$ .

A consequence of (1)–(3) above is that the boundary-retained MAE reported in Table 2 ( $\approx 1.92 \times 10^{-45}$  at  $(\lambda, N, \text{dps}) = (10, 120, 50)$ ) and the boundary-retained MAE reported in Table 8 at  $N = 400$  ( $\approx 1.20 \times 10^{-1}$  at  $(\lambda^2, \text{dps}) = (50, 120)$ ) are both numerical facts about the same model under different parameter choices; the two figures are not interchangeable in the logical use of Sections 6 and 7, and the small- $N$  precision figure is not, by itself, evidence that the boundary-retained model approximates  $\zeta$  systematically.

*Structural interpretation.* The boundary-free behaviour is consistent with the structural decomposition of Theorem 3.2. By (4) and (7), the energy decomposes as  $\mathcal{E}_\lambda = \mathcal{E}_{\infty, \lambda} + \mathcal{E}_{p, \lambda}$  with both summands individually nonnegative; the corresponding spectral weights  $\nu(dr)$  of (5) and  $\Lambda(n)n^{-1/2}$  for  $1 < n \leq \lambda^2$  are strictly positive, and each elementary energy contribution  $\|f - U_n f\|_{H_\lambda}^2$ ,  $\|f\|_{H_\lambda}^2 - \|U_n f\|_{H_\lambda}^2$ ,  $\|g(\cdot + r) - g\|_{L^2(\mathbb{R})}^2$  is itself nonnegative. This nonnegativity prevents catastrophic cancellation in the assembly of  $A_{\lambda, N}$ , so that parameter refinement transports cleanly into spectral refinement, with the error distribution showing no signs of extreme dispersion along the sweep.

By contrast, in the boundary-retained model the bounded finite-rank term  $B_\lambda$  of Lemma 3.3 contributes with indefinite sign and is precisely what produces, at  $(\lambda, N, \text{dps}) = (10, 120, 50)$ , the near-machine-precision agreement with  $\gamma_1, \dots, \gamma_{20}$ . That precision is consistent with arising from a sharply tuned cancellation between  $Q_{D, \lambda}$  and  $B_\lambda$ , whose effective conditioning is documented by the dramatic shrinkage of the MAE/GM ratio in Table 7 as  $\lambda^2$  increases from

13 to 50.<sup>3</sup> The ratio MAE/GM therefore plays the role of a scale-independent diagnostic for the dispersion of the per-row error distribution: a value close to unity signals an approximately log-normal distribution, while a value much larger than unity signals that the arithmetic mean is dominated by a small tail of outlier errors. Its behaviour along the sweep documented in Table 7 is thus a numerical observation of the present paper, separate from any precision claim made in [7].

We emphasize that this sensitivity comparison is not a statement about which model is closer to  $\zeta$  at any single  $(\lambda, N)$ ; the boundary-retained model retains its small-scale precision advantage. The point is that the two models differ in the *shape* of their error distributions and in their dependence on the Galerkin dimension: the boundary-free MAE shows an overall decrease over the tested range (Table 4), with no monotone trend in error magnitude across the zero-index (Table 6); the boundary-retained errors are highly dispersed at small scale and progressively concentrate as  $\lambda$  grows. Any global limit that recovers the Riemann  $\Xi$ -function must take place along a sequence in which the matched zeros approach the  $\gamma_k$  as  $(\lambda, N) \rightarrow (\infty, \infty)$ .

Finite parameter sweeps cannot, in any model, establish the existence of such an asymptotic limit; their only legitimate logical use is the falsification of candidate models exhibiting pathological behavior across the tested range. The diagnostics documented above record both signatures (extreme dispersion across the per-row error distribution and  $N$ -instability under fixed window scale) in the boundary-retained model, while detecting neither within the tested range of the boundary-free model. This is a falsification screening: it disqualifies the boundary-retained truncation as a candidate carrier under the present level of analysis, but does not, by itself, establish the boundary-free truncation as a candidate carrier. The latter question is decided by the analytical structure of Sections 3–5, not by the present sweeps.

Whether the boundary-retained model admits a meaningful diagonal limit, even with high-precision arithmetic, would require both an analytic verification of the simplicity-and-evenness condition required by [8, Theorem 6.1] and a stability analysis beyond the scope of the present paper. The corresponding global limit problem on the boundary-free side is formulated precisely in Section 7.

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<sup>3</sup>We emphasize the soft nature of this structural reading. A quantitative diagnostic of the conditioning of the boundary-retained Galerkin matrix  $A_\lambda^N$  – for instance the ratio of  $\|B_\lambda\|$  to the lowest Ritz value of  $Q_{D,\lambda}|_{E_\lambda^N}$ , or the spectral gap of  $A_\lambda^N$  above the lowest Ritz value – would be needed to establish the cancellation mechanism rigorously. Such a quantitative analysis is left to subsequent work; the MAE/GM contraction in Table 7 is consistent with, but does not by itself establish, this mechanism.

6.4. *Summary.* The numerical comparison between the two models, supplemented by the parameter sweep of Subsection 6.3, leads to four main observations.

- (i) In either model, the sequence of absolute errors over the first 20 matched zeros is not monotone in the index  $k$ . In the boundary-free case at  $(\lambda, N, \text{dps}) = (10, 120, 50)$ , the absolute error decreases at 11 of the 19 consecutive steps; in the boundary-retained case at the same parameters, it decreases at 10 of the 19 steps. Across the five-point sweep, the per-row index slope  $|\beta|$  remains below 0.05 and the Spearman rank correlation  $|\rho|$  below 0.22 (Table 6), so the rank correlation between the residual error and the zero-index  $k$  is not statistically significant at the 5% level.
- (ii) The boundary term has a decisive quantitative impact on the fit with the  $\zeta$ -zeros at  $(\lambda, N, \text{dps}) = (10, 120, 50)$ . For the first 20 zeros of  $\zeta$ , the mean absolute error is approximately  $3.40 \times 10^{-1}$  in the boundary-free case, whereas it drops to  $1.92 \times 10^{-45}$  in the boundary-retained case. The inclusion of the boundary term therefore improves the mean absolute error at this fixed  $(\lambda, N, \text{dps})$  by more than forty-four orders of magnitude. We emphasize that this gap is genuinely an effect of the boundary term and not of differing arithmetic precision: both models are evaluated under the identical mpmath setting  $\text{dps} = 50$ .
- (iii) The two models also differ in the *shape* of their error distributions and in their dependence on the Galerkin dimension. In the boundary-free model, the ratio MAE/GM remains in  $[1.18, 1.79]$  throughout the five-point sweep, consistent with a low-dispersion error distribution in which the arithmetic and geometric means are of comparable magnitude; the MAE itself is  $N$ -stable at  $\lambda^2 = 50$  to relative change of order  $10^{-3}$  over  $N \in [60, 120]$  (Table 5). In the boundary-retained model the same ratio ranges from  $1.3 \times 10^{11}$  at  $\lambda^2 = 13$  to 3.21 at  $\lambda^2 = 50$  (Table 7), reflecting an extreme outlier-driven dispersion at small scale and its progressive contraction with  $\lambda$ ; at  $\lambda^2 = 50$  and fixed  $\text{dps} = 120$ , the MAE varies by approximately 66 orders of magnitude between  $N = 100$  and  $N = 400$  (Table 8).
- (iv) The boundary-free MAE decreases overall across the geometric sweep  $\lambda^2 \in \{13, 25, 50, 100, 200\}$ , from 0.680 at  $\lambda^2 = 13$  to 0.267 at  $\lambda^2 = 200$  (Kendall  $\tau = -0.6$ , continuous-decrease ratio  $\text{CR} = 0.75$ ), with one non-monotone step at  $\lambda^2 = 100$  whose cause is not analytically explained. The worst-case statistic Max decreases strictly monotonically along the same sweep ( $\tau = -1$ ,  $\text{CR} = 1$ ); a descriptive log-log linear fit on the five sampled points gives  $R^2 = 0.98$ , reported as a within-range trend coefficient and not extrapolated. The four-point  $N$ -convergence sweep at  $\lambda^2 = 50$  shows that the refinement at that point is consistent with being driven by the window scale  $\lambda$  rather than by an unsaturated Galerkin dimension (Table 5); no



analogous  $N$ -convergence test is performed at the other sweep points, where  $N/L \approx 25$  is held fixed throughout. The functional form of the decay beyond the tested range is left open.

The structural and analytic implications of these observations, together with the global limiting problem they raise, are discussed in Section 7 below.

## 7. Outlook

The numerical experiments of Section 6 compare two finite-dimensional truncations: the boundary-free model studied throughout Sections 3–5, for which all theorems of this paper hold unconditionally, and the boundary-retained variant of [7], in which the bounded finite-rank boundary term  $B_\lambda$  is kept.

*7.1. Complementary virtues of the two truncations.* The case for the boundary-free model as a carrier for the diagonal limit problem of Subsection 7.2 rests on the analytical structure established in Sections 3–5; the parameter sweeps of Section 6 play only a falsification-screening role. We make these two functions explicit.

*Structural support for the boundary-free model.* Independently of any numerical evidence, the boundary-free construction yields:

- (i) a nonnegative closed symmetric Dirichlet form  $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$  on  $H_\lambda$  (Theorem 3.2 (1));
- (ii) irreducibility and positivity improvement of the associated semigroup and resolvent (Proposition 4.1 and Theorem 4.3);
- (iii) compactness of the resolvent and discreteness of the spectrum of the generator  $L_\lambda$  (Proposition 5.1);
- (iv) simplicity of the bottom eigenvalue together with strict positivity and inversion-evenness of the unique  $L^2$ -normalized ground state  $\xi_\lambda$  (Theorem 5.2 and Proposition 5.3);
- (v) the unconditional real-zeros theorem for the Fourier–Mellin transform  $\widehat{\xi}_\lambda$  (Theorem 5.7) and the zeta-regularized determinant identity together with the spectral identification of the zero set of  $\widehat{\xi}_{\lambda,N}^D$  with the spectrum of the explicit finite self-adjoint operator  $D_{\log}^{(\lambda,N)}$  (Theorem 5.16).

The simplicity-and-evenness hypotheses required by [8, Theorem 6.1] for the real-zeros conclusion are verified analytically by Theorem 5.2 and Proposition 5.3. Consequently, the finite-dimensional operator  $D_{\log}^{(\lambda,N)}$  is an *unconditional* Hilbert–Pólya-type candidate, in the sense that its self-adjointness and the reality of its spectrum are established without any assumption on RH and without any unverified hypothesis on the spectral form. This structural enumeration is, by itself, the principal justification for studying the diagonal limit problem of Subsection 7.2 on the boundary-free side.

*Conditional status of the boundary-retained variant.* At  $(\lambda, N, \text{dps}) = (10, 120, 50)$  the boundary-retained model of [7] approximates the first twenty nontrivial zeros of  $\zeta$  with mean absolute error  $\approx 1.92 \times 10^{-45}$ , some forty-four orders of magnitude smaller than the boundary-free MAE at the same parameters. The interpretation of the corresponding zeros as real, however, depends on the simplicity of the lowest Ritz eigenvalue together with the inversion-evenness of the corresponding eigenvector; neither hypothesis is verified analytically in the present work or in [7].<sup>4</sup> The real-zero conclusion on the boundary-retained side is therefore *conditional* on these unverified hypotheses, in contrast to the unconditional status of the boundary-free conclusion.

*A falsification reading of the boundary-retained  $N$ -instability.* A finite parameter sweep cannot, in any model, establish the existence of an infinite-dimensional limit. Its only legitimate logical use is *falsification*: a candidate model can be ruled out as a viable carrier for an asymptotic limit if its finite-dimensional approximations exhibit pathological behavior across the tested parameter range.

Table 8 provides such a falsification diagnostic for the boundary-retained model. At fixed  $(\lambda^2, \text{dps}) = (50, 120)$ , the boundary-retained MAE varies between  $1.6 \times 10^{-67}$  at  $N = 100$  and  $1.2 \times 10^{-1}$  at  $N = 400$  — a swing of approximately 66 orders of magnitude under a four-fold change in the Galerkin dimension. This is consistent with the structural reading offered in Section 6: the small-scale precision arises from a sharply tuned cancellation between  $Q_{D,\lambda}$  and the indefinite-sign boundary term  $B_\lambda$ , and is therefore a fragile artifact of a specific  $(\lambda, N)$  rather than a systematic approximation. The 66-order swing does not, by itself, prove the nonexistence of a diagonal limit on the boundary-retained side; it does, however, remove the small-scale precision figure from the list of admissible evidence that the boundary-retained model approximates  $\zeta$  systematically.

We make explicit the precise logical scope of this observation. The  $N$ -instability documented in Table 8 does *not* prove that the boundary-retained model fails the diagonal limit problem of Subsection 7.2. The diagonal limit only requires the existence of *some* sequence  $(\lambda_j, N_j) \rightarrow (\infty, \infty)$  along which a suitable renormalization  $a_j \widehat{\xi}_{\lambda_j, N_j}^W$  of the boundary-retained ground-state Mellin transform converges locally uniformly on compact subsets of  $S_{1/2}$  to a nonzero scalar multiple of  $\Xi$ ; both the renormalizing scalars  $a_j \in \mathbb{C}^\times$  and the relation  $N_j = N_j(\lambda_j)$  remain at our disposal in this formulation. A 66-order swing

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<sup>4</sup>The status of these two hypotheses on the boundary-retained side is acknowledged in [7] itself, where the convergence of the constructed spectra to the nontrivial zeros of  $\zeta$  as  $N, \lambda \rightarrow \infty$  is supported by numerical evidence and a rigorous proof has yet to be obtained.

of the MAE under a uniform-ratio choice  $N_j/L_j \equiv \text{const}$  at fixed  $\lambda^2 = 50$  does not exclude the existence of a non-uniform diagonal sequence on which the boundary-retained Mellin transforms  $a_j \widehat{\xi}_{\lambda_j, N_j}^W$  nevertheless approach  $c\Xi$ ; MAE-instability of a discrete zero-matching statistic is in principle compatible with locally uniform convergence of the underlying entire functions, since the latter allows zeros to migrate continuously while the former does not.

The instability does, however, raise the analytical bar on any boundary-retained completion of Subsection 7.2. Such a completion would have to:

- (1) rest on a diagonal sequence chosen explicitly to control the  $N$ -fluctuations recorded in Table 8, rather than on the uniform-ratio  $N/L$  used elsewhere in the present numerics;
- (2) be accompanied by a quantitative perturbation estimate, of Davis–Kahan or analogous type, uniform along that diagonal sequence, controlling the displacement of the boundary-retained ground state under  $N$ - and  $\lambda$ -refinement;
- (3) supply an analytic verification of the simplicity-and-evenness hypotheses of [8, Theorem 6.1] on the boundary-retained side – the verification absent from [7] itself, as recalled above.

None of (1)–(3) is supplied by the present numerical evidence, in either model. Their analytic counterparts on the boundary-free side are established in Sections 3–5 of the present paper; this is the primary structural asymmetry between the two truncations relevant to Subsection 7.2, and the sole basis on which we formulate the diagonal limit problem on the boundary-free side.

*The screening role of the boundary-free sweep.* The five-point geometric sweep of Table 3, the four-point  $N$ -convergence sweep of Table 5 at  $\lambda^2 = 50$ , and the per-row monotonicity diagnostics of Table 6 document, within the tested range, the absence of three pathological signatures:

- (1) no extreme dispersion in the per-row error distribution ( $\text{MAE}/\text{GM} \in [1.18, 1.79]$  across all five sweep points, against  $\geq 10^4$  at the smaller boundary-retained points);
- (2) no  $N$ -instability comparable to that of the boundary-retained variant: the boundary-free MAE is stable to relative change  $\leq 10^{-3}$  across  $N \in [60, 120]$  at  $\lambda^2 = 50$ ;
- (3) no statistically significant rank correlation between the absolute error  $e_k$  and the index  $k$  of the matched zero ( $|\beta| < 0.05$  and Spearman  $|\rho| < 0.22$  across the sweep).

We emphasize the precise scope of this evidence. It is a screening for the absence, within the tested range, of the very pathologies that disqualify the boundary-retained model. It is *not* evidence for the existence of any  $\lambda \rightarrow \infty$  limit; finite data cannot, in principle, decide an asymptotic question. The MAE

trends documented in Table 4, including the strict monotonicity of the worst-case statistic ( $\tau = -1$ ,  $R^2 = 0.98$  on the five-point log-log fit), are descriptive of the tested range only and are not extrapolated.

*Conclusion.* Of the two truncations considered, the boundary-free model is the only one for which (a) every analytical prerequisite of [8, Theorem 6.1] is verified, and (b) the screening for pathological signatures within the tested parameter range yields no falsifier of the kind documented in Table 8. We emphasize that (b) is the correct, screening-level reading of the present sweeps: it asserts the absence, on the tested grid, of the very pathologies that disqualify the boundary-retained model, and not the positive existence of any  $\lambda \rightarrow \infty$  limit. This is why Subsection 7.2 formulates the global Fourier-side limit problem on the boundary-free side. The existence and quantitative control of such a diagonal limit remains an analytical question, to be resolved by methods independent of the present parameter sweeps.

7.2. *A diagonal limit problem on the Fourier side.* The principal open problem suggested by the present work is the passage from the finite-dimensional real-zeros proxies of Theorem 5.16 to a global limiting object on the Fourier–Laplace side of the strip

$$S_{1/2} = \{z \in \mathbb{C} : |\Im z| < 1/2\}.$$

Concretely, one seeks a diagonal sequence  $(\lambda_j, N_j) \rightarrow (\infty, \infty)$  and nonzero scalars  $a_j \in \mathbb{C}^\times$  such that the entire functions

$$F_j(z) := a_j \widehat{\xi}_{\lambda_j, N_j}^D(z)$$

converge locally uniformly on compact subsets of  $S_{1/2}$  to a nonzero scalar multiple of  $\Xi$ .

By Theorem 5.16 (iv), each  $\widehat{\xi}_{\lambda_j, N_j}^D$  is an entire function whose zero set, counted with multiplicity, equals the spectrum of the self-adjoint operator  $D_{\log}^{(\lambda_j, N_j)}$  of Subsection 5.6 and therefore lies entirely on the real axis; hence each  $F_j$  has only real zeros. The nontrivial zeros of  $\Xi$  are non-real if and only if RH fails. If the locally uniform convergence above can be established rigorously, then Hurwitz’s theorem on limits of zero-free holomorphic functions (applied on each open half-strip  $S_{1/2} \cap \{\Im z > 0\}$  and  $S_{1/2} \cap \{\Im z < 0\}$ , where the limit  $c\Xi$  is nonzero except at its zeros) gives  $Z(\Xi) \cap S_{1/2} \subset \mathbb{R}$ , which is RH via the substitution  $s = \frac{1}{2} + iz$ .

The construction of such a diagonal sequence  $(\lambda_j, N_j)$  and of the renormalizing scalars  $a_j$ , together with quantitative control on the convergence of  $F_j$  to  $c\Xi$  on compact subsets of  $S_{1/2}$ , is left to subsequent work.

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### Data Availability Statement

All data and code that support the findings of Section 6 of this article are included as supplementary material with this submission, in the file `Numerical Experiments.zip`, which contains the complete Python source code (using `mpmath` for arbitrary-precision arithmetic) used to assemble the Galerkin matrices, compute the lowest Ritz vectors, evaluate the Fourier–Mellin transforms, match the resulting positive zeros to the nontrivial zeros of  $\zeta$ , and produce the statistics reported in Tables 1–8. The manuscript itself, together with the same supplementary archive, is also publicly deposited on Zenodo at <https://doi.org/10.5281/zenodo.19981531>. No third-party datasets were used. Any further information required to reproduce the numerical experiments is available from the author upon reasonable request.

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