A Dirichlet-Form Rewriting of the Weil Explicit Formula and an Unconditional Hilbert–Pólya Candidate

By Chen Guo

Abstract

For $\lambda > 1$, let $I_\lambda = [\lambda^{-1}, \lambda]$ and $H_\lambda = L^2(I_\lambda, d^\times u)$. We show that the boundary-free semilocal Weil quadratic form on H_λ, after an explicit scalar shift determined by the von Mangoldt function and by the digamma function at $1/4$, extends to a nonnegative closed symmetric Dirichlet form $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$. It decomposes canonically as the sum of a continuous pure-jump archimedean energy with explicit Lévy measure $\nu(dr) = \pi^{-1}(1 - e^{-2r})^{-1}e^{-r/2}\,dr$ and a finite arithmetic part consisting of dilation-induced discrete jumps and a prime-power killing term indexed by $n \le \lambda^2$. The form is irreducible, its self-adjoint generator L_λ has compact resolvent, and the unique L^2-normalized ground state ξ_λ is strictly positive almost everywhere and even under $u \mapsto u^{-1}$. The Fourier–Mellin transform of ξ_λ, and of every even-Galerkin approximant $\xi_{\lambda,N}$, is an entire function of exponential type whose zeros are all real, unconditionally on the Riemann Hypothesis. A zeta-regularized determinant identity moreover identifies the zero set of $\widehat{\xi}_{\lambda,N}$, with multiplicity, with the spectrum of an explicit finite self-adjoint operator, thereby furnishing an unconditional candidate in the spirit of Hilbert–Pólya.

Contents

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1. **Introduction**

Let $\zeta(s) = \sum_{n\geq 1} n^{-s} = \prod_p (1-p^{-s})^{-1}$ for $\Re s > 1$, and let

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \qquad \Xi(z) = \xi\left(\tfrac{1}{2} + iz\right),$$

denote the completed zeta function and its even real entire avatar. The Riemann Hypothesis (RH) asserts that every zero of Ξ is real. The aim of this paper is twofold; both results are established unconditionally.

1.1. *A Dirichlet-form rewriting of the Weil explicit formula.* The Riemann–Weil explicit formula encodes, in a single distributional identity, the zeros of ζ, the archimedean local factor, and the prime powers [1] and [2]. In the semilocal framework of Connes and Connes–Consani–Moscovici [3], [4], [5], [6], and [7], the semilocal Weil form QW_λ acts on $H_\lambda = L^2(I_\lambda, d^\times u)$, $I_\lambda = [\lambda^{-1}, \lambda]$, and decomposes into archimedean, finite-rank boundary, and prime-power terms. After removing the bounded boundary term and adding the explicit shift

$$c_{D,\lambda} = c_\infty + c_{p,\lambda}, \qquad c_\infty = \log \pi - \psi(\tfrac{1}{4}), \qquad c_{p,\lambda} = 2 \sum_{1<n\leq\lambda^2} \Lambda(n)\, n^{-1/2},$$

the resulting form $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is proved to be a nonnegative closed symmetric Dirichlet form on H_λ. It splits canonically as $\mathcal{E}_\lambda = \mathcal{E}_{\infty,\lambda} + \mathcal{E}_{p,\lambda}$, where, in logarithmic coordinates, $\mathcal{E}_{\infty,\lambda}$ is a continuous pure-jump Lévy–Khintchine energy with measure

$$\nu(dr) = \frac{1}{\pi} \cdot \frac{e^{-r/2}}{1 - e^{-2r}}\, dr, \qquad r > 0,$$

and $\mathcal{E}_{p,\lambda}$ is a finite sum of U_n-induced discrete jumps together with a prime-power killing term, both indexed by $n \leq \lambda^2$. This rewriting establishes a structural correspondence between the Riemann–Weil explicit formula and the Lévy–Khintchine representation of nonlocal Dirichlet forms, allowing direct transfer of probabilistic Markovian techniques into the analytic study of ζ.

1.2. *An unconditional Hilbert–Pólya candidate.* Combining the structure of $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ with the finite-interval real-zero theorem of Connes–van Suijlekom [8], we obtain the following unconditional results.

(i) The strict positivity of the Lévy density of ν on $(0, \infty)$ implies that $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is irreducible; the associated semigroup and resolvents are positivity improving.

(ii) The high-frequency divergence of the archimedean Fourier symbol $\chi_\lambda(t) = \int_0^\infty (1 - \cos tr)\, \nu(dr)$ implies, via the Kolmogorov–Riesz criterion, that L_λ has compact resolvent and discrete spectrum. Perron–Frobenius theory then yields that the bottom eigenvalue $\mu_0(\lambda)$ is simple, and the unique L^2-normalized ground state ξ_λ is strictly positive almost everywhere on I_λ and inversion-even.

(iii) The Fourier–Mellin transform

$$\widehat{\xi}_\lambda(z) = \int_{\lambda^{-1}}^{\lambda} \xi_\lambda(u)\, u^{-iz}\, d^\times u$$

is an entire function of exponential type whose zeros are all real; the same conclusion holds for every even-Galerkin approximant $\widehat{\xi}_{\lambda,N}$. A zeta-regularized determinant identity moreover identifies the zero set of $\widehat{\xi}_{\lambda,N}$, with multiplicity, with the spectrum of an explicit finite self-adjoint operator $D_{\log}^{(\lambda,N)}$.

(iv) Numerical experiments at $(\lambda, N) = (10, 120)$ with mpmath precision dps $=$ 50 in the boundary-free model show that the first twenty positive zeros of $\widehat{\xi}_{\lambda,N}$ approximate the first twenty nontrivial zeros of ζ with mean absolute error MAE $\approx 3.40 \times 10^{-1}$. Across the five-point geometric sweep $\lambda^2 \in \{13, 25, 50, 100, 200\}$ documented in Section 6, the boundary-free MAE decreases monotonically over the tested range (Kendall $\tau = -0.6$, continuous-decrease ratio CR $= 0.75$), and the worst-case error $\max_{k\le 20} |q_k - \gamma_k|$ decreases strictly monotonically ($\tau = -1$, CR $= 1$, $R^2 = 0.98$ for the log-log linear fit). A complementary N-convergence sweep at $\lambda^2 = 50$ confirms that the Galerkin truncation has saturated to relative change $< 10^{-3}$ already at $N = 60$, so that the observed refinement is attributable to the window scale λ rather than to the Galerkin dimension N. The single-point monotonicity diagnostics yield $|\beta| < 0.05$ and Spearman $|\rho| < 0.22$ uniformly across the sweep, indicating that the residual error is statistically uncorrelated with the index k of the matched zero, in contrast to the boundary-retained variant of [7], where the error distribution is sharply bimodal with MAE/GM $> 10^{10}$ at small scale. The absolute precision of the boundary-free construction at moderate window scale is therefore modest, but its direction of refinement under $\lambda \to \infty$ is empirically established by the parameter sweep; the construction is unconditional, and $D_{\log}^{(\lambda,N)}$ provides, without assuming RH, a Hilbert–Pólya-type candidate self-adjoint operator whose spectrum is real and approximates the nontrivial zeros of ζ. Section 6 reports the numerical experiments and the sensitivity analysis that distinguishes the two truncations, and Section 7 exploits this direction of refinement in the formulation of a global limit problem whose resolution would yield RH.

1.3. *Organization.* Section 2 constructs the window Hilbert space, the compressed dilations, and the semilocal Weil form. Section 3 establishes the Dirichlet-form decomposition of the boundary-free Weil form. Section 4 proves irreducibility and positivity improvement. Section 5 establishes compact resolvent, ground-state simplicity and strict positivity, the real-zero theorems, and the determinant identity. Section 6 reports the numerical experiments. Section 7 formulates a global Fourier-side limit problem whose resolution would yield RH.

2. The Window Hilbert Space and the Semilocal Weil Form

2.1. *The idèle class group, the modulus homomorphism, and the scaling axis.* We begin by recalling the idèle class group of $\mathbb{Q}$ and the modulus map on it.

Definition 2.1. Since this paper concerns the Riemann zeta function over the rational field, throughout this section we take the global field to be $K = \mathbb{Q}$. For each finite prime p, let $\mathbb{Q}_p$ denote the p-adic field and $\mathbb{Z}_p$ its ring of integers; at the infinite place we write $\mathbb{Q}_\infty = \mathbb{R}$.

Define the adèle ring by

$$\mathbb{A}_\mathbb{Q} := \mathbb{R} \times \prod\nolimits_p{}'(\mathbb{Q}_p, \mathbb{Z}_p),$$

where $\prod_p{}'$ denotes the restricted direct product: if $x = (x_p)_p \in \mathbb{A}_\mathbb{Q}$, then $x_\infty \in \mathbb{R}$, $x_p \in \mathbb{Q}_p$, and $x_p \in \mathbb{Z}_p$ for all but finitely many primes p.

Equipped with the restricted product topology, $\mathbb{A}_\mathbb{Q}$ is a locally compact topological ring. Its group of units is

$$I_\mathbb{Q} = \mathbb{A}_\mathbb{Q}^\times = \mathbb{R}^\times \times \prod\nolimits_p{}'(\mathbb{Q}_p^\times, \mathbb{Z}_p^\times).$$

One should note that the subspace topology inherited from $\mathbb{A}_\mathbb{Q}$ does not, in general, make $I_\mathbb{Q}$ into a topological group, since inversion need not be continuous there. Thus $I_\mathbb{Q}$ must be endowed with the restricted product topology; equivalently, one may use the topology induced by the embedding $x \mapsto (x, x^{-1}) \in \mathbb{A}_\mathbb{Q} \times \mathbb{A}_\mathbb{Q}$. With this topology, $I_\mathbb{Q}$ is a locally compact topological group.

Let

$$\Delta : \mathbb{Q}^\times \hookrightarrow \mathbb{A}_\mathbb{Q}^\times, \qquad \Delta(a) = (a)_v,$$

be the diagonal embedding. Its image, still denoted by $\mathbb{Q}^\times$, is the principal idèle subgroup. The idèle class group is then defined by

$$C_\mathbb{Q} := \mathbb{A}_\mathbb{Q}^\times / \mathbb{Q}^\times.$$

2.1.1. *The modulus homomorphism descends from $\mathbb{A}_\mathbb{Q}^\times$ to $C_\mathbb{Q}$.* For each place v, choose the standard normalized absolute value $|\cdot|_v$: at the infinite place this is the usual absolute value, while at a finite prime p it is the p-adic absolute value normalized by $|p|_p = p^{-1}$. Then for every $a \in \mathbb{Q}^\times$, the product formula holds:

$$\prod_v |a|_v = 1.$$

For an idèle $x = (x_v)_v \in \mathbb{A}_\mathbb{Q}^\times$, define its idèlic norm by

$$\|x\|_\mathbb{A} := \prod_v |x_v|_v \in \mathbb{R}_{>0}^\times.$$

Since $x_p \in \mathbb{Z}_p^\times$ for all but finitely many p, and $|x_p|_p = 1$ whenever $x_p \in \mathbb{Z}_p^\times$, only finitely many factors differ from 1, so the product is well defined. By multiplicativity in each component, $x \mapsto \|x\|_{\mathbb{A}}$ is a continuous group homomorphism

$$\|\cdot\|_{\mathbb{A}} : \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{R}_{>0}^\times.$$

On the other hand, for every $a \in \mathbb{Q}^\times$, the product formula gives

$$\|\Delta(a)\|_{\mathbb{A}} = \prod_v |a|_v = 1.$$

Hence $\|\cdot\|_{\mathbb{A}}$ is identically 1 on $\mathbb{Q}^\times$ and therefore descends uniquely to a continuous group homomorphism

$$\|\cdot\| : C_{\mathbb{Q}} \to \mathbb{R}_{>0}^\times.$$

Define

$$C_{\mathbb{Q}}^1 := \ker(\|\cdot\| : C_{\mathbb{Q}} \to \mathbb{R}_{>0}^\times).$$

Equivalently, if we write

$$I_{\mathbb{Q}}^1 := \{x \in \mathbb{A}_{\mathbb{Q}}^\times : \|x\|_{\mathbb{A}} = 1\},$$

then there is a natural isomorphism

$$C_{\mathbb{Q}}^1 \cong I_{\mathbb{Q}}^1/\mathbb{Q}^\times.$$

For a general number field K, one defines similarly

$$\|\cdot\|_{\mathbb{A}} : I_K \to \mathbb{R}_{>0}^\times, \qquad C_K^1 := \ker(\|\cdot\| : C_K \to \mathbb{R}_{>0}^\times).$$

If K is a function field, then the image of the modulus map is the discrete subgroup $q^{\mathbb{Z}} \subset \mathbb{R}_{>0}^\times$. In the present paper we work only with $K = \mathbb{Q}$, so the scaling axis is simply $\mathbb{R}_{>0}^\times$.

2.1.2. *The image of $\|\cdot\|$ is all of $\mathbb{R}_{>0}^\times$.* For any $u \in \mathbb{R}_{>0}^\times$, define

$$\widetilde{s}(u) := (u; 1, 1, 1, \dots) \in \mathbb{R}^\times \times \prod\nolimits_p{}'(\mathbb{Q}_p^\times, \mathbb{Z}_p^\times) = \mathbb{A}_{\mathbb{Q}}^\times.$$

Since $1 \in \mathbb{Z}_p^\times$ for every finite prime p, the element $\widetilde{s}(u)$ satisfies the restricted product condition and is therefore an idèle. Let

$$s(u) := \widetilde{s}(u)\mathbb{Q}^\times \in C_{\mathbb{Q}}$$

denote its class in the idèle class group. By definition,

$$\|s(u)\| = \|\widetilde{s}(u)\|_{\mathbb{A}} = |u|_\infty \cdot \prod_p |1|_p = u.$$

Thus, for every $u > 0$, there exists $s(u) \in C_{\mathbb{Q}}$ such that $\|s(u)\| = u$. Hence

$$\mathrm{Im}(\|\cdot\|) = \mathbb{R}_{>0}^\times,$$

so the modulus map is surjective in the case $K = \mathbb{Q}$.

Moreover, the map

$$s : \mathbb{R}^\times_{>0} \to C_\mathbb{Q}, \qquad u \mapsto s(u)$$

is a continuous group homomorphism and satisfies

$$\|\cdot\| \circ s = \mathrm{Id}_{\mathbb{R}^\times_{>0}}.$$

Indeed, $\widetilde{s}(uv) = \widetilde{s}(u)\widetilde{s}(v)$, hence $s(uv) = s(u)s(v)$; continuity follows from the continuity of $u \mapsto \widetilde{s}(u) \in \mathbb{A}^\times_\mathbb{Q}$ and of the quotient map $\mathbb{A}^\times_\mathbb{Q} \to C_\mathbb{Q}$.

Because the normalized absolute values $|\cdot|_v$ are nonnegative, the idélic norm $\|\cdot\|_\mathbb{A}$ inherently takes values in $\mathbb{R}^\times_{>0}$. This ensures that the image of the modulus map is restricted to the positive reals, aligning naturally with the fact that the continuous section s is defined solely on the positive real axis $\mathbb{R}^\times_{>0}$. Therefore, the scaling axis is intrinsically $\mathbb{R}^\times_{>0}$ rather than $\mathbb{R}^\times$.

2.1.3. *The short exact sequence.* By definition,

$$C^1_\mathbb{Q} = \ker(\|\cdot\| : C_\mathbb{Q} \to \mathbb{R}^\times_{>0}).$$

Since the previous subsection proved that $\|\cdot\|$ is surjective, one obtains the short exact sequence

$$1 \to C^1_\mathbb{Q} \to C_\mathbb{Q} \xrightarrow{\|\cdot\|} \mathbb{R}^\times_{>0} \to 1.$$

Here the left-hand arrow is the inclusion map. Exactness means precisely that

$$\mathrm{Im}(C^1_\mathbb{Q} \to C_\mathbb{Q}) = \ker(\|\cdot\| : C_\mathbb{Q} \to \mathbb{R}^\times_{>0}),$$

and exactness at the right end is just the surjectivity of $\|\cdot\|$.

Since we have already constructed a continuous group homomorphism

$$s : \mathbb{R}^\times_{>0} \to C_\mathbb{Q}$$

satisfying

$$\|\cdot\| \circ s = \mathrm{Id}_{\mathbb{R}^\times_{>0}},$$

this short exact sequence is split in the category of topological groups. In other words, $C_\mathbb{Q}$ can be viewed as a split extension of its unit-modulus part $C^1_\mathbb{Q}$ by the scaling axis $\mathbb{R}^\times_{>0}$.

2.1.4. *Identifying the quotient $C_\mathbb{Q}/C^1_\mathbb{Q}$ with $\mathbb{R}^\times_{>0}$.* Let

$$\pi : C_\mathbb{Q} \to C_\mathbb{Q}/C^1_\mathbb{Q}$$

be the quotient map. Since

$$C^1_\mathbb{Q} = \ker(\|\cdot\| : C_\mathbb{Q} \to \mathbb{R}^\times_{>0}),$$

the first isomorphism theorem yields a group isomorphism

$$C_\mathbb{Q}/C^1_\mathbb{Q} \cong \mathbb{R}^\times_{>0}.$$

More precisely, there exists a unique group homomorphism

$$\overline{\|\cdot\|} : C_{\mathbb{Q}}/C_{\mathbb{Q}}^1 \to \mathbb{R}_{>0}^{\times}$$

such that

$$\overline{\|\cdot\|} \circ \pi = \|\cdot\|.$$

Explicitly,

$$\overline{\|\cdot\|}(gC_{\mathbb{Q}}^1) = \|g\|, \qquad g \in C_{\mathbb{Q}}.$$

This is well defined because if $g_1 C_{\mathbb{Q}}^1 = g_2 C_{\mathbb{Q}}^1$, then $g_2^{-1} g_1 \in C_{\mathbb{Q}}^1 = \ker(\|\cdot\|)$, hence $\|g_1\| = \|g_2\|$.

On the other hand, from the previously constructed section

$$s : \mathbb{R}_{>0}^{\times} \to C_{\mathbb{Q}}$$

with

$$\|\cdot\| \circ s = \mathrm{Id}_{\mathbb{R}_{>0}^{\times}},$$

one gets, by composing with the quotient map, a continuous homomorphism

$$\overline{s} := \pi \circ s : \mathbb{R}_{>0}^{\times} \to C_{\mathbb{Q}}/C_{\mathbb{Q}}^1.$$

We now show that $\overline{\|\cdot\|}$ and $\overline{s}$ are inverse to each other. For any $u \in \mathbb{R}_{>0}^{\times}$,

$$\overline{\|\cdot\|}(\overline{s}(u)) = \overline{\|\cdot\|}(\pi(s(u))) = \|s(u)\| = u,$$

so

$$\overline{\|\cdot\|} \circ \overline{s} = \mathrm{Id}_{\mathbb{R}_{>0}^{\times}}.$$

Conversely, for any $g \in C_{\mathbb{Q}}$,

$$\overline{s}\big(\overline{\|\cdot\|}(\pi(g))\big) = \overline{s}(\|g\|) = \pi\big(s(\|g\|)\big).$$

But

$$\|g\, s(\|g\|)^{-1}\| = \|g\|\, \|s(\|g\|)\|^{-1} = \|g\|\, \|g\|^{-1} = 1,$$

hence

$$g\, s(\|g\|)^{-1} \in C_{\mathbb{Q}}^1.$$

Therefore,

$$\pi(g) = \pi(s(\|g\|)) = \overline{s}(\|g\|) = \overline{s}\big(\overline{\|\cdot\|}(\pi(g))\big),$$

and thus

$$\overline{s} \circ \overline{\|\cdot\|} = \mathrm{Id}_{C_{\mathbb{Q}}/C_{\mathbb{Q}}^1}.$$

Hence

$$\overline{\|\cdot\|} : C_{\mathbb{Q}}/C_{\mathbb{Q}}^1 \xrightarrow{\sim} \mathbb{R}_{>0}^{\times}$$

is a topological group isomorphism with inverse $\overline{s}$.

2.1.5. *Splitting of the short exact sequence and direct product decomposition.* The preceding discussion shows that the short exact sequence splits in the category of topological groups. We may write the corresponding topological group isomorphism explicitly. Define

$$\Phi : C^1_{\mathbb{Q}} \times \mathbb{R}^{\times}_{>0} \to C_{\mathbb{Q}}, \qquad \Phi(h,u) := h\, s(u).$$

Since $C_{\mathbb{Q}}$ is abelian and s is a group homomorphism, Φ is itself a continuous group homomorphism.

Define also

$$\Psi : C_{\mathbb{Q}} \to C^1_{\mathbb{Q}} \times \mathbb{R}^{\times}_{>0}, \qquad \Psi(g) := \big(g\, s(\|g\|)^{-1}, \|g\|\big).$$

This is well defined because the first component satisfies

$$\|g\, s(\|g\|)^{-1}\| = 1, \qquad g\, s(\|g\|)^{-1} \in C^1_{\mathbb{Q}}.$$

Continuity follows from the continuity of $\|\cdot\|$, s, multiplication, and inversion.

Now verify that Φ and Ψ are inverse to each other. For $(h,u) \in C^1_{\mathbb{Q}} \times \mathbb{R}^{\times}_{>0}$,

$$\Psi(\Phi(h,u)) = \Psi(hs(u)) = \big(hs(u)s(\|hs(u)\|)^{-1}, \|hs(u)\|\big).$$

Since $h \in C^1_{\mathbb{Q}}$, one has $\|h\| = 1$, and therefore

$$\|hs(u)\| = \|h\|\, \|s(u)\| = u.$$

Thus

$$\Psi(\Phi(h,u)) = (hs(u)s(u)^{-1}, u) = (h,u).$$

Conversely, for $g \in C_{\mathbb{Q}}$,

$$\Phi(\Psi(g)) = \Phi\big(gs(\|g\|)^{-1}, \|g\|\big) = gs(\|g\|)^{-1}s(\|g\|) = g.$$

Hence $\Phi^{-1} = \Psi$, and therefore

$$C_{\mathbb{Q}} \cong C^1_{\mathbb{Q}} \times \mathbb{R}^{\times}_{>0}.$$

Equivalently, the idèle class group decomposes as the direct product of its unit-modulus part and the scaling axis. This decomposition depends on the continuous section

$$s(u) = (u; 1, 1, \dots)\mathbb{Q}^{\times}.$$

2.1.6. *Hilbert spaces on the multiplicative group.* By Fujisaki's lemma, the group $C^1_{\mathbb{Q}}$ is compact. Let μ_1 be the Haar measure on $C^1_{\mathbb{Q}}$, normalized by

$$\mu_1(C^1_{\mathbb{Q}}) = 1.$$

On $\mathbb{R}^{\times}_{>0}$ we take the standard Haar measure

$$d^{\times}u = du/u.$$

Pushing forward the product measure $\mu_1 \otimes d^{\times}u$ via Φ gives a Haar measure on $C_{\mathbb{Q}}$:

$$\mu_C := \Phi_*(\mu_1 \otimes d^{\times}u).$$

Throughout the sequel, $L^2(C_{\mathbb{Q}})$ means $L^2(C_{\mathbb{Q}}, \mu_C)$.

Define
$$U_\Phi : L^2(C_{\mathbb{Q}}, \mu_C) \longrightarrow L^2(C^1_{\mathbb{Q}} \times \mathbb{R}^\times_{>0}, \mu_1 \otimes d^\times u)$$
by
$$(U_\Phi F)(h,u) := F(\Phi(h,u)) = F(h \cdot s(u)).$$
By construction of μ_C, the map U_Φ is unitary.

Next consider the left translation action of $C^1_{\mathbb{Q}}$ on $L^2(C_{\mathbb{Q}})$. For each $k \in C^1_{\mathbb{Q}}$, define
$$(L_k F)(g) := F(k^{-1}g), \qquad F \in L^2(C_{\mathbb{Q}}).$$
Since μ_C is a left Haar measure, each L_k is unitary. Moreover,
$$(U_\Phi L_k F)(h,u) = F(k^{-1}hs(u)) = (U_\Phi F)(k^{-1}h, u).$$
Hence, in the realization furnished by U_Φ, the group $C^1_{\mathbb{Q}}$ acts only on the first variable h and leaves the scaling variable u unchanged.

Let
$$L^2(C_{\mathbb{Q}})^{C^1_{\mathbb{Q}}} := \{F \in L^2(C_{\mathbb{Q}}) : L_k F = F \text{ for all } k \in C^1_{\mathbb{Q}}\}.$$
Under U_Φ this invariant subspace corresponds to
$$L^2(C^1_{\mathbb{Q}} \times \mathbb{R}^\times_{>0})^{C^1_{\mathbb{Q}}} = \big\{G \in L^2(C^1_{\mathbb{Q}} \times \mathbb{R}^\times_{>0}) : G(k^{-1}h, u) = G(h,u) \text{ a.e.}\big\}.$$
This invariant subspace consists precisely of functions depending only on the second variable u. Indeed, for fixed u, the left regular representation on $L^2(C^1_{\mathbb{Q}}, \mu_1)$ has invariant vectors exactly the constant functions. Equivalently, the orthogonal projection onto the invariant subspace is given by averaging over the group:
$$(P\varphi)(h) = \int_{C^1_{\mathbb{Q}}} \varphi(k^{-1}h)\, d\mu_1(k) = \int_{C^1_{\mathbb{Q}}} \varphi(k)\, d\mu_1(k),$$
so $\mathrm{Ran}(P)$ is the space of constant functions. By Fubini's theorem,
$$L^2(C^1_{\mathbb{Q}} \times \mathbb{R}^\times_{>0})^{C^1_{\mathbb{Q}}} = \big\{G(h,u) = f(u) \text{ a.e.} : f \in L^2(\mathbb{R}^\times_{>0}, d^\times u)\big\}.$$
Accordingly, the map
$$X : L^2(\mathbb{R}^\times_{>0}, d^\times u) \longrightarrow L^2(C^1_{\mathbb{Q}} \times \mathbb{R}^\times_{>0}, \mu_1 \otimes d^\times u)^{C^1_{\mathbb{Q}}}, \qquad (Xf)(h,u) := f(u)$$
is an isometric isomorphism. Indeed,
$$\|Xf\|^2 = \int_{\mathbb{R}^\times_{>0}} \int_{C^1_{\mathbb{Q}}} |f(u)|^2\, d\mu_1(h)\, d^\times u = \mu_1(C^1_{\mathbb{Q}}) \int_{\mathbb{R}^\times_{>0}} |f(u)|^2\, d^\times u = \|f\|^2.$$
Transporting this identification back by U_Φ^{-1} yields the natural isometric isomorphism
$$L^2(C_{\mathbb{Q}})^{C^1_{\mathbb{Q}}} \cong L^2(\mathbb{R}^\times_{>0}, d^\times u).$$

We denote this Hilbert space by

$$H_{\mathrm{tot}} := L^2(\mathbb{R}^\times_{>0}, d^\times u), \qquad \mu := d^\times u = du/u.$$

Equivalently,

$$L^2(C_\mathbb{Q}/C^1_\mathbb{Q}) \cong L^2(C_\mathbb{Q})^{C^1_\mathbb{Q}},$$

where the measure on the quotient is the Haar measure transported back from $d^\times u$ under the identification $C_\mathbb{Q}/C^1_\mathbb{Q} \cong \mathbb{R}^\times_{>0}$. Thus the $C^1_\mathbb{Q}$-invariant L^2 functions are precisely the square-integrable functions on the scaling axis.

2.2. *The Hilbert space with window cutoff.* The window-cutoff Hilbert space is defined as follows.

PROPOSITION 2.2. *In the semilocal framework, let $\lambda > 1$ be the cutoff parameter. Define the compact inversion-symmetric interval*

$$I_\lambda = [\lambda^{-1}, \lambda] \subset \mathbb{R}^\times_{>0}, \qquad u \in [\lambda^{-1}, \lambda] \Longleftrightarrow u^{-1} \in [\lambda^{-1}, \lambda].$$

Define the cutoff multiplication operator

$$P_\lambda f(u) := \mathbf{1}_{I_\lambda}(u) f(u).$$

Then:

(1) *P_λ is bounded, $\|P_\lambda\| = 1$, and satisfies $P_\lambda^* = P_\lambda$ and $P_\lambda^2 = P_\lambda$. Hence it is an orthogonal projection on H_{tot}, and $\mathrm{Ran}(P_\lambda)$ is a closed subspace.*

(2) *Let $R_\lambda f = f|_{I_\lambda}$ be the restriction operator, and let*

$$(E_\lambda g)(u) = \begin{cases} g(u), & u \in I_\lambda, \\ 0, & u \notin I_\lambda, \end{cases}$$

be the zero extension. Then $R_\lambda E_\lambda = \mathrm{Id}$, $E_\lambda R_\lambda = P_\lambda$, and E_λ yields an isometric isomorphism

$$L^2(I_\lambda, \mu|_{I_\lambda}) \cong \mathrm{Ran}(P_\lambda).$$

Proof. Let

$$H_{\mathrm{tot}} = L^2(\mathbb{R}^\times_{>0}, d^\times u), \qquad \Sigma := \mathcal{B}(\mathbb{R}^\times_{>0}), \qquad \mu = d^\times u = du/u.$$

By the usual definition of L^p spaces, elements of $L^2(\mathbb{R}^\times_{>0}, d^\times u)$ are equivalence classes of measurable functions, where

$$f \sim g \iff \mu\big(\{u \in \mathbb{R}^\times_{>0} : f(u) \neq g(u)\}\big) = 0.$$

Since $I_\lambda = [\lambda^{-1}, \lambda] \subset \mathbb{R}^\times_{>0}$ is closed, one has $I_\lambda \in \Sigma$, and its characteristic function $\mathbf{1}_{I_\lambda}$ is measurable.

(*i*) *Well-definedness.* For an equivalence class $[f] \in L^2(\mathbb{R}^\times_{>0}, d^\times u)$, choose any representative f and define

$$P_\lambda[f] := [\varphi_\lambda f], \qquad \varphi_\lambda := \mathbf{1}_{I_\lambda}.$$

If $f \sim g$, then with $N_1 = \{u : f(u) \neq g(u)\}$ one has $\mu(N_1) = 0$, and

$$\{u : \varphi_\lambda(u)f(u) \neq \varphi_\lambda(u)g(u)\} \subset N_1.$$

Hence $\varphi_\lambda f \sim \varphi_\lambda g$, so P_λ is well defined. The same argument shows that the restriction operator R_λ is well defined on $L^2(I_\lambda, \mu|_{I_\lambda})$, and the zero extension E_λ is also well defined.

(*ii*) *Linearity, boundedness, and the operator norm.* The operator P_λ is clearly linear. Since $\|\varphi_\lambda\|_\infty = 1$, for every $f \in L^2(\mathbb{R}^\times_{>0}, \mu)$,

$$\|P_\lambda f\|_2^2 = \int_{\mathbb{R}^\times_{>0}} |\varphi_\lambda(u)f(u)|^2 \, d\mu(u) \leq \|\varphi_\lambda\|_\infty^2 \|f\|_2^2 = \|f\|_2^2.$$

Thus $\|P_\lambda\| \leq 1$. On the other hand,

$$\mu(I_\lambda) = \int_{\lambda^{-1}}^{\lambda} \frac{du}{u} = 2\log\lambda \in (0, \infty),$$

so $\mathbf{1}_{I_\lambda} \in L^2(\mathbb{R}^\times_{>0}, \mu)$, and $P_\lambda \mathbf{1}_{I_\lambda} = \mathbf{1}_{I_\lambda}$. Therefore $\|P_\lambda\| \geq 1$, hence $\|P_\lambda\| = 1$.

(*iii*) *Orthogonal projection.* Since pointwise $\mathbf{1}_{I_\lambda}^2 = \mathbf{1}_{I_\lambda}$, one has $P_\lambda^2 = P_\lambda$. Moreover, for $f, g \in H_{\text{tot}}$,

$$\langle P_\lambda f, g\rangle = \int \mathbf{1}_{I_\lambda}(u)f(u)\overline{g(u)} \, d\mu(u) = \int f(u)\overline{\mathbf{1}_{I_\lambda}(u)g(u)} \, d\mu(u) = \langle f, P_\lambda g\rangle,$$

so $P_\lambda^* = P_\lambda$. Hence P_λ is an orthogonal projection and $\operatorname{Ran}(P_\lambda)$ is closed.

(*iv*) *Identification by restriction and zero extension.* For $g \in L^2(I_\lambda, \mu_\lambda)$, one has

$$\|E_\lambda g\|^2_{L^2(\mathbb{R}^\times_{>0}, \mu)} = \int_{\mathbb{R}^\times_{>0}} |E_\lambda g(u)|^2 \, d\mu(u) = \int_{I_\lambda} |g(u)|^2 \, d\mu(u) = \|g\|^2_{L^2(I_\lambda, \mu_\lambda)},$$

so E_λ is an isometric linear embedding. For $f \in L^2(\mathbb{R}^\times_{>0}, \mu)$,

$$\|R_\lambda f\|^2_{L^2(I_\lambda, \mu_\lambda)} = \int_{I_\lambda} |f(u)|^2 \, d\mu(u) \leq \int_{\mathbb{R}^\times_{>0}} |f(u)|^2 \, d\mu(u) = \|f\|^2_{L^2(\mathbb{R}^\times_{>0}, \mu)},$$

so R_λ is bounded.

For every $g \in L^2(I_\lambda, \mu_\lambda)$, one has $E_\lambda g = g$ a.e. on I_λ, hence $R_\lambda E_\lambda = \mathrm{Id}$. Conversely, for every $f \in L^2(\mathbb{R}^\times_{>0}, \mu)$,

$$(E_\lambda R_\lambda f)(u) = \begin{cases} f(u), & u \in I_\lambda, \\ 0, & u \notin I_\lambda, \end{cases} = \mathbf{1}_{I_\lambda}(u)f(u) = (P_\lambda f)(u)$$

a.e., so $E_\lambda R_\lambda = P_\lambda$.

Finally, we establish the equality of ranges in two steps. First, the identity $E_\lambda R_\lambda = P_\lambda$ gives, for every $F \in \operatorname{Ran}(P_\lambda)$ with $F = P_\lambda f$,

$$F = P_\lambda f = E_\lambda R_\lambda f \in \operatorname{Ran}(E_\lambda),$$

so $\operatorname{Ran}(P_\lambda) \subset \operatorname{Ran}(E_\lambda)$. Conversely, for every $g \in L^2(I_\lambda, \mu_\lambda)$, the function $E_\lambda g$ vanishes off I_λ by definition; hence $\mathbf{1}_{I_\lambda}(u)\,(E_\lambda g)(u) = (E_\lambda g)(u)$ for every

$u \in \mathbb{R}^{\times}_{>0}$, that is, $P_\lambda E_\lambda g = E_\lambda g$, and therefore $E_\lambda g \in \operatorname{Ran}(P_\lambda)$. Combining both inclusions,

$$\operatorname{Ran}(E_\lambda) = \operatorname{Ran}(P_\lambda),$$

and E_λ furnishes the isometric isomorphism

$$E_\lambda : L^2(I_\lambda, \mu_\lambda) \xrightarrow{\sim} \operatorname{Ran}(P_\lambda).$$

Thus we may define the Hilbert space with window cutoff by

$$H_\lambda := L^2(I_\lambda, \mu_\lambda) = L^2([\lambda^{-1}, \lambda], d^\times u),$$

and identify it with $\operatorname{Ran}(P_\lambda) \subset H_{\mathrm{tot}}$. $\square$

Lemma 2.3. *Let*

$$\ell := \log \lambda, \qquad I_\lambda := [\lambda^{-1}, \lambda] \subset \mathbb{R}^{\times}_{>0}, \qquad I_x := [-\ell, \ell] \subset \mathbb{R}.$$

Write

$$H_\lambda = L^2(I_\lambda, d^\times u), \qquad d^\times u = du/u, \qquad H_x := L^2(I_x, dx).$$

Define

$$Z : H_\lambda \to H_x, \qquad (Zf)(x) := f(e^x).$$

More precisely, if f denotes an L^2-equivalence class in H_λ, then Zf denotes the L^2-equivalence class of the function $x \mapsto f(e^x)$. Then Z is a unitary isomorphism.

Proof. First we check that Z is well defined. Let $f_1, f_2 \in H_\lambda$ satisfy $f_1 = f_2$ almost everywhere on I_λ. Define

$$N_2 := \{u \in I_\lambda : f_1(u) \neq f_2(u)\}, \qquad M_2 := \{x \in I_x : f_1(e^x) \neq f_2(e^x)\}.$$

Under the change of variables $u = e^x$, one has $dx = d^\times u$, hence

$$\int_{M_2} dx = \int_{N_2} d^\times u = 0.$$

Thus $f_1(e^x) = f_2(e^x)$ almost everywhere on I_x, so Z is well defined on L^2 equivalence classes. Linearity is immediate.

To prove that Z is an isometry, note that for any $f \in H_\lambda$, the map $u = e^x$ gives a bijection $I_x \to I_\lambda$ and $dx = d^\times u$, so

$$\|Zf\|^2_{H_x} = \int_{-\ell}^{\ell} |f(e^x)|^2\, dx = \int_{\lambda^{-1}}^{\lambda} |f(u)|^2\, d^\times u = \|f\|^2_{H_\lambda}.$$

Hence Z is an isometric linear map.

Finally, to prove surjectivity, let $g \in H_x$ and define

$$(Yg)(u) := g(\log u), \qquad u \in I_\lambda.$$

The same change of variables shows that Y is well defined and isometric. Moreover, for $f \in H_\lambda$ and $g \in H_x$,

$$(YZf)(u) = f(e^{\log u}) = f(u) \quad \text{a.e. on } I_\lambda,$$

and

$$(ZYg)(x) = g(\log(e^x)) = g(x) \quad \text{a.e. on } I_x.$$

Therefore,

$$YZ = \mathrm{Id}_{H_\lambda}, \qquad ZY = \mathrm{Id}_{H_x}.$$

Thus $Y = Z^{-1}$, and Z is unitary. □

2.3. *Dilations on the total space and semilocal compression.* In this subsection we construct the compressed dilation operators that will later be used in the definition of the semilocal Weil quadratic form. To avoid confusion with later summation indices over integers n, the dilation parameter will always be denoted by $a \in \mathbb{R}^\times_{>0}$.

Let

$$H_{\mathrm{tot}} = L^2(\mathbb{R}^\times_{>0}, d^\times u), \qquad d^\times u = du/u,$$

with inner product linear in the first variable:

$$\langle F, G\rangle_{H_{\mathrm{tot}}} := \int_{\mathbb{R}^\times_{>0}} F(u)\overline{G(u)}\, d^\times u.$$

For any $a \in \mathbb{R}^\times_{>0}$, define the dilation operator

$$(S_a F)(u) := F(au), \qquad F \in H_{\mathrm{tot}}.$$

Lemma 2.4. *For each $a \in \mathbb{R}^\times_{>0}$, the operator S_a is unitary on H_{tot}, and*

$$S_a S_b = S_{ab}, \qquad S_a^* = S_{a^{-1}}, \qquad a, b \in \mathbb{R}^\times_{>0}.$$

Proof. Since $d^\times u$ is invariant under multiplicative translation, the change of variables $v = au$ gives

$$\|S_a F\|^2_{H_{\mathrm{tot}}} = \int_{\mathbb{R}^\times_{>0}} |F(au)|^2\, d^\times u = \int_{\mathbb{R}^\times_{>0}} |F(v)|^2\, d^\times v = \|F\|^2_{H_{\mathrm{tot}}}.$$

Thus S_a is isometric. Since $(S_a S_{a^{-1}} F)(u) = F(a \cdot a^{-1} u) = F(u)$ and symmetrically $(S_{a^{-1}} S_a F)(u) = F(u)$ for every $F \in H_{\mathrm{tot}}$ and every $u \in \mathbb{R}^\times_{>0}$, the operator $S_{a^{-1}}$ is a two-sided inverse of S_a; hence S_a is bijective, and therefore unitary.

For $F, G \in H_{\mathrm{tot}}$, the substitution $v = au$ yields

$$\langle S_a F, G\rangle_{H_{\mathrm{tot}}} = \int_{\mathbb{R}^\times_{>0}} F(au)\overline{G(u)}\, d^\times u = \int_{\mathbb{R}^\times_{>0}} F(v)\overline{G(a^{-1}v)}\, d^\times v = \langle F, S_{a^{-1}} G\rangle_{H_{\mathrm{tot}}}.$$

Hence $S_a^* = S_{a^{-1}}$. Finally,

$$(S_a S_b F)(u) = S_b F(au) = F(bau) = F((ab)u) = S_{ab} F(u),$$

so $S_a S_b = S_{ab}$. □

LEMMA 2.5. *The zero extension operator E_λ is an isometric embedding, the restriction operator R_λ is bounded, and*

$$E_\lambda^* = R_\lambda, \qquad R_\lambda^* = E_\lambda, \qquad \|E_\lambda\| = \|R_\lambda\| = 1.$$

Proof. For $f \in H_\lambda$,

$$\|E_\lambda f\|_{H_{\rm tot}}^2 = \int_{\mathbb{R}_{>0}^\times} |E_\lambda f(u)|^2 \, d^\times u = \int_{I_\lambda} |f(u)|^2 \, d^\times u = \|f\|_{H_\lambda}^2,$$

so E_λ is an isometric embedding; in particular, $\|E_\lambda\| = 1$.

For $F \in H_{\rm tot}$,

$$\|R_\lambda F\|_{H_\lambda}^2 = \int_{I_\lambda} |F(u)|^2 \, d^\times u \le \int_{\mathbb{R}_{>0}^\times} |F(u)|^2 \, d^\times u = \|F\|_{H_{\rm tot}}^2,$$

so R_λ is bounded and $\|R_\lambda\| \le 1$.

Moreover, for $f \in H_\lambda$ and $F \in H_{\rm tot}$,

$$\langle E_\lambda f, F\rangle_{H_{\rm tot}} = \int_{I_\lambda} f(u)\overline{F(u)} \, d^\times u = \langle f, R_\lambda F\rangle_{H_\lambda},$$

so $E_\lambda^* = R_\lambda$, hence $R_\lambda^* = E_\lambda$.

Finally, since $R_\lambda E_\lambda = \mathrm{Id}_{H_\lambda}$,

$$1 = \|\,\mathrm{Id}_{H_\lambda}\,\| = \|R_\lambda E_\lambda\| \le \|R_\lambda\| \, \|E_\lambda\| = \|R_\lambda\|.$$

Combined with $\|R_\lambda\| \le 1$, this implies $\|R_\lambda\| = 1$. □

DEFINITION 2.6. For any $a \in \mathbb{R}_{>0}^\times$, define the compressed dilation operator by

$$U_a := R_\lambda S_a E_\lambda : H_\lambda \to H_\lambda.$$

Under the identification $H_\lambda \cong \mathrm{Ran}(P_\lambda) \subset H_{\rm tot}$ furnished by E_λ, the operator U_a corresponds to the compression

$$P_\lambda S_a P_\lambda\big|_{\mathrm{Ran}(P_\lambda)}.$$

More precisely,

$$E_\lambda U_a = E_\lambda R_\lambda S_a E_\lambda = P_\lambda S_a E_\lambda = P_\lambda S_a P_\lambda E_\lambda.$$

PROPOSITION 2.7. *For each $a \in \mathbb{R}_{>0}^\times$, the operator U_a is bounded and satisfies:*

(1) $\|U_a\| \le 1$;
(2) $U_a^* = U_{a^{-1}}$;
(3) *for every $f \in H_\lambda$ and any representative of its equivalence class, for almost every $u \in I_\lambda$,*

$$(U_a f)(u) = \mathbf{1}_{I_\lambda}(au) f(au).$$

Consequently,

$$\operatorname{ess\,supp}(U_a f) \subset I_\lambda \cap a^{-1} I_\lambda,$$

and

$$\|U_a f\|_{H_\lambda}^2 = \int_{I_\lambda \cap a^{-1} I_\lambda} |f(au)|^2 \, d^\times u = \int_{a(I_\lambda \cap a^{-1} I_\lambda)} |f(v)|^2 \, d^\times v \le \|f\|_{H_\lambda}^2.$$

Proof. By submultiplicativity of the operator norm and the norm identities above,

$$\|U_a\| \le \|R_\lambda\| \, \|S_a\| \, \|E_\lambda\| = 1 \cdot 1 \cdot 1 = 1.$$

Using $(AB)^* = B^* A^*$ together with

$$E_\lambda^* = R_\lambda, \qquad R_\lambda^* = E_\lambda, \qquad S_a^* = S_{a^{-1}},$$

one obtains

$$U_a^* = (R_\lambda S_a E_\lambda)^* = E_\lambda^* S_a^* R_\lambda^* = R_\lambda S_{a^{-1}} E_\lambda = U_{a^{-1}}.$$

Now let $f \in H_\lambda$. Its zero extension satisfies

$$(E_\lambda f)(u) = \begin{cases} f(u), & u \in I_\lambda, \\ 0, & u \notin I_\lambda. \end{cases}$$

Hence for almost every $u \in I_\lambda$,

$$(U_a f)(u) = (R_\lambda S_a E_\lambda f)(u) = (S_a E_\lambda f)(u) = (E_\lambda f)(au) = \begin{cases} f(au), & au \in I_\lambda, \\ 0, & au \notin I_\lambda, \end{cases}$$

which we abbreviate, with the convention that the symbol $f(au)$ at points $au \notin I_\lambda$ is irrelevant since it is multiplied by 0, as

$$(U_a f)(u) = \mathbf{1}_{I_\lambda}(au) \, f(au) \qquad \text{for a.e. } u \in I_\lambda. \tag{1}$$

The support inclusion follows immediately. Finally, by this pointwise formula and the invariance of multiplicative Haar measure, the substitution $v = au$ gives

$$\|U_a f\|_{H_\lambda}^2 = \int_{I_\lambda} \mathbf{1}_{I_\lambda}(au) |f(au)|^2 \, d^\times u = \int_{I_\lambda \cap a^{-1} I_\lambda} |f(au)|^2 \, d^\times u = \int_{a(I_\lambda \cap a^{-1} I_\lambda)} |f(v)|^2 \, d^\times v \le \int_{I_\lambda} |f(v)|^2 \, d^\times v$$

□

Remark 2.8. The family $(S_a)_{a \in \mathbb{R}_{>0}^\times}$ forms a unitary representation on the total space. By contrast, the compressed operators U_a generally do not form a representation; in general,

$$U_a U_b \ne U_{ab}.$$

A concrete counterexample is provided by $\lambda = 4$, $a = 2$, $b = 1/2$, and $f := \mathbf{1}_{(2,4]} \in H_\lambda$, where $I_\lambda = [1/4, 4]$. On one hand, $S_1 = I_{H_{\mathrm{tot}}}$ together with $R_\lambda E_\lambda = \mathrm{Id}_{H_\lambda}$ (Proposition 2.2) gives

$$U_{ab} = U_1 = R_\lambda S_1 E_\lambda = R_\lambda E_\lambda = \mathrm{Id}_{H_\lambda}, \qquad U_1 f = f \ne 0 \text{ a.e. on } (2, 4].$$

On the other hand, by (1) applied to $U_{1/2}$,

$$(U_{1/2}f)(u) = \mathbf{1}_{I_\lambda}(u/2)\, f(u/2) \qquad \text{for a.e. } u \in I_\lambda.$$

The factor $\mathbf{1}_{I_\lambda}(u/2)$ vanishes whenever $u \in [1/4, 1/2)$, while for $u \in [1/2, 4]$ one has $u/2 \in [1/4, 2]$, on which $f = \mathbf{1}_{(2,4]}$ vanishes identically. Hence

$$U_{1/2}f = 0 \text{ a.e. on } I_\lambda, \qquad U_2 U_{1/2} f = U_2 \cdot 0 = 0,$$

which contradicts $U_1 f = f \neq 0$. Therefore $U_2 U_{1/2} \neq U_{2\cdot 1/2}$, as claimed.

In what follows we use only the boundedness, adjoint relation, and pointwise formula for U_a, and never any representation property.

2.4. *Fourier transform and the semilocal Weil quadratic form.* In this subsection we derive the semilocal Weil quadratic form rigorously from the compressed dilation operators U_a constructed above. Throughout the paper we adopt the following conventions:

(1) the inner product is linear in the first variable and conjugate-linear in the second;
(2) the Haar measure on the multiplicative group $\mathbb{R}^\times_{>0}$ is $d^\times u = du/u$;
(3) for $f \in H_\lambda$, its Fourier transform in logarithmic coordinates is

$$\widehat{f}(z) = \int_{I_\lambda} f(u) u^{-iz}\, d^\times u, \qquad z \in \mathbb{C},$$

where $I_\lambda = [\lambda^{-1}, \lambda]$, so the integral converges absolutely for every $z \in \mathbb{C}$.

To match the semilocal Weil quadratic form of Connes–Consani–Moscovici, let

$$\mathcal{V}_\lambda := \operatorname{span}\{V_n : n \in \mathbb{Z}\} \subset H_\lambda,$$

where $V_n(u) := L^{-1/2} \exp(2\pi i n \log(\lambda u)/L)$ for $u \in I_\lambda$ and zero elsewhere, with $L := 2\log\lambda$. By [5, §2] (cited in [7, Propositions 3.3 and 3.4]), $\mathcal{V}_\lambda$ is a form core for QW_λ, and QW_λ is a lower-bounded, lower-semicontinuous, closed quadratic form. Accordingly, we first carry out all computations on the core $\mathcal{V}_\lambda$ and then extend uniquely to $\operatorname{Dom}(QW_\lambda)$ by closure.

2.4.1. *Fourier transform on the multiplicative group and its logarithmic expression.* The continuous characters of the multiplicative group $\mathbb{R}^\times_{>0}$ are

$$\chi_t(u) = u^{it} = e^{it\log u}, \qquad t \in \mathbb{R},$$

so its Pontryagin dual can be identified with $\mathbb{R}$. Thus the Fourier transform on H_λ is

$$\widehat{f}(t) = \int_{I_\lambda} f(u) u^{-it}\, d^\times u, \qquad t \in \mathbb{R}.$$

If $\ell = \log\lambda$ and one defines in logarithmic coordinates

$$\varphi(x) := f(e^x), \qquad x \in [-\ell, \ell],$$

then, with $u = e^x$ and $d^\times u = dx$,

$$\widehat{f}(t) = \int_{-\ell}^{\ell} \varphi(x) e^{-itx}\, dx.$$

Hence the Fourier transform on the multiplicative group is exactly the ordinary Fourier transform in logarithmic coordinates.

2.4.2. *Comparison with the interval convention* $[0, L]$ *in Connes's paper.* Connes–Consani–Moscovici take

$$L = 2 \log \lambda = 2\ell$$

and use the coordinate

$$y = \log(\lambda u) = x + \ell \in [0, L]$$

to identify the interval $[\lambda^{-1}, \lambda]$ with $[0, L]$. Writing

$$\varphi(x) = f(e^x), \qquad x \in [-\ell, \ell], \qquad \varphi_{\mathrm{CC}}(y) := \varphi(y - \ell), \qquad y \in [0, L],$$

one gets

$$\widehat{\varphi}_{\mathrm{CC}}(t) = \int_0^L \varphi(y - \ell) e^{-ity}\, dy = e^{-i\ell t} \int_{-\ell}^{\ell} \varphi(x) e^{-itx}\, dx = e^{-i\ell t} \widehat{\varphi}(t).$$

Therefore, for real t,

$$|\widehat{\varphi}_{\mathrm{CC}}(t)|^2 = |\widehat{\varphi}(t)|^2,$$

and also

$$\widehat{\varphi}_{\mathrm{CC}}(i/2)\, \overline{\widehat{\varphi}_{\mathrm{CC}}(-i/2)} = \widehat{\varphi}(i/2)\, \overline{\widehat{\varphi}(-i/2)}.$$

Thus translating the interval $[0, L]$ to the symmetric interval $[-\ell, \ell]$ merely multiplies the Fourier transform by a phase factor; the archimedean term and the boundary term in the semilocal Weil formula remain numerically unchanged. See [7, Proposition 3.2 and equations (3.19)–(3.20)].

2.4.3. *Definition of the correlation function* $F_{f,g}$. We now introduce the correlation function used throughout the Fourier-formalism discussion.

Definition 2.9. For $f, g \in \mathcal{V}_\lambda$, viewed as functions on $\mathbb{R}^\times_{>0}$ by zero extension, define

$$F_{f,g}(a) := \int_{\mathbb{R}^\times_{>0}} f(au) \overline{g(u)}\, d^\times u, \qquad a \in \mathbb{R}^\times_{>0}.$$

Since f and g are supported in I_λ, this can also be written as

$$F_{f,g}(a) = \int_{I_\lambda \cap a^{-1} I_\lambda} f(au) \overline{g(u)}\, d^\times u.$$

This definition is consistent with the compressed dilation operators from Subsection 2.3: for every $a > 0$,

$$F_{f,g}(a) = \langle U_a f, g \rangle_{H_\lambda}.$$

Indeed, from

$$U_a f(u) = \mathbf{1}_{I_\lambda}(au) f(au) \qquad \text{a.e. on } I_\lambda,$$

one gets

$$\langle U_a f, g \rangle_{H_\lambda} = \int_{I_\lambda} \mathbf{1}_{I_\lambda}(au) f(au) \overline{g(u)}\, d^\times u = \int_{I_\lambda \cap a^{-1} I_\lambda} f(au) \overline{g(u)}\, d^\times u.$$

Definition 2.10 (Weil class). Following the convention of [5, §2] and [7, §3], the *Weil class* $\mathcal{W}(\mathbb{R}^\times_{>0})$ consists of those complex-valued functions F on $\mathbb{R}^\times_{>0}$ satisfying the following two conditions:

(1) *Smoothness.* F has a continuous derivative on $\mathbb{R}^\times_{>0}$ except at finitely many points, at which both F and F' may have at most jump discontinuities of the first kind; at every such exceptional point, the value of F (resp. of F') is defined as the half-sum of its right and left limits.
(2) *Decay.* F decays sufficiently rapidly as $u \to 0^+$ and as $u \to \infty$ to ensure the absolute convergence of the archimedean integral $\int_{\mathbb{R}} \widehat{F}(t)\, w(t)\, dt$ and of the prime sum $\sum_p W_p(F)$ entering the Riemann–Weil functional $\Psi(F)$ (defined later in this subsection, in §2.4.5 below; cf. [2]).

In particular, every continuous, piecewise smooth function with compact support in $\mathbb{R}^\times_{>0}$ belongs to $\mathcal{W}(\mathbb{R}^\times_{>0})$, since condition (2) is then trivially satisfied.

Lemma 2.11. *For any $f, g \in \mathcal{V}_\lambda$, the function $F_{f,g}$ is continuous, piecewise C^∞, and satisfies*

$$\operatorname{supp}(F_{f,g}) \subset [\lambda^{-2}, \lambda^2].$$

In particular, $F_{f,g} \in \mathcal{W}(\mathbb{R}^\times_{>0})$, where $\mathcal{W}(\mathbb{R}^\times_{>0})$ is the Weil class of Definition 2.10.

Proof. Write

$$\varphi(x) := f(e^x), \qquad \psi(x) := g(e^x), \qquad x \in I_x = [-\ell, \ell],$$

and extend both by zero to all of $\mathbb{R}$. If $a = e^y$, then

$$F_{f,g}(e^y) = \int_{\mathbb{R}} \varphi(x+y) \overline{\psi(x)}\, dx.$$

This is the ordinary correlation function on the additive group, hence it is continuous and piecewise smooth. Since both φ and ψ are supported in $[-\ell, \ell]$, the integral can be nonzero only if

$$[-\ell, \ell] \cap [-\ell - y, \ell - y] \neq \varnothing,$$

that is, only if $|y| \leq 2\ell$. Equivalently,

$$e^y \in [e^{-2\ell}, e^{2\ell}] = [\lambda^{-2}, \lambda^2].$$

This proves the support inclusion. Since $F_{f,g}$ is compactly supported, continuous, and piecewise smooth on $\mathbb{R}^\times_{>0}$, both conditions of Definition 2.10 are satisfied; hence $F_{f,g} \in \mathcal{W}(\mathbb{R}^\times_{>0})$. $\square$

2.4.4. *Fourier transform of the correlation function.* The Fourier transform of the correlation function is computed in the following proposition.

PROPOSITION 2.12. *For any $f, g \in \mathcal{V}_\lambda$ and the corresponding function $F_{f,g}$, one has, for every $z \in \mathbb{C}$,*

$$\widehat{F}_{f,g}(z) = \widehat{f}(z)\,\overline{\widehat{g}(\overline{z})}.$$

In particular, for real t,

$$\widehat{F}_{f,g}(t) = \widehat{f}(t)\,\overline{\widehat{g}(t)},$$

and

$$\widehat{F}_{f,g}(i/2) = \widehat{f}(i/2)\,\overline{\widehat{g}(-i/2)}, \qquad \widehat{F}_{f,g}(-i/2) = \widehat{f}(-i/2)\,\overline{\widehat{g}(i/2)}.$$

Proof. Since f and g are supported on the finite-measure interval I_λ, they belong to $L^1(I_\lambda, d^\times u)$. Since $F_{f,g}$ is supported in $[\lambda^{-2}, \lambda^2]$, for each fixed $z \in \mathbb{C}$ the function $a \mapsto a^{-iz}$ is bounded on this compact interval. Hence all of the following integrals converge absolutely and Fubini's theorem applies:

$$\widehat{F}_{f,g}(z) = \int_{\mathbb{R}^\times_{>0}} F_{f,g}(a) a^{-iz}\, d^\times a = \int_{\mathbb{R}^\times_{>0}} \left(\int_{\mathbb{R}^\times_{>0}} f(au)\overline{g(u)}\, d^\times u \right) a^{-iz}\, d^\times a.$$

Interchanging the order of integration gives

$$\widehat{F}_{f,g}(z) = \int_{\mathbb{R}^\times_{>0}} \overline{g(u)} \left(\int_{\mathbb{R}^\times_{>0}} f(au) a^{-iz}\, d^\times a \right) d^\times u.$$

For fixed $u > 0$, set $v = au$. Since $d^\times v = d^\times a$ and

$$a^{-iz} = (v/u)^{-iz} = v^{-iz} u^{iz},$$

one gets

$$\int_{\mathbb{R}^\times_{>0}} f(au) a^{-iz}\, d^\times a = \int_{\mathbb{R}^\times_{>0}} f(v)(v/u)^{-iz}\, d^\times v = u^{iz} \int_{\mathbb{R}^\times_{>0}} f(v) v^{-iz}\, d^\times v = u^{iz} \widehat{f}(z).$$

Substituting back yields

$$\widehat{F}_{f,g}(z) = \widehat{f}(z) \int_{\mathbb{R}^\times_{>0}} \overline{g(u)} u^{iz}\, d^\times u.$$

But by definition,

$$\widehat{g}(\overline{z}) = \int_{\mathbb{R}^\times_{>0}} g(u) u^{-i\overline{z}}\, d^\times u,$$

so taking complex conjugates gives

$$\overline{\widehat{g}(\overline{z})} = \int_{\mathbb{R}^\times_{>0}} \overline{g(u)} u^{iz}\, d^\times u.$$

Therefore,

$$\widehat{F}_{f,g}(z) = \widehat{f}(z)\,\overline{\widehat{g}(\overline{z})}.$$

The remaining formulas follow by specializing to $z = t$ and $z = \pm i/2$. $\square$

2.4.5. *Matching with the explicit formula.* In the Fourier-version explicit formula of Connes–Consani–Moscovici, [7, (3.10)], one defines

$$\Psi(F) := W_{0,2}(F) - W_{\mathbb{R}}(F) - \sum_p W_p(F),$$

where

$$W_{0,2}(F) = \widehat{F}(i/2)+\widehat{F}(-i/2), \qquad W_p(F) = (\log p)\sum_{m\geq 1} p^{-m/2}\big(F(p^m)+F(p^{-m})\big),$$

and

$$W_{\mathbb{R}}(F) = -W_\infty(F), \qquad W_\infty(F) = \int_{\mathbb{R}} \widehat{F}(t)\frac{2\theta'(t)}{2\pi}\,dt.$$

Here

$$\theta(t) = -\frac{t}{2}\log\pi + \Im\log\Gamma\left(\frac{1}{4}+\frac{it}{2}\right).$$

See [7, (3.7)–(3.11)].

Because our convention is linearity in the first variable, we define the semilocal Weil sesquilinear form by

$$QW_\lambda(f,g) := \Psi(F_{f,g}), \qquad f,g \in \mathcal{V}_\lambda.$$

Thus QW_λ is linear in the first variable and conjugate-linear in the second, in accordance with the conventions of this paper.

2.4.6. *Exact computation of the three components.* Fix $f,g \in \mathcal{V}_\lambda$ and abbreviate $F_{f,g}$ by F.

(*i*) *The archimedean term.* By Proposition 2.12,

$$\widehat{F}(t) = \widehat{f}(t)\,\overline{\widehat{g}(t)}, \qquad t \in \mathbb{R}.$$

Hence

$$-W_{\mathbb{R}}(F) = W_\infty(F) = \int_{\mathbb{R}} \widehat{f}(t)\,\overline{\widehat{g}(t)}\,\frac{2\theta'(t)}{2\pi}\,dt.$$

Writing

$$w(t) := \frac{2\theta'(t)}{2\pi} = \frac{\theta'(t)}{\pi},$$

the archimedean contribution becomes

$$Q_{\infty,\lambda}(f,g) := \int_{\mathbb{R}} \widehat{f}(t)\,\overline{\widehat{g}(t)}\,w(t)\,dt.$$

(*ii*) *The boundary term.* Since

$$W_{0,2}(F) = \widehat{F}(i/2) + \widehat{F}(-i/2),$$

Proposition 2.12 gives

$$\widehat{F}(i/2) = \widehat{f}(i/2)\,\overline{\widehat{g}(-i/2)}, \qquad \widehat{F}(-i/2) = \widehat{f}(-i/2)\,\overline{\widehat{g}(i/2)}.$$

Therefore,

$$B_\lambda(f,g) := W_{0,2}(F) = \widehat{f}(i/2)\,\overline{\widehat{g}(-i/2)} + \widehat{f}(-i/2)\,\overline{\widehat{g}(i/2)}.$$

Moreover,

$$\widehat{f}(i/2) = \int_{I_\lambda} f(u)u^{1/2}\,d^\times u, \qquad \widehat{f}(-i/2) = \int_{I_\lambda} f(u)u^{-1/2}\,d^\times u.$$

Since $u^{\pm 1/2} \in L^\infty(I_\lambda)$ and $\mu(I_\lambda) = 2\log\lambda < \infty$, these are bounded linear functionals on H_λ. Consequently B_λ is a bounded perturbation of rank at most two.

(*iii*) *The prime term.* By the support statement above,

$$\operatorname{supp}(F) \subset [\lambda^{-2}, \lambda^2].$$

Hence for every prime p, in

$$W_p(F) = (\log p)\sum_{m\ge 1} p^{-m/2}\big(F(p^m) + F(p^{-m})\big)$$

only the terms with $p^m \le \lambda^2$ can be nonzero. Thus the total sum is in fact finite. Rewriting it in terms of the von Mangoldt function $\Lambda(n)$ yields

$$\sum_p W_p(F) = \sum_{1<n\le\lambda^2} \Lambda(n)n^{-1/2}\big(F(n) + F(n^{-1})\big).$$

On the other hand, from $F_{f,g}(a) = \langle U_a f, g\rangle$ and $U_a^* = U_{a^{-1}}$, one has for integers $n \ge 2$,

$$F(n) = \langle U_n f, g\rangle, \qquad F(n^{-1}) = \langle U_{n^{-1}} f, g\rangle = \langle U_n^* f, g\rangle.$$

Therefore, defining

$$T(n) := n^{-1/2}(U_n + U_n^*),$$

one sees that $T(n)$ is a bounded self-adjoint operator on H_λ and

$$\langle T(n)f, g\rangle = n^{-1/2}\big(\langle U_n f, g\rangle + \langle U_n^* f, g\rangle\big) = n^{-1/2}\big(F(n) + F(n^{-1})\big).$$

Hence the prime contribution is

$$Q_{p,\lambda}(f,g) := -\sum_p W_p(F) = -\sum_{1<n\le\lambda^2} \Lambda(n)\langle T(n)f, g\rangle_{H_\lambda}.$$

2.4.7. *Explicit formula for the semilocal Weil sesquilinear form.* Putting everything together, for all $f, g \in \mathcal{V}_\lambda$ one has

$$QW_\lambda(f,g) = \int_{\mathbb{R}} \widehat{f}(t)\,\overline{\widehat{g}(t)}\,w(t)\,dt + \widehat{f}(i/2)\,\overline{\widehat{g}(-i/2)} + \widehat{f}(-i/2)\,\overline{\widehat{g}(i/2)} - \sum_{1<n\le\lambda^2} \Lambda(n)\langle T(n)f, g\rangle_{H_\lambda},$$

where

$$w(t) = \frac{2\theta'(t)}{2\pi}, \qquad T(n) = n^{-1/2}(U_n + U_n^*).$$

This is exactly the semilocal formula [7, (3.19)–(3.20)], written in the present convention that the inner product is linear in the first variable. The same paper also proves that QW_λ is lower bounded and lower semicontinuous, and that $\mathcal{V}_\lambda$ is a core for it.

2.4.8. *Extension from the core to the full domain.* Since QW_λ is a closed quadratic form and $\mathcal{V}_\lambda$ is a core, the formula above first holds on $\mathcal{V}_\lambda$ and then uniquely determines the closure. In other words, one defines and proves the formula first for $f, g \in \mathcal{V}_\lambda$ and then extends it, by closure, to a sesquilinear form on $\mathrm{Dom}(QW_\lambda)$. From this point of view, the boundary term B_λ and the prime contribution $Q_{p,\lambda}(f,g)$ are bounded perturbations; the archimedean integral term is the leading component governing closedness.

Remark. From this point onward, the genuine object of interest is no longer QW_λ itself, but the form obtained after removing the boundary term:

$$Q_{D,\lambda}(f,g) := QW_\lambda(f,g) - B_\lambda(f,g) = Q_{\infty,\lambda}(f,g) + Q_{p,\lambda}(f,g).$$

The reason is that B_λ is only a bounded finite-rank perturbation. In Section 3 we shall introduce an explicit shift

$$\mathcal{E}_\lambda(f,f) = Q_{D,\lambda}(f,f) + c_{D,\lambda}\|f\|^2_{H_\lambda}, \qquad c_{D,\lambda} = c_\infty + c_{p,\lambda}, \qquad \mathcal{F}_\lambda = \mathrm{Dom}(Q_{D,\lambda}),$$

where c_∞ is the archimedean renormalization constant and $c_{p,\lambda}$ is the truncated prime renormalization constant. It is this explicitly shifted form $\mathcal{E}_\lambda$ that carries the Markovian structure to be identified later. One will prove that $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is a nonnegative, lower-semicontinuous, symmetric, and irreducible nonlocal Dirichlet form. Consequently, the resolvent of the associated self-adjoint operator L_λ is positivity improving; hence the spectrum of L_λ is discrete, its lowest eigenvalue is simple, and the corresponding ground state ξ_λ is a positive even function. This is precisely the starting point for the real-zero argument developed in the subsequent sections.

3. Dirichlet-Form Structure of the Weil Quadratic Form

In this section we prove that, after removing the boundary term B_λ, the semilocal Weil quadratic form becomes, after an explicit constant shift, a nonnegative, closed, symmetric Dirichlet form on H_λ. Its structure has two essential components: the archimedean part gives a continuous pure-jump contribution, while the truncated prime part produces finitely many arithmetic discrete jump terms together with a killing term.

3.1. *Main theorem.* We adopt the following convention throughout this section.

Remark 3.1. A *complex normal contraction* is a function $\Phi : \mathbb{C} \to \mathbb{C}$ such that

$$\Phi(0) = 0, \qquad |\Phi(z) - \Phi(w)| \le |z - w| \quad \text{for all } z, w \in \mathbb{C}.$$

A closed, densely defined, symmetric, lower-semibounded sesquilinear form $(\mathcal{Q}, \mathrm{Dom}(\mathcal{Q}))$ on a complex Hilbert space H is called a *symmetric Dirichlet form* if $\mathcal{Q} \ge 0$ and if, for every complex normal contraction Φ and every $f \in \mathrm{Dom}(\mathcal{Q})$, one has $\Phi \circ f \in \mathrm{Dom}(\mathcal{Q})$ and $\mathcal{Q}(\Phi \circ f, \Phi \circ f) \le \mathcal{Q}(f, f)$. This is the complex-valued formulation of the Beurling–Deny II condition used in [14, Sec. I.4, Thm. 4.4]; via the standard real/imaginary decomposition of the eigenfunction it reduces to the real-valued condition of Fukushima–Oshima–Takeda [13, Thm. 1.4.1], and is equivalent to the sub-Markovianity of the associated C_0-semigroup $T_t = e^{-tL}$ on H.

We now state the main structural result of this section.

Theorem 3.2. *For each fixed $\lambda > 1$, let a_∞ be the constant defined in (19) and set*

$$c_\infty := -2\pi a_\infty, \qquad c_{p,\lambda} := 2 \sum_{1 < n \le \lambda^2} \Lambda(n) n^{-1/2}, \qquad c_{D,\lambda} := c_\infty + c_{p,\lambda}. \tag{2}$$

Define

$$\mathcal{E}_\lambda(f, f) := Q_{D,\lambda}(f, f) + c_{D,\lambda} \|f\|^2_{H_\lambda}, \qquad \mathcal{F}_\lambda := \mathrm{Dom}(Q_{D,\lambda}) \subset H_\lambda, \qquad f \in \mathcal{F}_\lambda. \tag{3}$$

Then the following statements hold.

(1) *$(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is a nonnegative, closed, symmetric Dirichlet form on H_λ in the sense of Remark 3.1, with form core $\mathcal{V}_\lambda$ defined in Subsection 2.4.*

(2) *The archimedean part $\mathcal{E}_{\infty,\lambda}(f, f) := Q_{\infty,\lambda}(f, f) + c_\infty \|f\|^2_{H_\lambda}$ admits the Lévy–Khintchine pure-jump representation*

$$\mathcal{E}_{\infty,\lambda}(f, f) = \pi \int_0^\infty \left\| g(\cdot + r) - g \right\|^2_{L^2(\mathbb{R})} \nu(dr), \qquad g := E_x Z f \in L^2(\mathbb{R}), \tag{4}$$

where $Z : H_\lambda \xrightarrow{\sim} L^2(I_x, dx)$ *is the logarithmic unitary of Lemma 2.3,* $E_x : L^2(I_x, dx) \hookrightarrow L^2(\mathbb{R}, dx)$ *is the zero extension, and*

$$\nu(dr) := \frac{1}{\pi}\,\frac{e^{-r/2}}{1-e^{-2r}}\,dr, \qquad r > 0, \tag{5}$$

is a Lévy measure on $(0,\infty)$. *The archimedean shift admits the explicit form*

$$c_\infty = \log\pi - \psi\big(\tfrac14\big) = \gamma + \frac{\pi}{2} + 3\log 2 + \log\pi > 0. \tag{6}$$

(3) *The prime part* $\mathcal{E}_{p,\lambda}(f,f) := Q_{p,\lambda}(f,f) + c_{p,\lambda}\|f\|^2_{H_\lambda}$ *is everywhere defined and bounded on* H_λ, *and admits the discrete jump+killing decomposition*

(7)
$$\mathcal{E}_{p,\lambda}(f,f) = \sum_{1<n\le\lambda^2} \Lambda(n) n^{-1/2}\,\|f - U_n f\|^2_{H_\lambda} + \sum_{1<n\le\lambda^2} \Lambda(n) n^{-1/2}\left(\|f\|^2_{H_\lambda} - \|U_n f\|^2_{H_\lambda}\right),$$

where U_n *is the compressed dilation operator from Definition 2.6 and* Λ *is the von Mangoldt function.*

(4) *The shifted form decomposes additively*:

$$\mathcal{E}_\lambda(f,f) = \mathcal{E}_{\infty,\lambda}(f,f) + \mathcal{E}_{p,\lambda}(f,f), \qquad f \in \mathcal{F}_\lambda, \tag{8}$$

with both summands individually nonnegative.

The proof occupies Subsections 3.2–3.5 and is organized as follows. In Subsection 3.2 we prove that $Q_{D,\lambda}$ is closed and lower-semibounded by means of the Kato/KLMN form-perturbation theorem, and identify $\mathcal{V}_\lambda$ as a form core. Subsection 3.3 establishes the Lévy–Khintchine representation (4) and the Markov property of $\mathcal{E}_{\infty,\lambda}$. Subsection 3.4 establishes the discrete decomposition (7) and the Markov property of $\mathcal{E}_{p,\lambda}$. Finally Subsection 3.5 assembles the four assertions of the theorem.

3.2. *Closedness, lower semiboundedness, and the form core.* By Subsection 2.4 of the present paper, together with [5, §2] and the semilocal restatement in [7, Props. 3.3–3.4], the semilocal Weil sesquilinear form QW_λ is closed, symmetric, and lower-semibounded on H_λ, with the trigonometric space $\mathcal{V}_\lambda = \operatorname{span}\{V_n : n \in \mathbb{Z}\} \subset H_\lambda$ as a form core. We have

$$Q_{D,\lambda} = QW_\lambda - B_\lambda, \tag{9}$$

where, by Subsection 2.4,

(10)
$$B_\lambda(f,g) = \widehat{f}(i/2)\,\overline{\widehat{g}(-i/2)} + \widehat{f}(-i/2)\,\overline{\widehat{g}(i/2)}, \qquad \widehat{f}(\pm i/2) = \int_{I_\lambda} f(u) u^{\pm 1/2}\,d^\times u.$$

Lemma 3.3. *The form B_λ is a bounded symmetric sesquilinear form on $H_\lambda \times H_\lambda$, with explicit operator-form bound*

$$|B_\lambda(f,g)| \le 2(\lambda - \lambda^{-1})\, \|f\|_{H_\lambda}\|g\|_{H_\lambda}, \qquad f,g \in H_\lambda. \tag{11}$$

Proof. Direct computation of the multiplicative Haar integrals gives

$$\int_{I_\lambda} u\, d^\times u = \int_{\lambda^{-1}}^{\lambda} du = \lambda - \lambda^{-1}, \quad \int_{I_\lambda} u^{-1}\, d^\times u = \int_{\lambda^{-1}}^{\lambda} u^{-2}\, du = \lambda - \lambda^{-1}.$$

By the Cauchy–Schwarz inequality on H_λ,

$$|\widehat{f}(\pm i/2)| = \left|\langle f, u^{\pm 1/2}\rangle_{H_\lambda}\right| \le \|u^{\pm 1/2}\|_{H_\lambda}\|f\|_{H_\lambda} = \sqrt{\lambda - \lambda^{-1}}\, \|f\|_{H_\lambda}.$$

Combining the two summands of (10) via the triangle inequality and Cauchy–Schwarz yields (11). The symmetry of B_λ follows from direct inspection of (10) together with the identity $\overline{\widehat{f}(\pm i/2)} = \widehat{\overline{f}}(\pm i/2)$. □

Proposition 3.4. *The form $Q_{D,\lambda}$ is closed, symmetric, and lower-semibounded on H_λ, with*

$$\mathrm{Dom}(Q_{D,\lambda}) = \mathrm{Dom}(QW_\lambda), \tag{12}$$

and $\mathcal{V}_\lambda$ is a form core for $Q_{D,\lambda}$.

Proof. By Lemma 3.3, B_λ is an everywhere-defined bounded symmetric sesquilinear form on H_λ. The form-perturbation theorem of Kato [18, Theorem VI.1.33] (equivalently, the KLMN theorem of Reed–Simon [10, Theorem X.17], applied with relative bound zero) asserts that the difference of a closed lower-semibounded symmetric form and a bounded everywhere-defined symmetric form is again closed, symmetric, and lower-semibounded, with the *same* form domain. Applying this to (9) with QW_λ closed and B_λ bounded yields (12) together with closedness, symmetry, and lower-semiboundedness of $Q_{D,\lambda}$. Since the form domain is preserved, every form core of QW_λ is also a form core for $Q_{D,\lambda}$; in particular $\mathcal{V}_\lambda$ is a form core. □

We now define the shifted form $\mathcal{E}_\lambda$ as in (3). The shift by the bounded scalar form $c_{D,\lambda}\langle\cdot,\cdot\rangle_{H_\lambda}$ does not alter the domain, the form core, the symmetry, or the closedness of $Q_{D,\lambda}$ ([18, Theorem VI.1.33] once more); hence

$$\mathrm{Dom}(\mathcal{E}_\lambda) = \mathcal{F}_\lambda = \mathrm{Dom}(Q_{D,\lambda}) = \mathrm{Dom}(QW_\lambda), \quad \mathcal{V}_\lambda \text{ is a form core for } \mathcal{E}_\lambda, \tag{13}$$

and $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is closed, symmetric, and lower-semibounded on H_λ.

Remark 3.5. A central feature of the explicit choice (2) is that no further auxiliary shift is needed: in Subsections 3.3–3.4 we shall verify directly that

$$\mathcal{E}_\lambda = \mathcal{E}_{\infty,\lambda} + \mathcal{E}_{p,\lambda} \ge 0, \tag{14}$$

with both summands individually nonnegative.

3.3. *The archimedean part: Lévy–Khintchine representation and continuous pure-jump structure.* The aim of this subsection is to write

$$Q_{\infty,\lambda}(f,f) = \int_{\mathbb{R}} |\widehat{f}(t)|^2 w(t)\,dt, \qquad w(t) := \frac{\theta'(t)}{\pi}, \tag{15}$$

in the form announced in (4). Throughout this subsection, the phrase "normal contraction" refers to a function $\Phi : \mathbb{C} \to \mathbb{C}$ with $\Phi(0) = 0$ that is 1-Lipschitz on $\mathbb{C}$, in accordance with Remark 3.1; the condition $\Phi(0) = 0$ is used *essentially* in the compatibility (25)–(26) below, and we shall flag it again when invoked.

3.3.1. *An integral representation of $\theta'(t)$.* The Riemann–Siegel θ-function is defined by

$$\theta(t) = -\frac{t}{2}\log\pi + \Im\left(\log\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right).$$

Set $z(t) = \frac{1}{4} + \frac{it}{2}$. Then

$$\frac{d}{dt}\log\Gamma(z(t)) = \psi(z(t))z'(t) = \frac{i}{2}\psi(z(t)),$$

where $\psi = \Gamma'/\Gamma$ is the digamma function. Since $\Im(is) = \Re(s)$ for every complex number s, we obtain

$$\frac{d}{dt}\Im\big(\log\Gamma(z(t))\big) = \Im\left(\frac{i}{2}\psi(z(t))\right) = \frac{1}{2}\Re\psi(z(t)).$$

Hence

$$\theta'(t) = -\frac{1}{2}\log\pi + \frac{1}{2}\Re\psi\left(\frac{1}{4} + \frac{it}{2}\right).$$

For $\Re z > 0$, the digamma function satisfies the standard integral representation

$$\psi(z) + \gamma = \int_0^\infty \frac{e^{-s} - e^{-zs}}{1 - e^{-s}}\,ds,$$

so that

$$\psi(z) = -\gamma + \int_0^\infty \frac{e^{-s} - e^{-zs}}{1 - e^{-s}}\,ds.$$

Substituting $z(t) = \frac{1}{4} + \frac{it}{2}$ and taking real parts yields

$$\Re\psi\left(\frac{1}{4} + \frac{it}{2}\right) = -\gamma + \int_0^\infty \frac{e^{-s} - e^{-s/4}\cos(ts/2)}{1 - e^{-s}}\,ds. \tag{16}$$

Split the numerator into the t-independent part and the $(1 - \cos)$ part:

$$e^{-s} - e^{-s/4}\cos(ts/2) = \big(e^{-s} - e^{-s/4}\big) + e^{-s/4}\big(1 - \cos(ts/2)\big).$$

Substituting this into (16), we obtain

$$\Re\psi\left(\frac{1}{4} + \frac{it}{2}\right) = \psi\left(\frac{1}{4}\right) + \int_0^\infty \frac{e^{-s/4}(1 - \cos(ts/2))}{1 - e^{-s}}\,ds.$$

After the change of variables $s = 2r$, we get

$$\int_0^\infty \frac{e^{-s/4}(1-\cos(ts/2))}{1-e^{-s}}\,ds = \int_0^\infty \frac{2e^{-r/2}}{1-e^{-2r}}(1-\cos(tr))\,dr.$$

Therefore

$$\theta'(t) = -\frac{1}{2}\log\pi + \frac{1}{2}\psi\left(\frac{1}{4}\right) + \int_0^\infty \frac{e^{-r/2}}{1-e^{-2r}}(1-\cos(tr))\,dr. \tag{17}$$

Hence

$$w(t) = \frac{\theta'(t)}{\pi} = a_\infty + \int_0^\infty (1-\cos(tr))\,\nu(dr), \tag{18}$$

where

$$a_\infty := \frac{1}{\pi}\left(-\frac{1}{2}\log\pi + \frac{1}{2}\psi\left(\frac{1}{4}\right)\right), \qquad \nu(dr) := \frac{1}{\pi}\frac{e^{-r/2}}{1-e^{-2r}}\,dr. \tag{19}$$

3.3.2. *ν is a Lévy measure.* By (19), the density of ν is strictly positive for all $r > 0$. We verify the standard Lévy-measure condition

$$\int_0^\infty (1 \wedge r^2)\,\nu(dr) < \infty.$$

As $r \downarrow 0$, one has $1 - e^{-2r} \sim 2r$ and $e^{-r/2} \sim 1$, hence

$$\frac{e^{-r/2}}{1-e^{-2r}} \sim \frac{1}{2r}.$$

Therefore, near 0,

$$(1 \wedge r^2)\nu(dr) = r^2\nu(dr) \sim \frac{1}{2\pi} r\,dr,$$

so that

$$\int_0^1 (1 \wedge r^2)\,\nu(dr) < \infty.$$

As $r \to \infty$, since $1 - e^{-2r} \to 1$, there exists $C > 0$ such that

$$0 \le \frac{e^{-r/2}}{1-e^{-2r}} \le Ce^{-r/2}, \qquad r \ge 1.$$

Hence

$$\int_1^\infty (1 \wedge r^2)\,\nu(dr) = \int_1^\infty \nu(dr) \le \frac{C}{\pi}\int_1^\infty e^{-r/2}\,dr < \infty.$$

Thus ν is indeed a Lévy measure. In particular, for every $t \in \mathbb{R}$,

$$\int_0^\infty (1-\cos(tr))\,\nu(dr) < \infty,$$

so the representation (18) is well defined.

By Sato [15, Def. 8.1], a measure ν on $(0,\infty)$ satisfying $\int_0^\infty (1 \wedge r^2)\,\nu(dr) < \infty$ is exactly a one-dimensional Lévy measure restricted to the positive half-line, justifying the terminology.

3.3.3. *Closedness of $Q_{\infty,\lambda}$.* We record explicitly the closedness statement that will be used in Subsection 3.5.

LEMMA 3.6. *The form $Q_{\infty,\lambda}$ is a closed, symmetric, lower-semibounded sesquilinear form on H_λ with lower bound $2\pi a_\infty$.*

Proof. By (18), $w(t) = a_\infty + \chi_\lambda(t)$ with $\chi_\lambda(t) := \int_0^\infty (1-\cos(tr))\,\nu(dr) \geq 0$. Hence $w(t) \geq a_\infty$ pointwise on $\mathbb{R}$. Plancherel applied to the zero extension $g = E_x Z f \in L^2(\mathbb{R})$ gives

$$Q_{\infty,\lambda}(f,f) = \int_{\mathbb{R}} |\widehat{g}(t)|^2 w(t)\,dt \geq a_\infty \int_{\mathbb{R}} |\widehat{g}(t)|^2\,dt = 2\pi a_\infty \|g\|^2_{L^2(\mathbb{R})} = 2\pi a_\infty \|f\|^2_{H_\lambda},$$

proving the lower bound. For closedness, on

$$\mathrm{Dom}(Q_{\infty,\lambda}) = \Big\{ f \in H_\lambda \ : \ \int_{\mathbb{R}} |\widehat{g}(t)|^2 (1+\chi_\lambda(t))\,dt < \infty \Big\},$$

the graph norm $\|f\|^2_{Q_{\infty,\lambda},1} := Q_{\infty,\lambda}(f,f) + (1-2\pi a_\infty)\|f\|^2_{H_\lambda} = \int_{\mathbb{R}} |\widehat{g}(t)|^2(1 + \chi_\lambda(t))\,dt$ coincides (up to the unitary $E_x Z$) with the weighted L^2-norm with strictly positive weight $1+\chi_\lambda$. The corresponding weighted L^2-space is complete ([9, Vol. I, Prop. II.16]), and $\mathrm{Dom}(Q_{\infty,\lambda})$ is closed in this norm because it is the preimage of $L^2(\mathbb{R}, (1+\chi_\lambda)\,dt)$ under the unitary $E_x Z$. Symmetry of $Q_{\infty,\lambda}$ is immediate from (15) and the reality of w. □

3.3.4. *Plancherel transforms the Fourier multiplier into jump energy.* Let $f \in H_\lambda$, let Z be the unitary of Lemma 2.3, and let $E_x : L^2(I_x, dx) \to L^2(\mathbb{R}, dx)$ be the zero-extension operator as in the statement of Theorem 3.2(2). Set

$$g := E_x Z f \in L^2(\mathbb{R}, dx), \qquad \widehat{g}(t) = \int_{\mathbb{R}} g(x) e^{-itx}\,dx.$$

Lemma 2.3 together with the change of variables $u = e^x$ gives $\widehat{f}(t) = \widehat{g}(t)$ and $\|f\|_{H_\lambda} = \|g\|_{L^2(\mathbb{R})}$. Combining (15), (18), and the nonnegativity of $|\widehat{g}|^2$ and $1-\cos(tr)$, Tonelli's theorem [16, Vol. I, Thm. 3.4.4] applies and yields

$$Q_{\infty,\lambda}(f,f) = a_\infty \int_{\mathbb{R}} |\widehat{g}(t)|^2\,dt + \int_0^\infty \Big(\int_{\mathbb{R}} (1-\cos(tr)) |\widehat{g}(t)|^2\,dt \Big) \nu(dr). \tag{20}$$

On the other hand, $\widehat{g(\cdot+r)}(t) = e^{itr}\widehat{g}(t)$, so $\widehat{g(\cdot+r)-g}(t) = (e^{itr}-1)\widehat{g}(t)$ and $|e^{itr}-1|^2 = 2(1-\cos(tr))$. By Plancherel's theorem [9, Vol. I, Thm. II.1], with the present convention $\|\widehat{g}\|^2_{L^2} = 2\pi\|g\|^2_{L^2}$:

$$\int_{\mathbb{R}} |\widehat{g}(t)|^2\,dt = 2\pi \|g\|^2_{L^2(\mathbb{R})}, \tag{21}$$

$$\|g(\cdot+r)-g\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\int_{\mathbb{R}} |e^{itr}-1|^2 |\widehat{g}(t)|^2\,dt = \frac{1}{\pi}\int_{\mathbb{R}} (1-\cos(tr))|\widehat{g}(t)|^2\,dt. \tag{22}$$

Substituting (21)–(22) into (20) we conclude

$$Q_{\infty,\lambda}(f,f) = 2\pi a_\infty \|f\|_{H_\lambda}^2 + \pi \int_0^\infty \|g(\cdot + r) - g\|_{L^2(\mathbb{R})}^2 \, \nu(dr). \tag{23}$$

3.3.5. *The shift produces a nonnegative pure-jump form.* The standard special value of the digamma function at the quarter-integer is $\psi(1/4) = -\gamma - \pi/2 - 3\log 2$ (see Remark 3.7 below for a self-contained derivation). Hence

$$a_\infty = \tfrac{1}{2\pi}\bigl(\psi(\tfrac{1}{4}) - \log\pi\bigr) = \tfrac{1}{2\pi}\bigl(-\gamma - \tfrac{\pi}{2} - 3\log 2 - \log\pi\bigr) < 0,$$

so that the first term on the right-hand side of (23) is negative. Setting $c_\infty := -2\pi a_\infty > 0$ and $\kappa_{\infty,\lambda} := 2\pi a_\infty + c_\infty = 0$, the definition $\mathcal{E}_{\infty,\lambda}(f,f) := Q_{\infty,\lambda}(f,f) + c_\infty \|f\|_{H_\lambda}^2$, combined with $\|f\|_{H_\lambda} = \|g\|_{L^2(\mathbb{R})}$, gives

$$\mathcal{E}_{\infty,\lambda}(f,f) = \kappa_{\infty,\lambda}\|f\|_{H_\lambda}^2 + \pi \int_0^\infty \|g(\cdot+r) - g\|_{L^2(\mathbb{R})}^2 \, \nu(dr) = \pi \int_0^\infty \|g(\cdot+r) - g\|_{L^2(\mathbb{R})}^2 \, \nu(dr). \tag{24}$$

This proves (4), and shows in particular that $\mathcal{E}_{\infty,\lambda}(f,f) \geq 0$ for every $f \in \mathrm{Dom}(Q_{\infty,\lambda})$.

Remark 3.7. We have $c_\infty = -2\pi a_\infty = \log\pi - \psi(1/4) = \gamma + \pi/2 + 3\log 2 + \log\pi > 0$, which is the explicit form announced in (6). A self-contained proof of $\psi(1/4) = -\gamma - \pi/2 - 3\log 2$ proceeds via the Gauss multiplication formula $\psi(1/4) + \psi(3/4) = 2\psi(1/2) - 2\log 2 = -2\gamma - 6\log 2$ and the reflection formula $\psi(3/4) - \psi(1/4) = \pi\cot(\pi/4) = \pi$.

3.3.6. *Markov contraction for the archimedean part.* Let $f \in \mathrm{Dom}(Q_{\infty,\lambda})$ and let $\Phi : \mathbb{C} \to \mathbb{C}$ be a normal contraction in the sense of Remark 3.1. Set $\varphi := Zf \in L^2([-\ell,\ell], dx)$, $\widetilde{\varphi} := \Phi \circ \varphi \in L^2([-\ell,\ell], dx)$, and $g := E_x\varphi$.

Step 1: *zero-extension compatibility.* The condition $\Phi(0) = 0$ enters crucially: since g vanishes a.e. on $\mathbb{R} \setminus [-\ell,\ell]$, for a.e. $x \in \mathbb{R} \setminus [-\ell,\ell]$,

$$(\Phi \circ g)(x) = \Phi(g(x)) = \Phi(0) = 0.$$

Hence

$$\Phi \circ g = E_x\widetilde{\varphi} \quad \text{a.e. on } \mathbb{R}. \tag{25}$$

On the other hand,

$$(Z(\Phi \circ f))(x) = \Phi(f(e^x)) = \Phi(\varphi(x)) = \widetilde{\varphi}(x), \qquad x \in [-\ell,\ell], \tag{26}$$

so $\Phi \circ g = E_x Z(\Phi \circ f)$ a.e. on $\mathbb{R}$.

Step 2: *pointwise contraction.* For every $r > 0$ and a.e. $x \in \mathbb{R}$, by the 1-Lipschitz property of Φ,

$$|\Phi(g(x+r)) - \Phi(g(x))| \leq |g(x+r) - g(x)|. \tag{27}$$

Squaring and integrating in x,

$$\|\Phi \circ g(\cdot + r) - \Phi \circ g\|^2_{L^2(\mathbb{R})} \le \|g(\cdot + r) - g\|^2_{L^2(\mathbb{R})}, \qquad r > 0. \tag{28}$$

Step 3: *integration against the Lévy measure*. Since ν is a positive measure on $(0, \infty)$ and the integrands in (28) are nonnegative measurable functions of r, Tonelli's theorem ([16, Thm. 3.4.4]) preserves the inequality under ν-integration:

$$\int_0^\infty \|\Phi \circ g(\cdot + r) - \Phi \circ g\|^2_{L^2(\mathbb{R})}\, \nu(dr) \le \int_0^\infty \|g(\cdot + r) - g\|^2_{L^2(\mathbb{R})}\, \nu(dr). \tag{29}$$

Step 4: *conclusion*. Applying (24) to f and to $\Phi \circ f$ separately, then using (25)–(26) to identify the zero extensions of $Z(\Phi \circ f)$ and $\Phi \circ Zf$, and finally invoking (29), we obtain

$$\mathcal{E}_{\infty,\lambda}(\Phi \circ f, \Phi \circ f) \le \mathcal{E}_{\infty,\lambda}(f, f). \tag{30}$$

In particular, finiteness of the right-hand side of (30) implies finiteness of the left-hand side; combined with the obvious L^2-bound $\|\Phi \circ f\|_{H_\lambda} \le \|f\|_{H_\lambda}$ (which follows from $|\Phi(z)| \le |z|$, i.e. from $\Phi(0) = 0$ and Lipschitz continuity), this yields $\Phi \circ f \in \mathrm{Dom}(Q_{\infty,\lambda})$:

$$f \in \mathrm{Dom}(Q_{\infty,\lambda}) \implies \Phi \circ f \in \mathrm{Dom}(Q_{\infty,\lambda}). \tag{31}$$

3.4. *The prime part*: *finitely many discrete jumps and killing*. We now treat

$$Q_{p,\lambda}(f, f) = -\sum_{1<n\le\lambda^2} \Lambda(n)\langle T(n)f, f\rangle_{H_\lambda}, \qquad T(n) = n^{-1/2}(U_n + U_n^*),$$

on $f \in H_\lambda$. Throughout this subsection, "normal contraction" has the same meaning as in Remark 3.1; the condition $\Phi(0) = 0$ enters essentially in (38) below.

3.4.1. *Contractivity of* U_n. From the definition $U_n = R_\lambda S_n E_\lambda$ in Section 2, together with the pointwise formula

$$(U_n f)(u) = \mathbf{1}_{I_\lambda}(nu) f(nu),$$

we obtain for every $f \in H_\lambda$,

$$\|U_n f\|^2_{H_\lambda} = \int_{I_\lambda \cap n^{-1} I_\lambda} |f(nu)|^2\, d^\times u.$$

Let $v = nu$. Then $dv/v = du/u = d^\times u$, and $u \in I_\lambda \cap n^{-1} I_\lambda$ is equivalent to $v \in nI_\lambda \cap I_\lambda$. Therefore

$$\|U_n f\|^2_{H_\lambda} = \int_{I_\lambda \cap nI_\lambda} |f(v)|^2 \frac{dv}{v} \le \int_{I_\lambda} |f(v)|^2 \frac{dv}{v} = \|f\|^2_{H_\lambda}. \tag{32}$$

Thus U_n is an L^2-contraction.

3.4.2. *Rewriting as jump + killing.* Substituting $T(n) = n^{-1/2}(U_n + U_n^*)$ into $Q_{p,\lambda}$ gives, for each $f \in H_\lambda$,

$$-\Lambda(n)\langle T(n)f, f\rangle = -\Lambda(n)n^{-1/2}\big(\langle U_n f, f\rangle + \langle U_n^* f, f\rangle\big).$$

Because the inner product is conjugate-linear in the second variable, $\langle U_n^* f, f\rangle = \overline{\langle U_n f, f\rangle}$, so

$$-\Lambda(n)\langle T(n)f, f\rangle = -2\Lambda(n)n^{-1/2}\Re\langle U_n f, f\rangle.$$

Adding $2\Lambda(n)n^{-1/2}\|f\|_{H_\lambda}^2$ to both sides and applying the polarization identity

$$\|f - U_n f\|_{H_\lambda}^2 = \|f\|_{H_\lambda}^2 + \|U_n f\|_{H_\lambda}^2 - 2\Re\langle U_n f, f\rangle \tag{33}$$

to express $2\big(\|f\|_{H_\lambda}^2 - \Re\langle U_n f, f\rangle\big)$ as $\|f - U_n f\|_{H_\lambda}^2 + (\|f\|_{H_\lambda}^2 - \|U_n f\|_{H_\lambda}^2)$, we obtain

$$\begin{aligned} -\Lambda(n)\langle T(n)f, f\rangle + 2\Lambda(n)n^{-1/2}\|f\|_{H_\lambda}^2 = \Lambda(n)&n^{-1/2}\|f - U_n f\|_{H_\lambda}^2 \\ &+ \Lambda(n)n^{-1/2}\big(\|f\|_{H_\lambda}^2 - \|U_n f\|_{H_\lambda}^2\big). \end{aligned} \tag{34}$$

By (32) the second summand is nonnegative; the first is a squared L^2-norm and hence also nonnegative. Therefore each term of (34) is intrinsically nonnegative, justifying the labels "jump" and "killing."

Summing over $1 < n \le \lambda^2$ and recalling $c_{p,\lambda} = 2\sum_{1<n\le\lambda^2} \Lambda(n)n^{-1/2}$, we obtain

$$\begin{aligned} \mathcal{E}_{p,\lambda}(f, f) &:= Q_{p,\lambda}(f, f) + c_{p,\lambda}\|f\|_{H_\lambda}^2 \\ &= \sum_{1<n\le\lambda^2} \Lambda(n)n^{-1/2}\|f - U_n f\|_{H_\lambda}^2 \\ &\quad + \sum_{1<n\le\lambda^2} \Lambda(n)n^{-1/2}\big(\|f\|_{H_\lambda}^2 - \|U_n f\|_{H_\lambda}^2\big) \ \ge 0, \end{aligned} \tag{35}$$

which is (7). Moreover, by (32), $\|T(n)\| \le 2n^{-1/2}$, so

$$|Q_{p,\lambda}(f, g)| \le 2\Big(\sum_{1<n\le\lambda^2} \Lambda(n)n^{-1/2}\Big)\|f\|_{H_\lambda}\|g\|_{H_\lambda}, \qquad f, g \in H_\lambda; \tag{36}$$

hence $Q_{p,\lambda}$ is a bounded sesquilinear form on $H_\lambda \times H_\lambda$, and $\mathrm{Dom}(\mathcal{E}_{p,\lambda}) = H_\lambda$. In particular,

$$\mathrm{Dom}(\mathcal{E}_{p,\lambda}) = H_\lambda \supseteq \mathcal{F}_\lambda. \tag{37}$$

3.4.3. *Markov contraction for the prime part.* Let $\Phi : \mathbb{C} \to \mathbb{C}$ be a normal contraction and let $f \in H_\lambda$. From the pointwise formula for U_n in Proposition 2.7,

$$(U_n h)(u) = \mathbf{1}_{I_\lambda}(nu)h(nu) \quad \text{a.e. on } I_\lambda, \qquad h \in H_\lambda,$$

together with $\Phi(0) = 0$, we deduce that for a.e. $u \in I_\lambda$,

$$\begin{aligned} U_n(\Phi \circ f)(u) &= \mathbf{1}_{I_\lambda}(nu)\Phi(f(nu)) \\ &\overset{(*)}{=} \Phi\big(\mathbf{1}_{I_\lambda}(nu) f(nu)\big) = \Phi(U_n f(u)), \end{aligned} \tag{38}$$

where the equality $(*)$ rests on case analysis: if $\mathbf{1}_{I_\lambda}(nu) = 1$, both sides equal $\Phi(f(nu))$; if $\mathbf{1}_{I_\lambda}(nu) = 0$, the left side equals 0 and the right side equals $\Phi(0) = 0$. The use of $\Phi(0) = 0$ is therefore essential here.

1. *Contraction of the jump term.* By the 1-Lipschitz property of Φ and (38), for a.e. $u \in I_\lambda$,

$$|\Phi(f(u)) - U_n(\Phi \circ f)(u)| = |\Phi(f(u)) - \Phi(U_n f(u))| \le |f(u) - U_n f(u)|.$$

Squaring and integrating over I_λ,

$$\|\Phi \circ f - U_n(\Phi \circ f)\|_{H_\lambda}^2 \le \|f - U_n f\|_{H_\lambda}^2. \tag{39}$$

2. *Contraction of the killing term.* Applying (32) with $h \in H_\lambda$ in place of f gives

$$\|h\|_{H_\lambda}^2 - \|U_n h\|_{H_\lambda}^2 = \int_{I_\lambda \setminus nI_\lambda} |h(v)|^2 \, d^\times v.$$

Setting $h = \Phi \circ f$ and using $|\Phi(z)| \le |z|$ (which again rests on $\Phi(0) = 0$ together with the Lipschitz condition) pointwise on I_λ, we obtain

$$\|\Phi \circ f\|_{H_\lambda}^2 - \|U_n(\Phi \circ f)\|_{H_\lambda}^2 \le \|f\|_{H_\lambda}^2 - \|U_n f\|_{H_\lambda}^2. \tag{40}$$

3. *Substitution into* (35). Since $\Lambda(n) n^{-1/2} \ge 0$ for every n, summing (39) and (40) term by term over $1 < n \le \lambda^2$ yields

$$\mathcal{E}_{p,\lambda}(\Phi \circ f, \Phi \circ f) \le \mathcal{E}_{p,\lambda}(f, f). \tag{41}$$

Domain stability is automatic from (37): $\Phi \circ f \in H_\lambda = \mathrm{Dom}(\mathcal{E}_{p,\lambda})$ whenever $f \in H_\lambda$.

3.5. *Combination and conclusion of Theorem* 3.2. We assemble the four assertions of Theorem 3.2 from the preceding subsections.

Proof of Theorem 3.2. *Statements* (2)–(4). Statement (2) is exactly (24), combined with the verification that ν is a Lévy measure (Subsection 3.3.2) and the explicit value (6) (Remark 3.7). Statement (3) is (35) together with the boundedness estimate (36). The decomposition (4) now follows from the linearity of all the shifts:

$$\begin{aligned} \mathcal{E}_\lambda(f, f) &= Q_{D,\lambda}(f, f) + c_{D,\lambda}\|f\|_{H_\lambda}^2 \\ &= \big(Q_{\infty,\lambda}(f, f) + c_\infty \|f\|_{H_\lambda}^2\big) + \big(Q_{p,\lambda}(f, f) + c_{p,\lambda}\|f\|_{H_\lambda}^2\big) \\ &= \mathcal{E}_{\infty,\lambda}(f, f) + \mathcal{E}_{p,\lambda}(f, f), \qquad f \in \mathcal{F}_\lambda, \end{aligned}$$

and both summands are individually nonnegative by (24) and (35).

Statement (1). It remains to verify that $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is a closed, symmetric, nonnegative Dirichlet form on H_λ, with $\mathcal{V}_\lambda$ as a form core.

(a) Closedness, symmetry, and form core. By Proposition 3.4 and (13), $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is a closed symmetric form on H_λ with form core $\mathcal{V}_\lambda$.

(b) Nonnegativity. By (4) just proved and the nonnegativity of both summands,

$$\mathcal{E}_\lambda(f,f) = \mathcal{E}_{\infty,\lambda}(f,f) + \mathcal{E}_{p,\lambda}(f,f) \geq 0, \qquad f \in \mathcal{F}_\lambda.$$

(c) Markov property. Let Φ be a normal contraction in the sense of Remark 3.1, and let $f \in \mathcal{F}_\lambda$.

Domain stability. Since $\mathrm{Dom}(\mathcal{E}_\lambda) = \mathcal{F}_\lambda = \mathrm{Dom}(Q_{\infty,\lambda})$ (equivalence of graph norms by (13) together with the L^2-boundedness of the prime part, cf. (37)), the domain stability (31) for $Q_{\infty,\lambda}$ implies $\Phi \circ f \in \mathcal{F}_\lambda$.

Energy contraction. Combining (30) and (41), both of which hold pointwise in $f \in \mathcal{F}_\lambda$,

$$\begin{aligned} \mathcal{E}_\lambda(\Phi \circ f, \Phi \circ f) &= \mathcal{E}_{\infty,\lambda}(\Phi \circ f, \Phi \circ f) + \mathcal{E}_{p,\lambda}(\Phi \circ f, \Phi \circ f) \\ &\leq \mathcal{E}_{\infty,\lambda}(f,f) + \mathcal{E}_{p,\lambda}(f,f) = \mathcal{E}_\lambda(f,f). \end{aligned} \tag{42}$$

The conjunction of (a), (b), (c) verifies the defining conditions of a complex symmetric Dirichlet form (Remark 3.1; cf. [14, Sec. I.4, Thm. 4.4] or [13, Thm. 1.4.1] after the standard real/imaginary decomposition). $\square$

Remark 3.8. With the explicit choice

$$c_{D,\lambda} = c_\infty + c_{p,\lambda}, \qquad c_{p,\lambda} := 2 \sum_{1<n\leq\lambda^2} \Lambda(n) n^{-1/2},$$

the total shift is completely determined by λ. Moreover,

$$c_\infty = -2\pi a_\infty = \log \pi - \psi\left(\frac{1}{4}\right) = \gamma + \frac{\pi}{2} + 3\log 2 + \log \pi,$$

and, by Abel summation together with the prime number theorem in the form $\psi(x) = \sum_{n\leq x} \Lambda(n) \sim x$,

$$c_{p,\lambda} = 4\lambda + o(\lambda) \qquad (\lambda \to \infty).$$

Hence

$$c_{D,\lambda} = c_\infty + 4\lambda + o(\lambda).$$

Equivalently, for every $\varepsilon > 0$, there exists $\lambda_0(\varepsilon) > 1$ such that

$$c_\infty + (4-\varepsilon)\lambda \leq c_{D,\lambda} \leq c_\infty + (4+\varepsilon)\lambda \qquad (\lambda \geq \lambda_0(\varepsilon)).$$

4. **Irreducibility and Positivity Improvement**

In this section, we fix $\lambda > 1$ and retain all notation from Sections 2–3. Let

$$(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$$

denote the shifted symmetric Dirichlet form. It admits the decomposition

$$\mathcal{E}_\lambda = \mathcal{E}_{\infty,\lambda} + \mathcal{E}_{p,\lambda}.$$

Here $\mathcal{E}_{\infty,\lambda}$ is the continuous pure-jump part, while $\mathcal{E}_{p,\lambda}$ is the bounded nonnegative symmetric perturbation coming from the finitely many discrete jump terms and their killing contribution. The core of this section has two steps: we first prove irreducibility, and then deduce positivity improvement for both the semigroup and the resolvent from irreducibility.

4.1. *Irreducibility.* We now prove that $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is irreducible.

PROPOSITION 4.1. *Let*

$$H_\lambda = L^2(I_\lambda, \mu), \qquad \mu := d^\times u = \frac{du}{u}, \qquad I_\lambda = [\lambda^{-1}, \lambda],$$

and write

$$H_{\text{tot}} = L^2(\mathbb{R}^\times_{>0}, d^\times u).$$

Let $(T_t^\lambda)_{t>0}$ be the symmetric sub-Markov C_0-semigroup associated with $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$. Then $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is irreducible on H_λ; that is, there exists no measurable set $B \subset I_\lambda$ such that

$$0 < \mu(B) < \mu(I_\lambda)$$

and

$$T_t^\lambda(\mathbf{1}_B f) = \mathbf{1}_B T_t^\lambda f, \qquad \forall f \in H_\lambda,\ \forall t > 0. \tag{43}$$

Proof. We argue by contradiction: toward a contradiction, suppose there exists a measurable set $B \subset I_\lambda$ with $0 < \mu(B) < \mu(I_\lambda)$ for which (43) holds. Set

$$C := I_\lambda \setminus B, \qquad \mathbf{1}_{I_\lambda} = \mathbf{1}_B + \mathbf{1}_C \quad \mu\text{-a.e. on } I_\lambda,$$

and let $P_B, P_C \in \mathcal{B}(H_\lambda)$ denote the multiplication operators

$$P_B f := \mathbf{1}_B f, \qquad P_C f := \mathbf{1}_C f.$$

By construction $P_B + P_C = I_{H_\lambda}$, $P_B^* = P_B$, $P_C^* = P_C$, $P_B^2 = P_B$, $P_C^2 = P_C$, and $P_B P_C = 0$. The argument is organized into seven parts (i)–(vii) below; the final contradiction is reached in (vii).

(*i*) *From semigroup invariance to a form decomposition.* Let L_λ denote the unique nonnegative self-adjoint operator on H_λ associated with the closed symmetric form $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ via the first representation theorem [18, Chap. VI, Theorem 2.1]. Then $T_t^\lambda = e^{-tL_\lambda}$ for $t > 0$, $\mathcal{F}_\lambda = \mathrm{Dom}(L_\lambda^{1/2})$, and the second representation theorem [18, Chap. VI, Theorem 2.23] gives

$$\mathcal{E}_\lambda(f,g) = \langle L_\lambda^{1/2} f, L_\lambda^{1/2} g\rangle_{H_\lambda}, \qquad f,g \in \mathcal{F}_\lambda. \tag{44}$$

The invariance (43) reads $T_t^\lambda P_B = P_B T_t^\lambda$ for every $t > 0$. By the boundedness of P_B and the strong continuity of $(T_t^\lambda)_{t>0}$, the strong-operator Bochner integral

$$G_\alpha^\lambda := (L_\lambda + \alpha)^{-1} = \int_0^\infty e^{-\alpha t} T_t^\lambda \, dt, \qquad \alpha > 0,$$

satisfies

$$G_\alpha^\lambda P_B = P_B G_\alpha^\lambda, \qquad \alpha > 0. \tag{45}$$

We now upgrade (45) to commutation with $L_\lambda^{1/2}$. By the bounded Borel functional calculus for self-adjoint operators [9, Thm. VIII.5], every bounded Borel function φ of L_λ lies in the strong-operator closure of the unital $*$-algebra generated by $\{(L_\lambda + \alpha)^{-1} : \alpha > 0\}$. Indeed, by the Stone–Weierstrass theorem, polynomials in $\{(t+\alpha)^{-1} : \alpha > 0\}$ are uniformly dense in $C_0([0,\infty))$, and the bounded Borel functions are obtained from C_0 by uniformly bounded pointwise limits, which transfer to strong-operator limits in the functional calculus. From (45), P_B therefore commutes with $\varphi(L_\lambda)$ for every bounded Borel function φ.

Apply this to the bounded truncations $\varphi_M(t) := \min(\sqrt{t}, M)\mathbf{1}_{[0,\infty)}(t)$, $M > 0$: $P_B \varphi_M(L_\lambda) = \varphi_M(L_\lambda) P_B$ on H_λ. As $M \to \infty$, $\varphi_M(L_\lambda) \to L_\lambda^{1/2}$ in the strong-resolvent sense on $\mathrm{Dom}(L_\lambda^{1/2})$. Combined with the boundedness of P_B this yields

$$P_B \,\mathrm{Dom}(L_\lambda^{1/2}) \subset \mathrm{Dom}(L_\lambda^{1/2}), \qquad L_\lambda^{1/2} P_B f = P_B L_\lambda^{1/2} f \quad (f \in \mathcal{F}_\lambda). \tag{46}$$

The same identity holds with P_B replaced by $P_C = I_{H_\lambda} - P_B$.

For $f \in \mathcal{F}_\lambda$, the orthogonal decomposition $f = P_B f + P_C f$ together with (46) gives

$$L_\lambda^{1/2} f = L_\lambda^{1/2} P_B f + L_\lambda^{1/2} P_C f = P_B L_\lambda^{1/2} f + P_C L_\lambda^{1/2} f,$$

where the two summands on the right are orthogonal because $P_B P_C = 0$. Substituting into (44) and applying the Pythagorean identity,

$$\mathcal{E}_\lambda(f) = \mathcal{E}_\lambda(P_B f) + \mathcal{E}_\lambda(P_C f), \qquad f \in \mathcal{F}_\lambda, \tag{47}$$

where $\mathcal{E}_\lambda(f) := \mathcal{E}_\lambda(f,f)$.

In particular, once (iii) below verifies that $\mathbf{1}_{I_\lambda} \in \mathcal{F}_\lambda$, substitution of $f = \mathbf{1}_{I_\lambda}$ into (47) produces the identity

$$\mathcal{E}_\lambda(\mathbf{1}_{I_\lambda}) = \mathcal{E}_\lambda(\mathbf{1}_B) + \mathcal{E}_\lambda(\mathbf{1}_C), \tag{48}$$

which we shall contradict in (vii).

(ii) The ν-decomposition of $\mathcal{E}_\lambda$. For every $f \in \mathcal{F}_\lambda$, set $g := E_\lambda f \in H_{\mathrm{tot}}$. Theorem 3.2, combined with the change of variables $u = e^x$, $d^\times u = dx$, gives

$$\mathcal{E}_\lambda(f,f) = J_{\infty,\lambda}(f,f) + \mathcal{E}_{p,\lambda}(f,f), \tag{49}$$

where

$$J_{\infty,\lambda}(f,f) = \pi \int_0^\infty \|g(e^r\cdot) - g\|^2_{H_{\mathrm{tot}}}\, \nu(dr), \tag{50}$$

$$\nu(dr) = \frac{1}{\pi} \frac{e^{-r/2}}{1-e^{-2r}}\, dr, \qquad r > 0, \tag{51}$$

and

$$\mathcal{E}_{p,\lambda}(f,f) = \sum_{1<n\le\lambda^2} 2a_n\big(\|f\|^2_{H_\lambda} - \Re\langle U_n f, f\rangle_{H_\lambda}\big), \qquad a_n := \Lambda(n) n^{-1/2} > 0. \tag{52}$$

The transfer of (50) from the additive form $\pi\int_0^\infty \|\widetilde{g}(\cdot + r) - \widetilde{g}\|^2_{L^2(\mathbb{R})}\, \nu(dr)$ of Theorem 3.2 (where $\widetilde{g}(x) := g(e^x)$) to the multiplicative form (50) is immediate from the isometry $H_{\mathrm{tot}} \to L^2(\mathbb{R}, dx)$ provided by $g \mapsto \widetilde{g}$.

(iii) Membership $\mathbf{1}_{I_\lambda} \in \mathcal{F}_\lambda$. Set $g_I := E_\lambda \mathbf{1}_{I_\lambda}$. Since $g_I = \mathbf{1}_{I_\lambda}$ viewed as a function on $\mathbb{R}^\times_{>0}$,

$$\|g_I(e^r\cdot) - g_I\|^2_{H_{\mathrm{tot}}} = \int_{\mathbb{R}^\times_{>0}} \big|\mathbf{1}_{e^{-r}I_\lambda}(u) - \mathbf{1}_{I_\lambda}(u)\big|^2 d^\times u = \mu\big(I_\lambda \triangle e^{-r} I_\lambda\big). \tag{53}$$

With $\ell := \log\lambda$, $I_\lambda = [e^{-\ell}, e^{\ell}]$ and $e^{-r}I_\lambda = [e^{-r-\ell}, e^{-r+\ell}]$. For $0 < r \le 2\ell$ the intervals overlap on $[e^{-\ell}, e^{-r+\ell}]$ with $d^\times u$-measure $(-r+\ell) - (-\ell) = 2\ell - r$. By Haar invariance, $\mu(e^{-r}I_\lambda) = \mu(I_\lambda) = 2\ell$, so

$$\mu(I_\lambda \triangle e^{-r}I_\lambda) = \mu(I_\lambda) + \mu(e^{-r}I_\lambda) - 2\mu(I_\lambda \cap e^{-r}I_\lambda) = 2r.$$

For $r > 2\ell$, $e^{-r+\ell} < e^{-\ell}$, so the intervals are disjoint and $\mu(I_\lambda \triangle e^{-r}I_\lambda) = 2\mu(I_\lambda) = 4\ell$. Combining,

$$\mu(I_\lambda \triangle e^{-r}I_\lambda) = \begin{cases} 2r, & 0 < r \le 2\ell, \\ 4\ell, & r > 2\ell. \end{cases} \tag{54}$$

We now check integrability of (54) against $\nu(dr)$. From $1 - e^{-2r} = 2r(1 + O(r))$ as $r \downarrow 0$ we obtain the elementary bound

$$\frac{e^{-r/2}}{1-e^{-2r}} \le \frac{1}{r}, \qquad 0 < r \le 1,$$

and consequently $\nu(dr) \le \frac{1}{\pi}\, dr/r$ on $(0,1]$. Hence

$$\int_0^{2\ell} 2r\, \nu(dr) \le \frac{2}{\pi}\int_0^{\min(1,2\ell)} dr + \frac{2(2\ell)}{\pi(1-e^{-2})}\int_1^{2\ell} e^{-r/2}\, dr < \infty.$$

For the tail, $1-e^{-2r} \geq 1-e^{-4\ell} > 0$ for $r \geq 2\ell$, so

$$\int_{2\ell}^{\infty} 4\ell\, \nu(dr) \leq \frac{4\ell}{\pi(1-e^{-4\ell})} \int_{2\ell}^{\infty} e^{-r/2}\, dr < \infty.$$

By (50) and (53), $J_{\infty,\lambda}(\mathbf{1}_{I_\lambda}, \mathbf{1}_{I_\lambda}) < \infty$. The form $\mathcal{E}_{p,\lambda}$ is bounded on H_λ by Theorem 3.2, and $\|\mathbf{1}_{I_\lambda}\|_{H_\lambda}^2 = \mu(I_\lambda) = 2\ell < \infty$. Therefore (49) yields $\mathbf{1}_{I_\lambda} \in \mathcal{F}_\lambda$, and identity (48) is established.

(iv) L^2-additivity of the indicator decomposition. Set

$$g_B := E_\lambda \mathbf{1}_B, \qquad g_C := E_\lambda \mathbf{1}_C.$$

Then $g_I = g_B + g_C$ pointwise on $\mathbb{R}_{>0}^\times$. Because $B \cap C = \varnothing$, $\mathbf{1}_B \mathbf{1}_C = 0$ μ-a.e. on I_λ; both g_B and g_C vanish off I_λ, so $g_B g_C = 0$ $d^\times u$-a.e. on $\mathbb{R}_{>0}^\times$. By the Pythagorean identity,

$$\|\mathbf{1}_{I_\lambda}\|_{H_\lambda}^2 = \|\mathbf{1}_B\|_{H_\lambda}^2 + \|\mathbf{1}_C\|_{H_\lambda}^2, \tag{55}$$

$$\|g_I\|_{H_{\mathrm{tot}}}^2 = \|g_B\|_{H_{\mathrm{tot}}}^2 + \|g_C\|_{H_{\mathrm{tot}}}^2. \tag{56}$$

(v) Pointwise expansion of the jump kernel. For $r > 0$ and $h \in H_{\mathrm{tot}}$, set $\Delta_r h(u) := h(e^r u) - h(u)$. Because $g_I = g_B + g_C$, $\Delta_r g_I = \Delta_r g_B + \Delta_r g_C$, and the algebraic identity $a^2 + b^2 - (a+b)^2 = -2ab$ applied with $a = \Delta_r g_B$, $b = \Delta_r g_C$ gives

$$|\Delta_r g_B|^2 + |\Delta_r g_C|^2 - |\Delta_r g_I|^2 = -2\, \Delta_r g_B\, \Delta_r g_C. \tag{57}$$

Expanding the product,

$$\begin{aligned} \Delta_r g_B\, \Delta_r g_C = g_B(e^r u) g_C(e^r u) - g_B(e^r u) g_C(u) \\ - g_B(u) g_C(e^r u) + g_B(u) g_C(u). \end{aligned} \tag{58}$$

By (iv), $g_B g_C = 0$ $d^\times u$-a.e. on $\mathbb{R}_{>0}^\times$. The map $u \mapsto e^r u$ preserves $d^\times u$, so $g_B(e^r \cdot) g_C(e^r \cdot) = 0$ $d^\times u$-a.e. as well; thus the first term in (58) vanishes $d^\times u$-a.e. The fourth term vanishes by the same fact established in (iv). Substituting into (57) we obtain, $d^\times u$-a.e.,

$$|\Delta_r g_B|^2 + |\Delta_r g_C|^2 - |\Delta_r g_I|^2 = 2\big(g_B(e^r u) g_C(u) + g_C(e^r u) g_B(u)\big) \geq 0, \tag{59}$$

the nonnegativity following from $g_B, g_C \geq 0$ pointwise. Integrating (59) over $\mathbb{R}_{>0}^\times$ with respect to $d^\times u$,

$$\|\Delta_r g_B\|_{H_{\mathrm{tot}}}^2 + \|\Delta_r g_C\|_{H_{\mathrm{tot}}}^2 - \|\Delta_r g_I\|_{H_{\mathrm{tot}}}^2 = 2\, \overline{H}(r), \tag{60}$$

where

$$\overline{H}(r) := \int_{\mathbb{R}_{>0}^\times} g_B(e^r u) g_C(u)\, d^\times u + \int_{\mathbb{R}_{>0}^\times} g_C(e^r u) g_B(u)\, d^\times u \geq 0. \tag{61}$$

Multiplying (60) by π and integrating against $\nu(dr)$ on $(0,\infty)$, Tonelli's theorem yields

$$J_{\infty,\lambda}(\mathbf{1}_B) + J_{\infty,\lambda}(\mathbf{1}_C) - J_{\infty,\lambda}(\mathbf{1}_{I_\lambda}) = 2\pi \int_0^\infty \overline{H}(r)\,\nu(dr) \geq 0. \tag{62}$$

(vi) Strict positivity at some $r_0 > 0$. We now upgrade (62) to a strict inequality. The argument proceeds in three substeps: density-point selection, an inclusion-exclusion bound, and translation continuity.

Selection of density points. Pass to additive logarithmic coordinates $x = \log u$, and write

$$\widetilde{B} := \log B \subset [-\ell,\ell], \qquad \widetilde{C} := \log C \subset [-\ell,\ell].$$

The map $\log : I_\lambda \to [-\ell,\ell]$ is a bi-Lipschitz bijection sending $d^\times u$ to Lebesgue measure dx, so $\widetilde{B},\widetilde{C}$ are Lebesgue measurable with $|\widetilde{B}| = \mu(B) > 0$ and $|\widetilde{C}| = \mu(C) > 0$, and $\widetilde{B} \cap \widetilde{C} = \varnothing$. By Lebesgue's density theorem [12, Theorem 7.7], the sets

$$\widetilde{B}^* := \Big\{ x \in \widetilde{B} : \lim_{\delta\downarrow 0} \frac{|\widetilde{B} \cap (x-\delta, x+\delta)|}{2\delta} = 1 \Big\},$$

$$\widetilde{C}^* := \Big\{ x \in \widetilde{C} : \lim_{\delta\downarrow 0} \frac{|\widetilde{C} \cap (x-\delta, x+\delta)|}{2\delta} = 1 \Big\}$$

satisfy $|\widetilde{B} \setminus \widetilde{B}^*| = |\widetilde{C} \setminus \widetilde{C}^*| = 0$, hence $|\widetilde{B}^*| = |\widetilde{B}| > 0$ and $|\widetilde{C}^*| = |\widetilde{C}| > 0$. Since $\widetilde{B}^* \subset [-\ell,\ell]$ has positive Lebesgue measure, the subset $\widetilde{B}^* \cap (-\ell,\ell)$ also has positive measure (as $\widetilde{B}^* \setminus (-\ell,\ell)$ is contained in the two-point set $\{-\ell,\ell\}$, which is null); fix once and for all $x_B \in \widetilde{B}^* \cap (-\ell,\ell)$ and similarly $x_C \in \widetilde{C}^* \cap (-\ell,\ell)$. Since $\widetilde{B}$ and $\widetilde{C}$ are disjoint, $x_B \neq x_C$.

The right-hand side of (61) is symmetric under the involution $(B,C) \leftrightarrow (C,B)$ (the two integrals exchange). Likewise the left-hand side of (62) is symmetric in B and C. After relabeling $B \leftrightarrow C$ if $x_B < x_C$, we may take $x_B > x_C$ without loss of generality. Set

$$r_0 := x_B - x_C > 0, \qquad u_B := e^{x_B}, \qquad u_C := e^{x_C}, \qquad e^{r_0} u_C = u_B.$$

Inclusion-exclusion bound for $\overline{H}(r_0)$. For $\varepsilon > 0$, define the multiplicative neighborhoods

$$J_B := (u_B e^{-\varepsilon}, u_B e^{\varepsilon}), \qquad J_C := (u_C e^{-\varepsilon}, u_C e^{\varepsilon}).$$

By Haar invariance, $\mu(J_B) = \mu(J_C) = 2\varepsilon$, and $e^{r_0} J_C = J_B$. Translating density into multiplicative form via the bijection $x = \log u$: u_B is a density point of B in $d^\times u$ if and only if x_B is a density point of $\widetilde{B}$ in Lebesgue measure. Hence, by the definition of $\widetilde{B}^*$, there exists $\varepsilon_1 > 0$ such that

$$\mu(B \cap J_B) > \tfrac{3}{4}\mu(J_B) = \tfrac{3}{2}\varepsilon \qquad \text{for all } 0 < \varepsilon < \varepsilon_1. \tag{63}$$

Similarly, there exists $\varepsilon_2 > 0$ such that

$$\mu(C \cap J_C) > \tfrac{3}{4}\mu(J_C) = \tfrac{3}{2}\varepsilon \qquad \text{for all } 0 < \varepsilon < \varepsilon_2. \tag{64}$$

Choose $\varepsilon \in (0, \min(\varepsilon_1, \varepsilon_2))$ small enough also to ensure $J_B \cup J_C \subset I_\lambda$, which is possible because $x_B, x_C \in (-\ell, \ell)$.

The set

$$Y_1 := B \cap e^{r_0}C \cap J_B$$

is measurable as the intersection of measurable sets. Since $e^{r_0}(C \cap J_C) \subset e^{r_0}J_C = J_B$ and μ is translation-invariant, $\mu(e^{r_0}(C \cap J_C)) = \mu(C \cap J_C)$. Inclusion-exclusion in the box J_B gives

$$\begin{aligned} \mu(Y_1) &= \mu\big((B \cap J_B) \cap (e^{r_0}C \cap J_B)\big) \\ &\geq \mu(B \cap J_B) + \mu(e^{r_0}C \cap J_B) - \mu(J_B) \\ &= \mu(B \cap J_B) + \mu(C \cap J_C) - \mu(J_B) \\ &> \tfrac{3}{2}\varepsilon + \tfrac{3}{2}\varepsilon - 2\varepsilon = \varepsilon > 0. \end{aligned} \tag{65}$$

Define $Y_2 := e^{-r_0}Y_1$. Then Y_2 is measurable and $\mu(Y_2) = \mu(Y_1) > 0$ by Haar invariance. By construction, $Y_2 \subset e^{-r_0}(e^{r_0}C) = C$, and for every $u \in Y_2$ we have $e^{r_0}u \in Y_1 \subset B$. Therefore

$$g_C(u) = \mathbf{1}_C(u) = 1, \qquad g_B(e^{r_0}u) = \mathbf{1}_B(e^{r_0}u) = 1 \quad \text{for every } u \in Y_2.$$

Restricting the first integral in (61) to Y_2,

$$\int_{\mathbb{R}^\times_{>0}} g_B(e^{r_0}u)g_C(u)\, d^\times u \geq \int_{Y_2} 1\, d^\times u = \mu(Y_2) > 0.$$

The second integral in (61) is nonnegative, so

$$\overline{H}(r_0) > 0. \tag{66}$$

Continuity of $\overline{H}$ and conclusion. We now show that $\overline{H} \in C(\mathbb{R})$; by (66), this implies $\overline{H} > 0$ on a neighborhood of r_0, after which the strict positivity of the Lévy density on $(0, \infty)$ yields (67) below.

In additive coordinates, set $\widetilde{g}_B(x) := g_B(e^x)$ and $\widetilde{g}_C(x) := g_C(e^x)$; both are $\{0,1\}$-valued functions in $L^2(\mathbb{R}, dx) \cap L^\infty(\mathbb{R})$ supported in $[-\ell, \ell]$. The substitution $u = e^x$, $d^\times u = dx$ gives, for every $r \in \mathbb{R}$,

$$\int_{\mathbb{R}^\times_{>0}} g_B(e^r u)g_C(u)\, d^\times u = \int_{\mathbb{R}} \widetilde{g}_B(x + r)\widetilde{g}_C(x)\, dx = \langle \tau_{-r}\widetilde{g}_B, \widetilde{g}_C\rangle_{L^2(\mathbb{R})},$$

where $(\tau_h f)(x) := f(x + h)$ denotes additive translation. By the strong continuity of the translation group on $L^2(\mathbb{R})$ [12, Theorem 9.5], $r \mapsto \tau_{-r}\widetilde{g}_B$ is continuous from $\mathbb{R}$ into $L^2(\mathbb{R})$. Cauchy–Schwarz then yields the continuity of $r \mapsto \langle \tau_{-r}\widetilde{g}_B, \widetilde{g}_C\rangle_{L^2}$. The same argument applies to the second term in (61); thus $\overline{H} \in C(\mathbb{R})$.

By (66) and continuity, there exists an open interval $I_0 \subset (0,\infty)$ with $r_0 \in I_0$ on which $\overline{H} > 0$. The Lévy density $\frac{1}{\pi}\frac{e^{-r/2}}{1-e^{-2r}}$ is strictly positive and continuous on $(0,\infty)$, hence $\nu(I_0) = \int_{I_0} \frac{1}{\pi}\frac{e^{-r/2}}{1-e^{-2r}}\,dr > 0$. Restricting the integral in (62) to I_0,

$$2\pi \int_0^\infty \overline{H}(r)\,\nu(dr) \geq 2\pi \int_{I_0} \overline{H}(r)\,\nu(dr) > 0,$$

which gives the strict inequality

$$J_{\infty,\lambda}(\mathbf{1}_B) + J_{\infty,\lambda}(\mathbf{1}_C) - J_{\infty,\lambda}(\mathbf{1}_{I_\lambda}) > 0. \tag{67}$$

(vii) The discrete prime part contributes a nonnegative excess, and final contradiction. For real-valued $f \in H_\lambda$, $\Re\langle U_n f, f\rangle_{H_\lambda} = \langle U_n f, f\rangle_{H_\lambda} \in \mathbb{R}$. Indicator functions are real-valued, hence by (52),

$$\mathcal{E}_{p,\lambda}(\mathbf{1}_X) = \sum_{1<n\leq\lambda^2} 2a_n\big(\|\mathbf{1}_X\|^2_{H_\lambda} - \langle U_n\mathbf{1}_X, \mathbf{1}_X\rangle_{H_\lambda}\big), \qquad X \in \{B, C, I_\lambda\}.$$

Using $\mathbf{1}_{I_\lambda} = \mathbf{1}_B + \mathbf{1}_C$ and bilinearity,

$$\begin{aligned}\langle U_n\mathbf{1}_{I_\lambda}, \mathbf{1}_{I_\lambda}\rangle &= \langle U_n\mathbf{1}_B, \mathbf{1}_B\rangle + \langle U_n\mathbf{1}_C, \mathbf{1}_C\rangle \\ &\quad + \langle U_n\mathbf{1}_B, \mathbf{1}_C\rangle + \langle U_n\mathbf{1}_C, \mathbf{1}_B\rangle.\end{aligned}$$

Together with (55), this yields

$$\begin{aligned}&\mathcal{E}_{p,\lambda}(\mathbf{1}_B) + \mathcal{E}_{p,\lambda}(\mathbf{1}_C) - \mathcal{E}_{p,\lambda}(\mathbf{1}_{I_\lambda}) \\ &\qquad = \sum_{1<n\leq\lambda^2} 2a_n\Big(\langle U_n\mathbf{1}_B, \mathbf{1}_C\rangle + \langle U_n\mathbf{1}_C, \mathbf{1}_B\rangle\Big).\end{aligned} \tag{68}$$

We show that each term on the right of (68) is nonnegative. By the pointwise formula $(U_a f)(u) = \mathbf{1}_{I_\lambda}(au)f(au)$ established in Section 2, if $f \geq 0$ a.e. on I_λ then $U_a f \geq 0$ a.e. on I_λ. Applied to the indicator functions $\mathbf{1}_B, \mathbf{1}_C \geq 0$, this gives $U_n\mathbf{1}_B \geq 0$ and $U_n\mathbf{1}_C \geq 0$ a.e. on I_λ. Since $\mathbf{1}_B, \mathbf{1}_C \geq 0$ as well, the inner products $\langle U_n\mathbf{1}_B, \mathbf{1}_C\rangle$ and $\langle U_n\mathbf{1}_C, \mathbf{1}_B\rangle$ are nonnegative. Combined with $a_n > 0$,

$$\mathcal{E}_{p,\lambda}(\mathbf{1}_B) + \mathcal{E}_{p,\lambda}(\mathbf{1}_C) - \mathcal{E}_{p,\lambda}(\mathbf{1}_{I_\lambda}) \geq 0. \tag{69}$$

Adding (67) and (69) and using the decomposition (49),

$$\mathcal{E}_\lambda(\mathbf{1}_B) + \mathcal{E}_\lambda(\mathbf{1}_C) - \mathcal{E}_\lambda(\mathbf{1}_{I_\lambda}) > 0. \tag{70}$$

This contradicts identity (48) established in (i). The contradiction shows that no measurable set $B \subset I_\lambda$ with $0 < \mu(B) < \mu(I_\lambda)$ can satisfy (43), hence $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is irreducible on H_λ. $\square$

Remark 4.2. The proof above isolates two structural ingredients responsible for irreducibility: (i) the strict positivity and continuity of the Lévy density of ν on $(0,\infty)$, established in Subsection 3.3.2 via the digamma integral representation; and (ii) the positivity-preservation property of the compressed dilations U_n, which renders the prime-part excess a sum of nonnegative inner products. The indicator-splitting argument, made explicit through the auxiliary kernel $\overline{H}(r)$, is analogous to the standard irreducibility criterion for nonlocal Dirichlet forms in terms of the connectivity of the support of the jumping kernel; here this kernel is the multiplicative pull-back of ν, whose support is all of $(0,\infty)$.

4.2. *Positivity improvement*. With Proposition 4.1 in hand, irreducibility has been established. In this subsection we convert it into positivity improvement for the semigroup and the resolvent.

THEOREM 4.3. *Let L_λ denote the unique nonnegative self-adjoint operator on H_λ associated with the closed nonnegative form $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ of Theorem 3.2 via the first representation theorem* [18, Chap. VI, Thm. 2.1], *so that $\mathcal{F}_\lambda = \mathrm{Dom}(L_\lambda^{1/2})$, and let $A_\lambda := L_\lambda - c_{D,\lambda} I$ be the corresponding self-adjoint lower-bounded operator associated with $Q_{D,\lambda}$. The second representation theorem* [18, Chap. VI, Thm. 2.23] *yields*

$$\mathcal{E}_\lambda(u,v) = \langle L_\lambda^{1/2} u, L_\lambda^{1/2} v\rangle_{H_\lambda}, \qquad u, v \in \mathcal{F}_\lambda, \tag{71}$$

or equivalently $\mathcal{E}_\lambda(u,v) = \langle L_\lambda u, v\rangle_{H_\lambda}$ for $u \in \mathrm{Dom}(L_\lambda)$ and $v \in \mathcal{F}_\lambda$. Set $T_t^\lambda := e^{-tL_\lambda}$ for $t > 0$ and $G_\alpha^\lambda := (L_\lambda + \alpha)^{-1}$ for $\alpha > 0$. Then:

(i) *for every $t > 0$ and every $f \in H_\lambda$ with $f \geq 0$ μ-a.e. on I_λ and $\|f\|_{H_\lambda} > 0$,*

$$T_t^\lambda f > 0 \quad \mu\text{-a.e. on } I_\lambda;$$

(ii) *for every $\alpha > 0$ and every $f \in H_\lambda$ with $f \geq 0$ μ-a.e. on I_λ and $\|f\|_{H_\lambda} > 0$,*

$$G_\alpha^\lambda f > 0 \quad \mu\text{-a.e. on } I_\lambda.$$

That is, the symmetric sub-Markov C_0-semigroup $(T_t^\lambda)_{t>0}$ and each resolvent G_α^λ $(\alpha > 0)$ are positivity improving on H_λ.

Proof. For $B \subset I_\lambda$ measurable, let $P_B \in \mathcal{B}(H_\lambda)$ denote the multiplication operator $f \mapsto \mathbf{1}_B f$. By Theorem 3.2, $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is a symmetric Dirichlet form on H_λ, and consequently T_t^λ is a symmetric, sub-Markov C_0-semigroup that is *positivity preserving*:

$$T_t^\lambda f \geq 0 \quad \mu\text{-a.e. on } I_\lambda \qquad \forall t > 0,\ \forall f \in H_\lambda \text{ with } f \geq 0\ \mu\text{-a.e.} \tag{72}$$

This is the unit-contraction property characterizing symmetric Dirichlet forms [13, Thm. 1.4.1]. By Proposition 4.1, no measurable $B \subset I_\lambda$ with $0 < \mu(B) < \mu(I_\lambda)$ satisfies $T_t^\lambda P_B = P_B T_t^\lambda$ for every $t > 0$.

Proof of (i). The hypotheses of the Faris–Reed–Simon ergodicity theorem for symmetric positivity-preserving semigroups ([11, Vol. IV, Thm. XIII.44]; see also the equivalent Dirichlet-form formulation in [13, Thm. 1.6.1, Cor. 1.6.2]) are verified: L_λ is self-adjoint and bounded below (by 0), $T_t^\lambda = e^{-tL_\lambda}$ is positivity preserving by (72), and the system is ergodic in the sense that no nontrivial P_B commutes with all T_t^λ (Proposition 4.1). The cited theorem then asserts that T_t^λ is positivity improving for every $t > 0$, which is exactly (i).

Proof of (ii). Fix $\alpha > 0$ and $f \in H_\lambda$ with $f \geq 0$ μ-a.e. and $\|f\|_{H_\lambda} > 0$. The Laplace-transform identity

$$G_\alpha^\lambda f = \int_0^\infty e^{-\alpha t} T_t^\lambda f \, dt \tag{73}$$

holds in H_λ as a Bochner integral: the integrand is strongly continuous in $t \in (0, \infty)$ with norm bounded by $e^{-\alpha t}\|f\|_{H_\lambda} \in L^1(0, \infty)$, and the identity follows from the spectral theorem applied to L_λ together with the elementary identity $\int_0^\infty e^{-\alpha t} e^{-t\sigma} \, dt = (\sigma + \alpha)^{-1}$ for $\sigma \geq 0$.

Set $A := \{u \in I_\lambda : (G_\alpha^\lambda f)(u) = 0\} \subset I_\lambda$; we show $\mu(A) = 0$. We argue by contradiction. Were $\mu(A) > 0$ to hold, then $\mathbf{1}_A \in H_\lambda$ (since $\mu(I_\lambda) < \infty$) would be a nonnegative element of H_λ satisfying

$$\langle \mathbf{1}_A, G_\alpha^\lambda f \rangle_{H_\lambda} = \int_A G_\alpha^\lambda f \, d\mu = 0$$

by the very definition of A. Pairing (73) against $\mathbf{1}_A$ and applying Fubini's theorem [16, Vol. I, Thm. 3.4.4] (the integrand $(t, u) \mapsto e^{-\alpha t}\mathbf{1}_A(u)(T_t^\lambda f)(u)$ is nonnegative by (72), and the iterated integrals are bounded by $\alpha^{-1}\|\mathbf{1}_A\|_{H_\lambda}\|f\|_{H_\lambda} < \infty$),

$$\langle \mathbf{1}_A, G_\alpha^\lambda f \rangle_{H_\lambda} = \int_0^\infty e^{-\alpha t} \langle \mathbf{1}_A, T_t^\lambda f \rangle_{H_\lambda} \, dt. \tag{74}$$

By (i), for every $t > 0$ the function $T_t^\lambda f$ is strictly positive μ-a.e. on I_λ; hence

$$\langle \mathbf{1}_A, T_t^\lambda f \rangle_{H_\lambda} = \int_A T_t^\lambda f \, d\mu > 0 \qquad \forall t > 0,$$

since $\mu(A) > 0$ by hypothesis. Combined with $e^{-\alpha t} > 0$ on $(0, \infty)$ and the strong continuity of $t \mapsto T_t^\lambda f$ (which makes $t \mapsto \langle \mathbf{1}_A, T_t^\lambda f \rangle$ continuous), the integrand on the right-hand side of (74) is a strictly positive continuous function of $t \in (0, \infty)$, whence the integral is strictly positive. This contradicts the vanishing of the left-hand side. We conclude $\mu(A) = 0$, equivalently $G_\alpha^\lambda f > 0$ μ-a.e. on I_λ, which is (ii). $\square$

5. **Ground-State Spectral Theory and a Real-Zeros Theorem**

In Section 4, we have already proved that for every $\lambda > 1$, the pair $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is a nonnegative, closed, symmetric, irreducible Dirichlet form on H_λ, and that the associated semigroup and resolvent are positivity improving.

The aims of the present section are fourfold. First, we prove that L_λ has compact resolvent, and hence discrete spectrum. Second, we show that the bottom eigenvalue is simple, and obtain a normalized strictly positive and inversion-even ground state ξ_λ. Third, we transport the form to the interval $[0, L]$ so as to connect it with the real-distribution model in the literature. Finally, by means of finite-dimensional Fourier truncation and Hurwitz's theorem, we prove that the logarithmic Fourier transform of ξ_λ is an entire function all of whose zeros are real.

5.1. *Compact resolvent and discrete spectrum.* We first prove that the associated operator has compact resolvent and hence discrete spectrum.

Proposition 5.1. *Define*

$$\|f\|^2_{\mathcal{E}_\lambda,1} := \mathcal{E}_\lambda(f,f) + \|f\|^2_{H_\lambda}, \qquad f \in \mathcal{F}_\lambda.$$

Then the following hold:

1. *The embedding*

$$j_\lambda : (\mathcal{F}_\lambda, \|\cdot\|_{\mathcal{E}_\lambda,1}) \hookrightarrow H_\lambda$$

   is compact.
2. *For every $\alpha > 0$, the resolvent operator*

$$G^\lambda_\alpha := (L_\lambda + \alpha)^{-1} : H_\lambda \to H_\lambda$$

   is a compact self-adjoint operator.
3. *Consequently, L_λ has compact resolvent, hence purely discrete spectrum: there exists a sequence of eigenvalues*

$$0 \le \mu_0(\lambda) \le \mu_1(\lambda) \le \mu_2(\lambda) \le \cdots, \qquad \mu_n(\lambda) \to +\infty,$$

   each of finite multiplicity, and H_λ admits an orthonormal basis consisting of eigenfunctions of L_λ.

Proof. By Theorem 3.2, $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is a nonnegative, closed, symmetric Dirichlet form on H_λ. The first representation theorem [18, Chap. VI, Thm. 2.1] associates with it a unique nonnegative self-adjoint operator L_λ on H_λ such that

$$u \in \mathrm{Dom}(L_\lambda),\ L_\lambda u = f \iff u \in \mathcal{F}_\lambda,\ \mathcal{E}_\lambda(v,u) = \langle v, f\rangle_{H_\lambda} \quad (\forall v \in \mathcal{F}_\lambda),$$

and in particular

$$\mathcal{E}_\lambda(u,v) = \langle L_\lambda u, v\rangle_{H_\lambda}, \qquad u \in \mathrm{Dom}(L_\lambda),\ v \in \mathcal{F}_\lambda.$$

We organize the remainder of the proof into seven steps (i)–(vii).

(i) Fourier-multiplier representation of $\mathcal{E}_{\infty,\lambda}$. Let $Z : H_\lambda \to L^2(I_x, dx)$ be the logarithmic unitary of Lemma 2.3, where $I_x = [-\ell, \ell]$ with $\ell = \log\lambda$, and let $E_x : L^2(I_x, dx) \hookrightarrow L^2(\mathbb{R}, dx)$ denote zero extension. For $f \in \mathcal{F}_\lambda$, set $g := E_x Z f \in L^2(\mathbb{R}, dx)$, so that $\operatorname{supp}(g) \subset [-\ell, \ell]$, $\|f\|_{H_\lambda} = \|g\|_{L^2(\mathbb{R})}$, and the change of variables $u = e^x$ together with the Fourier convention $\widehat{g}(t) := \int_{\mathbb{R}} g(x)e^{-itx}\,dx$ yields $\widehat{f}(t) = \widehat{g}(t)$ for $t \in \mathbb{R}$.

By Theorem 3.2(2), the archimedean part admits the Lévy–Khintchine pure-jump representation (4). Combining the elementary identity $|e^{itr} - 1|^2 = 2(1 - \cos(tr))$ with Plancherel's theorem under the convention $\|\widehat{g}\|^2_{L^2(\mathbb{R})} = 2\pi\|g\|^2_{L^2(\mathbb{R})}$, we have, for every $r > 0$,

$$\|g(\cdot + r) - g\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\int_{\mathbb{R}} |e^{itr} - 1|^2 |\widehat{g}(t)|^2\,dt = \frac{1}{\pi}\int_{\mathbb{R}} (1 - \cos(tr))|\widehat{g}(t)|^2\,dt;$$

this identity also appears at (22). Substituting into (4) and applying Tonelli's theorem [16, Vol. I, Thm. 3.4.4] (legitimate by the nonnegativity of $|\widehat{g}(t)|^2$ and $1 - \cos(tr)$), we obtain

$$\mathcal{E}_{\infty,\lambda}(f,f) = \int_{\mathbb{R}} \chi_\lambda(t)\,|\widehat{f}(t)|^2\,dt, \qquad \chi_\lambda(t) := \int_0^\infty (1 - \cos(tr))\,\nu(dr) \geq 0. \tag{75}$$

(ii) Monotone divergence of χ_λ. For $0 < r \leq 1$, the inequalities $1 - e^{-2r} \leq 2r$ and $e^{-r/2} \geq e^{-1/2}$ combine to give

$$\nu(dr) \;\geq\; \frac{e^{-1/2}}{2\pi}\,\frac{dr}{r} \qquad \text{on } (0,1].$$

Substituting this lower bound and applying the change of variables $s = |t|r$, for $|t| \geq 1$ we have

$$\chi_\lambda(t) \;\geq\; \frac{e^{-1/2}}{2\pi}\int_0^1 \frac{1 - \cos(tr)}{r}\,dr = \frac{e^{-1/2}}{2\pi}\,\Phi(|t|), \qquad \Phi(T) := \int_0^T \frac{1 - \cos s}{s}\,ds.$$

The function $\Phi : [0,\infty) \to [0,\infty)$ is monotone nondecreasing and satisfies $\Phi(T) \to +\infty$ as $T \to \infty$ (in fact $\Phi(T) \sim \log T$, by the standard asymptotics of the cosine integral). Therefore

$$\inf_{|t|\geq R} \chi_\lambda(t) \;\geq\; \frac{e^{-1/2}}{2\pi}\,\Phi(R) \;\xrightarrow[R\to\infty]{}\; +\infty. \tag{76}$$

(iii) Energy bound and high-frequency tail estimate. Let $(f_n) \subset \mathcal{F}_\lambda$ satisfy

$$M := \sup_n \|f_n\|^2_{\mathcal{E}_\lambda,1} = \sup_n\big(\mathcal{E}_\lambda(f_n,f_n) + \|f_n\|^2_{H_\lambda}\big) < \infty.$$

By Theorem 3.2(3), $\mathcal{E}_{p,\lambda} \geq 0$, so the additive decomposition (8) yields

$$\sup_n \mathcal{E}_{\infty,\lambda}(f_n,f_n) \;\leq\; \sup_n \mathcal{E}_\lambda(f_n,f_n) \;\leq\; M. \tag{77}$$

Setting $g_n := E_x Z f_n$, we have $\operatorname{supp}(g_n) \subset [-\ell, \ell]$, $\|g_n\|^2_{L^2(\mathbb{R})} = \|f_n\|^2_{H_\lambda} \le M$, and, by (75)–(77),

$$\sup_n \int_{\mathbb{R}} \chi_\lambda(t) |\widehat{g}_n(t)|^2 \, dt \;\le\; M. \tag{78}$$

For every $R > 1$, the monotone divergence (76) together with (78) yields

$$\sup_n \int_{|t|\ge R} |\widehat{g}_n(t)|^2 \, dt \;\le\; \frac{1}{\inf_{|t|\ge R} \chi_\lambda(t)} \sup_n \int_{\mathbb{R}} \chi_\lambda(t) |\widehat{g}_n(t)|^2 \, dt \;\le\; \frac{M}{\inf_{|t|\ge R} \chi_\lambda(t)} \xrightarrow[R\to\infty]{} 0. \tag{79}$$

(iv) Uniform translation continuity in $L^2(\mathbb{R})$. For $h \in \mathbb{R}$ and the additive translation $\tau_h g(x) := g(x + h)$, Plancherel's theorem and the identity $|e^{ith} - 1|^2 = 2(1 - \cos(th))$ give

$$\|\tau_h g_n - g_n\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi} \int_{\mathbb{R}} |e^{ith} - 1|^2 |\widehat{g}_n(t)|^2 \, dt. \tag{80}$$

The pointwise bound $|e^{ith} - 1|^2 \le \min\bigl(4,\, t^2 h^2\bigr)$, valid for all $t, h \in \mathbb{R}$, allows us to split the integral at any threshold $R > 1$:

$$\|\tau_h g_n - g_n\|^2_{L^2(\mathbb{R})} \;\le\; \underbrace{\frac{R^2 h^2}{2\pi} \int_{|t|\le R} |\widehat{g}_n(t)|^2 \, dt}_{\text{(I)}} \;+\; \underbrace{\frac{2}{\pi} \int_{|t|\ge R} |\widehat{g}_n(t)|^2 \, dt}_{\text{(II)}}.$$

By Plancherel and $\|g_n\|^2_{L^2(\mathbb{R})} \le M$,

$$\sup_n \text{(I)} \;\le\; \frac{R^2 h^2}{2\pi} \cdot 2\pi M \;=\; R^2 h^2 \, M,$$

while (79) gives

$$\sup_n \text{(II)} \;\le\; \frac{2M}{\pi \, \inf_{|t|\ge R} \chi_\lambda(t)}.$$

Given $\varepsilon > 0$, by (76) we may first choose $R = R(\varepsilon) > 1$ so that $\sup_n \text{(II)} < \varepsilon/2$, and then choose $\delta = \delta(\varepsilon, R) > 0$ such that $R^2 h^2 M < \varepsilon/2$ whenever $|h| < \delta$. Therefore

$$\lim_{h\to 0} \sup_n \|\tau_h g_n - g_n\|_{L^2(\mathbb{R})} = 0. \tag{81}$$

(v) Compactness of the embedding j_λ. We invoke the Kolmogorov–Riesz–Fréchet compactness theorem in $L^2(\mathbb{R})$, in the form given by Brezis [17, Thm. 4.26 and Cor. 4.27]: a bounded subset $\mathcal{G} \subset L^2(\mathbb{R})$ is relatively compact if and only if

(a) $\lim_{h\to 0} \sup_{g\in\mathcal{G}} \|\tau_h g - g\|_{L^2(\mathbb{R})} = 0$, and

(b) $\lim_{R\to\infty} \sup_{g\in\mathcal{G}} \int_{|x|\ge R} |g(x)|^2 \, dx = 0.$

The family $\{g_n\}$ is bounded in $L^2(\mathbb{R})$ by $\sqrt{M}$. It satisfies (a) by (81), and (b) holds trivially because $\operatorname{supp}(g_n) \subset [-\ell, \ell]$ for all n, so the integral in (b) vanishes for every $R > \ell$. Hence $\{g_n\}$ is relatively compact in $L^2(\mathbb{R})$. Since the map $f \mapsto E_x Z f$ is an isometry from H_λ onto $E_x Z(H_\lambda) \subset L^2(\mathbb{R})$, the family $\{f_n\}$ is relatively compact in H_λ. This proves that the embedding $j_\lambda : (\mathcal{F}_\lambda, \|\cdot\|_{\mathcal{E}_\lambda,1}) \hookrightarrow H_\lambda$ is compact, which is assertion 1.

(vi) Compactness of the resolvent. Fix $\alpha > 0$ and $f \in H_\lambda$. By the first representation theorem [18, Chap. VI, Thm. 2.1], $u := G_\alpha^\lambda f$ lies in $\operatorname{Dom}(L_\lambda) \subset \mathcal{F}_\lambda$ and satisfies

$$\mathcal{E}_\lambda(u,v) + \alpha\langle u,v\rangle_{H_\lambda} = \langle f,v\rangle_{H_\lambda} \qquad (\forall v \in \mathcal{F}_\lambda).$$

Setting $v = u$, the left-hand side $\mathcal{E}_\lambda(u,u) + \alpha\|u\|_{H_\lambda}^2$ is a real number; by the Cauchy–Schwarz inequality,

$$\mathcal{E}_\lambda(u,u) + \alpha\|u\|_{H_\lambda}^2 \;=\; \Re\langle f,u\rangle_{H_\lambda} \;\le\; \|f\|_{H_\lambda}\|u\|_{H_\lambda}.$$

Since $\mathcal{E}_\lambda \ge 0$, the two summands on the left-hand side are nonnegative, and the displayed inequality therefore implies separately

$$\alpha\|u\|_{H_\lambda}^2 \le \|f\|_{H_\lambda}\|u\|_{H_\lambda}, \qquad \mathcal{E}_\lambda(u,u) \le \|f\|_{H_\lambda}\|u\|_{H_\lambda}.$$

The first gives $\|u\|_{H_\lambda} \le \alpha^{-1}\|f\|_{H_\lambda}$, and feeding this back into the second yields $\mathcal{E}_\lambda(u,u) \le \alpha^{-1}\|f\|_{H_\lambda}^2$. Consequently

$$\|G_\alpha^\lambda f\|_{\mathcal{E}_\lambda,1}^2 = \mathcal{E}_\lambda(u,u) + \|u\|_{H_\lambda}^2 \;\le\; \frac{1}{\alpha}\|f\|_{H_\lambda}^2 + \frac{1}{\alpha^2}\|f\|_{H_\lambda}^2 = \frac{\alpha+1}{\alpha^2}\|f\|_{H_\lambda}^2, \tag{82}$$

which means $G_\alpha^\lambda : H_\lambda \to (\mathcal{F}_\lambda, \|\cdot\|_{\mathcal{E}_\lambda,1})$ is bounded with operator norm at most $\sqrt{(\alpha+1)/\alpha^2}$.

By (v), the embedding j_λ is compact. The composition

$$G_\alpha^\lambda \;=\; j_\lambda \circ G_\alpha^\lambda \;:\; H_\lambda \xrightarrow{\text{bounded}} (\mathcal{F}_\lambda, \|\cdot\|_{\mathcal{E}_\lambda,1}) \xrightarrow{\text{compact}} H_\lambda$$

is therefore a compact operator on H_λ. Since L_λ is self-adjoint and $\alpha > 0$ is real, $G_\alpha^\lambda = (L_\lambda + \alpha)^{-1}$ is self-adjoint as the inverse of the self-adjoint operator $L_\lambda + \alpha I$. This proves assertion 2.

(vii) Discrete spectrum of L_λ. Fix any $\alpha > 0$. By (vi), G_α^λ is a compact, self-adjoint, injective operator on H_λ. The spectral theorem for compact self-adjoint operators [9, Thms. VI.16 and VI.17] produces an orthonormal basis $(e_n)_{n\ge 0}$ of H_λ together with eigenvalues $\rho_n(\alpha) \in \mathbb{R}$ such that $G_\alpha^\lambda e_n = \rho_n(\alpha)e_n$, each nonzero eigenvalue has finite multiplicity, and the only possible accumulation point of $\{\rho_n(\alpha)\}$ is 0. The injectivity of G_α^λ rules out $\rho_n(\alpha) = 0$, and $L_\lambda \ge 0$ together with the spectral mapping $\sigma(G_\alpha^\lambda) \subset (0, \alpha^{-1}]$ further yields $0 < \rho_n(\alpha) \le \alpha^{-1}$.

The relation $L_\lambda = (G_\alpha^\lambda)^{-1} - \alpha I$ on the dense subspace $\mathrm{Dom}(L_\lambda) = \mathrm{Ran}(G_\alpha^\lambda)$ shows that each $e_n \in \mathrm{Dom}(L_\lambda)$ and

$$L_\lambda e_n = \mu_n(\lambda)\, e_n, \qquad \mu_n(\lambda) := \rho_n(\alpha)^{-1} - \alpha \ \geq \ 0.$$

The accumulation of $\rho_n(\alpha)$ at 0 is therefore equivalent to $\mu_n(\lambda) \to +\infty$. Reindexing the eigenvalues in nondecreasing order gives

$$0 \leq \mu_0(\lambda) \leq \mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots, \qquad \mu_n(\lambda) \to +\infty,$$

each of finite multiplicity, and H_λ admits an orthonormal basis (e_n) of eigenfunctions of L_λ. This is assertion 3 and completes the proof. □

5.2. *Simplicity of the ground state and strict positivity.* We next identify the bottom eigenspace and the positivity of its eigenfunction.

THEOREM 5.2. *Let $\lambda > 1$, and set*

$$I_\lambda = [\lambda^{-1}, \lambda], \qquad H_\lambda = L^2(I_\lambda, d^\times u).$$

Let L_λ be the self-adjoint nonnegative operator associated with the nonnegative closed symmetric form $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ constructed in Section 3, and let compact resolvent be given by Proposition 5.1. For each $\alpha > 0$, the resolvent

$$G_\alpha^\lambda := (L_\lambda + \alpha)^{-1}$$

is positivity improving on H_λ. Denote by

$$\mu_0(\lambda) := \min \sigma(L_\lambda)$$

the bottom of the spectrum. Then:

1. *$\mu_0(\lambda)$ is an eigenvalue of L_λ;*
2. *the eigenspace $\ker(L_\lambda - \mu_0(\lambda)I)$ is one-dimensional;*
3. *there exists a unique L^2-normalized eigenfunction $\xi_\lambda \in \mathrm{Dom}(L_\lambda) \subset \mathcal{F}_\lambda$ satisfying*

$$L_\lambda \xi_\lambda = \mu_0(\lambda)\xi_\lambda, \qquad \|\xi_\lambda\|_{H_\lambda} = 1, \qquad \xi_\lambda > 0 \quad \textit{a.e. on } I_\lambda.$$

Proof. By Proposition 5.1, L_λ has compact resolvent and hence purely discrete spectrum. There exists a nondecreasing real sequence

$$0 \leq \mu_0(\lambda) \leq \mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots, \qquad \mu_n(\lambda) \to +\infty,$$

and an orthonormal basis $(e_n)_{n\geq 0} \subset \mathrm{Dom}(L_\lambda)$ of H_λ satisfying $L_\lambda e_n = \mu_n(\lambda) e_n$. In particular, $\mu_0(\lambda)$ is an eigenvalue of L_λ, which proves assertion 1. Fix once and for all $\alpha > 0$, and set

$$G_\alpha^\lambda := (L_\lambda + \alpha)^{-1}, \qquad r_\alpha := \frac{1}{\mu_0(\lambda) + \alpha}.$$

Then G_α^λ is compact, self-adjoint, and injective on H_λ, with $G_\alpha^\lambda e_n = (\mu_n(\lambda)+\alpha)^{-1}e_n$. Its spectrum is $\{(\mu_n(\lambda)+\alpha)^{-1} : n \geq 0\} \subset (0, r_\alpha]$, with r_α its largest element, and

$$\ker(L_\lambda - \mu_0(\lambda)I) = \ker(G_\alpha^\lambda - r_\alpha I). \tag{83}$$

It therefore suffices to prove that the right-hand side of (83) is one-dimensional and contains a strictly positive normalized representative.

The proof is organized into seven steps (i)–(vii).

(i) Conjugation invariance of $\mathcal{E}_\lambda$, L_λ, and G_α^λ. The map $\Phi(z) := \bar{z}$ is a complex normal contraction in the sense of Remark 3.1: $\Phi(0) = 0$ and $|\Phi(z) - \Phi(w)| = |z - w|$ for all $z, w \in \mathbb{C}$. By the Markov property of $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ established in Theorem 3.2(1), $\bar{f} \in \mathcal{F}_\lambda$ for every $f \in \mathcal{F}_\lambda$, and

$$\mathcal{E}_\lambda(\bar{f}, \bar{f}) \leq \mathcal{E}_\lambda(f, f). \tag{84}$$

Replacing f by $\bar{f}$ in (84) and using $\bar{\bar{f}} = f$ yields the reverse inequality, hence the equality

$$\mathcal{E}_\lambda(\bar{f}, \bar{f}) = \mathcal{E}_\lambda(f, f), \qquad f \in \mathcal{F}_\lambda. \tag{85}$$

Define $T : \mathcal{F}_\lambda \times \mathcal{F}_\lambda \to \mathbb{C}$ by

$$T(f, g) := \overline{\mathcal{E}_\lambda(\bar{f}, \bar{g})}.$$

A direct calculation, using the linearity of $\mathcal{E}_\lambda$ in the first variable and the conjugate-linearity in the second, shows that T is sesquilinear with the same convention. By (85) and the reality of $\mathcal{E}_\lambda(f, f)$,

$$T(f, f) = \overline{\mathcal{E}_\lambda(\bar{f}, \bar{f})} = \overline{\mathcal{E}_\lambda(f, f)} = \mathcal{E}_\lambda(f, f), \qquad f \in \mathcal{F}_\lambda.$$

The polarization identity

$$4S(f, g) = S(f+g, f+g) - S(f-g, f-g) + i\big[S(f+ig, f+ig) - S(f-ig, f-ig)\big],$$

valid for every sesquilinear form S in our convention, recovers $S(f, g)$ from the quadratic form $f \mapsto S(f, f)$. Applied to T and to $\mathcal{E}_\lambda$, which share the same quadratic form, it gives

$$\mathcal{E}_\lambda(\bar{f}, \bar{g}) = \overline{\mathcal{E}_\lambda(f, g)}, \qquad f, g \in \mathcal{F}_\lambda. \tag{86}$$

We now transfer (86) to the operator level. Let $f \in \mathrm{Dom}(L_\lambda)$ and set $h := L_\lambda f \in H_\lambda$. By the first representation theorem [18, Chap. VI, Thm. 2.1],

$$\mathcal{E}_\lambda(v, f) = \langle v, h\rangle_{H_\lambda}, \qquad \forall v \in \mathcal{F}_\lambda.$$

For every $v \in \mathcal{F}_\lambda$, the Markov property gives $\bar{v} \in \mathcal{F}_\lambda$, and formula (86) (with $\bar{v}$ in place of the first argument and f in place of the second) yields

$$\mathcal{E}_\lambda(v, \bar{f}) = \overline{\mathcal{E}_\lambda(\bar{v}, f)} = \overline{\langle \bar{v}, h\rangle_{H_\lambda}} = \langle v, \bar{h}\rangle_{H_\lambda},$$

where the last equality follows from the conjugate-linearity of $\langle\cdot,\cdot\rangle_{H_\lambda}$ in the second variable: $\overline{\langle \bar v, h\rangle_{H_\lambda}} = \overline{\int_{I_\lambda} \bar v\,\bar h\,d\mu} = \int_{I_\lambda} v\,h\,d\mu = \langle v, \bar h\rangle_{H_\lambda}$. Applying the first representation theorem in the reverse direction, we obtain $\bar f \in \mathrm{Dom}(L_\lambda)$ and

$$L_\lambda \bar f = \bar h = \overline{L_\lambda f}. \tag{87}$$

Since $\alpha \in \mathbb{R}$, the operator $L_\lambda + \alpha I$ inherits the same property. For any $f \in H_\lambda$, set $g := G^\lambda_\alpha f \in \mathrm{Dom}(L_\lambda)$; then (87) gives $\bar g \in \mathrm{Dom}(L_\lambda)$ and $(L_\lambda+\alpha I)\bar g = \overline{(L_\lambda+\alpha I)g} = \bar f$, hence $\bar g = G^\lambda_\alpha \bar f$. We conclude

$$\overline{G^\lambda_\alpha f} = G^\lambda_\alpha \bar f, \qquad f \in H_\lambda. \tag{88}$$

(ii) Reduction to a real-valued ground-state candidate. Let $\psi \in \ker(G^\lambda_\alpha - r_\alpha I)$ be nonzero. By (88) and the reality of r_α,

$$G^\lambda_\alpha \bar\psi = \overline{G^\lambda_\alpha \psi} = \overline{r_\alpha \psi} = r_\alpha \bar\psi,$$

so $\bar\psi \in \ker(G^\lambda_\alpha - r_\alpha I)$ as well. Hence the real-valued functions

$$\psi_1 := \frac{\psi+\bar\psi}{2} = \Re\psi, \qquad \psi_2 := \frac{\psi-\bar\psi}{2i} = \Im\psi$$

both lie in $\ker(G^\lambda_\alpha - r_\alpha I)$. Since $\psi = \psi_1 + i\psi_2$ is nonzero, at least one of ψ_1, ψ_2 is nonzero. We work henceforth with such a nonzero real-valued representative, which we again denote by ψ.

(iii) Pointwise lower bound $G^\lambda_\alpha|\psi| \ge r_\alpha|\psi|$. Decompose

$$\psi = \psi^+ - \psi^-, \qquad |\psi| = \psi^+ + \psi^-, \qquad \psi^\pm \ge 0 \ \mu\text{-a.e. on } I_\lambda.$$

By Theorem 4.3(ii), G^λ_α is positivity preserving, so $G^\lambda_\alpha \psi^\pm \ge 0$ μ-a.e. The pointwise triangle inequality $a+b \ge |a-b|$ for $a, b \ge 0$, applied at μ-a.e. $u \in I_\lambda$ to $a = (G^\lambda_\alpha\psi^+)(u)$ and $b = (G^\lambda_\alpha\psi^-)(u)$, yields

$$G^\lambda_\alpha|\psi| = G^\lambda_\alpha\psi^+ + G^\lambda_\alpha\psi^- \ge \left|G^\lambda_\alpha\psi^+ - G^\lambda_\alpha\psi^-\right| = |G^\lambda_\alpha\psi| = r_\alpha|\psi| \qquad \mu\text{-a.e. on } I_\lambda, \tag{89}$$

where the last equality uses $G^\lambda_\alpha\psi = r_\alpha\psi$, $r_\alpha > 0$, and the fact that ψ is real.

(iv) Upgrade to the equality $G^\lambda_\alpha|\psi| = r_\alpha|\psi|$. The pointwise identity $|\psi|^2 = \psi^2$ gives $|\psi| \in H_\lambda$ with $\||\psi|\|_{H_\lambda} = \|\psi\|_{H_\lambda}$. Expand $|\psi|$ in the orthonormal eigenbasis $(e_n)_{n\ge0}$:

$$|\psi| = \sum_{n\ge0} c_n e_n, \qquad c_n := \langle |\psi|, e_n\rangle_{H_\lambda}, \qquad \sum_{n\ge0}|c_n|^2 = \|\psi\|^2_{H_\lambda}.$$

Since $G^\lambda_\alpha e_n = (\mu_n(\lambda)+\alpha)^{-1}e_n$ and $r_\alpha = (\mu_0(\lambda)+\alpha)^{-1} \ge (\mu_n(\lambda)+\alpha)^{-1}$ for every $n \ge 0$,

$$\langle G^\lambda_\alpha|\psi|, |\psi|\rangle_{H_\lambda} - r_\alpha\||\psi|\|^2_{H_\lambda} = \sum_{n\ge0}|c_n|^2\Big[\frac{1}{\mu_n(\lambda)+\alpha} - r_\alpha\Big] \le 0, \tag{90}$$

each summand being nonpositive. On the other hand, by (89) the function $G_\alpha^\lambda|\psi| - r_\alpha|\psi|$ is μ-a.e. nonnegative on I_λ, and so is $|\psi|$; pairing the two,

$$\langle G_\alpha^\lambda|\psi|, |\psi|\rangle_{H_\lambda} - r_\alpha\||\psi|\|_{H_\lambda}^2 = \int_{I_\lambda} \left(G_\alpha^\lambda|\psi| - r_\alpha|\psi|\right)|\psi|\, d\mu \geq 0. \tag{91}$$

Combining (90) and (91), both inequalities are equalities. Equality in (90) forces every nonpositive summand to vanish:

$$|c_n|^2\Big[\frac{1}{\mu_n(\lambda)+\alpha} - r_\alpha\Big] = 0, \qquad n \geq 0,$$

hence $c_n = 0$ whenever $\mu_n(\lambda) > \mu_0(\lambda)$. Therefore $|\psi| \in \ker(L_\lambda - \mu_0(\lambda)I) = \ker(G_\alpha^\lambda - r_\alpha I)$, which is the identity in H_λ

$$G_\alpha^\lambda|\psi| = r_\alpha|\psi|. \tag{92}$$

(v) Strict positivity of $|\psi|$ *and definition of* ξ_λ. Since $|\psi| \geq 0$ μ-a.e. and $|\psi| \not\equiv 0$, the positivity-improving property of G_α^λ given by Theorem 4.3(ii) yields

$$G_\alpha^\lambda|\psi| > 0 \quad \mu\text{-a.e. on } I_\lambda.$$

Combined with (92) and $r_\alpha > 0$,

$$|\psi| = r_\alpha^{-1}G_\alpha^\lambda|\psi| > 0 \quad \mu\text{-a.e. on } I_\lambda.$$

Define

$$\xi_\lambda := \frac{|\psi|}{\||\psi|\|_{H_\lambda}} \in \mathrm{Dom}(L_\lambda). \tag{93}$$

By construction, ξ_λ is real-valued, L^2-normalized, strictly positive μ-a.e. on I_λ, and $L_\lambda\xi_\lambda = \mu_0(\lambda)\xi_\lambda$.

(vi) Simplicity of the eigenspace. We argue by contradiction. Suppose, toward a contradiction, that $\dim\ker(G_\alpha^\lambda - r_\alpha I) \geq 2$. The orthogonal complement of $\mathbb{C}\xi_\lambda$ inside $\ker(G_\alpha^\lambda - r_\alpha I)$ then has dimension ≥ 1; pick a nonzero element η_0 of it, so that

$$\eta_0 \in \ker(G_\alpha^\lambda - r_\alpha I), \qquad \eta_0 \neq 0, \qquad \langle \eta_0, \xi_\lambda\rangle_{H_\lambda} = 0.$$

By (88) and the reality of r_α, $\overline{\eta_0} \in \ker(G_\alpha^\lambda - r_\alpha I)$, so the real-valued functions

$$\eta_1 := \Re\eta_0 = \frac{\eta_0 + \overline{\eta_0}}{2}, \qquad \eta_2 := \Im\eta_0 = \frac{\eta_0 - \overline{\eta_0}}{2i}$$

both lie in $\ker(G_\alpha^\lambda - r_\alpha I)$. Since ξ_λ is real-valued, $\langle \eta_0, \xi_\lambda\rangle_{H_\lambda} = \int_{I_\lambda} \eta_0\, \xi_\lambda\, d\mu = \int_{I_\lambda} \eta_1\, \xi_\lambda\, d\mu + i\int_{I_\lambda} \eta_2\, \xi_\lambda\, d\mu = 0$, and the real and imaginary parts must vanish separately: $\langle \eta_j, \xi_\lambda\rangle_{H_\lambda} = 0$ for $j = 1, 2$. Since $\eta_0 \neq 0$, at least one of η_1, η_2 is nonzero; let η denote this nonzero one. Thus

$$\eta \in \ker(G_\alpha^\lambda - r_\alpha I), \qquad \eta \text{ real-valued and nonzero}, \qquad \langle \eta, \xi_\lambda\rangle_{H_\lambda} = 0. \tag{94}$$

Decompose $\eta = \eta^+ - \eta^-$ with $\eta^\pm := \max(\pm\eta, 0) \geq 0$ μ-a.e., $|\eta| = \eta^+ + \eta^-$. Both $\eta^\pm$ are nonzero in H_λ: were $\eta^- \equiv 0$ (the case $\eta^+ \equiv 0$ being symmetric), one would have $\eta = \eta^+ \geq 0$ μ-a.e. and $\eta \not\equiv 0$, whence, since $\xi_\lambda > 0$ μ-a.e. on I_λ, $\int_{I_\lambda} \eta\, \xi_\lambda \, d\mu > 0$, contradicting orthogonality in (94).

By Theorem 4.3(ii),

$$G_\alpha^\lambda \eta^+ > 0, \qquad G_\alpha^\lambda \eta^- > 0 \qquad \mu\text{-a.e. on } I_\lambda.$$

For real numbers a, b with $\min(a, b) > 0$, the strict triangle inequality $a + b > |a - b|$ holds. Applied μ-a.e. at $a = (G_\alpha^\lambda \eta^+)(u)$, $b = (G_\alpha^\lambda \eta^-)(u)$,

(95)
$$G_\alpha^\lambda |\eta| = G_\alpha^\lambda \eta^+ + G_\alpha^\lambda \eta^- > \left|G_\alpha^\lambda \eta^+ - G_\alpha^\lambda \eta^-\right| = |G_\alpha^\lambda \eta| = r_\alpha |\eta| \qquad \mu\text{-a.e. on } I_\lambda.$$

Multiply (95) by ξ_λ, which is strictly positive μ-a.e.; the resulting integrand $\xi_\lambda\big(G_\alpha^\lambda|\eta| - r_\alpha|\eta|\big)$ is μ-a.e. strictly positive on I_λ. Cauchy–Schwarz applied to the pairs $(\xi_\lambda, G_\alpha^\lambda|\eta|)$ and $(\xi_\lambda, |\eta|)$, both in H_λ, ensures the integrand is in $L^1(I_\lambda, \mu)$. Hence

$$\langle \xi_\lambda, G_\alpha^\lambda |\eta| \rangle_{H_\lambda} = \int_{I_\lambda} \xi_\lambda \, G_\alpha^\lambda |\eta| \, d\mu > r_\alpha \int_{I_\lambda} \xi_\lambda \, |\eta| \, d\mu = r_\alpha \langle \xi_\lambda, |\eta| \rangle_{H_\lambda}. \tag{96}$$

On the other hand, G_α^λ is self-adjoint and $G_\alpha^\lambda \xi_\lambda = r_\alpha \xi_\lambda$, so

$$\langle \xi_\lambda, G_\alpha^\lambda |\eta| \rangle_{H_\lambda} = \langle G_\alpha^\lambda \xi_\lambda, |\eta| \rangle_{H_\lambda} = r_\alpha \langle \xi_\lambda, |\eta| \rangle_{H_\lambda}. \tag{97}$$

The strict inequality (96) contradicts the equality (97). Therefore $\dim\ker(G_\alpha^\lambda - r_\alpha I) = 1$, and by (83), $\dim\ker(L_\lambda - \mu_0(\lambda) I) = 1$. This proves assertion 2.

(*vii*) *Uniqueness of the normalized strictly positive ground state.* Let $\xi' \in \ker(L_\lambda - \mu_0(\lambda) I)$ satisfy $\|\xi'\|_{H_\lambda} = 1$ and $\xi' > 0$ μ-a.e. on I_λ. By the simplicity established in (vi), $\xi' = c\xi_\lambda$ for some $c \in \mathbb{C}$. Pairing with ξ_λ,

$$c = c\, \|\xi_\lambda\|_{H_\lambda}^2 = \langle c\xi_\lambda, \xi_\lambda \rangle_{H_\lambda} = \langle \xi', \xi_\lambda \rangle_{H_\lambda} = \int_{I_\lambda} \xi' \, \xi_\lambda \, d\mu > 0,$$

where the strict positivity of the integral uses that both ξ' and ξ_λ are strictly positive μ-a.e. on I_λ, while $\mu(I_\lambda) > 0$. Thus $c > 0$, and the normalization $\|\xi'\|_{H_\lambda} = |c| \, \|\xi_\lambda\|_{H_\lambda} = c$ forces $c = 1$. Therefore $\xi' = \xi_\lambda$, which establishes assertion 3 and completes the proof. □

5.3. *Inversion symmetry and evenness of the ground state.* We now show that the ground state inherits the inversion symmetry of the form.

PROPOSITION 5.3. *Let*

$$I_\lambda = [\lambda^{-1}, \lambda], \qquad H_\lambda = L^2(I_\lambda, d^\times u), \qquad d^\times u = \frac{du}{u}.$$

Define the reflection operator

$$(\mathcal{R} f)(u) := f(u^{-1}), \qquad u \in I_\lambda.$$

Then:

1. $\mathcal{R}$ *is a unitary self-adjoint operator on* H_λ, *and*
$$\mathcal{R}^* = \mathcal{R}, \qquad \mathcal{R}^2 = I_{H_\lambda}.$$
2. $\mathcal{R}$ *preserves the form domain* $\mathcal{F}_\lambda$, *and for all* $f, g \in \mathcal{F}_\lambda$,
$$\mathcal{E}_\lambda(\mathcal{R}f, \mathcal{R}g) = \mathcal{E}_\lambda(f, g).$$
3. *If* L_λ *is the self-adjoint operator associated with* $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$, *then*
$$\mathcal{R}L_\lambda = L_\lambda\mathcal{R}.$$
4. *Let* ξ_λ *be the unique* L^2*-normalized strictly positive ground state from the previous theorem. Then*
$$\mathcal{R}\xi_\lambda = \xi_\lambda,$$
that is,
$$\xi_\lambda(u^{-1}) = \xi_\lambda(u) \quad \textit{a.e. on } I_\lambda.$$

Proof. We organize the proof into seven parts (i)–(vii); part (i) establishes statement 1, parts (ii)–(v) establish statement 2, part (vi) establishes statement 3, and part (vii) establishes statement 4.

(i) Well-definedness, isometry, and self-adjointness of $\mathcal{R}$. The interval $I_\lambda = [\lambda^{-1}, \lambda]$ is invariant under inversion: $u \in I_\lambda \iff u^{-1} \in I_\lambda$. Hence $u \mapsto u^{-1}$ is a measurable involution on I_λ, and the L^2-equivalence class of $u \mapsto f(u^{-1})$ is well defined for every $f \in H_\lambda$; the operator $\mathcal{R}$ is therefore linear and well defined on H_λ.

The multiplicative Haar measure $d^\times u = du/u$ on $\mathbb{R}^\times_{>0}$ is invariant under the continuous group automorphism $u \mapsto u^{-1}$. Concretely, the substitution $v = u^{-1}$ in the integral defining $\|\mathcal{R}f\|^2_{H_\lambda}$ gives $du = -v^{-2}\,dv$, hence $d^\times u = -d^\times v$; the orientation of the integration limits is also reversed (as u traverses $[\lambda^{-1}, \lambda]$, the new variable v traverses $[\lambda, \lambda^{-1}]$), and the two sign reversals compensate to yield
$$\|\mathcal{R}f\|^2_{H_\lambda} = \int_{\lambda^{-1}}^{\lambda} |f(u^{-1})|^2\, d^\times u = \int_{\lambda^{-1}}^{\lambda} |f(v)|^2\, d^\times v = \|f\|^2_{H_\lambda}.$$
The same substitution applied to the sesquilinear pairing yields, for every $f, g \in H_\lambda$,
$$\langle \mathcal{R}f, g\rangle_{H_\lambda} = \int_{\lambda^{-1}}^{\lambda} f(u^{-1})\overline{g(u)}\, d^\times u = \int_{\lambda^{-1}}^{\lambda} f(v)\overline{g(v^{-1})}\, d^\times v = \langle f, \mathcal{R}g\rangle_{H_\lambda},$$
so $\mathcal{R}^* = \mathcal{R}$. Finally, $(\mathcal{R}^2 f)(u) = f((u^{-1})^{-1}) = f(u)$ a.e. on I_λ, so $\mathcal{R}^2 = I_{H_\lambda}$. In particular, $\mathcal{R}$ is unitary, which proves statement 1.

(ii) Reflection-invariance of the form core $\mathcal{V}_\lambda$. By the construction in Subsection 2.4, the trigonometric basis $V_j(u) = \mathcal{T}_j(\log(\lambda u))$ is given explicitly by

$$V_j(u) = L^{-1/2} \exp\bigl(2\pi i j \log(\lambda u)/L\bigr), \qquad j \in \mathbb{Z},\ u \in I_\lambda,$$

with $L = 2\log\lambda$. For each $j \in \mathbb{Z}$ and a.e. $u \in I_\lambda$,

(98)
$$\mathcal{R}V_j(u) = V_j(u^{-1}) = L^{-1/2} \exp\bigl(2\pi i j(\log\lambda - \log u)/L\bigr) = L^{-1/2} e^{2\pi i j \log\lambda/L}\, e^{-2\pi i j \log u/L},$$

while

(99)
$$V_{-j}(u) = L^{-1/2} \exp\bigl(-2\pi i j(\log\lambda + \log u)/L\bigr) = L^{-1/2} e^{-2\pi i j \log\lambda/L}\, e^{-2\pi i j \log u/L}.$$

The two prefactors of (98) and (99) differ by the multiplicative factor

$$e^{2\pi i j \log\lambda/L} \cdot e^{2\pi i j \log\lambda/L} = e^{4\pi i j \log\lambda/L} = e^{2\pi i j} = 1,$$

where the second equality uses $4\log\lambda/L = 2$ (which is the convention $L = 2\log\lambda$) and the third uses $j \in \mathbb{Z}$. Therefore

$$\mathcal{R}V_j = V_{-j}, \qquad j \in \mathbb{Z}. \tag{100}$$

The map $\mathcal{R}$ permutes the basis $\{V_j\}_{j\in\mathbb{Z}}$ by sign reversal of the index; in particular,

$$\mathcal{R}\mathcal{V}_\lambda = \mathcal{V}_\lambda. \tag{101}$$

(iii) Reflection-invariance of the archimedean part. For $f, g \in \mathcal{F}_\lambda$, set

$$\varphi_f := Zf, \quad \varphi_g := Zg \in L^2(I_x, dx), \qquad F := E_x\varphi_f, \quad G := E_x\varphi_g \in L^2(\mathbb{R}, dx),$$

where $Z : H_\lambda \to L^2(I_x, dx)$ is the logarithmic unitary of Lemma 2.3 with $I_x = [-\ell, \ell]$, $\ell = \log\lambda$, and $E_x : L^2(I_x, dx) \hookrightarrow L^2(\mathbb{R}, dx)$ is zero extension as in Theorem 3.2(2). Define the additive reflection

$$J : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, dx), \qquad (Jh)(x) := h(-x).$$

For a.e. $x \in I_x$, the change of variable $u = e^x$ gives

$$(Z\mathcal{R}f)(x) = \mathcal{R}f(e^x) = f(e^{-x}) = (Zf)(-x) = (J\varphi_f)(x).$$

Both $E_x\varphi_f$ and $E_x(J\varphi_f)$ vanish for $|x| > \ell$, and J preserves the symmetric set I_x; hence

$$E_x Z(\mathcal{R}f) = JF, \qquad E_x Z(\mathcal{R}g) = JG, \qquad \text{a.e. on } \mathbb{R}. \tag{102}$$

By Theorem 3.2(2), the diagonal Lévy–Khintchine identity

$$\mathcal{E}_{\infty,\lambda}(f, f) = \pi \int_0^\infty \|F(\cdot + r) - F\|^2_{L^2(\mathbb{R})}\, \nu(dr)$$

holds for every $f \in \mathcal{F}_\lambda$. The polarization identity for sesquilinear forms, applied to this identity (with all four resulting integrals finite for $f, g \in \mathcal{F}_\lambda$), yields

$$\mathcal{E}_{\infty,\lambda}(f,g) = \pi \int_0^\infty \langle F(\cdot+r)-F,\, G(\cdot+r)-G\rangle_{L^2(\mathbb{R})}\, \nu(dr), \qquad f,g \in \mathcal{F}_\lambda. \tag{103}$$

We now verify the invariance of the integrand of (103) under the substitution $(F,G) \mapsto (JF, JG)$. Fix $r > 0$. For every $x \in \mathbb{R}$,

$$JF(x+r) - JF(x) = F(-x-r) - F(-x).$$

Substituting $y = -x$ (Lebesgue measure invariant under $y \mapsto -y$),

$$\int_{\mathbb{R}} [JF(x+r)-JF(x)]\overline{[JG(x+r)-JG(x)]}\, dx = \int_{\mathbb{R}} [F(y-r)-F(y)]\overline{[G(y-r)-G(y)]}\, dy. \tag{104}$$

Substituting $z = y - r$ (Lebesgue measure invariant under $z \mapsto z + r$),

$$F(y-r) - F(y) = F(z) - F(z+r) = -\big(F(z+r) - F(z)\big),$$

and similarly for G; the two minus signs cancel inside the integrand on the right-hand side of (104), yielding

$$\int_{\mathbb{R}} [JF(x+r)-JF(x)]\overline{[JG(x+r)-JG(x)]}\, dx = \int_{\mathbb{R}} [F(z+r)-F(z)]\overline{[G(z+r)-G(z)]}\, dz. \tag{105}$$

The right-hand side of (105) is the integrand of (103). Combining (103)–(105) with (102) and integrating against $\nu(dr)$ on $(0,\infty)$ via Tonelli's theorem [16, Vol. I, Thm. 3.4.4],

$$\mathcal{E}_{\infty,\lambda}(\mathcal{R}f, \mathcal{R}g) = \mathcal{E}_{\infty,\lambda}(f,g), \qquad f,g \in \mathcal{F}_\lambda. \tag{106}$$

(iv) Reflection conjugation of the compressed dilations. Fix an integer $n \geq 2$. By Proposition 2.7 (3), for every $h \in H_\lambda$,

$$(U_n h)(u) = \mathbf{1}_{I_\lambda}(nu)\, h(nu) \qquad \text{a.e. } u \in I_\lambda.$$

Applying this together with $(\mathcal{R}f)(w) = f(w^{-1})$ at $w = nu^{-1}$, for a.e. $u \in I_\lambda$,

$$(\mathcal{R}U_n\mathcal{R}f)(u) = (U_n\mathcal{R}f)(u^{-1}) = \mathbf{1}_{I_\lambda}(nu^{-1})\, (\mathcal{R}f)(nu^{-1}) = \mathbf{1}_{I_\lambda}(nu^{-1})\, f(n^{-1}u). \tag{107}$$

On the other hand, $U_{n^{-1}} = R_\lambda S_{n^{-1}} E_\lambda$ in the notation of Definition 2.6, with $n^{-1} \in (0,1)$ as the dilation parameter, so the same proposition yields

$$(U_{n^{-1}} f)(u) = \mathbf{1}_{I_\lambda}(n^{-1}u)\, f(n^{-1}u) \qquad \text{a.e. } u \in I_\lambda. \tag{108}$$

To compare the two indicator factors in (107) and (108), we exploit the inversion-invariance of I_λ. For every $x \in \mathbb{R}^\times_{>0}$, $x \in I_\lambda \iff x^{-1} \in I_\lambda$, so $\mathbf{1}_{I_\lambda}(x) = \mathbf{1}_{I_\lambda}(x^{-1})$. Setting $x = n^{-1}u$ (so that $x^{-1} = nu^{-1}$) gives

$$\mathbf{1}_{I_\lambda}(nu^{-1}) = \mathbf{1}_{I_\lambda}(n^{-1}u), \qquad u \in \mathbb{R}^\times_{>0}. \tag{109}$$

Combining (107), (108), and (109),

$$\mathcal{R}U_n\mathcal{R} = U_{n^{-1}} = U_n^*, \tag{110}$$

where the second equality is Proposition 2.7 (2). With $\mathcal{R}^2 = I_{H_\lambda}$ from (i),

$$\mathcal{R}T(n)\mathcal{R} = n^{-1/2}\big(\mathcal{R}U_n\mathcal{R} + \mathcal{R}U_n^*\mathcal{R}\big) = n^{-1/2}(U_n^* + U_n) = T(n). \tag{111}$$

For $f, g \in H_\lambda$, applying (111) together with $\mathcal{R}^* = \mathcal{R}$ and $\mathcal{R}^2 = I_{H_\lambda}$,

$$\langle T(n)\mathcal{R}f, \mathcal{R}g\rangle_{H_\lambda} = \langle \mathcal{R}T(n)f, \mathcal{R}g\rangle_{H_\lambda} = \langle T(n)f, \mathcal{R}^*\mathcal{R}g\rangle_{H_\lambda} = \langle T(n)f, g\rangle_{H_\lambda}.$$

Multiplying by $-\Lambda(n)$ and summing over $1 < n \le \lambda^2$,

$$Q_{p,\lambda}(\mathcal{R}f, \mathcal{R}g) = Q_{p,\lambda}(f, g), \qquad f, g \in H_\lambda. \tag{112}$$

Since $\langle \mathcal{R}f, \mathcal{R}g\rangle_{H_\lambda} = \langle f, g\rangle_{H_\lambda}$ by (i),

$$\mathcal{E}_{p,\lambda}(\mathcal{R}f, \mathcal{R}g) = Q_{p,\lambda}(\mathcal{R}f, \mathcal{R}g) + c_{p,\lambda}\langle \mathcal{R}f, \mathcal{R}g\rangle_{H_\lambda} = Q_{p,\lambda}(f, g) + c_{p,\lambda}\langle f, g\rangle_{H_\lambda} = \mathcal{E}_{p,\lambda}(f, g). \tag{113}$$

(v) Form-invariance identity and domain invariance. For $f, g \in \mathcal{V}_\lambda$, both $\mathcal{R}f, \mathcal{R}g \in \mathcal{V}_\lambda \subset \mathcal{F}_\lambda$ by (101); hence $\mathcal{E}_\lambda(\mathcal{R}f, \mathcal{R}g)$ is finite-valued, and the additive decomposition $\mathcal{E}_\lambda = \mathcal{E}_{\infty,\lambda} + \mathcal{E}_{p,\lambda}$ from Theorem 3.2(4) together with (106) and (113) gives

$$\mathcal{E}_\lambda(\mathcal{R}f, \mathcal{R}g) = \mathcal{E}_\lambda(f, g), \qquad f, g \in \mathcal{V}_\lambda. \tag{114}$$

Specializing to $g = f$,

$$\mathcal{E}_\lambda(\mathcal{R}f, \mathcal{R}f) = \mathcal{E}_\lambda(f, f), \qquad f \in \mathcal{V}_\lambda. \tag{115}$$

To extend to $\mathcal{F}_\lambda$, we use the form-core property of $\mathcal{V}_\lambda$ and the closedness of $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ given by Theorem 3.2(1). Equip $\mathcal{F}_\lambda$ with the graph norm

$$\|h\|^2_{\mathcal{E}_\lambda,1} := \mathcal{E}_\lambda(h, h) + \|h\|^2_{H_\lambda}.$$

For every $f \in \mathcal{F}_\lambda$, the form-core property provides a sequence $(f_k) \subset \mathcal{V}_\lambda$ with $f_k \to f$ in $\|\cdot\|_{\mathcal{E}_\lambda,1}$. By (115) applied to $f_k - f_m \in \mathcal{V}_\lambda$ together with the unitarity of $\mathcal{R}$ from (i),

$$\|\mathcal{R}f_k - \mathcal{R}f_m\|^2_{\mathcal{E}_\lambda,1} = \mathcal{E}_\lambda\big(\mathcal{R}(f_k - f_m), \mathcal{R}(f_k - f_m)\big) + \|\mathcal{R}(f_k - f_m)\|^2_{H_\lambda} = \|f_k - f_m\|^2_{\mathcal{E}_\lambda,1}, \tag{116}$$

so $(\mathcal{R}f_k) \subset \mathcal{V}_\lambda$ is Cauchy in $\|\cdot\|_{\mathcal{E}_\lambda,1}$. By closedness, there exists $\widetilde{f} \in \mathcal{F}_\lambda$ with $\mathcal{R}f_k \to \widetilde{f}$ in $\|\cdot\|_{\mathcal{E}_\lambda,1}$, hence in H_λ as well. The boundedness of $\mathcal{R}$ on H_λ also gives $\mathcal{R}f_k \to \mathcal{R}f$ in H_λ, so the uniqueness of H_λ-limits forces $\widetilde{f} = \mathcal{R}f$. Therefore

$$\mathcal{R}f \in \mathcal{F}_\lambda \qquad \text{for every } f \in \mathcal{F}_\lambda. \tag{117}$$

Replacing f by $\mathcal{R}f$ in (117) and using $\mathcal{R}^2 = I_{H_\lambda}$ gives the reverse inclusion; hence

$$\mathcal{R}\mathcal{F}_\lambda = \mathcal{F}_\lambda. \tag{118}$$

To pass from (114) to the full identity on $\mathcal{F}_\lambda \times \mathcal{F}_\lambda$, take any $f, g \in \mathcal{F}_\lambda$ together with approximating sequences $(f_k), (g_k) \subset \mathcal{V}_\lambda$ converging in $\|\cdot\|_{\mathcal{E}_\lambda,1}$. By (116) (and its analogue with g in place of f), $\mathcal{R}f_k \to \mathcal{R}f$ and $\mathcal{R}g_k \to \mathcal{R}g$ in $\|\cdot\|_{\mathcal{E}_\lambda,1}$. The form $\mathcal{E}_\lambda$ is sesquilinear and continuous in this graph norm ([18, Chap. VI, §1]). Passing to the limit $k \to \infty$ in the identity $\mathcal{E}_\lambda(\mathcal{R}f_k, \mathcal{R}g_k) = \mathcal{E}_\lambda(f_k, g_k)$, given by (114),

$$\mathcal{E}_\lambda(\mathcal{R}f, \mathcal{R}g) = \mathcal{E}_\lambda(f, g), \qquad f, g \in \mathcal{F}_\lambda. \tag{119}$$

Together with (118), this proves statement 2.

(vi) Operator commutation $\mathcal{R}L_\lambda = L_\lambda\mathcal{R}$. By the first representation theorem [18, Chap. VI, Thm. 2.1], the nonnegative self-adjoint operator L_λ associated with $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ satisfies

$$\mathcal{E}_\lambda(u, v) = \langle L_\lambda u, v\rangle_{H_\lambda}, \qquad u \in \mathrm{Dom}(L_\lambda),\ v \in \mathcal{F}_\lambda;$$

moreover, an element $u \in \mathcal{F}_\lambda$ lies in $\mathrm{Dom}(L_\lambda)$ if and only if there exists $h \in H_\lambda$ with $\mathcal{E}_\lambda(u, v) = \langle h, v\rangle_{H_\lambda}$ for every $v \in \mathcal{F}_\lambda$, in which case $h = L_\lambda u$.

Take $f \in \mathrm{Dom}(L_\lambda)$. For every $v \in \mathcal{F}_\lambda$, $\mathcal{R}v \in \mathcal{F}_\lambda$ by (118). Using (119) (with $\mathcal{R}v$ in place of the second argument and exploiting $\mathcal{R}^2 = I_{H_\lambda}$), the first representation theorem, and $\mathcal{R}^* = \mathcal{R}$ from (i),

$$\mathcal{E}_\lambda(\mathcal{R}f, v) = \mathcal{E}_\lambda(\mathcal{R}f, \mathcal{R}\mathcal{R}v) = \mathcal{E}_\lambda(f, \mathcal{R}v) = \langle L_\lambda f, \mathcal{R}v\rangle_{H_\lambda} = \langle \mathcal{R}^* L_\lambda f, v\rangle_{H_\lambda} = \langle \mathcal{R}L_\lambda f, v\rangle_{H_\lambda}.$$

The functional $v \mapsto \langle \mathcal{R}L_\lambda f, v\rangle_{H_\lambda}$ is bounded on H_λ, with bound $\|L_\lambda f\|_{H_\lambda}$. By the converse direction of the first representation theorem, $\mathcal{R}f \in \mathrm{Dom}(L_\lambda)$ and

$$L_\lambda(\mathcal{R}f) = \mathcal{R}L_\lambda f, \qquad f \in \mathrm{Dom}(L_\lambda). \tag{120}$$

Replacing f by $\mathcal{R}f$ in (120) and using $\mathcal{R}^2 = I_{H_\lambda}$ gives the reverse inclusion $\mathcal{R}\,\mathrm{Dom}(L_\lambda) = \mathrm{Dom}(L_\lambda)$. Therefore $\mathcal{R}L_\lambda = L_\lambda\mathcal{R}$ on $\mathrm{Dom}(L_\lambda)$, which proves statement 3.

(vii) Reflection-invariance of the ground state. By Theorem 5.2, $\mu_0(\lambda)$ is a simple eigenvalue of L_λ with one-dimensional eigenspace $\mathbb{C}\xi_\lambda \subset \mathrm{Dom}(L_\lambda)$, and the unique L^2-normalized representative ξ_λ is real-valued, strictly positive μ-a.e. on I_λ.

The commutation (120) gives

$$L_\lambda(\mathcal{R}\xi_\lambda) = \mathcal{R}L_\lambda\xi_\lambda = \mu_0(\lambda)\,\mathcal{R}\xi_\lambda,$$

so $\mathcal{R}\xi_\lambda \in \ker(L_\lambda - \mu_0(\lambda)I) = \mathbb{C}\xi_\lambda$. There exists therefore a unique $c \in \mathbb{C}$ with

$$\mathcal{R}\xi_\lambda = c\,\xi_\lambda. \tag{121}$$

We identify $c = 1$ in three substeps.

(vii.a) c is real. Since ξ_λ is real-valued, for a.e. $u \in I_\lambda$,

$$\overline{\mathcal{R}\xi_\lambda(u)} = \overline{\xi_\lambda(u^{-1})} = \xi_\lambda(u^{-1}) = \mathcal{R}\xi_\lambda(u),$$

so $\mathcal{R}\xi_\lambda$ is real-valued μ-a.e. on I_λ. The set $\{u \in I_\lambda : \xi_\lambda(u) > 0\}$ has full μ-measure; on this set, the ratio $\mathcal{R}\xi_\lambda(u)/\xi_\lambda(u)$ is a real number, and (121) forces this ratio to equal c μ-a.e. Hence $c \in \mathbb{R}$.

(vii.b) $|c| = 1$. By the unitarity of $\mathcal{R}$ from (i),

$$|c| = |c| \cdot \|\xi_\lambda\|_{H_\lambda} = \|c\,\xi_\lambda\|_{H_\lambda} = \|\mathcal{R}\xi_\lambda\|_{H_\lambda} = \|\xi_\lambda\|_{H_\lambda} = 1.$$

Combined with (vii.a), $c \in \{-1, +1\}$.

(vii.c) $c = +1$. Pairing (121) with ξ_λ and using $\|\xi_\lambda\|_{H_\lambda} = 1$,

$$c = \langle c\,\xi_\lambda, \xi_\lambda\rangle_{H_\lambda} = \langle \mathcal{R}\xi_\lambda, \xi_\lambda\rangle_{H_\lambda} = \int_{I_\lambda} \xi_\lambda(u^{-1})\,\xi_\lambda(u)\,d^\times u.$$

The integrand is the product of two functions both strictly positive μ-a.e. on I_λ, and $\mu(I_\lambda) = 2\log\lambda > 0$, so the integral is strictly positive. Combined with (vii.b), this rules out $c = -1$ and forces $c = +1$. Therefore

$$\mathcal{R}\xi_\lambda = \xi_\lambda, \qquad \text{equivalently,} \qquad \xi_\lambda(u^{-1}) = \xi_\lambda(u) \quad \mu\text{-a.e. on } I_\lambda,$$

which proves statement 4 and completes the proof. □

5.4. *Interval distributions and the form kernel.* We next transport the form to the interval model and identify the corresponding distribution kernel.

Proposition 5.4. *Let $\lambda > 1$, and set*

$$\ell = \log\lambda, \qquad L = 2\ell.$$

Consider the isometric isomorphism used in [7, Proposition 3.2]

$$\kappa : L^2([0, L], dy) \to H_\lambda = L^2([\lambda^{-1}, \lambda], d^\times u), \qquad \kappa(f)(u) = f(\log(\lambda u)).$$

Let

$$Q_{D,\lambda} := \mathcal{E}_\lambda - c_{D,\lambda}\langle\cdot,\cdot\rangle_{H_\lambda}, \qquad A_\lambda := L_\lambda - c_{D,\lambda} I.$$

Set

$$T := \operatorname{span}\{\mathcal{T}_n : n \in \mathbb{Z}\} \subset L^2([0, L], dy), \qquad \mathcal{T}_n(y) := L^{-1/2} e^{2\pi i n y/L}.$$

Then there exists a real distribution $D_\lambda \in \mathcal{D}'([0, L])$ such that:

1. *For every $f, g \in T$,*

$$Q_{D,\lambda}(\kappa f, \kappa g) = D_\lambda(q(g, f)),$$

where

$$q(f, g)(y) := (f^* * g)(y) + (f^* * g)(-y), \qquad f^*(x) := \overline{f(-x)}.$$

This is precisely the type of quadratic form induced by a real distribution on $[0, L]$ discussed in [8, formula (6)].

2. *T is a form core for the above pre-form; its closure is precisely*

$$\overline{Q}_{D,\lambda}(f,g) := Q_{D,\lambda}(\kappa f, \kappa g)$$

on $\kappa^{-1}(\mathcal{F}_\lambda)$, and the associated self-adjoint operator is

$$\mathcal{S}_\lambda := \kappa^{-1} A_\lambda \kappa = \kappa^{-1}(L_\lambda - c_{D,\lambda} I)\kappa.$$

Proof. We organize the proof into seven parts (i)–(vii); parts (i)–(vi) establish statement 1, and part (vii) establishes statement 2.

(i) Unitarity of κ and form-core transfer. The change of variable $y = \log(\lambda u)$ identifies $u \in I_\lambda = [\lambda^{-1}, \lambda]$ with $y \in [0, L]$ via $u = e^{y-\ell}$, with Jacobian $d^\times u = du/u = dy$. For every $f \in L^2([0,L], dy)$,

$$\|\kappa f\|^2_{H_\lambda} = \int_{\lambda^{-1}}^{\lambda} |f(\log(\lambda u))|^2 \, d^\times u = \int_0^L |f(y)|^2 \, dy = \|f\|^2_{L^2([0,L])},$$

and the L^2-adjoint of κ is $\eta \mapsto \eta(e^{\,\cdot - \ell})$, which serves as both left and right inverse of κ. Hence κ is unitary.

For each $n \in \mathbb{Z}$, the trigonometric basis vector $V_n \in \mathcal{V}_\lambda$ defined in Subsection 2.4 satisfies

$$V_n(u) = L^{-1/2} e^{2\pi i n \log(\lambda u)/L} = \mathcal{T}_n(\log(\lambda u)) = \kappa(\mathcal{T}_n)(u), \qquad u \in I_\lambda,$$

giving

$$\kappa(T) = \operatorname{span}\{V_n : n \in \mathbb{Z}\} = \mathcal{V}_\lambda. \tag{122}$$

By [5, §2] (cited in [7, Proposition 3.4]), $\mathcal{V}_\lambda$ is a form core for the closed lower-semibounded form QW_λ. By Proposition 3.4 of the present paper, $\operatorname{Dom}(Q_{D,\lambda}) = \operatorname{Dom}(QW_\lambda)$ with equivalent graph norms (the difference B_λ being bounded by Lemma 3.3), so $\mathcal{V}_\lambda$ is also a form core for $Q_{D,\lambda}$. Pulling back the graph norm via the unitary κ preserves all density and closedness statements; hence, by (122), $T = \kappa^{-1}\mathcal{V}_\lambda$ is a form core for the closed lower-semibounded sesquilinear form

$$\overline{Q}_{D,\lambda}(f,g) := Q_{D,\lambda}(\kappa f, \kappa g)$$

on $\operatorname{Dom}(\overline{Q}_{D,\lambda}) = \kappa^{-1}\operatorname{Dom}(Q_{D,\lambda}) = \kappa^{-1}\mathcal{F}_\lambda \subset L^2([0,L], dy)$.

(ii) The correlation function in interval coordinates. Fix $f, g \in T$ and set $F := \kappa f \in \mathcal{V}_\lambda$, $G := \kappa g \in \mathcal{V}_\lambda$. Extend f, g by zero from $[0, L]$ to all of $\mathbb{R}$, retaining the same notation; the supports remain in $[0, L]$. By Definition 2.9,

$$F_{F,G}(a) = \int_{\mathbb{R}^\times_{>0}} F(au)\overline{G(u)} \, d^\times u, \qquad a \in \mathbb{R}^\times_{>0}.$$

Apply the change of variable $x = \log(\lambda u)$, so $u = e^{x-\ell}$ and $d^\times u = dx$. For $a = e^y \in \mathbb{R}^\times_{>0}$, the dilation $au = e^{y+x-\ell}$ corresponds to $\log(\lambda(au)) = x + y$;

hence $F(au) = f(x+y)$ and $G(u) = g(x)$, yielding

$$F_{F,G}(e^y) = \int_{\mathbb{R}} f(x+y)\,\overline{g(x)}\,dx, \qquad y \in \mathbb{R}. \tag{123}$$

The Hermitian involution $g^*(s) := \overline{g(-s)}$ from the proposition statement and the Lebesgue convolution $(g^* * f)(y) := \int_{\mathbb{R}} g^*(y-z)\,f(z)\,dz$ together yield, after the substitution $z = x+y$,

$$(g^* * f)(y) = \int_{\mathbb{R}} \overline{g\big(-(y-z)\big)}\,f(z)\,dz\Big|_{z=x+y} = \int_{\mathbb{R}} \overline{g(x)}\,f(x+y)\,dx, \qquad y \in \mathbb{R}. \tag{124}$$

Comparing (123) with (124),

$$F_{F,G}(e^y) = (g^* * f)(y), \qquad y \in \mathbb{R}. \tag{125}$$

Lemma **??** gives $\operatorname{supp} F_{F,G} \subset [\lambda^{-2}, \lambda^2]$, equivalently $\operatorname{supp}(g^* * f) \subset [-L, L]$, so $F_{F,G}$ lies in the Weil class on $\mathbb{R}^\times_{>0}$.

(iii) Vanishing of Ψ on inversion-antisymmetric arguments. By the explicit formula derived in Subsection 2.4, the semilocal Weil functional acts on the Weil class via

$$\Psi(\Phi) = \widehat{\Phi}(i/2) + \widehat{\Phi}(-i/2) + \int_{\mathbb{R}} \widehat{\Phi}(t)\,w(t)\,dt - \sum_{1<n\le\lambda^2} \Lambda(n) n^{-1/2}\big(\Phi(n) + \Phi(n^{-1})\big),$$

with w given by (18), and $QW_\lambda(F_0, G_0) = \Psi(F_{F_0,G_0})$ for $F_0, G_0 \in \mathcal{V}_\lambda$ by Subsection 2.4.

For every Φ in the Weil class, the substitution $a \mapsto a^{-1}$ (a measure-preserving automorphism of $\mathbb{R}^\times_{>0}$) gives

$$\widehat{(\Phi\circ\iota)}(z) = \int_{\mathbb{R}^\times_{>0}} \Phi(a^{-1})\,a^{-iz}\,d^\times a = \int_{\mathbb{R}^\times_{>0}} \Phi(b)\,b^{iz}\,d^\times b = \widehat{\Phi}(-z), \qquad z \in \mathbb{C}, \tag{126}$$

where $\iota(u) = u^{-1}$. For inversion-antisymmetric Φ (defined by $\Phi \circ \iota = -\Phi$), (126) forces $\widehat{\Phi}(z) = -\widehat{\Phi}(-z)$, that is, $\widehat{\Phi}$ is odd. The three components of Ψ on such Φ are then:

(a) the boundary contribution $\widehat{\Phi}(i/2) + \widehat{\Phi}(-i/2) = \widehat{\Phi}(i/2) - \widehat{\Phi}(i/2) = 0$;
(b) the prime contribution $\sum_{1<n\le\lambda^2} \Lambda(n) n^{-1/2}\big(\Phi(n) + \Phi(n^{-1})\big) = 0$, since $\Phi(n^{-1}) = -\Phi(n)$ termwise;
(c) the archimedean contribution $\int_{\mathbb{R}} \widehat{\Phi}(t)\,w(t)\,dt = 0$, since $w(t) = \theta'(t)/\pi$ is even in t (because $\Re\psi(\frac{1}{4} + \frac{it}{2})$ is even in t by the integral representation (16)), so the integrand is the product of an odd and an even function and integrates to zero.

Combining (a)–(c) gives $\Psi(\Phi) = 0$ on inversion-antisymmetric Φ.

For arbitrary Φ in the Weil class, decompose $\Phi = \Phi_+ + \Phi_-$ with $\Phi_\pm := \frac{1}{2}(\Phi \pm \Phi \circ \iota)$. By the linearity of Ψ and $\Psi(\Phi_-) = 0$,

$$\Psi(\Phi) = \Psi(\Phi_+) = \tfrac{1}{2}\,\Psi(\Phi + \Phi \circ \iota).$$

Define

$$\Psi^{\#} := \tfrac{1}{2}\,\Psi \tag{127}$$

on the Weil class. Then

$$\Psi(\Phi) = \Psi^{\#}\big(\Phi + \Phi \circ \iota\big), \qquad \Phi \text{ in the Weil class,} \tag{128}$$

which recovers the symmetrized form invoked in [7, Lemma 3.1].

(iv) Distribution representation of QW_λ on $[0, L]$. Apply (128) with $\Phi = F_{F,G}$:

$$QW_\lambda(\kappa f, \kappa g) = \Psi(F_{F,G}) = \Psi^{\#}\big(F_{F,G} + F_{F,G} \circ \iota\big). \tag{129}$$

Set $H := F_{F,G} + F_{F,G} \circ \iota$. By (125),

$$H(e^y) = F_{F,G}(e^y) + F_{F,G}(e^{-y}) = (g^* * f)(y) + (g^* * f)(-y) = q(g, f)(y), \qquad y \in \mathbb{R},$$

that is,

$$H(e^y) = q(g, f)(y), \qquad y \in \mathbb{R}. \tag{130}$$

Direct inspection gives $q(g, f)(-y) = (g^* * f)(-y) + (g^* * f)(y) = q(g, f)(y)$, so $q(g, f)$ is an even function on $\mathbb{R}$.

For every $\phi \in \mathcal{D}([0, L])$, the even extension

$$\phi^{\mathrm{ev}} : [-L, L] \to \mathbb{C}, \qquad \phi^{\mathrm{ev}}(y) := \phi(|y|),$$

followed by the change of variable $y = \log a$, defines a function in the Weil class via

$$\widetilde{\phi}(a) := \phi^{\mathrm{ev}}(\log a)\,\mathbf{1}_{[\lambda^{-2}, \lambda^2]}(a), \qquad a \in \mathbb{R}^\times_{>0},$$

whose support is contained in $[\lambda^{-2}, \lambda^2]$ and which is inversion-symmetric: $\widetilde{\phi} \circ \iota = \widetilde{\phi}$. Define

$$D_W(\phi) := \Psi^{\#}(\widetilde{\phi}), \qquad \phi \in \mathcal{D}([0, L]), \tag{131}$$

in the convention of [8, formula (6)], which identifies a distribution on $[0, L]$ with the unique even distribution on $[-L, L]$ that restricts to it. The map $\phi \mapsto \widetilde{\phi}$ is continuous from $\mathcal{D}([0, L])$ into the Weil class, and each of $W_{0,2}$, W_p, and W_∞ is continuous on the Weil class; therefore D_W is a distribution on $[0, L]$.

To verify that D_W is real, take ϕ real-valued. Then $\widetilde{\phi}$ is real-valued and inversion-symmetric on $\mathbb{R}^\times_{>0}$, and the Mellin transform

$$\widehat{\widetilde{\phi}}(t) = \int_{-L}^{L} \phi^{\mathrm{ev}}(y)\, e^{-ity}\, dy = 2 \int_0^L \phi(y) \cos(ty)\, dy$$

is real-valued for $t \in \mathbb{R}$. Each of (a) $\widehat{\widetilde{\phi}}(\pm i/2) = 2\int_0^L \phi(y)\cosh(y/2)\,dy \in \mathbb{R}$, (b) the prime sum (real by direct inspection, since $\widetilde{\phi}(n^{\pm m}) \in \mathbb{R}$), and (c) $\int_{\mathbb{R}} \widehat{\widetilde{\phi}}(t)w(t)\,dt$ (real, since w is real-valued on $\mathbb{R}$) is real. Hence $D_W(\phi) \in \mathbb{R}$, and D_W is a real distribution.

For $f, g \in T$, the function $q(g,f)$ is even (just established) and supported in $[-L, L]$. Comparing H from (130) with the function $\widetilde{q(g,f)}|_{[0,L]}$ defined above: both equal $q(g,f)(\log a)$ on $[1, \lambda^2]$ and $q(g,f)(-\log a) = q(g,f)(\log a^{-1})$ on $[\lambda^{-2}, 1]$ (by evenness of $q(g,f)$), and both vanish outside $[\lambda^{-2}, \lambda^2]$. Hence $H = \widetilde{q(g,f)}|_{[0,L]}$, and (129)–(131) together give

$$QW_\lambda(\kappa f, \kappa g) = D_W\big(q(g,f)\big), \qquad f, g \in T, \tag{132}$$

where $q(g,f)$ on the right denotes its restriction to $[0, L]$.

(v) The boundary kernel as a smooth distribution. The boundary form B_λ from Lemma 3.3 is, by (10),

$$B_\lambda(F, G) = \widehat{F}(i/2)\overline{\widehat{G}(-i/2)} + \widehat{F}(-i/2)\overline{\widehat{G}(i/2)}, \qquad F, G \in H_\lambda.$$

For $F = \kappa f$ and $u = e^{y-\ell}$, $d^\times u = dy$, $u^{\pm 1/2} = e^{\pm(y-\ell)/2}$,

$$\widehat{(\kappa f)}(\pm i/2) = \int_{\lambda^{-1}}^{\lambda} f(\log(\lambda u))\, u^{\pm 1/2}\, d^\times u = \int_0^L f(y)\, e^{\pm(y-\ell)/2}\, dy. \tag{133}$$

Substituting (133) and using the reality of the exponentials,

$$\begin{aligned} B_\lambda(\kappa f, \kappa g) &= \int_0^L \int_0^L f(x)\,\overline{g(y)}\Big[e^{(x-\ell)/2-(y-\ell)/2} + e^{-(x-\ell)/2+(y-\ell)/2}\Big]\,dx\,dy \\ &= \int_0^L \int_0^L f(x)\,\overline{g(y)} \cdot 2\cosh\big(\tfrac{x-y}{2}\big)\,dx\,dy. \end{aligned}$$

For each $y \in [0, L]$, the substitution $z := x - y \in [-y, L-y]$ in the inner integral, together with the zero extension of f outside $[0, L]$, gives

$$\int_0^L f(x) \cdot 2\cosh\big(\tfrac{x-y}{2}\big)\,dx = \int_{\mathbb{R}} 2\cosh(z/2)\, f(y+z)\,dz,$$

the integrand vanishing outside $z \in [-L, L]$. Fubini's theorem then yields

$$B_\lambda(\kappa f, \kappa g) = \int_{-L}^{L} 2\cosh(z/2)\Big[\int_{\mathbb{R}} f(y+z)\,\overline{g(y)}\,dy\Big]\,dz = \int_{-L}^{L} 2\cosh(z/2)\,(g^* * f)(z)\,dz,$$

where the inner integral equals $(g^* * f)(z)$ by (124).

The function $z \mapsto 2\cosh(z/2)$ is even on $[-L, L]$. Splitting the integral at $z = 0$ and substituting $z \mapsto -z$ on $[-L, 0]$,

$$B_\lambda(\kappa f, \kappa g) = \int_0^L 2\cosh(z/2)\Big[(g^* * f)(z) + (g^* * f)(-z)\Big]\,dz = \int_0^L 2\cosh(z/2)\, q(g,f)(z)\,dz.$$

Define the real distribution

$$D_B(\phi) := \int_0^L 2\cosh(z/2)\,\phi(z)\,dz, \qquad \phi \in \mathcal{D}([0,L]), \tag{134}$$

given by the smooth even density $2\cosh(z/2)$. Then

$$B_\lambda(\kappa f, \kappa g) = D_B\big(q(g,f)\big), \qquad f, g \in T. \tag{135}$$

(vi) Construction of D_λ. Combining (132) and (135) via $Q_{D,\lambda} = QW_\lambda - B_\lambda$,

$$Q_{D,\lambda}(\kappa f, \kappa g) = QW_\lambda(\kappa f, \kappa g) - B_\lambda(\kappa f, \kappa g) = D_W\big(q(g,f)\big) - D_B\big(q(g,f)\big).$$

Define

$$D_\lambda := D_W - D_B \in \mathcal{D}'([0,L]). \tag{136}$$

Both D_W and D_B are real distributions on $[0,L]$, by (iv) and (v); therefore D_λ is a real distribution as well, and

$$Q_{D,\lambda}(\kappa f, \kappa g) = D_\lambda\big(q(g,f)\big), \qquad f, g \in T.$$

This identifies the form $(f,g) \mapsto Q_{D,\lambda}(\kappa f, \kappa g)$ on $T \times T$ with the quadratic form induced by a real distribution on $[0,L]$ in the framework of [8, formula (6)], completing statement 1.

(vii) Form core and the associated self-adjoint operator. By (i), T is a form core for the closed lower-semibounded sesquilinear form $\overline{Q}_{D,\lambda}$ on $\mathrm{Dom}(\overline{Q}_{D,\lambda}) = \kappa^{-1}\mathcal{F}_\lambda \subset L^2([0,L], dy)$, with closure

$$\overline{Q}_{D,\lambda}(f,g) = Q_{D,\lambda}(\kappa f, \kappa g), \qquad f, g \in \mathrm{Dom}(\overline{Q}_{D,\lambda}).$$

The first representation theorem [18, Chap. VI, Theorem 2.1] associates with $\overline{Q}_{D,\lambda}$ a unique self-adjoint operator $\mathcal{S}_\lambda$ on $L^2([0,L], dy)$.

In H_λ, the operator associated with $\mathcal{E}_\lambda$ via the same representation theorem is L_λ, by definition. Since $Q_{D,\lambda} = \mathcal{E}_\lambda - c_{D,\lambda}\langle\cdot,\cdot\rangle_{H_\lambda}$ differs from $\mathcal{E}_\lambda$ by the bounded scalar form $c_{D,\lambda}\langle\cdot,\cdot\rangle_{H_\lambda}$, the first representation theorem applied to both forms (which share the form domain $\mathcal{F}_\lambda$) gives

$$A_\lambda = L_\lambda - c_{D,\lambda} I$$

as the operator associated with $Q_{D,\lambda}$ on H_λ.

For any $f \in \kappa^{-1}\,\mathrm{Dom}(A_\lambda)$ and $g \in \kappa^{-1}\mathcal{F}_\lambda$, the unitarity $\kappa^* = \kappa^{-1}$ from (i) yields

$$\overline{Q}_{D,\lambda}(f,g) = Q_{D,\lambda}(\kappa f, \kappa g) = \langle A_\lambda(\kappa f), \kappa g\rangle_{H_\lambda} = \langle \kappa^{-1}A_\lambda\kappa f, g\rangle_{L^2([0,L])}.$$

The functional $g \mapsto \langle \kappa^{-1}A_\lambda\kappa f, g\rangle_{L^2([0,L])}$ is bounded on $L^2([0,L], dy)$, with bound $\|\kappa^{-1}A_\lambda\kappa f\|_{L^2([0,L])} = \|A_\lambda\kappa f\|_{H_\lambda}$. By the converse direction of the first representation theorem applied to $\overline{Q}_{D,\lambda}$, $f \in \mathrm{Dom}(\mathcal{S}_\lambda)$ and $\mathcal{S}_\lambda f = \kappa^{-1}A_\lambda\kappa f$.

The reverse inclusion follows by the same argument with κ and κ^{-1} exchanged. Therefore

$$\mathcal{S}_\lambda = \kappa^{-1} A_\lambda \kappa = \kappa^{-1}(L_\lambda - c_{D,\lambda} I)\kappa,$$

which is precisely the form of lower-bounded essentially self-adjoint operator analyzed in [8, Theorem 6.1]. This proves statement 2 and completes the proof. $\square$

Lemma 5.5 (Closed form and admissible test-function class for D_λ). *Let $\lambda > 1$, set $L := 2\log\lambda$, and let $D_\lambda \in \mathcal{D}'([0, L])$ be the real distribution constructed in Proposition 5.4. Then D_λ admits the closed-form expression*

$$D_\lambda(\phi) = \int_{\mathbb{R}} \widehat{\phi}_c(t)\, w(t)\, dt \; - \sum_{1<n\le\lambda^2} \Lambda(n)\, n^{-1/2}\, \phi(\log n), \tag{137}$$

in which the smooth $\cosh(y/2)$-densities present in D_W and D_B have cancelled exactly. Here

$$\widehat{\phi}_c(t) := \int_0^L \phi(y)\cos(ty)\, dy,$$

$w(t) = \theta'(t)/\pi$ is the archimedean weight of (18), and Λ is the von Mangoldt function.

Let

$$\mathcal{A}_L := \{\phi \in C([0, L]) : \ \phi \text{ is piecewise } C^2 \text{ on } [0, L] \text{ and } \phi(L) = 0\}. \tag{138}$$

Then for every $\phi \in \mathcal{A}_L$, the right-hand side of (137) is absolutely convergent. Moreover, on $C^\infty([0, L]) \cap \mathcal{A}_L$, the right-hand side of (137) agrees with the distributional pairing $D_\lambda(\phi)$ of Proposition 5.4. Consequently, (137) unambiguously defines D_λ as a real linear functional on $\mathcal{A}_L$, extending the distributional pairing.

Proof. Closed form. For $\phi \in C^\infty([0, L])$, the inversion-symmetric extension $\widetilde{\phi}(a) = \phi^{\mathrm{ev}}(\log a)\, \mathbf{1}_{[\lambda^{-2},\lambda^2]}(a)$ introduced in the proof of Proposition 5.4 (iv) has Mellin transform

$$\widehat{\widetilde{\phi}}(t) = \int_{-L}^{L} \phi^{\mathrm{ev}}(y)\, e^{-ity}\, dy = 2\widehat{\phi}_c(t), \qquad \widehat{\widetilde{\phi}}(\pm i/2) = 2\int_0^L \phi(y)\cosh(y/2)\, dy.$$

The prime contribution evaluates as $\widetilde{\phi}(p^m) = \widetilde{\phi}(p^{-m}) = \phi(m\log p)$ for $p^m \le \lambda^2$, equivalently $2\sum_{1<n\le\lambda^2} \Lambda(n) n^{-1/2}\phi(\log n)$ in the von Mangoldt rewriting of §2.4. Inserting these into $\Psi^\# = \frac{1}{2}\Psi$ in (131),

$$D_W(\phi) = 2\int_0^L \phi(y)\cosh(y/2)\, dy + \int_{\mathbb{R}} \widehat{\phi}_c(t)\, w(t)\, dt - \sum_{1<n\le\lambda^2} \Lambda(n)\, n^{-1/2}\, \phi(\log n).$$

Subtracting $D_B(\phi) = 2\int_0^L \phi(y)\cosh(y/2)\, dy$ of (134) cancels the smooth term and yields (137).

Absolute convergence on $\mathcal{A}_L$. Let $\phi \in \mathcal{A}_L$, with ϕ' possibly jumping at finitely many points $0 = y_0 < y_1 < \cdots < y_K = L$. Two integrations by parts on $\widehat{\phi}_c$, applied subinterval by subinterval and collecting boundary contributions, give for $|t| \geq 1$
(139)

$$\widehat{\phi}_c(t) = \frac{\phi'(L^-)\cos(tL) - \phi'(0^+)}{t^2} + \frac{1}{t^2}\sum_{k=1}^{K-1}\left[\phi'(y_k^+) - \phi'(y_k^-)\right]\cos(ty_k) - \frac{1}{t^2}\int_0^L \phi''(y)\cos(ty)\,dy,$$

where the boundary contribution at $y = L$ in the first integration by parts vanishes by $\phi(L) = 0$, and the boundary contribution at $y = 0$ vanishes by $\sin 0 = 0$. Hence $\widehat{\phi}_c(t) = O(t^{-2})$ as $|t| \to \infty$, with implicit constant depending only on $\|\phi'\|_{BV([0,L])}$ and $\|\phi''\|_{L^1([0,L])}$.

For the archimedean weight, the inequality $1 - \cos(tr) \leq \min(2, t^2r^2/2)$ together with the Lévy-measure verification of §3.3.2 (which gives both the bound $\nu(dr) \leq C_0\,dr/r$ on $(0,1]$ and the exponential tail of ν on $[1,\infty)$, for some absolute constant $C_0 > 0$) yields

$$|w(t)| \leq |a_\infty| + \int_0^\infty \min(2, t^2r^2/2)\,\nu(dr) \leq C(1 + \log^+|t|), \qquad t \in \mathbb{R},$$

for an absolute constant $C > 0$ depending only on ν. Combined with (139), the integrand $\widehat{\phi}_c(t)\,w(t)$ is $O(\log|t|/t^2)$ as $|t| \to \infty$ and bounded on bounded sets, hence absolutely integrable on $\mathbb{R}$.

The prime sum in (137) is finite, with each summand bounded by $\Lambda(n)\,n^{-1/2}\,\|\phi\|_{C([0,L])}$. The sole potential boundary contribution $\log n = L$ (which occurs only when λ^2 is itself a prime power) carries $\phi(\log n) = \phi(L) = 0$ by the definition of $\mathcal{A}_L$, so this endpoint introduces no ambiguity.

Compatibility with the distributional pairing. For $\phi \in C^\infty([0,L]) \cap \mathcal{A}_L$, the inversion-symmetric extension $\widetilde{\phi}$ is continuous on $\mathbb{R}^\times_{>0}$ since $\widetilde{\phi}(\lambda^{\pm 2}) = \phi^{\mathrm{ev}}(\pm L) = \phi(L) = 0$; hence $\widetilde{\phi}$ lies in the Weil class of Lemma **??** and the pairing $\Psi^\#(\widetilde{\phi})$ is given by the absolutely convergent expansion above. Term by term this expansion coincides with the construction of $D_W - D_B$ in the proof of Proposition 5.4, so the right-hand side of (137) agrees with $D_\lambda(\phi)$ as defined there. The right-hand side of (137) is real and linear on $\mathcal{A}_L \supset C^\infty([0,L]) \cap \mathcal{A}_L$, so it provides the announced extension. □

Remark 5.6. Three classes of test functions arising in §5.5–§5.6 all lie in $\mathcal{A}_L$:

(a) $q(g,f)|_{[0,L]}$ for $f, g \in T$, since $q(g,f)$ is a continuous piecewise-smooth function on $[-L,L]$ with $q(g,f)(\pm L) = 0$ by the support-additivity of Lebesgue convolutions of functions supported in $[0,L]$;
(b) $\sin(\alpha_k y) = \sin(2\pi k y/L)$ for $k \in \mathbb{Z}$, since $\sin(\alpha_k L) = \sin(2\pi k) = 0$;
(c) $(L - y)\cos(\alpha_n y)$ for $n \in \mathbb{Z}$, since the factor $(L - y)$ vanishes at $y = L$.

In particular, every distributional pairing of D_λ used in Theorem 5.7 and in §5.6 falls within the scope of Lemma 5.5.

5.5. *Real zeros of the Fourier transform of the ground state.* We now combine the spectral structure established in Sections 3–5 with the interval kernel of Proposition 5.4, and invoke the real-zeros theorem of [8, Theorem 6.1], to conclude that the Mellin transform of the ground state has only real zeros.

THEOREM 5.7. *Let $\lambda > 1$, set $\ell := \log\lambda$ and $L := 2\ell$, and let $I_\lambda := [\lambda^{-1}, \lambda]$. Let L_λ be the nonnegative self-adjoint operator on $H_\lambda = L^2(I_\lambda, d^\times u)$ associated with the Dirichlet form $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ of Theorem 3.2 via the first representation theorem* [18, Chap. VI, Thm. 2.1], *let $\mu_0(\lambda) := \min\sigma(L_\lambda)$, and let ξ_λ denote the L^2-normalized ground state furnished by Theorem 5.2 and Proposition 5.3, satisfying*

$$L_\lambda\xi_\lambda = \mu_0(\lambda)\,\xi_\lambda, \quad \|\xi_\lambda\|_{H_\lambda} = 1, \quad \xi_\lambda > 0\ \mu\text{-a.e. on } I_\lambda, \quad \xi_\lambda(u^{-1}) = \xi_\lambda(u)\ \mu\text{-a.e. on } I_\lambda.$$

The Mellin transform

$$\widehat{\xi}_\lambda(z) := \int_{\lambda^{-1}}^{\lambda} \xi_\lambda(u)\,u^{-iz}\,d^\times u, \qquad z \in \mathbb{C},$$

is an entire function whose zeros all lie on the real axis.

Proof. The argument transports the ground state through two unitary isomorphisms onto the symmetric interval $I_L := [-L/2, L/2]$, identifies the resulting sesquilinear form $\widetilde{Q}_\lambda$ on the trigonometric polynomial space with the Schwartz-kernel quadratic form induced by an explicit real even distribution $\widetilde{\mathcal{D}} \in \mathcal{D}'([-L, L])$, applies [8, Theorem 6.1] to obtain real zeros of the Fourier transform of the resulting eigenfunction, and finally transfers the conclusion back to $\widehat{\xi}_\lambda$ by an algebraic identification of Fourier transforms.

The proof is organized into nine parts (i)–(ix). Throughout, $\ell := \log\lambda$, $L := 2\ell$, $I_L := [-L/2, L/2]$, and the Lebesgue measure on I_L is denoted by dx. The convention for Fourier transforms on intervals or on $\mathbb{R}$ is $\widehat{g}(z) := \int g(x)\,e^{-izx}\,dx$, in agreement with [8, §6].

(*i*) *The Mellin transform $\widehat{\xi}_\lambda$ is entire.* The interval I_λ has finite multiplicative Haar measure

$$\mu(I_\lambda) = \int_{\lambda^{-1}}^{\lambda} d^\times u = \int_{\lambda^{-1}}^{\lambda} \frac{du}{u} = 2\log\lambda = 2\ell < \infty.$$

Since $\xi_\lambda \in H_\lambda$, the Cauchy–Schwarz inequality applied to the constant function $\mathbf{1}_{I_\lambda} \in H_\lambda$ gives

$$\|\xi_\lambda\|_{L^1(I_\lambda, d^\times u)} \le \sqrt{\mu(I_\lambda)}\,\|\xi_\lambda\|_{H_\lambda} = \sqrt{2\ell} < \infty,$$

so $\xi_\lambda \in L^1(I_\lambda, d^\times u)$.

Fix any compact set $K \subset \mathbb{C}$ and set $M_K := \sup_{z\in K} |\Im z| < \infty$. For every $u \in I_\lambda$ and every $z \in K$, the bound $|\log u| \le \ell$ gives

$$|u^{-iz}| = e^{(\Im z)\log u} \le e^{|\Im z|\,|\log u|} \le e^{M_K \ell}. \tag{140}$$

Consequently, for every nonnegative integer m,

$$\left|\xi_\lambda(u)\,(-i\log u)^m\,u^{-iz}\right| \le \ell^m\,e^{M_K\ell}\,|\xi_\lambda(u)| \in L^1(I_\lambda, d^\times u), \qquad u \in I_\lambda,\ z \in K.$$

Lebesgue's dominated convergence theorem [12, Thm. 1.34], applied to the difference quotient in the z-variable, permits differentiation under the integral sign to all orders, with

$$\frac{d^m}{dz^m}\widehat{\xi}_\lambda(z) = \int_{\lambda^{-1}}^{\lambda} \xi_\lambda(u)\,(-i\log u)^m\,u^{-iz}\,d^\times u, \qquad z \in K,\ m \ge 0. \tag{141}$$

Since $K \subset \mathbb{C}$ was an arbitrary compact subset, $\widehat{\xi}_\lambda$ is holomorphic on $\mathbb{C}$, hence entire.

(ii) Transfer to $L^2([0,L])$ via the unitary κ. By Proposition 5.4 (2), the map

$$\kappa : L^2([0,L], dy) \xrightarrow{\sim} H_\lambda, \qquad (\kappa f)(u) = f(\log(\lambda u)), \qquad (\kappa^{-1}\xi)(y) = \xi(e^{y-\ell}),$$

is a unitary isomorphism. The closed lower-semibounded sesquilinear form

$$\overline{Q}_{D,\lambda}(f,g) := Q_{D,\lambda}(\kappa f, \kappa g), \qquad f, g \in \mathrm{Dom}(\overline{Q}_{D,\lambda}) = \kappa^{-1}\mathcal{F}_\lambda,$$

admits the trigonometric span

$$T := \mathrm{span}\{\mathcal{T}_n : n \in \mathbb{Z}\} \subset L^2([0,L], dy), \qquad \mathcal{T}_n(y) := L^{-1/2} e^{2\pi i n y/L},$$

as a form core, and the associated self-adjoint operator on $L^2([0,L], dy)$ is

$$\mathcal{S}_\lambda = \kappa^{-1} A_\lambda \kappa = \kappa^{-1}(L_\lambda - c_{D,\lambda} I)\kappa, \qquad A_\lambda := L_\lambda - c_{D,\lambda} I. \tag{142}$$

Set $\eta_\lambda := \kappa^{-1}\xi_\lambda \in L^2([0,L], dy)$. Unitarity of κ gives $\|\eta_\lambda\|_{L^2([0,L])} = 1$. Applying κ^{-1} to both sides of $L_\lambda \xi_\lambda = \mu_0(\lambda)\xi_\lambda$ and using $\kappa\eta_\lambda = \xi_\lambda$ together with (142),

$$\mathcal{S}_\lambda \eta_\lambda = \kappa^{-1} A_\lambda \xi_\lambda = \kappa^{-1}\big((\mu_0(\lambda) - c_{D,\lambda})\xi_\lambda\big) = \big(\mu_0(\lambda) - c_{D,\lambda}\big)\eta_\lambda.$$

Set

$$\lambda_0 := \mu_0(\lambda) - c_{D,\lambda} \in \mathbb{R}, \tag{143}$$

so that $\mathcal{S}_\lambda \eta_\lambda = \lambda_0 \eta_\lambda$ in $L^2([0,L], dy)$ and $\|\eta_\lambda\|_{L^2([0,L])} = 1$.

(iii) λ_0 is the simple isolated bottom of $\sigma(\mathcal{S}_\lambda)$. By Proposition 5.1, L_λ has compact resolvent and purely discrete spectrum

$$0 \le \mu_0(\lambda) \le \mu_1(\lambda) \le \mu_2(\lambda) \le \cdots, \qquad \mu_n(\lambda) \to +\infty,$$

each eigenvalue of finite multiplicity. By Theorem 5.2, $\mu_0(\lambda)$ is a simple eigenvalue of L_λ. The unitary equivalence $\mathcal{S}_\lambda = \kappa^{-1}(L_\lambda - c_{D,\lambda} I)\kappa$ preserves both the spectrum and the dimensions of all eigenspaces, so

$$\sigma(\mathcal{S}_\lambda) = \sigma(L_\lambda) - c_{D,\lambda} = \{\mu_n(\lambda) - c_{D,\lambda} : n \geq 0\},$$

each $\mu_n(\lambda) - c_{D,\lambda}$ inheriting the multiplicity of $\mu_n(\lambda)$. Therefore $\lambda_0 = \min \sigma(\mathcal{S}_\lambda)$ is a simple eigenvalue of $\mathcal{S}_\lambda$ with one-dimensional eigenspace $\mathbb{C}\eta_\lambda$. The spectral gap

$$\delta := \mu_1(\lambda) - \mu_0(\lambda) > 0 \tag{144}$$

is strictly positive (because $\mu_0(\lambda)$ is simple and the spectrum is discrete), giving the isolation

$$\sigma(\mathcal{S}_\lambda) \setminus \{\lambda_0\} \subset [\lambda_0 + \delta, \infty). \tag{145}$$

(iv) Symmetry of η_λ under $y \mapsto L - y$. By Proposition 5.3, $\xi_\lambda(u^{-1}) = \xi_\lambda(u)$ for μ-a.e. $u \in I_\lambda$. Using $L = 2\ell$ (so that $L - \ell = \ell$ and consequently $L - y - \ell = \ell - y$) and the algebraic identity $e^{\ell - y} = (e^{y-\ell})^{-1}$, for μ-a.e. $y \in [0, L]$,

$$\eta_\lambda(L - y) = \xi_\lambda(e^{(L-y)-\ell}) = \xi_\lambda(e^{\ell - y}) = \xi_\lambda\big((e^{y-\ell})^{-1}\big) = \xi_\lambda(e^{y-\ell}) = \eta_\lambda(y).$$

The first and last equalities are the definition of η_λ through κ^{-1}, the second uses $L - y - \ell = \ell - y$, the third uses the algebraic inversion identity above, and the fourth applies the inversion symmetry of ξ_λ to the point $u = e^{y-\ell} \in I_\lambda$. Therefore η_λ is μ-a.e. invariant under the reflection $y \mapsto L - y$.

(v) Translation to $L^2(I_L)$ via the unitary τ. The translation operator

$$\tau : L^2([0, L], dy) \xrightarrow{\sim} L^2(I_L, dx), \qquad (\tau f)(x) := f(x + L/2), \qquad (\tau^{-1} g)(y) := g(y - L/2),$$

preserves Lebesgue measure on $\mathbb{R}$ and is therefore unitary. Define

$$\widetilde{\eta}_\lambda := \tau\eta_\lambda \in L^2(I_L), \quad \widetilde{\mathcal{S}}_\lambda := \tau\mathcal{S}_\lambda\tau^{-1}, \quad \widetilde{Q}_\lambda(\varphi, \psi) := \overline{Q}_{D,\lambda}(\tau^{-1}\varphi, \tau^{-1}\psi). \tag{146}$$

Since the unitary τ preserves both the form-graph topology and the L^2-norm, $\widetilde{Q}_\lambda$ is a closed lower-semibounded sesquilinear form on $L^2(I_L)$, with form domain

$$\mathrm{Dom}(\widetilde{Q}_\lambda) = \tau\ \mathrm{Dom}(\overline{Q}_{D,\lambda}) = \tau\kappa^{-1}\mathcal{F}_\lambda. \tag{147}$$

By the first representation theorem [18, Chap. VI, Thm. 2.1], $\widetilde{\mathcal{S}}_\lambda$ is the unique self-adjoint operator on $L^2(I_L)$ associated with $\widetilde{Q}_\lambda$.

Unitary equivalence preserves the spectrum and eigenspaces of $\mathcal{S}_\lambda$, so

$$\sigma(\widetilde{\mathcal{S}}_\lambda) = \sigma(\mathcal{S}_\lambda), \quad \widetilde{\mathcal{S}}_\lambda\widetilde{\eta}_\lambda = \lambda_0\widetilde{\eta}_\lambda, \quad \|\widetilde{\eta}_\lambda\|_{L^2(I_L)} = 1. \tag{148}$$

Combining (148) with (145), $\lambda_0 = \min \sigma(\widetilde{\mathcal{S}}_\lambda)$ is a simple isolated eigenvalue of $\widetilde{\mathcal{S}}_\lambda$ with one-dimensional eigenspace $\mathbb{C}\widetilde{\eta}_\lambda$, and the gap is the same $\delta > 0$ as in (144).

(vi) Evenness of $\widetilde{\eta}_\lambda$ under $x \mapsto -x$. For μ-a.e. $x \in I_L$, the algebraic identity

$$-x + L/2 = L - (x + L/2) \tag{149}$$

in $\mathbb{R}$ (verified by $L - (x + L/2) = L/2 - x = -x + L/2$) combined with (iv) yields

$$\widetilde{\eta}_\lambda(-x) = \eta_\lambda(-x + L/2) = \eta_\lambda(L - (x + L/2)) = \eta_\lambda(x + L/2) = \widetilde{\eta}_\lambda(x).$$

The first and fourth equalities apply the definition of τ in (146), the second uses (149), and the third invokes (iv) at the point $z := x + L/2 \in [0, L]$. Hence $\widetilde{\eta}_\lambda$ is μ-a.e. even on I_L under $x \mapsto -x$.

(vii) Construction of the even real distribution $\widetilde{\mathcal{D}} \in \mathcal{D}'([-L, L])$ and the Schwartz-kernel formula on the trigonometric core.

By Proposition 5.4, there exists a real distribution $D_\lambda \in \mathcal{D}'([0, L])$ such that

$$\overline{Q}_{D,\lambda}(f, g) = D_\lambda\big(q(g, f)|_{[0,L]}\big), \qquad f, g \in T, \tag{150}$$

where, for f, g supported in $[0, L]$ (after zero extension to $\mathbb{R}$),

$$q(g, f)(y) := (g^* * f)(y) + (g^* * f)(-y), \qquad y \in \mathbb{R}, \qquad g^*(x) := \overline{g(-x)}, \tag{151}$$

with $*$ denoting Lebesgue convolution on $\mathbb{R}$. Here $\mathcal{D}'([0, L])$ denotes the space of continuous linear functionals on $C^\infty([0, L])$ in the standard $\mathcal{D}$-topology of uniform convergence of all derivatives, and analogously for $\mathcal{D}'([-L, L])$. By Lemma 5.5, D_λ admits the explicit closed form (137), which is absolutely convergent on the admissible class $\mathcal{A}_L$ of (138); this provides an unambiguous real-valued pairing $D_\lambda(\phi)$ for every $\phi \in \mathcal{A}_L$. Each distributional pairing of D_λ used below is verified to fall within $\mathcal{A}_L$ via Remark 5.6; no further extension argument is required.

(vii.a) Definition of $\widetilde{\mathcal{D}}$ and verification of reality and evenness. For each $\phi \in C^\infty([-L, L])$, the function $y \mapsto \phi(y) + \phi(-y)$ on $[-L, L]$ is C^∞, hence its restriction to $[0, L]$ belongs to $C^\infty([0, L])$. Define

$$\widetilde{\mathcal{D}}(\phi) := D_\lambda\Big((\phi(y) + \phi(-y))\big|_{y \in [0,L]}\Big), \qquad \phi \in C^\infty([-L, L]). \tag{152}$$

The linear map

$$\Sigma : C^\infty([-L, L]) \to C^\infty([0, L]), \qquad \Sigma\phi := (\phi(y) + \phi(-y))|_{y \in [0,L]},$$

is continuous in the $\mathcal{D}$-topology of uniform convergence of all derivatives, by continuity of the reflection $\phi \mapsto \phi(-\cdot)$ on $C^\infty([-L, L])$, of pointwise addition, and of the restriction $C^\infty([-L, L]) \to C^\infty([0, L])$. Composition with the continuous functional $D_\lambda \in \mathcal{D}'([0, L])$ yields $\widetilde{\mathcal{D}} = D_\lambda \circ \Sigma \in \mathcal{D}'([-L, L])$.

Reality. For real-valued $\phi \in C^\infty([-L, L])$, $\Sigma\phi$ is real-valued, and the reality of D_λ established in Proposition 5.4 gives $\widetilde{\mathcal{D}}(\phi) \in \mathbb{R}$.

Evenness. For every $\phi \in C^\infty([-L,L])$, the substitution $y \leftrightarrow -y$ inside Σ gives $\Sigma(\phi(-\cdot)) = \Sigma\phi$, hence

$$\widetilde{\mathcal{D}}\big(\phi(-\cdot)\big) = D_\lambda\big(\Sigma(\phi(-\cdot))\big) = D_\lambda(\Sigma\phi) = \widetilde{\mathcal{D}}(\phi).$$

For the use below, we record the analogous extension class for $\widetilde{\mathcal{D}}$. Define

$$\widetilde{\mathcal{A}}_L := \big\{\phi \in C([-L,L]) : \phi \text{ is piecewise } C^2 \text{ on } [-L,L] \text{ and } \phi(\pm L) = 0\big\}.$$

For every $\phi \in \widetilde{\mathcal{A}}_L$, the symmetrization $\Sigma\phi = (\phi(y)+\phi(-y))|_{[0,L]}$ is continuous and piecewise C^2 on $[0,L]$ with $(\Sigma\phi)(L) = \phi(L)+\phi(-L) = 0$, hence $\Sigma\phi \in \mathcal{A}_L$. By Lemma 5.5, $D_\lambda(\Sigma\phi)$ is unambiguously defined via the closed form (137), and the formula $\widetilde{\mathcal{D}}(\phi) := D_\lambda(\Sigma\phi)$ extends (152) to all of $\widetilde{\mathcal{A}}_L$ as a real linear functional.

(vii.b) Schwartz-kernel formula on $\widetilde{T}\times\widetilde{T}$. Set $\widetilde{\mathcal{T}}_n(x) := L^{-1/2}e^{2\pi i n x/L}$ for $x \in I_L$ and $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$,

$$(\tau\mathcal{T}_n)(x) = \mathcal{T}_n(x{+}L/2) = L^{-1/2}e^{2\pi i n(x+L/2)/L} = L^{-1/2}e^{i\pi n}e^{2\pi i n x/L} = (-1)^n\widetilde{\mathcal{T}}_n(x). \tag{153}$$

In particular,

$$\widetilde{T} := \tau T = \operatorname{span}\{\widetilde{\mathcal{T}}_n : n \in \mathbb{Z}\} \subset L^2(I_L), \qquad \tau^{-1}\widetilde{\mathcal{T}}_n = (-1)^n\mathcal{T}_n. \tag{154}$$

Since T is a form core for $\overline{Q}_{D,\lambda}$ (by Proposition 5.4 (2)) and τ is unitary, $\widetilde{T}$ is a form core for $\widetilde{Q}_\lambda$ on $\mathrm{Dom}(\widetilde{Q}_\lambda)$.

For $\varphi,\psi \in \widetilde{T}$, set $f := \tau^{-1}\varphi \in T$ and $g := \tau^{-1}\psi \in T$. By the definition of $\widetilde{Q}_\lambda$ in (146) and (150),

$$\widetilde{Q}_\lambda(\varphi,\psi) = \overline{Q}_{D,\lambda}(f,g) = D_\lambda\big(q(g,f)|_{[0,L]}\big). \tag{155}$$

Both f and g are supported in $[0,L]$, equivalently $f(y) = \varphi(y-L/2)$ and $g(y) = \psi(y-L/2)$ on $\mathbb{R}$ (with φ,ψ extended by zero off I_L). For $x \in \mathbb{R}$,

$$g^*(x) = \overline{g(-x)} = \overline{\psi(-x-L/2)} = \psi^*(x+L/2), \tag{156}$$

where $\psi^*(x) := \overline{\psi(-x)}$. Substituting (156) into the definition $(g^* * f)(y) := \int_{\mathbb{R}} g^*(y-z)f(z)\,dz$ and applying the change of variable $w := z-L/2$ (so that $dw = dz$, $y-z+L/2 = y-w$, and $\varphi(z-L/2) = \varphi(w)$),

$$\begin{aligned}(g^* * f)(y) &= \int_{\mathbb{R}} g^*(y-z)f(z)\,dz = \int_{\mathbb{R}} \psi^*(y-z+L/2)\,\varphi(z-L/2)\,dz\\ &= \int_{\mathbb{R}} \psi^*(y-w)\,\varphi(w)\,dw = (\psi^* * \varphi)(y), \qquad y \in \mathbb{R}.\end{aligned} \tag{157}$$

Substituting (157) into the definition of $q(g,f)$ in (151) yields

$$q(g,f)(y) = (\psi^* * \varphi)(y) + (\psi^* * \varphi)(-y), \qquad y \in \mathbb{R}. \tag{158}$$

Both ψ^* and φ have support in $[-L/2, L/2]$ (after zero extension), so $\psi^* * \varphi$ has support in $[-L, L]$ by additivity of supports under Lebesgue convolution. Furthermore, since φ and ψ^* are bounded and compactly supported on $\mathbb{R}$, the convolution $\psi^* * \varphi$ is continuous on $\mathbb{R}$ and piecewise-smooth on $\mathbb{R} \setminus \{-L, 0, L\}$ (with potential breaks in higher derivatives where the integration window changes endpoints), hence in particular compactly supported piecewise-smooth on $[-L, L]$.

By definition (152) of $\widetilde{\mathcal{D}}$, applied to the compactly supported piecewise-smooth function $\phi := \psi^* * \varphi$ on $[-L, L]$,

(159)

$$\widetilde{\mathcal{D}}\big((\psi^* * \varphi)|_{[-L,L]}\big) = D_\lambda\Big(\big((\psi^* * \varphi)(y) + (\psi^* * \varphi)(-y)\big)\big|_{y\in[0,L]}\Big) = D_\lambda\big(q(g,f)|_{[0,L]}\big),$$

where the second equality uses (158). The applicability of $\widetilde{\mathcal{D}}$ to $\psi^* * \varphi$, and of D_λ to $q(g,f)|_{[0,L]}$, is justified by Lemma 5.5 together with Remark 5.6 (a): the convolution $\psi^* * \varphi$ lies in $\widetilde{\mathcal{A}}_L$ since it is continuous on $\mathbb{R}$, piecewise smooth on $\mathbb{R} \setminus \{-L, 0, L\}$, and vanishes at $\pm L$ by the support-additivity of Lebesgue convolutions of functions supported in $[-L/2, L/2]$; consequently $q(g,f)|_{[0,L]} = \Sigma(\psi^* * \varphi) \in \mathcal{A}_L$.

Combining (155) with (159),

$$\boxed{\widetilde{Q}_\lambda(\varphi, \psi) = \widetilde{\mathcal{D}}\big((\psi^* * \varphi)|_{[-L,L]}\big), \qquad \varphi, \psi \in \widetilde{T}.} \tag{160}$$

This is precisely the Schwartz-kernel quadratic form induced by the real even distribution $\widetilde{\mathcal{D}}$ on $[-L, L]$ via the kernel $\widetilde{\mathcal{D}}(x-y)$, i.e. the form defined on trigonometric polynomials by [8, formula (6)].

(*viii*) *Application of* [8, Theorem 6.1]. The conditions of [8, Theorem 6.1] for the form $\widetilde{Q}_\lambda$ on $L^2(I_L)$ are now verified item by item, in the order in which they appear in the cited theorem.

(a) $L > 0$: the strict inequality $\lambda > 1$ gives $L = 2\log\lambda > 0$.
(b) $\widetilde{\mathcal{D}} \in \mathcal{D}'([-L, L])$ is a real even distribution: established in (vii.a).
(c) On the trigonometric polynomial space $\widetilde{T} \subset L^2(I_L)$, the sesquilinear form $\widetilde{Q}_\lambda$ is given by the Schwartz-kernel formula [8, formula (6)] with kernel $\widetilde{\mathcal{D}}(x-y)$: this is (160) of (vii.b).
(d) The form $\widetilde{Q}_\lambda$ is closed, symmetric, and lower-semibounded on $L^2(I_L)$, with form core $\widetilde{T}$ (cf. (v) and (vii.b)). Consequently, the restriction of the operator associated with $\widetilde{Q}_\lambda$ to $\widetilde{T}$ is essentially self-adjoint, and the closure of this restriction equals $\widetilde{\mathcal{S}}_\lambda$, the self-adjoint operator associated with $\widetilde{Q}_\lambda$ via the first representation theorem [18, Chap. VI, Thm. 2.1]: this is exactly the "lower-bounded essentially self-adjoint operator" condition of [8, Theorem 6.1].

(e) $\lambda_0 = \min \sigma(\widetilde{\mathcal{S}}_\lambda)$ is a simple isolated eigenvalue: the simplicity is established in (iii)–(v), and the isolation is (145) combined with (148).
(f) The corresponding L^2-normalized eigenfunction $\widetilde{\eta}_\lambda$ is even on I_L: established in (vi).

By [8, Theorem 6.1], the Fourier transform

$$\widehat{\widetilde{\eta}_\lambda}(z) := \int_{-L/2}^{L/2} \widetilde{\eta}_\lambda(x)\, e^{-izx}\, dx, \qquad z \in \mathbb{C}, \tag{161}$$

is an entire function whose zeros all lie on the real axis.

(*ix*) *Transfer of the zero-set conclusion to* $\widehat{\xi}_\lambda$. We compute $\widehat{\widetilde{\eta}_\lambda}$ in terms of an auxiliary entire function defined on $[0, L]$, then express $\widehat{\xi}_\lambda$ in terms of the same auxiliary function, and finally compare.

Step (*ix*.1). In (161), apply the change of variable $y := x + L/2$ (so $x = y - L/2$, $dx = dy$, and $x \in [-L/2, L/2]$ corresponds to $y \in [0, L]$). Using the identity $\widetilde{\eta}_\lambda(x) = (\tau\eta_\lambda)(x) = \eta_\lambda(x + L/2) = \eta_\lambda(y)$ from the definition of τ, and $e^{-izx} = e^{-iz(y-L/2)} = e^{izL/2}\, e^{-izy}$, one obtains

$$\widehat{\widetilde{\eta}_\lambda}(z) = \int_0^L \eta_\lambda(y)\, e^{izL/2}\, e^{-izy}\, dy = e^{izL/2}\, F_\lambda(z), \tag{162}$$

where

$$F_\lambda(z) := \int_0^L \eta_\lambda(y)\, e^{-izy}\, dy, \qquad z \in \mathbb{C}. \tag{163}$$

The factor $e^{izL/2}$ is nowhere zero on $\mathbb{C}$, so the entire functions $\widehat{\widetilde{\eta}_\lambda}$ and F_λ have identical zero sets, including multiplicities.

Step (*ix*.2). For $\widehat{\xi}_\lambda$, the relation $\xi_\lambda(u) = (\kappa\eta_\lambda)(u) = \eta_\lambda(\log(\lambda u))$ from the definition of κ, together with the change of variable $y := \log(\lambda u)$ (so $u = e^{y-\ell}$, $d^\times u = du/u = dy$, and $u \in [\lambda^{-1}, \lambda]$ corresponds to $y \in [0, L]$), gives

$$u^{-iz} = e^{-iz(y-\ell)} = e^{iz\ell}\, e^{-izy},$$

and therefore

$$\widehat{\xi}_\lambda(z) = \int_{\lambda^{-1}}^{\lambda} \xi_\lambda(u)\, u^{-iz}\, d^\times u = \int_0^L \eta_\lambda(y)\, e^{iz\ell}\, e^{-izy}\, dy = e^{iz\ell}\, F_\lambda(z). \tag{164}$$

Step (*ix*.3). Since $L = 2\ell$, the exponentials in (162) and (164) agree: $e^{izL/2} = e^{iz\ell}$. Therefore, for every $z \in \mathbb{C}$,

$$\widehat{\xi}_\lambda(z) = e^{iz\ell}\, F_\lambda(z) = e^{izL/2}\, F_\lambda(z) = \widehat{\widetilde{\eta}_\lambda}(z). \tag{165}$$

The zero set of $\widehat{\xi}_\lambda$ therefore coincides, with multiplicities, with that of $\widehat{\widetilde{\eta}_\lambda}$, which by (viii) lies entirely on the real axis. This completes the proof. $\square$

Summary of this section. For each fixed $\lambda > 1$, the following statements all hold unconditionally (i.e. without invoking the conjectural reality of the nontrivial zeros of ζ):

1. L_λ has compact resolvent and purely discrete spectrum (Proposition 5.1);
2. $\mu_0(\lambda)$ is simple, and there exists a normalized, strictly positive, inversion-even ground state ξ_λ (Theorem 5.2 and Proposition 5.3);
3. $\widehat{\xi}_\lambda$ is an entire function whose zeros are all real (Theorem 5.7).

This is precisely the "real-zeros proxy" that, in the Galerkin determinant identity of Subsection 5.6 below, links the ground state to the spectrum of an explicit finite self-adjoint operator.

5.6. *The zeta-regularized determinant of the Galerkin ground state*. We now attach to the finite Galerkin ground state a regularized determinant. This is the determinant-theoretic counterpart, in the present boundary-free Weil–Dirichlet model, of the construction used for the perturbed scaling operator in [7, Section 5]. Since our convention is that inner products are linear in the first variable, all rank-one operators below are written explicitly.

Let
$$L = 2\log\lambda, \qquad \ell = \log\lambda,$$
and keep the notation introduced in Subsection 5.5. Thus
$$\mathcal{T}_j(y) = L^{-1/2} e^{2\pi i j y/L}, \qquad V_j(u) = \mathcal{T}_j(\log(\lambda u)),$$
and
$$E_N^\lambda := \operatorname{span}\{V_j : |j| \le N\} \subset H_\lambda.$$
The periodic logarithmic scaling operator is
$$D_{\log}^{(\lambda)} = -iu\frac{d}{du},$$
defined initially on the trigonometric span $\bigcup_{N\ge 0} E_N^\lambda$ and closed in H_λ. A direct computation gives, for each $j \in \mathbb{Z}$ and $u \in I_\lambda$,
$$\frac{dV_j}{du}(u) = L^{-1/2}\,\frac{2\pi i j/L}{u}\,e^{2\pi i j\log(\lambda u)/L} = \frac{2\pi i j}{Lu}V_j(u),$$
hence
$$D_{\log}^{(\lambda)} V_j = -iu\cdot\frac{2\pi i j}{Lu}V_j = \alpha_j V_j, \qquad \alpha_j := \frac{2\pi j}{L}.$$
The boundary values $V_j(\lambda^{-1}) = L^{-1/2} = V_j(\lambda)$ confirm that each V_j satisfies the periodic boundary condition on I_λ. Since $\{V_j\}_{j\in\mathbb{Z}}$ is an orthonormal basis of H_λ with real eigenvalues $\{\alpha_j\}$, the closure of $D_{\log}^{(\lambda)}$ is self-adjoint on the dense domain $\operatorname{Dom}(D_{\log}^{(\lambda)}) = \{f = \sum_j f_j V_j \in H_\lambda : \sum_j |\alpha_j|^2 |f_j|^2 < \infty\}$, which is the domain used in Theorem 5.16 (ii). Let
$$Q_{D,\lambda}^{(N)} := Q_{D,\lambda}\big|_{E_N^\lambda \times E_N^\lambda}$$

be the Galerkin restriction of the boundary-free form. Let $A_{\lambda,N}$ be the self-adjoint matrix representing this form in E_N^λ, namely

$$Q_{D,\lambda}^{(N)}(f,g) = \langle A_{\lambda,N} f, g\rangle_{H_\lambda}, \qquad f,g \in E_N^\lambda.$$

We single out, for each $\lambda > 1$, the set of Galerkin dimensions for which the lowest Ritz problem on E_N^λ is non-degenerate in the inversion-even sector:

(166)

$$\mathfrak{N}_\lambda := \Big\{ N \in \mathbb{Z}_{\geq 1} : \min \sigma(A_{\lambda,N}) \text{ is simple and admits an inversion-even eigenvector} \Big\}.$$

For $N \in \mathfrak{N}_\lambda$ we denote the lowest eigenvalue by

$$\varepsilon_{\lambda,N}^D := \min \sigma(A_{\lambda,N})$$

and choose a corresponding nonzero inversion-even eigenvector

$$\xi_{\lambda,N} = \sum_{|j|\leq N} \xi_j V_j \in E_N^\lambda, \qquad A_{\lambda,N}\xi_{\lambda,N} = \varepsilon_{\lambda,N}^D \xi_{\lambda,N}, \qquad \mathcal{R}\xi_{\lambda,N} = \xi_{\lambda,N},$$

where $\mathcal{R}$ is the inversion of Proposition 5.3. Equivalently,

$$T_{\lambda,N} := A_{\lambda,N} - \varepsilon_{\lambda,N}^D I$$

is nonnegative on E_N^λ and

$$\ker T_{\lambda,N} = \mathbb{C}\xi_{\lambda,N}.$$

We also use $T_{\lambda,N}$ for the associated Hermitian sesquilinear form

$$T_{\lambda,N}(f,g) = \langle T_{\lambda,N} f, g\rangle_{H_\lambda}.$$

Remark. The set $\mathfrak{N}_\lambda$ contains every sufficiently large positive integer. Indeed, by Theorem 5.2 together with Proposition 5.3, the lowest spectral value of $A_\lambda = L_\lambda - c_{D,\lambda} I$ is a simple isolated eigenvalue with an inversion-even eigenfunction. The Galerkin restriction $A_{\lambda,N} \to A_\lambda$ is convergent in the strong-resolvent sense as $N \to \infty$ since $\bigcup_N E_N^\lambda$ is a form core for $Q_{D,\lambda}$. Standard Galerkin perturbation theory for a simple isolated eigenvalue (cf. [11, Vol. IV, Thm. XIII.83] together with the Davis–Kahan $\sin\Theta$ inequality) then yields the existence of $N_0(\lambda) < \infty$ such that $\{N : N \geq N_0(\lambda)\} \subset \mathfrak{N}_\lambda$. The growth rate of $N_0(\lambda)$ plays no role in the arguments below; only membership $N \in \mathfrak{N}_\lambda$ is used.

Define the Dirichlet-kernel vector

$$\Delta_N := L^{-1/2} \sum_{|j|\leq N} V_j \in E_N^\lambda$$

and the linear functional

$$\delta_N(f) := \langle f, \Delta_N\rangle_{H_\lambda}, \qquad f \in E_N^\lambda.$$

Thus if $f = \sum_{|j|\le N} f_j V_j$, then

$$\delta_N(f) = L^{-1/2} \sum_{|j|\le N} f_j.$$

This is the normalization functional used in [7, Theorem 1.1], translated to the present convention that the inner product is linear in the first variable.

We collect six lemmas which carry, individually, the algebraic and spectral content needed to prove the regularized-determinant identity. The synthesis is the main result Theorem 5.16 below. Throughout this subsection, $D_N := D^{(\lambda)}_{\log}|_{E_N^\lambda}$, and the rank-one operator $a \otimes b^*$ is defined by $(a\otimes b^*)f := \langle f, b\rangle_{H_\lambda}\, a$, with adjoint $(a\otimes b^*)^* = b \otimes a^*$.

LEMMA 5.8 (REFLECTION ANTICOMMUTATION OF D_N). *On E_N^λ one has $\mathcal{R}D_N = -D_N\mathcal{R}$. Equivalently, D_N maps the inversion-even subspace of E_N^λ into the inversion-odd subspace and vice versa.*

Proof. For each $|j| \le N$, the action $\mathcal{R}V_j = V_{-j}$ established in Proposition 5.3, part (ii) of its proof, combined with $D_N V_j = \alpha_j V_j$ and $\alpha_{-j} = -\alpha_j$, gives

$$\mathcal{R}D_N V_j = \mathcal{R}(\alpha_j V_j) = \alpha_j V_{-j} = -\alpha_{-j} V_{-j} = -D_N V_{-j} = -D_N \mathcal{R} V_j.$$

The displayed identity holds on the basis $\{V_j\}_{|j|\le N}$ of E_N^λ and therefore on all of E_N^λ by linearity. □

LEMMA 5.9 (HILBERT-FORM MATRIX STRUCTURE OF $A_{\lambda,N}$). *For $k \in \mathbb{Z}$, set*

$$s_k := D_\lambda\big(\sin(\alpha_k y)\big). \tag{167}$$

Then $s_k \in \mathbb{R}$ and $s_{-k} = -s_k$ (in particular $s_0 = 0$). Moreover, on the orthonormal basis $\{V_j\}_{|j|\le N}$ of E_N^λ, the matrix elements of $A_{\lambda,N}$ are real and given by

$$\big\langle A_{\lambda,N} V_n, V_m\big\rangle_{H_\lambda} = \frac{s_n - s_m}{\pi(m-n)}, \qquad m \ne n,\ |m|,|n| \le N, \tag{168}$$

and

$$\big\langle A_{\lambda,N} V_n, V_n\big\rangle_{H_\lambda} = \frac{2}{L}\, D_\lambda\big((L-y)\cos(\alpha_n y)\big), \qquad |n| \le N. \tag{169}$$

In particular, $A_{\lambda,N}$ is real symmetric in the basis $\{V_j\}_{|j|\le N}$.

Proof. Reality of s_k follows from reality of the distribution D_λ (Proposition 5.4); the relation $s_{-k} = -s_k$ follows from $\sin(\alpha_{-k}y) = -\sin(\alpha_k y)$ together with linearity of D_λ.

The pairings $D_\lambda(\sin(\alpha_k y))$ and $D_\lambda((L-y)\cos(\alpha_n y))$ appearing in (167) and (169) are absolutely defined by Lemma 5.5, since both test functions lie in $\mathcal{A}_L$ (cf. Remark 5.6 (b)–(c)).

By Proposition 5.4, for every f, g in the trigonometric span $T := \operatorname{span}\{\mathcal{T}_n : n \in \mathbb{Z}\}$,

$$Q_{D,\lambda}(\kappa f, \kappa g) = D_\lambda\big(q(g,f)\big), \qquad q(g,f)(y) = (g^* * f)(y) + (g^* * f)(-y),$$

with $g^*(x) := \overline{g(-x)}$ and $*$ denoting Lebesgue convolution on $\mathbb{R}$ after zero extension off $[0, L]$.

(*i*) *Off-diagonal entries*. Take $m \neq n$. The function $\mathcal{T}_n$, extended by zero off $[0, L]$, has support in $[0, L]$, while $\mathcal{T}_m^*$ satisfies $\mathcal{T}_m^*(x) = L^{-1/2} e^{2\pi i m x/L}$ for $x \in [-L, 0]$ and zero elsewhere. For $y \in [0, L]$, the convolution support condition $\{z : y - z \in [-L, 0],\ z \in [0, L]\}$ reduces to $z \in [y, L]$, giving

$$(\mathcal{T}_m^* * \mathcal{T}_n)(y) = \int_y^L L^{-1} e^{2\pi i m(y-z)/L} e^{2\pi i n z/L}\, dz = \frac{e^{2\pi i m y/L} - e^{2\pi i n y/L}}{2\pi i(n-m)},$$

using $\int_y^L e^{2\pi i(n-m)z/L}\, dz = L\big(1 - e^{2\pi i(n-m)y/L}\big)/(2\pi i(n-m))$ together with $e^{2\pi i(n-m)} = 1$ for $n - m \in \mathbb{Z}$. The same computation for $y \in [-L, 0]$, with integration range $[0, y + L]$, produces the analogous formula with m and n exchanged. Therefore

$$q(\mathcal{T}_m, \mathcal{T}_n)(y) = (\mathcal{T}_m^* * \mathcal{T}_n)(y) + (\mathcal{T}_m^* * \mathcal{T}_n)(-y) = \frac{\sin(\alpha_m y) - \sin(\alpha_n y)}{\pi(n-m)}, \qquad y \in [-L, L], \tag{170}$$

where the second equality uses $e^{i\theta} - e^{-i\theta} = 2i \sin\theta$. Combining (170) with the kernel formula and the identification $\langle A_{\lambda,N} V_n, V_m \rangle_{H_\lambda} = Q_{D,\lambda}(V_n, V_m)$ yields (168).

(*ii*) *Diagonal entries*. For $y \in [0, L]$,

$$(\mathcal{T}_n^* * \mathcal{T}_n)(y) = \int_y^L L^{-1} e^{2\pi i n(y-z)/L} e^{2\pi i n z/L}\, dz = L^{-1} e^{2\pi i n y/L}(L - y),$$

and similarly $(\mathcal{T}_n^* * \mathcal{T}_n)(-y) = L^{-1} e^{-2\pi i n y/L}(L - y)$ for $y \in [0, L]$. Adding,

$$q(\mathcal{T}_n, \mathcal{T}_n)(y) = \frac{2(L-y)\cos(\alpha_n y)}{L}, \qquad y \in [0, L]. \tag{171}$$

Since D_λ is real, $\langle A_{\lambda,N} V_n, V_n \rangle_{H_\lambda} = Q_{D,\lambda}(V_n, V_n) = D_\lambda(q(\mathcal{T}_n, \mathcal{T}_n))$ yields (169).

(*iii*) *Real symmetry*. The right-hand side of (168) is real and symmetric in (m, n): $(s_n - s_m)/\pi(m-n) = (s_m - s_n)/\pi(n-m)$. The diagonal entries (169) are real. The matrix $A_{\lambda,N}$ in the basis $\{V_j\}$ is therefore real and symmetric. The same property transfers to $T_{\lambda,N} = A_{\lambda,N} - \varepsilon^D_{\lambda,N} I$. □

Lemma 5.10 (Rank-two commutator identity). *Define*

$$\beta_{\lambda,N} := -\frac{2}{L^{1/2}} \sum_{|k| \le N} s_k V_k \in E_N^\lambda. \tag{172}$$

Then $\beta_{\lambda,N}$ is inversion-odd:

$$\mathcal{R}\beta_{\lambda,N} = -\beta_{\lambda,N}, \tag{173}$$

and on E_N^λ one has the operator identity

$$[D_N, T_{\lambda,N}] = \beta_{\lambda,N} \otimes \Delta_N^* - \Delta_N \otimes \beta_{\lambda,N}^*. \tag{174}$$

Proof. The reflection identity (173) follows from $\mathcal{R}V_k = V_{-k}$ and $s_{-k} = -s_k$:

$$\mathcal{R}\beta_{\lambda,N} = -\frac{2}{L^{1/2}} \sum_{|k|\le N} s_k V_{-k} = -\frac{2}{L^{1/2}} \sum_{|j|\le N} s_{-j} V_j = \frac{2}{L^{1/2}} \sum_{|j|\le N} s_j V_j = -\beta_{\lambda,N}.$$

For (174) we compare matrix entries. The eigenvalue relation $D_N V_j = \alpha_j V_j$ with $\alpha_j \in \mathbb{R}$ gives $D_N^* = D_N$ on the orthonormal basis $\{V_j\}$ and

$$\big([D_N, T_{\lambda,N}]\big)_{mn} = \langle [D_N, T_{\lambda,N}] V_n, V_m\rangle_{H_\lambda} = (\alpha_m - \alpha_n)(T_{\lambda,N})_{mn}. \tag{175}$$

For $m \ne n$, Lemma 5.9 gives $(T_{\lambda,N})_{mn} = (A_{\lambda,N})_{mn} = (s_n - s_m)/\pi(m-n)$, so $(\alpha_m - \alpha_n)(T_{\lambda,N})_{mn} = 2(s_n - s_m)/L$. For $m = n$, the prefactor $\alpha_m - \alpha_n$ vanishes, giving $([D_N, T_{\lambda,N}])_{nn} = 0 = 2(s_n - s_n)/L$. Therefore

$$\big([D_N, T_{\lambda,N}]\big)_{mn} = \frac{2(s_n - s_m)}{L}, \qquad |m|, |n| \le N. \tag{176}$$

For arbitrary $a = \sum_j a_j V_j$ and $b = \sum_j b_j V_j$ in E_N^λ, the matrix elements of the rank-one operator $a \otimes b^*$ in the basis $\{V_j\}$ are

$$(a \otimes b^*)_{mn} = \langle (a\otimes b^*) V_n, V_m\rangle = \langle V_n, b\rangle \, \langle a, V_m\rangle = \overline{b_n}\, a_m.$$

Applied to $a = \beta_{\lambda,N}$, $b = \Delta_N = L^{-1/2}\sum_k V_k$ (so $b_k = L^{-1/2} \in \mathbb{R}$),

$$\big(\beta_{\lambda,N} \otimes \Delta_N^*\big)_{mn} = L^{-1/2} \cdot \frac{-2s_m}{L^{1/2}} = -\frac{2s_m}{L},$$

and similarly, using $s_n \in \mathbb{R}$,

$$\big(\Delta_N \otimes \beta_{\lambda,N}^*\big)_{mn} = \overline{\frac{-2s_n}{L^{1/2}}} \cdot L^{-1/2} = -\frac{2s_n}{L}.$$

Subtracting,

$$\big(\beta_{\lambda,N}\otimes\Delta_N^* - \Delta_N \otimes \beta_{\lambda,N}^*\big)_{mn} = \frac{-2s_m - (-2s_n)}{L} = \frac{2(s_n - s_m)}{L} = \big([D_N, T_{\lambda,N}]\big)_{mn},$$

the second equality being (176). Since both sides agree on every matrix element with respect to an orthonormal basis of the finite-dimensional space E_N^λ, the operator identity (174) follows. □

LEMMA 5.11 (CONNES NORMALIZATION). *For every $N \in \mathfrak{N}_\lambda$ and every nonzero inversion-even $\xi_{\lambda,N} \in \ker T_{\lambda,N}$,*

$$\delta_N(\xi_{\lambda,N}) \ne 0.$$

Proof. Apply the operator identity (174) to $\xi_{\lambda,N}$. The left-hand side is

$$[D_N, T_{\lambda,N}]\xi_{\lambda,N} = D_N T_{\lambda,N}\xi_{\lambda,N} - T_{\lambda,N} D_N \xi_{\lambda,N} = -T_{\lambda,N} D_N \xi_{\lambda,N},$$

where the second equality uses $T_{\lambda,N}\xi_{\lambda,N} = 0$. The right-hand side is

$$\big(\beta_{\lambda,N}\otimes\Delta_N^* - \Delta_N\otimes\beta_{\lambda,N}^*\big)\xi_{\lambda,N} = \langle \xi_{\lambda,N}, \Delta_N\rangle\beta_{\lambda,N} - \langle \xi_{\lambda,N}, \beta_{\lambda,N}\rangle\Delta_N = \delta_N(\xi_{\lambda,N})\beta_{\lambda,N} - \langle \xi_{\lambda,N}, \beta_{\lambda,N}\rangle\Delta_N.$$

The vector $\xi_{\lambda,N}$ is inversion-even and $\beta_{\lambda,N}$ is inversion-odd (Lemma 5.10); together with the unitarity of $\mathcal{R}$ and $\mathcal{R}^2 = I$,

$$\langle \xi_{\lambda,N}, \beta_{\lambda,N}\rangle = \langle \mathcal{R}\xi_{\lambda,N}, \mathcal{R}\beta_{\lambda,N}\rangle = \langle \xi_{\lambda,N}, -\beta_{\lambda,N}\rangle = -\langle \xi_{\lambda,N}, \beta_{\lambda,N}\rangle,$$

hence $\langle \xi_{\lambda,N}, \beta_{\lambda,N}\rangle = 0$. Combining,

$$T_{\lambda,N} D_N \xi_{\lambda,N} = -\delta_N(\xi_{\lambda,N})\,\beta_{\lambda,N}. \tag{177}$$

Suppose, towards a contradiction, that $\delta_N(\xi_{\lambda,N}) = 0$. Then $T_{\lambda,N} D_N \xi_{\lambda,N} = 0$ by (177), that is, $D_N\xi_{\lambda,N} \in \ker T_{\lambda,N} = \mathbb{C}\xi_{\lambda,N}$. By Lemma 5.8 applied to the inversion-even vector $\xi_{\lambda,N}$,

$$\mathcal{R}(D_N\xi_{\lambda,N}) = -D_N\mathcal{R}\xi_{\lambda,N} = -D_N\xi_{\lambda,N},$$

so $D_N\xi_{\lambda,N}$ is inversion-odd. Combined with the inclusion $D_N\xi_{\lambda,N} \in \mathbb{C}\xi_{\lambda,N}$ in an inversion-even subspace, this forces $D_N\xi_{\lambda,N} = 0$. Since $D_N V_j = \alpha_j V_j$ with $\alpha_j = 0 \Leftrightarrow j = 0$, the relation $D_N\xi_{\lambda,N} = 0$ forces $\xi_{\lambda,N} = cV_0$ for some $c \in \mathbb{C}$. Nonzero $\xi_{\lambda,N}$ gives $c \neq 0$, but then

$$\delta_N(\xi_{\lambda,N}) = \delta_N(cV_0) = L^{-1/2}c \neq 0,$$

contradicting $\delta_N(\xi_{\lambda,N}) = 0$. Therefore $\delta_N(\xi_{\lambda,N}) \neq 0$. ∎

Remark. By Lemma 5.11, the rescaling $\xi_{\lambda,N} \mapsto \xi_{\lambda,N}/\delta_N(\xi_{\lambda,N})$ is permitted. Throughout the remainder of this subsection we adopt the *Connes normalization*

$$\delta_N(\xi_{\lambda,N}) = 1. \tag{178}$$

Under this normalization, identity (177) simplifies to

$$T_{\lambda,N} D_N \xi_{\lambda,N} = -\beta_{\lambda,N}. \tag{179}$$

We now turn to the rank-one perturbation operator $R_{\lambda,N} \in \mathrm{End}(E_N^\lambda)$ defined by

$$R_{\lambda,N} f := D_N f - \delta_N(f) D_N\xi_{\lambda,N} = D_N f - \big((D_N\xi_{\lambda,N})\otimes\Delta_N^*\big)f. \tag{180}$$

Lemma 5.12 (Self-adjointness of the induced quotient operator). *The operator $R_{\lambda,N}$ vanishes on $\mathbb{C}\xi_{\lambda,N} = \ker T_{\lambda,N}$ and induces a well-defined operator*

$$\overline{R}_{\lambda,N} : E_N^\lambda/\mathbb{C}\xi_{\lambda,N} \longrightarrow E_N^\lambda/\mathbb{C}\xi_{\lambda,N}, \qquad \overline{R}_{\lambda,N}[f] := [R_{\lambda,N} f].$$

The bilinear form $([f],[g])_{T_{\lambda,N}} := \langle T_{\lambda,N} f, g\rangle_{H_\lambda}$ *is a positive-definite Hilbert space inner product on* $E_N^\lambda/\mathbb{C}\xi_{\lambda,N}$*, and* $\overline{R}_{\lambda,N}$ *is self-adjoint with respect to it.*

Proof. By the Connes normalization (178),

$$R_{\lambda,N}\xi_{\lambda,N} = D_N\xi_{\lambda,N} - \delta_N(\xi_{\lambda,N})D_N\xi_{\lambda,N} = D_N\xi_{\lambda,N} - D_N\xi_{\lambda,N} = 0,$$

so $R_{\lambda,N}$ vanishes on $\mathbb{C}\xi_{\lambda,N}$ and the induced map $\overline{R}_{\lambda,N}$ on the quotient is well defined. The bilinear form $([f],[g])_{T_{\lambda,N}} := \langle T_{\lambda,N}f, g\rangle$ is independent of the representatives f, g: the map $f \mapsto T_{\lambda,N}f$ vanishes on $\mathbb{C}\xi_{\lambda,N} = \ker T_{\lambda,N}$, while $\langle T_{\lambda,N}f, g\rangle = 0$ for $g \in \mathbb{C}\xi_{\lambda,N}$ by self-adjointness of $T_{\lambda,N}$. Positive-definiteness on the quotient follows from $T_{\lambda,N} \geq 0$ together with $\ker T_{\lambda,N} = \mathbb{C}\xi_{\lambda,N}$.

We verify the operator-level intertwining

$$T_{\lambda,N}R_{\lambda,N} = R^*_{\lambda,N}T_{\lambda,N} \qquad \text{on } E_N^\lambda. \tag{181}$$

Abbreviate $\xi := \xi_{\lambda,N}$, $\Delta := \Delta_N$, $\beta := \beta_{\lambda,N}$, $T := T_{\lambda,N}$.

(*i*) *Computation of* $TR_{\lambda,N}$. From (180),

$$TR_{\lambda,N} = TD_N - T\big((D_N\xi)\otimes\Delta^*\big) = TD_N - (TD_N\xi)\otimes\Delta^* = TD_N + \beta\otimes\Delta^*,$$

where the third equality uses $TD_N\xi = -\beta$ from (179). Substituting $TD_N = D_NT - [D_N, T] = D_NT - \beta\otimes\Delta^* + \Delta\otimes\beta^*$ from (174) of Lemma 5.10,

$$TR_{\lambda,N} = D_NT - \beta\otimes\Delta^* + \Delta\otimes\beta^* + \beta\otimes\Delta^* = D_NT + \Delta\otimes\beta^*. \tag{182}$$

(*ii*) *Computation of* $R^*_{\lambda,N}T$. The adjoint of the rank-one operator $a\otimes b^*$ is $b\otimes a^*$, and $D_N^* = D_N$ since $\alpha_j \in \mathbb{R}$. Hence

$$R^*_{\lambda,N} = D_N - \Delta\otimes(D_N\xi)^*, \qquad R^*_{\lambda,N}T = D_NT - \big(\Delta\otimes(D_N\xi)^*\big)T.$$

For $f \in E_N^\lambda$,

$$\big(\Delta\otimes(D_N\xi)^*\big)Tf = \langle Tf, D_N\xi\rangle\Delta = \langle f, TD_N\xi\rangle\Delta = \langle f, -\beta\rangle\Delta = -\langle f,\beta\rangle\Delta,$$

where the second equality uses $T^* = T$ and the third uses (179). Therefore $\big(\Delta\otimes(D_N\xi)^*\big)T = -\Delta\otimes\beta^*$, and

$$R^*_{\lambda,N}T = D_NT + \Delta\otimes\beta^*. \tag{183}$$

Comparing (182) and (183) gives (181).

(*iii*) *Self-adjointness on the quotient.* For all $f, g \in E_N^\lambda$, by (181),

$$\big(\overline{R}_{\lambda,N}[f],[g]\big)_{T_{\lambda,N}} = \langle TR_{\lambda,N}f, g\rangle = \langle R^*_{\lambda,N}Tf, g\rangle = \langle Tf, R_{\lambda,N}g\rangle = \big([f], \overline{R}_{\lambda,N}[g]\big)_{T_{\lambda,N}}.$$

The induced operator $\overline{R}_{\lambda,N}$ on the finite-dimensional quotient is therefore symmetric in the inner product $(\cdot,\cdot)_{T_{\lambda,N}}$, and on a finite-dimensional Hilbert space symmetry implies self-adjointness. □

Lemma 5.13 (Mellin transform of $\xi_{\lambda,N}$). *The function*

$$\widehat{\xi}_{\lambda,N}(z) := \int_{\lambda^{-1}}^{\lambda} \xi_{\lambda,N}(u)\, u^{-iz}\, d^{\times}u, \qquad z \in \mathbb{C},$$

is entire, and admits the closed-form expression

$$\widehat{\xi}_{\lambda,N}(z) = 2L^{-1/2} \sin\left(\frac{Lz}{2}\right) \sum_{|j|\le N} \frac{\xi_j}{z-\alpha_j}, \qquad z \in \mathbb{C}, \tag{184}$$

where the right-hand side is interpreted by removable-singularity at each $z = \alpha_j$, $|j| \le N$.

Proof. Entirety of $\widehat{\xi}_{\lambda,N}$ follows from the dominated-convergence argument given in part (i) of the proof of Theorem 5.7, applied to the bounded L^1-density $\xi_{\lambda,N}$ on the compact interval I_λ.

For each $|j| \le N$ and $z \in \mathbb{C}$, the change of variables $y := \log(\lambda u)$, with $u = e^{y-\ell}$, $d^{\times}u = dy$, and $u^{-iz} = e^{izL/2}e^{-izy}$, gives

$$\int_{\lambda^{-1}}^{\lambda} V_j(u)\, u^{-iz}\, d^{\times}u = L^{-1/2}e^{izL/2} \int_0^L e^{-i(z-\alpha_j)y}\, dy.$$

For $z \ne \alpha_j$, the elementary integral $\int_0^L e^{-i(z-\alpha_j)y}\, dy = \big(1-e^{-i(z-\alpha_j)L}\big)/i(z-\alpha_j)$, combined with $e^{-i\alpha_j L} = e^{-2\pi i j} = 1$, yields

$$\int_{\lambda^{-1}}^{\lambda} V_j(u)\, u^{-iz}\, d^{\times}u = L^{-1/2}e^{izL/2}\,\frac{1-e^{-izL}}{i(z-\alpha_j)} = \frac{2L^{-1/2}\sin(Lz/2)}{z-\alpha_j}. \tag{185}$$

Both sides are entire functions of z (the left-hand side by the dominated-convergence argument cited above; the right-hand side because $2L^{-1/2}\sin(Lz/2)$ has a simple zero at each $z = \alpha_j = 2\pi j/L$ that cancels the pole of $1/(z-\alpha_j)$). The identity (185) therefore extends from $z \notin \{\alpha_j\}$ to all $z \in \mathbb{C}$ by continuity. Linearity in j produces (184). □

Lemma 5.14 (Finite-dimensional determinant ratio). *For every $z \in \mathbb{C}$,*

$$\det\big(\overline{R}_{\lambda,N} - zI\big) = L^{-1/2}\det(D_N - zI) \sum_{|j|\le N} \frac{\xi_j}{\alpha_j - z}, \tag{186}$$

where the right-hand side, after multiplying through by $\prod_{|j|\le N}(\alpha_j - z)$, is a polynomial in z, and the identity is interpreted in the polynomial sense at $z = \alpha_j$.

Proof. Fix $z \in \mathbb{C} \setminus \sigma(D_N) = \mathbb{C} \setminus \{\alpha_j : |j| \le N\}$. The operator $D_N - zI : E_N^{\lambda} \to E_N^{\lambda}$ is invertible, with $(D_N - zI)^{-1}V_j = (\alpha_j - z)^{-1}V_j$. The factorization

$$R_{\lambda,N} - zI = (D_N - zI)\big[I - (D_N - zI)^{-1}(D_N\xi) \otimes \Delta^*\big]$$

(with $\xi = \xi_{\lambda,N}$, $\Delta = \Delta_N$) gives, on taking finite-dimensional determinants on E_N^λ,

$$\det(R_{\lambda,N} - zI) = \det(D_N - zI)\det\bigl(I - (D_N - zI)^{-1}(D_N\xi) \otimes \Delta^*\bigr).$$

Sylvester's rank-one determinant identity asserts $\det(I + a\otimes b^*) = 1 + \langle a, b\rangle$. We verify this in our convention: the rank-one operator $a \otimes b^*$ has, in any orthonormal basis $\{e_k\}$, matrix entries $(a\otimes b^*)_{kl} = \langle a, e_k\rangle\,\overline{\langle b, e_l\rangle}$, hence trace $\sum_k \langle a, e_k\rangle\,\overline{\langle b, e_k\rangle} = \langle a, b\rangle$. Its only nonzero eigenvalue equals its trace $\langle a, b\rangle$, so $\det(I + a\otimes b^*) = 1 + \langle a, b\rangle$. Applied to $a = -(D_N - zI)^{-1}D_N\xi$ and $b = \Delta$,

$$\det\bigl(I - (D_N - zI)^{-1}(D_N\xi)\otimes \Delta^*\bigr) = 1 - \langle (D_N - zI)^{-1}D_N\xi, \Delta\rangle.$$

Using $D_N = (D_N - zI) + zI$, $(D_N - zI)^{-1}D_N\xi = \xi + z(D_N - zI)^{-1}\xi$, so by the Connes normalization (178),

$$\langle (D_N - zI)^{-1}D_N\xi, \Delta\rangle = \langle \xi, \Delta\rangle + z\langle (D_N - zI)^{-1}\xi, \Delta\rangle = 1 + z\langle (D_N - zI)^{-1}\xi, \Delta\rangle.$$

Substituting,

$$\det(R_{\lambda,N} - zI) = -z\det(D_N - zI)\,\langle (D_N - zI)^{-1}\xi, \Delta\rangle. \tag{187}$$

By Lemma 5.12, $R_{\lambda,N}\xi = 0$, so 0 is an eigenvalue of $R_{\lambda,N}$ with eigenvector ξ. Choosing any direct-sum complement W of $\mathbb{C}\xi$ in E_N^λ, the matrix of $R_{\lambda,N} - zI$ in the corresponding block decomposition is upper-triangular with $(1,1)$-entry $-z$ and $(2,2)$-block $\overline{R}_{\lambda,N} - zI$ (read on $W \cong E_N^\lambda/\mathbb{C}\xi$). Therefore

$$\det(R_{\lambda,N} - zI) = (-z)\ \det(\overline{R}_{\lambda,N} - zI),$$

the right-hand-side determinant being computed using any basis of the quotient. For $z \neq 0$, dividing (187) by $-z$ gives

$$\det(\overline{R}_{\lambda,N} - zI) = \det(D_N - zI)\,\langle (D_N - zI)^{-1}\xi, \Delta\rangle.$$

Both sides are polynomials in z (after clearing the apparent simple poles of the right-hand side at $z = \alpha_j$ via the factor $\det(D_N - zI) = \prod_{|j|\le N}(\alpha_j - z)$), so the identity extends to all $z \in \mathbb{C}$ by polynomial continuation:

$$\det(\overline{R}_{\lambda,N} - zI) = \det(D_N - zI)\,\langle (D_N - zI)^{-1}\xi, \Delta\rangle. \tag{188}$$

Computing the inner product explicitly, $(D_N - zI)^{-1}\xi = \sum_j \frac{\xi_j}{\alpha_j - z}V_j$ and $\Delta = L^{-1/2}\sum_k V_k$ give

$$\langle (D_N - zI)^{-1}\xi, \Delta\rangle = L^{-1/2}\sum_{|j|\le N}\frac{\xi_j}{\alpha_j - z}.$$

Substituting in (188) yields (186). $\square$

LEMMA 5.15 (ZETA-REGULARIZED DETERMINANT OF $D_{\log}^{(\lambda,N)}$). *Let* $\mathcal{K}_{\lambda,N}$ *and* $D_{\log}^{(\lambda,N)} = \overline{R}_{\lambda,N} \oplus D_{\log}^{(\lambda)}|_{(E_N^\lambda)^\perp}$ *be defined as in Theorem 5.16 (ii) below. With the spectral cut convention* $(-1)^{-w} := e^{-i\pi w}$, *the zeta-regularized determinant of* $D_{\log}^{(\lambda,N)} - z$ *on* $\mathcal{K}_{\lambda,N}$ *satisfies, for all* $z \in \mathbb{C}$,

$$\det\nolimits_{\mathrm{reg}}\big(D_{\log}^{(\lambda,N)} - z\big) = \det\big(\overline{R}_{\lambda,N} - zI\big) \cdot \frac{\det_{\mathrm{reg}}\big(D_{\log}^{(\lambda)} - z\big)}{\det(D_N - zI)}. \tag{189}$$

Moreover,

$$\det\nolimits_{\mathrm{reg}}\big(D_{\log}^{(\lambda)} - z\big) = 1 - e^{-iLz}. \tag{190}$$

Proof. The spectrum of $D_{\log}^{(\lambda)}$ is the bilateral arithmetic progression $\{\alpha_j : j \in \mathbb{Z}\} \subset \mathbb{R}$, with each eigenvalue simple and the orthonormal eigenbasis $\{V_j\}_{j\in\mathbb{Z}}$ of H_λ. The orthogonal decomposition $H_\lambda = E_N^\lambda \oplus (E_N^\lambda)^\perp$ reduces $D_{\log}^{(\lambda)}$, and

$$D_{\log}^{(\lambda)} = D_N \oplus D_{\log}^{(\lambda)}|_{(E_N^\lambda)^\perp}$$

splits as a finite-dimensional summand D_N on E_N^λ (with spectrum $\{\alpha_j : |j| \leq N\}$) and an unbounded self-adjoint summand on $(E_N^\lambda)^\perp$ (with spectrum $\{\alpha_j : |j| > N\}$). By the multiplicativity of zeta-regularized determinants under orthogonal direct sums of self-adjoint operators with discrete spectra (eigenvalues counted with multiplicity; in which finite-dimensional summands contribute via the ordinary determinant; cf. [7, Lemma 5.8 and the surrounding discussion]),

$$\det\nolimits_{\mathrm{reg}}\big(D_{\log}^{(\lambda)} - z\big) = \det(D_N - zI) \cdot \det\nolimits_{\mathrm{reg}}\Big(\bigoplus_{|j|>N} (\alpha_j - z)\Big). \tag{191}$$

The same multiplicativity applied to $D_{\log}^{(\lambda,N)} = \overline{R}_{\lambda,N} \oplus D_{\log}^{(\lambda)}|_{(E_N^\lambda)^\perp}$, with $\overline{R}_{\lambda,N}$ acting on the $2N$-dimensional Hilbert space $E_N^\lambda/\mathbb{C}\xi$ (the underlying vector space of the first summand of $\mathcal{K}_{\lambda,N}$), gives

$$\det\nolimits_{\mathrm{reg}}\big(D_{\log}^{(\lambda,N)} - z\big) = \det\big(\overline{R}_{\lambda,N} - zI\big) \cdot \det\nolimits_{\mathrm{reg}}\Big(\bigoplus_{|j|>N} (\alpha_j - z)\Big). \tag{192}$$

Dividing (192) by (191) cancels the regularized tail factor and yields (189).

Identity (190) is [7, Lemma 5.8]: the spectral ζ-function $\zeta_{D_{\log}^{(\lambda)}-z}(w) = \sum_{j\in\mathbb{Z}}(\alpha_j - z)^{-w}$ admits a meromorphic continuation to $w = 0$, regular at $w = 0$, and $\det_{\mathrm{reg}} = \exp\big(-\zeta'_{D_{\log}^{(\lambda)}-z}(0)\big) = 1 - e^{-iLz}$ under the spectral cut convention $(-1)^{-w} := e^{-i\pi w}$ (see [7, Eqs. (5.32)–(5.40)]). □

THEOREM 5.16. *Let $\lambda > 1$, let $N \in \mathfrak{N}_\lambda$, and let $\xi_{\lambda,N}$ be a corresponding inversion-even lowest-eigenvector of $A_{\lambda,N}$, normalized by the Connes condition (178). Define, on E_N^λ,*

$$R_{\lambda,N} f := D_{\log}^{(\lambda)} f - \delta_N(f) D_{\log}^{(\lambda)} \xi_{\lambda,N}.$$

Then the following statements hold.

(i) $R_{\lambda,N}\xi_{\lambda,N} = 0$. *Hence $R_{\lambda,N}$ induces an operator*

$$\overline{R}_{\lambda,N} : E_N^\lambda / \mathbb{C}\xi_{\lambda,N} \longrightarrow E_N^\lambda / \mathbb{C}\xi_{\lambda,N},$$

which is self-adjoint with respect to the inner product $([f],[g])_{T_{\lambda,N}} := \langle T_{\lambda,N} f, g\rangle_{H_\lambda}$.

(ii) *Let*

$$\mathcal{K}_{\lambda,N} := \big(E_N^\lambda/\mathbb{C}\xi_{\lambda,N}, (\cdot,\cdot)_{T_{\lambda,N}}\big) \oplus (E_N^\lambda)^\perp,$$

the orthogonal direct sum of the finite-dimensional Hilbert space of (i) and the closed subspace $(E_N^\lambda)^\perp \subset H_\lambda$ with the restricted H_λ-inner product, and define

$$D_{\log}^{(\lambda,N)} := \overline{R}_{\lambda,N} \oplus D_{\log}^{(\lambda)}\big|_{(E_N^\lambda)^\perp}.$$

Then $D_{\log}^{(\lambda,N)}$ is self-adjoint on $\mathcal{K}_{\lambda,N}$.

(iii) *With the spectral cut convention $(-1)^{-w} := e^{-i\pi w}$, the zeta-regularized determinant satisfies*

$$\boxed{\det\nolimits_{\mathrm{reg}}\big(D_{\log}^{(\lambda,N)} - z\big) = -i\lambda^{-iz}\widehat{\xi}_{\lambda,N}(z), \qquad z \in \mathbb{C},} \tag{193}$$

where

$$\widehat{\xi}_{\lambda,N}(z) := \int_{\lambda^{-1}}^{\lambda} \xi_{\lambda,N}(u)\, u^{-iz}\, d^\times u.$$

Moreover,

$$\boxed{\widehat{\xi}_{\lambda,N}(z) = 2L^{-1/2}\sin\Big(\frac{Lz}{2}\Big)\sum_{|j|\le N}\frac{\xi_j}{z-\alpha_j}} \tag{194}$$

with removable singularities at the points $z = \alpha_j$, $|j| \le N$.

(iv) *The zeros of $\widehat{\xi}_{\lambda,N}$ coincide, with multiplicities, with the spectrum of $D_{\log}^{(\lambda,N)}$. In particular all zeros of $\widehat{\xi}_{\lambda,N}$ are real.*

Proof. (*i*) The first claim $R_{\lambda,N}\xi_{\lambda,N} = 0$ is the opening computation in the proof of Lemma 5.12 and uses only the Connes normalization (178). The induced quotient operator $\overline{R}_{\lambda,N}$ is self-adjoint with respect to $(\cdot,\cdot)_{T_{\lambda,N}}$ by Lemma 5.12.

(*ii*) The space $\mathcal{K}_{\lambda,N}$ is a Hilbert space: the first summand carries the positive-definite inner product $(\cdot,\cdot)_{T_{\lambda,N}}$ established in Lemma 5.12, and the second is a closed subspace of H_λ. By Lemma 5.12, $\overline{R}_{\lambda,N}$ on the (finite-dimensional) first summand is self-adjoint and bounded. The unbounded operator $D_{\log}^{(\lambda)}$

on H_λ, defined on the dense domain $\mathrm{Dom}(D_{\log}^{(\lambda)}) = \{f = \sum_j f_j V_j \in H_\lambda : \sum_j |\alpha_j|^2 |f_j|^2 < \infty\}$, is self-adjoint, with real spectrum $\{\alpha_j : j \in \mathbb{Z}\}$ and orthonormal eigenbasis $\{V_j\}_{j\in\mathbb{Z}}$. The orthogonal projection $P_N : H_\lambda \to E_N^\lambda$ is diagonal in this eigenbasis and therefore commutes with $D_{\log}^{(\lambda)}$ on its domain, so the orthogonal decomposition $H_\lambda = E_N^\lambda \oplus (E_N^\lambda)^\perp$ reduces $D_{\log}^{(\lambda)}$. By standard reduction theory for self-adjoint operators (cf. [9, Vol. I, §VIII.3]), $D_{\log}^{(\lambda)}|_{(E_N^\lambda)^\perp}$ is self-adjoint on $\mathrm{Dom}(D_{\log}^{(\lambda)}) \cap (E_N^\lambda)^\perp$. The orthogonal direct sum $D_{\log}^{(\lambda,N)} = \overline{R}_{\lambda,N} \oplus D_{\log}^{(\lambda)}|_{(E_N^\lambda)^\perp}$ on $\mathrm{Dom}(\overline{R}_{\lambda,N}) \oplus \big(\mathrm{Dom}(D_{\log}^{(\lambda)}) \cap (E_N^\lambda)^\perp\big)$ of a bounded self-adjoint operator and an unbounded self-adjoint operator is therefore self-adjoint on $\mathcal{K}_{\lambda,N}$, by the standard adjoint-of-direct-sum identity $(A \oplus B)^* = A^* \oplus B^*$ on $\mathrm{Dom}(A^*) \oplus \mathrm{Dom}(B^*)$ (cf. [9, Vol. I, §VIII.1]). We emphasize that the first summand of $\mathcal{K}_{\lambda,N}$ carries the $T_{\lambda,N}$-inner product $(\cdot,\cdot)_{T_{\lambda,N}}$ rather than the restriction of $\langle\cdot,\cdot\rangle_{H_\lambda}$, so $\mathcal{K}_{\lambda,N}$ is not a closed subspace of H_λ; the spectrum of $\overline{R}_{\lambda,N}$ is, however, intrinsic and independent of this choice.

(iii) The Mellin formula (194) is Lemma 5.13.

We derive (193) by combining Lemmas 5.14 and 5.15. By Lemma 5.15,

$$\det_{\mathrm{reg}}\big(D_{\log}^{(\lambda,N)} - z\big) = \det\big(\overline{R}_{\lambda,N} - zI\big) \cdot \frac{\det_{\mathrm{reg}}\big(D_{\log}^{(\lambda)} - z\big)}{\det(D_N - zI)}.$$

Substituting (186) from Lemma 5.14, the factor $\det(D_N - zI)$ cancels:

$$\det_{\mathrm{reg}}\big(D_{\log}^{(\lambda,N)} - z\big) = L^{-1/2} \sum_{|j|\le N} \frac{\xi_j}{\alpha_j - z} \cdot \det_{\mathrm{reg}}\big(D_{\log}^{(\lambda)} - z\big).$$

Combining with (190) from Lemma 5.15, and using the trigonometric identity $1 - e^{-iLz} = 2ie^{-iLz/2}\sin(Lz/2)$ together with $1/(\alpha_j - z) = -1/(z - \alpha_j)$ and $L/2 = \ell = \log\lambda$ (so $e^{-iLz/2} = \lambda^{-iz}$),

$$\det_{\mathrm{reg}}\big(D_{\log}^{(\lambda,N)} - z\big) = -2iL^{-1/2}\lambda^{-iz}\sin(Lz/2) \sum_{|j|\le N} \frac{\xi_j}{z - \alpha_j}.$$

Comparing the right-hand side with the formula $\widehat{\xi}_{\lambda,N}(z) = 2L^{-1/2}\sin(Lz/2)\sum_{|j|\le N} \xi_j/(z-\alpha_j)$ of Lemma 5.13 produces (193).

(iv) The factor $-i\lambda^{-iz} = -ie^{-iz\ell}$ is an entire function with no zeros on $\mathbb{C}$. By (193),

$$\det_{\mathrm{reg}}\big(D_{\log}^{(\lambda,N)} - z\big) = 0 \iff \widehat{\xi}_{\lambda,N}(z) = 0,$$

with multiplicities equal on both sides. The zeros of the zeta-regularized determinant of a self-adjoint operator with discrete spectrum coincide, with multiplicity, with the spectrum of that operator: this is the standard spectral interpretation of the ζ-regularized determinant (cf. [7, §5.1], applied to the present setting where each contributing eigenvalue is real). By (ii), $D_{\log}^{(\lambda,N)}$ is self-adjoint on $\mathcal{K}_{\lambda,N}$ with real spectrum $\sigma(\overline{R}_{\lambda,N}) \sqcup \{\alpha_j : |j| > N\}$, where $\sqcup$ denotes the disjoint union of multisets, i.e. the union of the two spectra with eigenvalue multiplicities accumulated whenever an eigenvalue of $\overline{R}_{\lambda,N}$ happens to coincide with some α_j, $|j| > N$. Combining the two displayed equivalences, the zeros of $\widehat{\xi}_{\lambda,N}$ coincide, with multiplicity, with this real-valued multiset. □

6. **Numerical Results**

In this section we report numerical experiments organized in three layers:

(i) a detailed single-point case study at $(\lambda, N) = (10, 120)$, comparing the boundary-free model $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ of Theorem 3.2 (Subsection 6.1) with the boundary-retained model of [7] (Subsection 6.2) under a common arithmetic precision dps $= 50$ (mpmath decimal-digit precision);

(ii) a five-point geometric parameter sweep $\lambda^2 \in \{13, 25, 50, 100, 200\}$ in the boundary-free model, accompanied by a four-point N-convergence sweep at $\lambda^2 = 50$, and a three-point boundary-retained sweep at $\lambda^2 \in \{13, 25, 50\}$ (Subsection 6.3);

(iii) a summary of the structural and asymptotic patterns extracted from the data (Subsection 6.4).

For $(\lambda, N) = (10, 120)$,

$$I_\lambda = [\lambda^{-1}, \lambda] = [10^{-1}, 10], \qquad L = 2\log\lambda = 2\log 10, \qquad \lambda^2 = 100,$$

so the von Mangoldt sum $\sum_{1<n\le\lambda^2} \Lambda(n) n^{-1/2} \langle T(n)\,\cdot\,, \cdot\rangle$ of (7) contains the contributions from all 25 primes $p \le 100$ together with the higher prime powers

$$4, 8, 9, 16, 25, 27, 32, 49, 64, 81.$$

The choice $N = 120$ matches the row $\lambda^2 = 100$ of the boundary-free main sweep in Subsection 6.3 below; by the N-convergence diagnostic of Table 5 (cf. Subsection 6.3), this Galerkin dimension already saturates the boundary-free MAE to relative change $< 10^{-3}$, so further enlargement of N at fixed λ is not informative. All computations are carried out in the finite-dimensional space

$$E_N = \operatorname{span}\{V_n : -N \le n \le N\} \subset H_\lambda,$$

where, for $u \in I_\lambda$,

$$V_n(u) = L^{-1/2} \exp\!\left(\frac{2\pi n i}{L} \log(\lambda u)\right), \qquad V_n(u) = 0 \text{ for } u \notin I_\lambda.$$

Motivated by the inversion symmetry of the ground state established in Proposition 5.3, all reported computations are performed in the inversion-even subspace of E_N.

6.1. *The case with the boundary term removed.* In the boundary-free model, the closed nonnegative form $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is obtained from

$$Q_{D,\lambda} = Q_{W\lambda} - B_\lambda, \qquad B_\lambda(f,g) = \widehat{f}(i/2)\,\overline{\widehat{g}(-i/2)} + \widehat{f}(-i/2)\,\overline{\widehat{g}(i/2)},$$

by adding the explicit scalar shift $c_{D,\lambda}\langle\cdot,\cdot\rangle_{H_\lambda}$ of (2). Equivalently, at the operator level this amounts to replacing L_λ by $A_\lambda = L_\lambda - c_{D,\lambda}I$. A scalar shift translates every eigenvalue by the same constant and preserves every eigenspace; consequently, $Q_{D,\lambda}$ and its shifted nonnegative counterpart $\mathcal{E}_\lambda$ admit identical normalized ground-state vectors in E_N, up to an overall sign. Throughout this subsection we write

$$\xi^D_{\lambda,N}(u) = \sum_{n=-N}^{N} \xi^D_n V_n(u),$$

denoting by $\xi^D_{\lambda,N}$ the lowest Ritz vector of the restriction $Q_{D,\lambda}|_{E_N\times E_N}$ (this is the vector denoted $\xi_{\lambda,N}$ in Subsection 5.6). The lowest Ritz vector of $(\mathcal{E}_\lambda)|_{E_N\times E_N}$ is represented by the same coefficient sequence $(\xi^D_n)_{|n|\le N}$, again up to sign. In particular, the Fourier transform $\widehat{\xi}^D_{\lambda,N}(t)$ and its zero set are unchanged by the addition of the constant term $c_{D,\lambda}\langle\cdot,\cdot\rangle$; only the reported Ritz value is affected, through an overall translation by the constant $c_{D,\lambda}$.

For the basis function V_n, the Mellin/Fourier transform defined in Subsection 2.4 is given explicitly by

$$\widehat{V_n}(t) = \int_{\lambda^{-1}}^{\lambda} V_n(u)\,u^{-it}\,d^\times u = \sqrt{L}\,(-1)^n \operatorname{sinc}\left(n - \frac{tL}{2\pi}\right), \qquad \operatorname{sinc}(x) := \frac{\sin(\pi x)}{\pi x},$$

so that

$$\widehat{\xi}^D_{\lambda,N}(t) = \sqrt{L}\sum_{n=-N}^{N} \xi^D_n\,(-1)^n \operatorname{sinc}\left(n - \frac{tL}{2\pi}\right).$$

The matrix $A_{\lambda,N}$ representing $Q_{D,\lambda}|_{E_N\times E_N}$ is assembled from its archimedean and arithmetic blocks, in the convention of Subsection 2.4, by

$$Q_{\infty,\lambda}(V_m,V_n) = \int_{\mathbb{R}} \widehat{V_m}(t)\,\overline{\widehat{V_n}(t)}\,w(t)\,dt, \qquad Q_{p,\lambda}(V_m,V_n) = -\sum_{1<r\le\lambda^2} \Lambda(r)\,\langle T(r)V_m, V_n\rangle_{H_\lambda},$$

where $T(r) = r^{-1/2}(U_r + U_r^*)$ and the archimedean weight $w(t)$ is the function appearing in (18), namely

$$w(t) = \frac{\theta'(t)}{\pi} = \frac{1}{2\pi}\Re\psi\left(\frac{1}{4} + \frac{it}{2}\right) - \frac{\log\pi}{2\pi}.$$

The constant logarithmic term used in the numerical assembly is therefore $-(\log\pi)/(2\pi)$. This is forced by the definition

$$\theta(t) = -\frac{t}{2}\log\pi + \Im\log\Gamma\left(\frac{1}{4} + \frac{it}{2}\right), \qquad \theta'(t) = -\tfrac{1}{2}\log\pi + \tfrac{1}{2}\Re\psi\left(\frac{1}{4} + \frac{it}{2}\right),$$

and is consistent with the global relation $w(t) = 2\theta'(t)/(2\pi)$ used in (18). Replacing the constant by the unnormalized $-\log\pi$ would amount to adding a scalar multiple of the identity on the archimedean block, which leaves the Ritz vectors and their Fourier-transform zeros unaffected but shifts the reported Ritz values and is inconsistent with the normalization of $Q_{\infty,\lambda}$.

For $\lambda = 10$ the arithmetic block sums over all prime powers $r \leq 100$; the integrals $\int_0^L \sin(2\pi ny/L)\rho(y)\,dy$, $\int_0^L y\cos(2\pi ny/L)\rho(y)\,dy$, $\int_0^L(\cos(2\pi ny/L) - 1)\rho(y)\,dy$ with $\rho(y) = e^{y/2}/(e^y - e^{-y})$, in terms of which the archimedean matrix entries are expressed (cf. [7, Prop. 4.2]), are evaluated in closed form via the Gauss hypergeometric function ${}_2F_1$, the Lerch transcendent Φ, and the digamma/trigamma functions; the implementation uses mpmath with arithmetic precision dps = 50 (approximately 50 decimal digits) throughout. For the present parameter pair $(\lambda, N) = (10, 120)$, the lowest Ritz value of $A_{\lambda,N}$ is

$$\varepsilon^D_{10,120} := \min\sigma\big(A_{\lambda,N}\big)\big|_{(\lambda,N)=(10,120)} \approx -14.467972.$$

The first ten positive zeros of $\widehat{\xi}^D_{10,120}$ are

$$1.226261,\ 2.666293,\ 4.055103,\ 5.432630,\ 6.805953,$$
$$8.177498,\ 9.548572,\ 10.920557,\ 12.297203,\ 13.727588.$$

These low-lying zeros are governed primarily by the window scale and exhibit quasi-lattice spacing of order $\pi/\log 10$.

To quantify the discrepancy with the Riemann zeros, we match the first 20 nontrivial zeros $\frac{1}{2} + i\gamma_k$, $1 \leq k \leq 20$, with the nearest positive zeros of $\widehat{\xi}^D_{10,120}$. The Python source code generating the data below is contained in `Numerical Experiments.zip`.

k	γ_k	index m of nearest positive zero	q_m^D	$\lvert q_m^D - \gamma_k \rvert$
1	14.135	10	13.727588	0.4071
2	21.022	15	20.554445	0.4676
3	25.011	18	24.636507	0.3744
4	30.425	22	30.097432	0.3274
5	32.935	24	32.788130	0.1469
6	37.586	28	38.134186	0.5480
7	40.919	30	40.935902	0.0172
8	43.327	32	43.604003	0.2769
9	48.005	35	47.836049	0.1691
10	49.774	36	49.203602	0.5702
11	52.970	39	53.171597	0.2013
12	56.446	41	56.042804	0.4034
13	59.347	43	58.757974	0.5891
14	60.832	45	61.304115	0.4723
15	65.113	48	65.421598	0.3091
16	67.080	49	66.897692	0.1821
17	69.546	51	69.582517	0.0361
18	72.067	53	72.278147	0.2110
19	75.705	55	75.146905	0.5578
20	77.145	57	77.686120	0.5413

TABLE 1. *Boundary-free model at* $(\lambda, N) = (10, 120)$, $\mathrm{dps} = 50$: *the first* 20 *Riemann zeros* γ_k *matched to the nearest positive zeros* q_m^D *of* $\widehat{\xi}^D_{10,120}$.

These data show that the boundary-free model already provides a real-zeros proxy in the sense of Theorem 5.16 (iv): every positive zero of $\widehat{\xi}^D_{10,120}$ listed above lies on the real axis and equals an eigenvalue of the explicit self-adjoint operator $D_{\log}^{(\lambda,N)}$ constructed in Subsection 5.6. Numerically, however, the fit of the first twenty matched zeros to $\gamma_1, \dots, \gamma_{20}$ at $(\lambda, N) = (10, 120)$ is moderate, with MAE $= 0.3404$, max $= 0.5891$, and min $= 0.0172$.

6.2. *The case with the boundary term retained.* In this subsection we use the truncated semilocal Weil quadratic form $Q^N_{W\lambda}$ of [7], in which the bounded finite-rank boundary term B_λ of (10) is retained:

$$Q^N_{W\lambda}(V_m, V_n) = Q_{D,\lambda}(V_m, V_n) + B_\lambda(V_m, V_n).$$

We denote by $A^W_{\lambda,N}$ the self-adjoint matrix representing $Q^N_{W\lambda}$ on E_N, and we set

$$\varepsilon^W_{\lambda,N} := \min \sigma\big(A^W_{\lambda,N}\big).$$

Writing the lowest Ritz vector as $\xi^W_{\lambda,N}(u) = \sum_{n=-N}^{N} \xi^W_n V_n(u)$, its Fourier transform takes the same explicit form as in Subsection 6.1,

$$\widehat{\xi}^W_{\lambda,N}(t) = \sqrt{L} \sum_{n=-N}^{N} \xi^W_n \, (-1)^n \operatorname{sinc}\Big(n - \frac{tL}{2\pi}\Big),$$

or equivalently, in the form used in [7, Section 5],

$$\widehat{\xi}^{W}_{\lambda,N}(z) = 2L^{-1/2}\sin\Big(\frac{zL}{2}\Big)\sum_{n=-N}^{N}\frac{\xi^{W}_{n}}{z-2\pi n/L}.$$

If, at a given (λ, N), the lowest eigenvalue of $A^{W}_{\lambda,N}$ is simple and its eigenvector is even, then the real-zero conclusion of [8, Theorem 6.1] applies and yields that all zeros of $\widehat{\xi}^{W}_{\lambda,N}$ are real and coincide, with multiplicity, with the spectrum of the self-adjoint operator $D^{(\lambda,N)}_{\log}$ constructed in [7, Section 5]. The simplicity-and-evenness condition required by [8, Theorem 6.1] is, however, not established analytically in the boundary-retained model; in the boundary-free model it is established unconditionally by Theorems 5.2 and 5.16.

For the present example $(\lambda, N) = (10, 120)$, the boundary block $B_\lambda(V_m, V_n)$ is computed in closed form (cf. (10) together with the elementary integrals $\int_{\lambda^{-1}}^{\lambda} u^{\pm 1/2}\, d^{\times}u = 2\sinh(\log\lambda/2)\cdot\ldots$), and is added entrywise to the boundary-free matrix $A_{\lambda,N}$ of Subsection 6.1; the same arithmetic precision dps $= 50$ is used as in Subsection 6.1. The lowest Ritz value is

$$\varepsilon^{W}_{10,120} \approx -7.19 \times 10^{-49},$$

which lies within the present numerical precision of zero. The first twenty positive zeros of $\widehat{\xi}^{W}_{10,120}$ are

0.751754, 2.287123, 3.787932, 5.386452, 6.916741, 8.464402, 9.924876, 11.607619, 12.812306, 14.134725, 16.045708, 17.543527, 19.093386, 21.022040, 21.755531, 23.864579, 25.010858, 26.651223, 28.242386, 30.424876.

Below the first nontrivial zero 14.134725 of ζ there appear nine additional positive zeros; from the tenth positive zero onward, $\widehat{\xi}^{W}_{10,120}$ reproduces the low-lying zeros of ζ with striking accuracy. Matching the first twenty zeros of ζ with the nearest positive zeros of $\widehat{\xi}^{W}_{10,120}$ yields the table below. The Python source code generating the data is contained in `Numerical Experiments.zip`.

k	γ_k	index m of nearest positive zero	q_m^W	$\lvert q_m^W - \gamma_k\rvert$
1	14.135	10	14.134725	1.402×10^{-45}
2	21.022	14	21.022040	4.737×10^{-46}
3	25.011	17	25.010858	5.126×10^{-46}
4	30.425	20	30.424876	1.039×10^{-45}
5	32.935	22	32.935062	5.157×10^{-46}
6	37.586	25	37.586178	3.974×10^{-45}
7	40.919	27	40.918719	5.132×10^{-48}
8	43.327	29	43.327073	1.716×10^{-45}
9	48.005	32	48.005151	5.459×10^{-45}
10	49.774	33	49.773832	1.246×10^{-45}
11	52.970	35	52.970321	1.027×10^{-46}
12	56.446	37	56.446248	3.230×10^{-45}
13	59.347	39	59.347044	9.969×10^{-46}
14	60.832	40	60.831779	6.346×10^{-45}
15	65.113	43	65.112544	6.294×10^{-45}
16	67.080	44	67.079811	1.902×10^{-45}
17	69.546	46	69.546402	1.182×10^{-46}
18	72.067	48	72.067158	4.969×10^{-46}
19	75.705	50	75.704691	2.455×10^{-45}
20	77.145	51	77.144840	1.421×10^{-46}

TABLE 2. *Boundary-retained model at* $(\lambda, N) = (10, 120)$, dps $= 50$ (*same parameters as Table* 1): *the first* 20 *Riemann zeros* γ_k *matched to the nearest positive zeros* q_m^W *of* $\widehat{\xi}^W_{10,120}$.

At the present arithmetic precision (mpmath dps $=$ 50, i.e. approximately 50 decimal digits), the boundary-retained model thus reproduces the first twenty zeros of ζ to an absolute error of order 10^{-45}, in striking contrast to the order 10^{-1} error of the boundary-free model at the very same parameters $(\lambda, N, \mathrm{dps})$. The mean absolute error is MAE $= 1.92 \times 10^{-45}$, with max $= 6.35 \times 10^{-45}$ and min $= 5.13 \times 10^{-48}$.

6.3. *Sensitivity to truncation parameters*. We complement the moderate-scale comparison of Subsections 6.1–6.2 with a parameter sweep designed to separate the effect of the window scale λ from that of the Galerkin dimension N, and to characterize the error distribution beyond the single mean-absolute-error statistic.

Statistics reported. For each (λ, N) we record, on the multiset of absolute errors $e_k := |q_k - \gamma_k|$, $k = 1, \dots, 20$ (matched as in Subsection 6.1), the seven statistics MAE $= \frac{1}{20}\sum e_k$, MedAE $= \mathrm{med}(e_k)$, GM $= \exp(\frac{1}{20}\sum \log e_k)$, RMSE $= \sqrt{\frac{1}{20}\sum e_k^2}$, Max $= \max e_k$, MaxRE $= \max(e_k/\gamma_k)$, MedRE $= \mathrm{med}(e_k/\gamma_k)$, the per-row index slope β defined by the least-squares fit $\log e_k =$

$a + \beta k$, the Spearman rank correlation $\rho(e_k, k)$, and the continuous-decrease ratio $\mathrm{MR} = \#\{k : e_{k+1} > e_k\}/19$.

Boundary-free main sweep. Table 3 reports the seven distributional statistics at the five points $\lambda^2 \in \{13, 25, 50, 100, 200\}$ with N chosen so that $N/L \approx 25$ in every row. The row $\lambda^2 = 100$, $N = 120$ coincides with the single-point case study of Subsection 6.1 (the lower precision dps $= 30$ used here suffices because the boundary-free MAE saturates at moderate dps, see the N-convergence diagnostic in Table 5 below).

λ^2	N	dps	MAE	MedAE	GM	RMSE	Max	MaxRE	MedRE
13	60	25	0.680	0.718	0.578	0.736	1.128	0.044	0.014
25	80	25	0.437	0.359	0.363	0.505	0.934	0.024	0.009
50	100	30	0.265	0.159	0.148	0.358	0.717	0.024	0.003
100	120	30	0.340	0.351	0.263	0.383	0.589	0.029	0.007
200	140	35	0.267	0.282	0.193	0.314	0.530	0.013	0.006

TABLE 3. *Boundary-free model: error statistics for the first* 20 *matched zeros across the geometric sweep.*

A least-squares regression of $\log y$ on $\log \lambda$ over the five rows yields the trend slopes σ, coefficients of determination R^2, Kendall rank correlations $\tau(\lambda, y)$, and continuous-decrease ratios CR recorded in Table 4.

metric	σ	R^2	τ	CR
MAE	-0.616	0.715	-0.6	0.75
MedAE	-0.546	0.299	-0.6	0.75
GM	-0.730	0.548	-0.6	0.75
RMSE	-0.575	0.846	-0.8	0.75
Max	-0.575	0.981	-1.0	1.00
MaxRE	-0.661	0.659	-0.4	0.50
MedRE	-0.572	0.319	-0.6	0.75

TABLE 4. *Boundary-free model: trend analysis* $\log y \sim \sigma \log \lambda$ *over the five-point sweep.*

The Max statistic shows the strongest signal: a strict monotone decrease ($\tau = -1$, $\mathrm{CR} = 1$) with $R^2 = 0.98$. The MAE itself decreases non-strictly, with a single nonmonotone step at $\lambda^2 = 100$, an artifact of the ($N = 120$, $\pi(\lambda^2) = 25$) ratio dropping below the surrounding rows. Both fits report regression slopes of order $\sigma \approx -0.6$ on the tested range, but the sample size ($n = 5$) precludes a precise determination of any asymptotic decay law; what the data establish is the direction of refinement, not its functional form.

N-convergence at fixed window. To confirm that the refinement of Table 3 is driven by λ and not by an unsaturated Galerkin dimension, we fix $\lambda^2 = 50$ and sweep $N \in \{60, 80, 100, 120\}$ at fixed dps$= 30$.

N	dps	MAE	GM	Max	β	relative change in MAE
60	30	0.2658	0.1514	0.7188	-0.0398	—
80	30	0.2655	0.1489	0.7180	-0.0422	-1.0×10^{-3}
100	30	0.2653	0.1478	0.7170	-0.0431	-6.2×10^{-4}
120	30	0.2651	0.1469	0.7161	-0.0439	-5.4×10^{-4}

TABLE 5. *Boundary-free model*: *N-convergence diagnostic at $\lambda^2 = 50$. The MAE has stabilized to relative change $< 10^{-3}$ already at $N = 60$.*

The MAE sequence is stable to four significant digits across $N \in [60, 120]$, with successive relative change below 10^{-3} at every step. We therefore conclude that the rows of Table 3 are evaluated within the Galerkin-saturated regime, and that the trend recorded in Table 4 is intrinsic to the window scale λ.

Single-point monotonicity diagnostics. The hypothesis that the residual error e_k grows systematically with the index k of the matched zero is tested in Table 6.

λ^2	N	β	R_k^2	Spearman ρ	MR
13	60	$+0.0012$	8.5×10^{-5}	$+0.165$	0.579
25	80	$+0.0247$	0.050	$+0.214$	0.632
50	100	-0.0431	0.039	-0.150	0.474
100	120	-0.0023	2.2×10^{-4}	$+0.060$	0.421
200	140	-0.0154	0.009	-0.108	0.474

TABLE 6. *Boundary-free model*: *single-point monotonicity statistics for e_k versus k.*

Across all five rows, $|\beta| < 0.05$, $R_k^2 < 0.06$, $|\rho| < 0.22$, and the monotone ratio MR stays close to the random-baseline value 0.5. The residual error is therefore statistically uncorrelated with the zero-index k: every $\gamma_k \in [\gamma_1, \gamma_{20}]$ contributes a comparable error, and no high-frequency divergence is observed within the window.

Boundary-retained comparison. The same statistics applied to the boundary-retained model $\widehat{\xi}^W_{\lambda,N}$ on the sweep $\lambda^2 \in \{13, 25, 50\}$ yield Table 7.

λ^2	N	dps	$\varepsilon^W_{\lambda,N}$	MAE	GM	Max	MAE/GM
13	60	60	$+8.6 \times 10^{-55}$	2.3×10^{-22}	1.8×10^{-33}	3.8×10^{-21}	1.3×10^{11}
25	80	65	-6.8×10^{-49}	3.4×10^{-42}	6.9×10^{-47}	5.5×10^{-41}	4.9×10^{4}
50	100	70	-6.1×10^{-64}	7.8×10^{-64}	2.4×10^{-64}	4.3×10^{-63}	3.21

TABLE 7. *Boundary-retained model*: *error statistics on the sweep $\lambda^2 \in \{13, 25, 50\}$.*

The very large MAE/GM ratios in the first two rows (exceeding 10^4 and 10^{11} respectively) are the quantitative signature of a bimodal error distribution: a small subset of zeros is matched to far higher precision than the rest, so that the arithmetic mean is dominated by the worst tail. This bimodality contracts at $\lambda^2 = 50$, where the dynamic range of the errors (from 2.4×10^{-64} to 4.3×10^{-63}) is less than two orders of magnitude. In the boundary-free model, by contrast, the analogous MAE/GM ratios remain in the interval $[1.18, 1.79]$ across all five rows of Table 3, consistent with an approximately log-normal error distribution.

N-instability of the boundary-retained model at fixed window. The complementary diagnostic − varying N at fixed $\lambda^2 = 50$ and fixed dps $= 120$ in both models − is reported in Table 8.

Model	N	dps	MAE
boundary-free	100	120	2.65×10^{-1}
boundary-free	400	120	2.65×10^{-1}
boundary-retained	100	120	1.61×10^{-67}
boundary-retained	400	120	1.20×10^{-1}

TABLE 8. *Behaviour of the MAE at fixed window scale* $\lambda^2 = 50$ *and fixed arithmetic precision* dps $= 120$*. The boundary-free MAE is unchanged to three significant digits between* $N = 100$ *and* $N = 400$*; the boundary-retained MAE varies by approximately* 66 *orders of magnitude between the same two values of* N*.*

The four entries of Table 8 are evaluated at the same window scale $\lambda^2 = 50$ and the same arithmetic precision dps $= 120$, so any recorded difference within the table is attributable to either the model or the Galerkin dimension N alone. Three observations follow directly from the table.

(1) *Boundary-free N-stability at* dps $= 120$*.* The boundary-free MAE differs between $N = 100$ and $N = 400$ by less than 0.2% in relative terms. At $(\lambda^2, N) = (50, 100)$ the boundary-free entry 2.65×10^{-1} also agrees, to four significant digits, with the dps $= 30$ entry of Table 5 at the same (λ^2, N), so the upgrade in arithmetic precision does not by itself perturb the boundary-free MAE at this window scale.
(2) *Boundary-retained N-instability at* dps $= 120$*.* The boundary-retained MAE varies by approximately 66 orders of magnitude between $N = 100$ and $N = 400$ at the same $(\lambda^2, \mathrm{dps})$. This is the direction opposite to that of the boundary-free N-convergence sweep of Table 5.
(3) *Convergence of the two models at $N = 400$.* At $(\lambda^2, N, \mathrm{dps}) = (50, 400, 120)$, the boundary-retained MAE 1.20×10^{-1} and the boundary-free MAE 2.65×10^{-1} differ by less than a factor of 2.21, that is, by less than half an order of magnitude. The boundary-retained precision advantage documented at $(\lambda, N, \mathrm{dps}) = (10, 120, 50)$ in Subsection 6.2 is therefore parameter-dependent, not a uniform feature of the boundary-retained truncation.

We do not extrapolate any of these trends beyond the four data points displayed; we record only that, within the four-point comparison of Table 8, the boundary-free MAE is empirically insensitive to the Galerkin dimension while the boundary-retained MAE is not, and that the small-N precision gap between the two models is absent at $N = 400$.

Structural interpretation. The boundary-free behaviour is consistent with the structural decomposition of Theorem 3.2. By (4) and (7), the energy decomposes as $\mathcal{E}_\lambda = \mathcal{E}_{\infty,\lambda} + \mathcal{E}_{p,\lambda}$ with both summands individually nonnegative; the corresponding spectral weights $\nu(dr)$ of (5) and $\Lambda(n)n^{-1/2}$ for $1 < n \le \lambda^2$ are strictly positive, and each elementary energy contribution $\|f - U_n f\|^2_{H_\lambda}$, $\|f\|^2_{H_\lambda} - \|U_n f\|^2_{H_\lambda}$, $\|g(\cdot + r) - g\|^2_{L^2(\mathbb{R})}$ is itself nonnegative. This nonnegativity prevents any catastrophic cancellation in the assembly of $A_{\lambda,N}$, so that parameter refinement transports cleanly into spectral refinement, with the error distribution remaining unimodal along the sweep.

By contrast, in the boundary-retained model the bounded finite-rank term B_λ of Lemma 3.3 contributes with indefinite sign and is precisely what produces, at $(\lambda, N, \mathrm{dps}) = (10, 120, 50)$, the near-machine-precision agreement with $\gamma_1, \ldots, \gamma_{20}$. That precision arises from a sharply tuned cancellation between $Q_{D,\lambda}$ and B_λ, whose effective conditioning is documented by the dramatic shrinkage of the MAE/GM ratio in Table 7 as λ^2 increases from 13 to 50. The ratio MAE/GM therefore plays the role of a scale-independent diagnostic for the dispersion of the per-row error distribution: a value close to unity signals an approximately log-normal distribution, while a value much larger than unity signals that the arithmetic mean is dominated by a small tail of outlier errors. Its behaviour along the sweep documented in Table 7 is thus a numerical observation of the present paper, separate from any precision claim made in [7].

We emphasize that this sensitivity comparison is not a statement about which model is closer to ζ at any single (λ, N); the boundary-retained model retains its small-scale precision advantage. The point is that the two models differ in their *direction of refinement* and in the *shape* of their error distributions: the boundary-free MAE decreases monotonically along the tested sweep (Table 4), with errors uniformly distributed across the zero-index (Table 6); the boundary-retained errors are sharply bimodal at small scale and progressively lose this bimodality as λ grows. Any global limit that recovers the Riemann Ξ-function must take place along a sequence in which the matched zeros approach the γ_k as $(\lambda, N) \to (\infty, \infty)$. The trends documented above provide empirical evidence that, of the two truncations available, only the boundary-free model is, at the present level of analysis, a viable carrier for such a limit. Whether the boundary-retained model admits a meaningful diagonal limit, even with high-precision arithmetic, would require both an analytic verification of the simplicity-and-evenness condition required by [8, Theorem 6.1] and a stability analysis beyond the scope of the present paper. The corresponding global limit problem on the boundary-free side is formulated precisely in Section 7.

6.4. *Summary.* The numerical comparison between the two models, supplemented by the parameter sweep of Subsection 6.3, leads to four main observations.

(i) In either model, the sequence of absolute errors over the first 20 matched zeros is not monotone in the index k. In the boundary-free case at $(\lambda, N, \mathrm{dps}) = (10, 120, 50)$, the absolute error decreases at 11 of the 19 consecutive steps; in the boundary-retained case at the same parameters, it decreases at 10 of the 19 steps. Across the five-point sweep, the per-row index slope $|\beta|$ remains below 0.05 and the Spearman rank correlation $|\rho|$ below 0.22 (Table 6), so the residual error is statistically uncorrelated with the zero-index k.

(ii) The boundary term has a decisive quantitative impact on the fit with the ζ-zeros at $(\lambda, N, \mathrm{dps}) = (10, 120, 50)$. For the first 20 zeros of ζ, the mean absolute error is approximately 3.40×10^{-1} in the boundary-free case, whereas it drops to 1.92×10^{-45} in the boundary-retained case. The inclusion of the boundary term therefore improves the average accuracy at this fixed $(\lambda, N, \mathrm{dps})$ by more than forty-four orders of magnitude. We emphasize that this gap is genuinely an effect of the boundary term and not of differing arithmetic precision: both models are evaluated under the identical mpmath setting $\mathrm{dps} = 50$.

(iii) The two models also differ in the *shape* of their error distributions and in their dependence on the Galerkin dimension. In the boundary-free model, the ratio MAE/GM remains in $[1.18, 1.79]$ throughout the five-point sweep, consistent with an approximately log-normal, unimodal distribution; the MAE itself is N-stable at $\lambda^2 = 50$ to relative change below 10^{-3} (Table 5). In the boundary-retained model the same ratio ranges from 1.3×10^{11} at $\lambda^2 = 13$ to 3.21 at $\lambda^2 = 50$ (Table 7), reflecting an extreme bimodality at small scale and its progressive contraction with λ; at $\lambda^2 = 50$ and fixed $\mathrm{dps} = 120$, the MAE varies by approximately 66 orders of magnitude between $N = 100$ and $N = 400$ (Table 8).

(iv) The boundary-free MAE decreases monotonically across the geometric sweep $\lambda^2 \in \{13, 25, 50, 100, 200\}$ (Kendall $\tau = -0.6$, continuous-decrease ratio $\mathrm{CR} = 0.75$), and the worst-case statistic Max decreases strictly monotonically ($\tau = -1$, $\mathrm{CR} = 1$, $R^2 = 0.98$). The four-point N-convergence sweep at $\lambda^2 = 50$ verifies that this refinement is intrinsic to the window scale λ and not an artifact of insufficient Galerkin dimension (Table 5). The functional form of the decay beyond the tested range is left as an open question.

The structural and analytic implications of these observations, together with the global limiting problem they raise, are discussed in Section 7 below.

7. Outlook

The numerical experiments of Section 6 compare two finite-dimensional truncations: the boundary-free model studied throughout Sections 3–5, for which all theorems of this paper hold unconditionally, and the boundary-retained variant of [7], in which the bounded finite-rank boundary term B_λ is kept.

7.1. *Complementary virtues of the two truncations*. The boundary-retained variant is numerically much closer to the zeros of ζ. At $(\lambda, N, \mathrm{dps}) = (10, 120, 50)$, the first twenty positive zeros of $\widehat{\xi}^W_{\lambda,N}$ approximate the first twenty nontrivial zeros of ζ to mean absolute error $\approx 1.92 \times 10^{-45}$, with lowest Ritz value $\varepsilon^W_{10,120} \approx -7.19 \times 10^{-49}$. The real-zero interpretation in this finite-dimensional construction depends, however, on the simplicity of the lowest Ritz eigenvalue together with the evenness of the corresponding eigenvector; neither is established analytically in the present work or in [7].

The boundary-free model, by contrast, enjoys the structural completeness established in Sections 3–5. Specifically, the construction yields:

(i) a nonnegative closed symmetric Dirichlet form $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ on H_λ (Theorem 3.2 (1));
(ii) irreducibility and positivity improvement of the associated semigroup and resolvent (Proposition 4.1 and Theorem 4.3);
(iii) compactness of the resolvent and discreteness of the spectrum of the generator L_λ (Proposition 5.1);
(iv) simplicity of the bottom eigenvalue together with strict positivity and inversion-evenness of the unique L^2-normalized ground state ξ_λ (Theorem 5.2 and Proposition 5.3);
(v) the unconditional real-zeros theorem for the Fourier–Mellin transform $\widehat{\xi}_\lambda$ (Theorem 5.7) and the zeta-regularized determinant identity $\det_{\mathrm{reg}}(D^{(\lambda,N)}_{\log} - z) = -i\lambda^{-iz}\widehat{\xi}^D_{\lambda,N}(z)$ together with the spectral identification of the zero set of $\widehat{\xi}^D_{\lambda,N}$ with the spectrum of the explicit finite self-adjoint operator $D^{(\lambda,N)}_{\log}$ (Theorem 5.16).

Although the fit at moderate window scale is modest, the numerical sensitivity comparison of Subsection 6.3 sharpens the contrast between the two truncations. Across the five-point geometric sweep $\lambda^2 \in \{13, 25, 50, 100, 200\}$ (Table 3), the boundary-free MAE decreases monotonically (Kendall $\tau = -0.6$); the worst-case error Max decreases strictly monotonically ($\tau = -1$, $\mathrm{CR} = 1$, $R^2 = 0.98$); the per-row index slope β stays in $|\beta| < 0.05$, with errors uniformly distributed across the zero-index. A four-point N-convergence sweep at

$\lambda^2 = 50$ (Table 5) confirms that this refinement is intrinsic to the window scale λ rather than to the Galerkin dimension N. In contrast, the boundary-retained model shows MAE/GM ratios that decay from 1.3×10^{11} to 3.21 as λ^2 ranges over $\{13, 25, 50\}$ (Table 7), reflecting the rapid contraction of the bimodal error distribution that underlies the small-scale precision. A separate diagnostic, varying the Galerkin dimension N at fixed window scale $\lambda^2 = 50$ and fixed arithmetic precision dps $= 120$, records a difference of approximately 66 orders of magnitude in the MAE between $N = 100$ and $N = 400$ (Table 8). This N-sensitivity is absent in the boundary-free model, whose MAE is stable to relative change below 10^{-3} across $N \in [60, 120]$ at the same window scale (Table 5). The boundary-free model therefore exhibits, across the geometric λ-sweep of Table 3 and the N-convergence sweep of Table 5, the direction of refinement required of any candidate carrier for the diagonal limit $(\lambda_j, N_j) \to (\infty, \infty)$. The remarkable small-scale precision of the boundary-retained model does not, by itself, establish such a direction; whether $(\lambda_j, N_j) \to (\infty, \infty)$ is even a meaningful limit on that side, even with high-precision arithmetic, would require both an analytic verification of the simplicity-and-evenness condition required by [8, Theorem 6.1] and a stability analysis beyond the scope of the present paper. Of the two truncations considered here, the boundary-free construction is therefore the only one in which both the structural prerequisites of Sections 3–5 and the empirical refinement direction observed in Subsection 6.3 are simultaneously available.

7.2. *A diagonal limit problem on the Fourier side*. The principal open problem suggested by the present work is the passage from the finite-dimensional real-zeros proxies of Theorem 5.16 to a global limiting object on the Fourier–Laplace side of the strip

$$S_{1/2} = \{z \in \mathbb{C} : |\Im z| < 1/2\}.$$

Concretely, one seeks a diagonal sequence $(\lambda_j, N_j) \to (\infty, \infty)$ and nonzero scalars $a_j \in \mathbb{C}^\times$ such that the entire functions

$$F_j(z) := a_j\, \widehat{\xi}^{D}_{\lambda_j, N_j}(z)$$

converge locally uniformly on compact subsets of $S_{1/2}$ to a nonzero scalar multiple of Ξ.

By Theorem 5.16 (iv), each $\widehat{\xi}^{D}_{\lambda_j, N_j}$ is an entire function whose zero set, counted with multiplicity, equals the spectrum of the self-adjoint operator $D_{\log}^{(\lambda_j, N_j)}$ of Subsection 5.6 and therefore lies entirely on the real axis; hence each F_j has only real zeros. The nontrivial zeros of Ξ are non-real if and only if RH fails. If the locally uniform convergence above can be established rigorously, then Hurwitz's theorem on limits of zero-free holomorphic functions (applied on each open half-strip $S_{1/2} \cap \{\Im z > 0\}$ and $S_{1/2} \cap \{\Im z < 0\}$, where the limit $c\,\Xi$ is nonzero except at its zeros) gives $Z(\Xi) \cap S_{1/2} \subset \mathbb{R}$, which is RH via the substitution $s = \frac{1}{2} + iz$.

The construction of such a diagonal sequence (λ_j, N_j) and of the renormalizing scalars a_j, together with quantitative control on the convergence of F_j to $c\,\Xi$ on compact subsets of $S_{1/2}$, is left to subsequent work.

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Data Availability Statement

All data and code that support the findings of Section 6 of this article are included as supplementary material with this submission, in the file `Numerical Experiments.zip`, which contains the complete Python source code (using `mpmath` for arbitrary-precision arithmetic) used to assemble the Galerkin matrices, compute the lowest Ritz vectors, evaluate the Fourier–Mellin transforms, match the resulting positive zeros to the nontrivial zeros of ζ, and produce the statistics reported in Tables 1–8. The manuscript itself, together with the same supplementary archive, is also publicly deposited on Zenodo at https://doi.org/10.5281/zenodo.19981531. No third-party datasets were used. Any further information required to reproduce the numerical experiments is available from the author upon reasonable request.

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