Total Defect Calculus for the Residual Erdős–Straus Obstruction

Projective Scale, Gerbe Sectors, Split-Zero Positivity, Fricke Completion, and Binary-Quadratic Envelopes

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Abstract

This note constructs the integrated obstruction object underlying the six-class residual Erdős–Straus problem. The fixed-shell divisor equation is first normalized projectively, replacing the raw target $-N$ by the invariant target -1. This removes the scalar scale but leaves a second target $-p^{-1}$, so the actual shell coefficient is not an unweighted edge but the weighted two-channel expression

$$\widetilde{c}_R(a;-1) + 2\widetilde{c}_R(a;-p^{-1}).$$

The six surviving modulo-840 classes are written simultaneously as a C_6 Coxeter fiber, a $C_3 \times C_2$ gerbe sector, and a snowflake of nonzero isotropic 3-torsion endpoints in the A_8^3 discriminant group. The total target defect for $p \equiv 289^k \pmod{840}$ is

$$\mathfrak{D}_k = E_{0,-} + 2E_{-k,-(-1)^k} \in \mathbb{Z}[C_3 \times C_2].$$

We compute its Fourier transform, circulant determinant, Smith type, inverse defect kernel, mixed arm–sign tensor, and augmentation frame. We then keep the same support object through the split-zero coefficient semiring, the $744, 747, 750$ completed A_8 arm, the A_8^3 and $A_5^4 D_4$ Fricke/Eisenstein sheets, and the genus-two binary-quadratic theta/paramodular envelope. The main conclusion is a total obstruction theorem: the target defect has full Fourier support and is nondegenerate over $\mathbb{F}_{107}$; remaining shell failure is precisely bounded positive-mass failure of the normalized divisor box against this full-support defect, with the two failure modes separated as unsupported absence and supported zero.

Contents

1 The simultaneous obstruction datum

The calculation is governed by the champion residual datum

$$p_* = 8\,803\,369, \qquad R_* = 107, \qquad a_* := \frac{p_* + 107}{4} = 2\,200\,869,$$

with

$$a_* = 3^2 \cdot 11^2 \cdot 43 \cdot 47.$$

Modulo 107 one has

$$p_* \equiv 51, \qquad a_* \equiv 93, \qquad N_* := p_* a_* \equiv 35.$$

The successful certificate is carried by

$$11^2 \equiv 121 \equiv 14 \equiv -a_* \pmod{107},$$

and therefore

$$p_* \cdot 11^2 \equiv -N_* \pmod{107}.$$

The residual square fiber after the $R = 3, 7$ sieve is

$$S_{840} = \{1, 121, 169, 289, 361, 529\} = \langle 289 \rangle \cong C_6,$$

with cyclic order

$$289^0, 289^1, 289^2, 289^3, 289^4, 289^5 \equiv 1, 289, 361, 169, 121, 529 \pmod{840}.$$

The snowflake note proves that the same six classes are the six nonzero isotropic endpoints

$$\{\pm 3e_0, \pm 3e_1, \pm 3e_2\}$$

in the discriminant group of A_8^3, and that a prime class $p \equiv 289^k \pmod{840}$ determines the target edge

$$E_k = \{v_3, v_{3-k}\}$$

from the distinguished hub $v_3 = -3e_0$ [1, 2]. The finite verification report records the congruence order of S_{840}, the triangular residues modulo 9, the $107^2 \equiv 529 \pmod{840}$ computation, and the modular coefficient rows used below [3].

The notation of this note keeps the following objects present simultaneously:

$$\boxed{\begin{array}{c}
\text{raw divisor equation} \quad d \equiv -N \pmod{R} \\
\downarrow \quad \text{projective normalization} \\
\text{target } -1 \text{ and divisor-ratio support } \mathcal{U}_R(p) \\
\downarrow \quad \text{three } p\text{-origins and inversion symmetry} \\
\text{two weighted targets } (-1) + 2(-p^{-1}) \\
\downarrow \quad S_{840} \cong C_6 \cong C_3 \times C_2 \\
\text{sector defect } \mathfrak{D}_k = E_{0,-} + 2E_{-k,-(-1)^k} \\
\downarrow \quad G(\mathbb{Z}) \text{ split-zero coefficients} \\
\tau, \\
0_{\mathbb{Z}}, \\
\mathbb{Z}_{>0} \text{ support trichotomy} \\
\downarrow \quad T_9, T_6 \text{ Fricke/Eisenstein completions} \\
\text{critical-arm support resolvents and positivity tests.}
\end{array}}$$

The semiring layer uses

$$G(\mathbb{Z}) = \mathbb{Z} \sqcup \{\tau\},$$

where τ is the semiring zero and $0_{\mathbb{Z}} \in \mathbb{Z}$ is the supported arithmetic zero. The globalization note proves the strict separation of τ and $0_{\mathbb{Z}}$, the classification of ideals and prime ideals, and the prime chain

$$P_\tau \subsetneq P_0 \subsetneq P_\ell, \qquad P_0 = \{0, \tau\}$$

[6]. This prime chain is used throughout: unsupported absence and supported zero are different states.

2 Projective fixed-shell normalization

Let $p \equiv 1 \pmod{24}$ be prime, let $R \equiv 3 \pmod 4$, and set

$$a = a_R(p) := \frac{p+R}{4}, \qquad N = pa.$$

Then

$$\frac{4}{p} - \frac{1}{a} = \frac{4a-p}{pa} = \frac{R}{N}.$$

Assume in this section that $\gcd(R, N) = 1$.

Theorem 2.1 (Projective residual normalization)**.** *Let*

$$\mathcal{U}_R(p) := \{dN^{-1} \bmod R : d \mid N^2\} \subseteq (\mathbb{Z}/R\mathbb{Z})^\times.$$

Then

$$\boxed{R \text{ gives an Erdős–Straus completion with first denominator } a \iff -1 \in \mathcal{U}_R(p).}$$

Proof. The equation

$$\frac{R}{N} = \frac{1}{b} + \frac{1}{c}$$

is equivalent to

$$Rbc = N(b+c).$$

Completing the product gives

$$(Rb-N)(Rc-N) = N^2.$$

Thus a completion determines a divisor

$$d := Rb - N \mid N^2$$

with

$$d \equiv -N \pmod R.$$

Conversely, if $d \mid N^2$ and $d \equiv -N \pmod R$, set

$$e := \frac{N^2}{d}.$$

Since $d \equiv -N$ and N is invertible modulo R,

$$e = N^2 d^{-1} \equiv N^2(-N)^{-1} \equiv -N \pmod R.$$

Hence

$$b = \frac{N+d}{R}, \qquad c = \frac{N+e}{R}$$

are positive integers and give the desired decomposition.

Multiplication by N^{-1} is a bijection of $(\mathbb{Z}/R\mathbb{Z})^\times$, so

$$d \equiv -N \pmod R \iff dN^{-1} \equiv -1 \pmod R.$$

This proves the assertion. $\square$

The projective hidden-scale note proves an analogous reduction in the q-Springborn transport setting: an unreduced determinant contains a raw factor $(1-q)^2$, while after projective normalization only the intrinsic factor $(1-q)$ remains [8]. The present shell normalization is the arithmetic analogue

$$-N \quad \rightsquigarrow \quad -1.$$

It accounts for the scale/sign sheet, but not for the three-arm coordinate. The latter is the C_3 component of the gerbe/snowflake sector.

3 The bounded signed divisor box and the weighted two-target coefficient

Factor

$$a = \prod_q q^{e_q}.$$

Define the signed divisor box

$$\mathcal{B}_R(a) := \left\{ \prod_{q^{e_q} \| a} q^{\beta_q} \bmod R : -e_q \le \beta_q \le e_q \right\}.$$

Every divisor $d \mid N^2 = p^2 a^2$ has the form

$$d = p^i u, \qquad i \in \{0,1,2\}, \qquad u \mid a^2.$$

Writing

$$u = \prod q^{\alpha_q}, \qquad 0 \le \alpha_q \le 2e_q,$$

one has

$$ua^{-1} = \prod q^{\alpha_q - e_q} \in \mathcal{B}_R(a).$$

Therefore

$$dN^{-1} = p^i u (pa)^{-1} = p^{i-1} u a^{-1}.$$

The normalized target condition $dN^{-1} \equiv -1$ gives the three signed-box targets

$$ua^{-1} \equiv -p, \qquad ua^{-1} \equiv -1, \qquad ua^{-1} \equiv -p^{-1}.$$

Because $\mathcal{B}_R(a)$ is inversion-invariant, the counts for $-p$ and $-p^{-1}$ agree. Thus the actual target is not a three-term object; it is a two-term object with weights 1 and 2.

Definition 3.1 (Split-zero signed count)**.** *For $t \in (\mathbb{Z}/R\mathbb{Z})^\times$, let*

$$\widetilde{c}_R(a;t) \in G(\mathbb{Z}_{\ge 0})$$

be defined as follows:

$$\widetilde{c}_R(a;t) = \tau$$

if t is not in the subgroup generated by the prime support of a modulo R;

$$\widetilde{c}_R(a;t) = 0_{\mathbb{Z}}$$

if t lies in that subgroup but no bounded exponent vector β in the box reaches it; and

$$\widetilde{c}_R(a;t) \in \mathbb{Z}_{>0}$$

if at least one bounded exponent vector reaches it.

Theorem 3.2 (Two-channel shell coefficient)**.** *For a coprime shell,*

$$\boxed{\Omega_R(p) := ildec_R(a; -1) + 2\tilde{c}_R(a; -p^{-1})}$$

is positive in $G(\mathbb{Z}_{\geq 0})$ if and only if the shell gives an Erdős–Straus certificate.

Proof. The three divisor origins p^0, p^1, p^2 give the signed-box targets

$$-p, \qquad -1, \qquad -p^{-1}.$$

The middle channel occurs once. The two edge channels occur once each and have equal bounded counts by inversion of the signed box. Thus the total number of successful normalized divisor ratios is

$$\tilde{c}_R(a; -1) + 2\tilde{c}_R(a; -p^{-1}).$$

The shell works exactly when this split-zero value is positive supported. □

The coefficient $1 + 2$ is therefore forced by the divisor algebra. It will be the source of the nondegeneracy of the sector defect.

4 The projective–gerbe sector group

The six-square fiber has the decomposition

$$C_6 \cong C_3 \times C_2, \qquad g^k \longmapsto (k \bmod 3, (-1)^k).$$

The C_2 factor is the projective endpoint sign. The C_3 factor is the gerbe arm.

The gerbe construction on

$$T^3 = S^1_t \times T^2_{x,y}$$

constructs a degree-N bundle gerbe G_N with curvature

$$H_N = 2\pi i N \, dt \wedge dx \wedge dy,$$

Dixmier–Douady class

$$DD(G_N) = N[dt] \smile [dx \wedge dy],$$

transgressed sector modules

$$(E_{\phi,\psi})_n = L_N^{\otimes n} \otimes F_{\phi,\psi},$$

and magnetic translations

$$U^{(n)}_{a,b} : H_{n,\phi,\psi} \xrightarrow{\sim} H_{n,\phi+nNb,\psi+nNa}.$$

For $N = 3, n = 1, \psi = 0$ it gives a distinguished $\mathbb{Z}_3 \times \mathbb{Z}_2$ six-sector slice with one-dimensional N-ality summands [9]. Explicitly, the gerbe note uses characteristics

$$\phi_{k,\epsilon} := \frac{k}{3} + \frac{\epsilon}{2} \pmod 1, \qquad (k, \epsilon) \in \mathbb{Z}_3 \times \mathbb{Z}_2.$$

Let

$$E_{r,+}, E_{r,-} \qquad (r \in \mathbb{Z}/3\mathbb{Z})$$

be the six basis sector vectors. The distinguished target hub is

$$v_3 = -3e_0 = (0, -),$$

so its basis vector is $E_{0,-}$.

If

$$p \equiv g^k \pmod{840},$$

then

$$-p^{-1} \longleftrightarrow v_{3-k} = (-k, -(-1)^k).$$

Thus the total sector defect is

$$\boxed{\mathfrak{D}_k = E_{0,-} + 2E_{-k,-(-1)^k} \in \mathbb{Z}[C_3 \times C_2].}$$

Theorem 4.1 (The total target defect)**.** *The two-channel shell coefficient for $p \equiv g^k \pmod{840}$ is the split-zero pairing of the signed divisor mass with*

$$\mathfrak{D}_k = E_{0,-} + 2E_{-k,-(-1)^k}.$$

Consequently the projective sign sheet and the gerbe arm sheet are both necessary to place the two target channels.

Proof. The normalized target -1 corresponds to the distinguished hub $v_3 = (0, -)$, hence to $E_{0,-}$. The second target $-p^{-1}$ corresponds to $v_{3-k} = (-k, -(-1)^k)$, hence to $E_{-k,-(-1)^k}$. The weights are 1 and 2 by the two-channel divisor theorem. Thus the total defect is the displayed element. $\square$

5 Explicit arm–sign matrices and mixed defect

Use row order $r = 0, 1, 2$ and column order $(+, -)$. The six defects are:

k	$\mathfrak{D}_k$
0	$\begin{pmatrix} 0 & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
1	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 0 \end{pmatrix}$
2	$\begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
4	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 2 \end{pmatrix}$
5	$\begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}$

Define arm and sign marginals by

$$A_k(r) := \sum_{\epsilon=\pm} \mathfrak{D}_k(r, \epsilon), \qquad B_k(\epsilon) := \sum_{r=0}^{2} \mathfrak{D}_k(r, \epsilon).$$

Then

k	A_k	B_k
0	$(3, 0, 0)$	$(0, 3)$
1	$(1, 0, 2)$	$(2, 1)$
2	$(1, 2, 0)$	$(0, 3)$
3	$(3, 0, 0)$	$(2, 1)$
4	$(1, 0, 2)$	$(0, 3)$
5	$(1, 2, 0)$	$(2, 1)$

Theorem 5.1 (Both projections are necessary)**.** *The sign marginal alone does not determine $\mathfrak{D}_k$. The arm marginal alone does not determine $\mathfrak{D}_k$. The product grading $C_3 \times C_2$ is the minimal grading among these two projections that carries all six target placements.*

Proof. The sign marginals satisfy

$$B_0 = B_2 = B_4 = (0, 3),$$

but

$$A_0 = (3,0,0), \quad A_2 = (1,2,0), \quad A_4 = (1,0,2).$$

Thus the sign projection loses arm information.

Similarly,

$$A_0 = A_3 = (3,0,0),$$

but

$$B_0 = (0,3), \qquad B_3 = (2,1).$$

Thus the arm projection loses sign information. The product records both. □

Define the independence tensor

$$\mathfrak{D}_k^{\mathrm{ind}}(r,\epsilon) := \frac{A_k(r)B_k(\epsilon)}{3}$$

and the mixed arm–sign defect

$$\mathfrak{M}_k := \mathfrak{D}_k - \mathfrak{D}_k^{\mathrm{ind}}.$$

Theorem 5.2 (Mixed arm–sign defect)**.**

$$\boxed{\mathfrak{M}_k = 0 \iff k \notin \{1,5\}.}$$

For the mixed sectors,

$$\mathfrak{M}_1 = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 \\ \frac{2}{3} & -\frac{2}{3} \end{pmatrix}, \qquad \mathfrak{M}_5 = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \\ 0 & 0 \end{pmatrix}.$$

Proof. For $k = 0$, $A_0 = (3,0,0)$ and $B_0 = (0,3)$, so $A_0 \otimes B_0/3 = \mathfrak{D}_0$. Direct computation gives equality also for $k = 2,3,4$.

For $k = 1$,

$$A_1 = (1,0,2), \qquad B_1 = (2,1),$$

so

$$\frac{A_1 \otimes B_1}{3} = \begin{pmatrix} 2/3 & 1/3 \\ 0 & 0 \\ 4/3 & 2/3 \end{pmatrix}.$$

Subtracting this from

$$\mathfrak{D}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 0 \end{pmatrix}$$

gives the displayed $\mathfrak{M}_1$. The computation for $k = 5$ is identical with arms 1 and 2 interchanged. □

Thus the sectors split into interaction types:

k	type
0	loop defect
2, 4	cross-arm same-sign defect
3	same-arm opposite-sign defect
1, 5	genuinely mixed arm–sign defect.

The champion prime is

$$p_* \equiv 169 \equiv 289^3 \pmod{840},$$

so $k = 3$ and the champion defect is

$$\mathfrak{D}_3 = E_{0,-} + 2E_{0,+},$$

the same-arm opposite-sign defect.

6 Fourier nondegeneracy, circulant determinant, and Smith type

Let

$$\omega = e^{2\pi i/3}.$$

Characters of $C_3 \times C_2$ are indexed by

$$(u, v) \in \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

with

$$(r, \epsilon) \mapsto \omega^{ur} \epsilon^v.$$

Theorem 6.1 (Fourier transform of the total defect)**.** *For*

$$\mathfrak{D}_k = E_{0,-} + 2E_{-k,-(-1)^k},$$

one has

$$\boxed{\widehat{\mathfrak{D}_k}(u, v) = (-1)^v \left(1 + 2(-1)^{kv}\omega^{-uk}\right).}$$

Moreover

$$\widehat{\mathfrak{D}_k}(u, v) \neq 0$$

for every character (u, v).

Proof. The first endpoint $E_{0,-}$ evaluates to $(-1)^v$. The second endpoint evaluates to

$$\omega^{-uk} \left(-(-1)^k\right)^v = \omega^{-uk}(-1)^v(-1)^{kv}.$$

With weight 2, the total character value is

$$(-1)^v + 2\omega^{-uk}(-1)^v(-1)^{kv} = (-1)^v \left(1 + 2(-1)^{kv}\omega^{-uk}\right).$$

The prefactor is nonzero. The second summand in the parenthesis has absolute value 2, while 1 has absolute value 1, so their sum cannot vanish. □

This proves that the target defect has no missing Fourier mode. The target is not where degeneracy can occur.

Let $\mathfrak{d}_k = e_3 + 2e_{3-k} \in \mathbb{Z}[C_6]$ be the same defect in cyclic coordinates. Let $C(\mathfrak{d}_k)$ be the 6×6 circulant matrix whose rows are translates of $\mathfrak{d}_k$.

Theorem 6.2 (Defect determinant)**.**

$$\boxed{\det C(\mathfrak{d}_k) = -\left(1-(-2)^{6/\gcd(k,6)}\right)^{\gcd(k,6)}.}$$

Consequently

k	$\det C(\mathfrak{d}_k)$	$\det C(\mathfrak{d}_k) \bmod 107$
0	-729	20
1	63	63
2	-81	26
3	27	27
4	-81	26
5	63	63

Proof. The defect $\mathfrak{d}_k$ is e_3 times $e_0 + 2e_{-k}$. Translation by e_3 on C_6 has determinant -1. The Fourier eigenvalues of convolution by $e_0 + 2e_{-k}$ are

$$1 + 2\zeta^{-km}, \qquad m = 0, dots, 5,$$

where $\zeta = e^{2\pi i/6}$. If $g_k = \gcd(k, 6)$, the multiset $\{\zeta^{-km}\}$ consists of all $(6/g_k)$-th roots of unity, each repeated g_k times. Thus

$$\prod_{m=0}^{5}(1 + 2\zeta^{-km}) = \left(1-(-2)^{6/g_k}\right)^{g_k}.$$

Multiplying by -1 gives the formula. □

Corollary 6.3. *For every k,*

$$C(\mathfrak{d}_k) \in GL_6(\mathbb{F}_{107}).$$

Theorem 6.4 (Integral Smith types)**.** *The Smith normal forms over $\mathbb{Z}$ are*

k	SNF $C(\mathfrak{d}_k)$	coker $C(\mathfrak{d}_k)$
0	$(3, 3, 3, 3, 3, 3)$	$(\mathbb{Z}/3)^6$
1	$(1, 1, 1, 1, 1, 63)$	$\mathbb{Z}/63$
2	$(1, 1, 1, 1, 9, 9)$	$(\mathbb{Z}/9)^2$
3	$(1, 1, 1, 3, 3, 3)$	$(\mathbb{Z}/3)^3$
4	$(1, 1, 1, 1, 9, 9)$	$(\mathbb{Z}/9)^2$
5	$(1, 1, 1, 1, 1, 63)$	$\mathbb{Z}/63$.

Proof. The determinantal divisors are

k	D_1	D_2	D_3	D_4	D_5	D_6
0	3	9	27	81	243	729
1	1	1	1	1	1	63
2	1	1	1	1	9	81
3	1	1	1	3	9	27
4	1	1	1	1	9	81
5	1	1	1	1	1	63.

The Smith factors satisfy

$$s_1 = D_1, \qquad s_j = D_j/D_{j-1}.$$

This gives the displayed normal forms. $\square$

The finite-field statement and the integral statement interact as follows: the target frame is invertible over the champion residue field $\mathbb{F}_{107}$, while its integral cokernel records the arithmetic torsion type of the sector placement.

7 Inverse defect kernels and response operators

Since $C(\mathfrak{d}_k)$ is invertible over $\mathbb{Q}$, there is a unique inverse defect

$$\rho_k \in \mathbb{Q}[C_6]$$

such that

$$\mathfrak{d}_k \rho_k = e_0.$$

These inverse kernels reconstruct sector mass from translated defect responses.

Theorem 7.1 (Rational inverse defects)**.**

k	ρ_k
0	$\frac{1}{3}e_3$
1	$\frac{1}{63}(8e_0 - 4e_1 + 2e_2 - e_3 + 32e_4 - 16e_5)$
2	$-\frac{2}{9}e_1 + \frac{1}{9}e_3 + \frac{4}{9}e_5$
3	$\frac{2}{3}e_0 - \frac{1}{3}e_3$
4	$\frac{4}{9}e_1 + \frac{1}{9}e_3 - \frac{2}{9}e_5$
5	$\frac{1}{63}(8e_0 - 16e_1 + 32e_2 - e_3 + 2e_4 - 4e_5).$

Proof. Each row is verified by cyclic convolution. For example, for $k = 3$,

$$\mathfrak{d}_3 = e_3 + 2e_0, \qquad \rho_3 = \frac{2}{3}e_0 - \frac{1}{3}e_3.$$

Then

$$(e_3 + 2e_0)\left(\frac{2}{3}e_0 - \frac{1}{3}e_3\right) = \frac{2}{3}e_3 - \frac{1}{3}e_0 + \frac{4}{3}e_0 - \frac{2}{3}e_3 = e_0.$$

The other cases are the same finite cyclic multiplication. Uniqueness follows from nonzero determinant. $\square$

Let $W_F = F[C_6]$ and define the translated-defect response map

$$\mathcal{R}_{k,F} : W_F \to F^6, \qquad \mathcal{R}_{k,F}(x) = \left(\langle x, \tau^r \mathfrak{d}_k \rangle\right)_{r=0}^{5}.$$

Theorem 7.2 (Defect response is informationally complete over $\mathbb{F}_{107}$)**.** *For every k,*

$$\boxed{\mathcal{R}_{k,\mathbb{F}_{107}} \textit{ is an isomorphism.}}$$

Proof. The matrix of $\mathcal{R}_{k,F}$ is a transpose of the circulant matrix $C(\mathfrak{d}_k)$. Since $C(\mathfrak{d}_k)$ is invertible over $\mathbb{F}_{107}$, so is the response map. $\square$

For a shell computation, the scalar $\Omega_R(p)$ records only the original target response. The full response vector

$$\big(\langle M_R(p), \tau^r \mathfrak{D}_k \rangle\big)_{r=0}^{5}$$

recovers the entire sector mass distribution over $\mathbb{F}_{107}$. Positivity, however, remains a split-zero semiring condition: linear inversion over $\mathbb{F}_{107}$ reconstructs residues, not positivity. Thus full-response storage is an algebraic diagnostic, while shell success remains the original support-positive coordinate condition.

8 Augmentation defect and the local A_5 frame

Let

$$u = e_0 + e_1 + e_2 + e_3 + e_4 + e_5.$$

Every defect has total mass 3, so define

$$\delta_k := \mathfrak{d}_k - \frac{1}{2} u \in \operatorname{Aug} \mathbb{Q}[C_6]$$

and the doubled integral defect

$$\Delta_k := 2\delta_k = 2\mathfrak{d}_k - u \in \operatorname{Aug} \mathbb{Z}[C_6].$$

Theorem 8.1 (Augmentation Gram matrix)**.** *With respect to $\langle e_i, e_j \rangle = \delta_{ij}$,*

$$\|\delta_0\|^2 = \frac{15}{2}, \qquad \|\delta_k\|^2 = \frac{7}{2} \quad (k = 1, \ldots, 5).$$

The Gram matrix $G = (\langle \delta_i, \delta_j \rangle)_{0 \le i,j \le 5}$ is

$$\begin{pmatrix} 15/2 & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 \\ 3/2 & 7/2 & -1/2 & -1/2 & -1/2 & -1/2 \\ 3/2 & -1/2 & 7/2 & -1/2 & -1/2 & -1/2 \\ 3/2 & -1/2 & -1/2 & 7/2 & -1/2 & -1/2 \\ 3/2 & -1/2 & -1/2 & -1/2 & 7/2 & -1/2 \\ 3/2 & -1/2 & -1/2 & -1/2 & -1/2 & 7/2 \end{pmatrix}.$$

Its eigenvalues are

$$9, \quad 4, 4, 4, 4, \quad 0.$$

Thus the six centered defects form a rank-five frame in

$$\operatorname{Aug} \mathbb{C}[C_6] \cong A_5 \otimes \mathbb{C}.$$

Proof. For $k = 0$, $\mathfrak{d}_0 = 3e_3$, so

$$\delta_0 = 3e_3 - \frac{1}{2} u,$$

and

$$\|\delta_0\|^2 = 9 - 3 + \frac{6}{4} = \frac{15}{2}.$$

For $k \neq 0$, $\mathfrak{d}_k = e_3 + 2e_{3-k}$ has distinct endpoints, so

$$\|\mathfrak{d}_k\|^2 = 1^2 + 2^2 = 5, \qquad \langle \mathfrak{d}_k, \tfrac{1}{2}u\rangle = \frac{3}{2}, \qquad \|\tfrac{1}{2}u\|^2 = \frac{3}{2}.$$

Hence

$$\|\delta_k\|^2 = 5 - 3 + \frac{3}{2} = \frac{7}{2}.$$

The off-diagonal entries follow from endpoint overlap. The characteristic polynomial of the displayed matrix is

$$\lambda(\lambda - 9)(\lambda - 4)^4.$$

□

This is the support-completed local A_5: the split-zero boundary turns a five-coordinate exponent interval into the six-point augmented frame whose augmentation is A_5.

9 The completed critical arm: $744, 747, 750$

Define

$$s(n) := \frac{n - 744}{6}, \qquad v(n) := 2s(n) - 1 = \frac{n - 747}{3}.$$

Then

$$v(744) = -1, \qquad v(747) = 0, \qquad v(750) = 1.$$

Modulo 9,

$$744 \equiv -3, \qquad 747 \equiv 0, \qquad 750 \equiv 3.$$

Thus

$$744, \quad 747, \quad 750$$

realize the completed A_8 arm

$$-3e_0, \quad 0_0, \quad +3e_0.$$

The involution

$$n \longmapsto 1494 - n$$

acts as

$$v \longmapsto -v.$$

The center 747 is the supported-zero hub.

10 The A_8^3 raw-zero sheet and nonlinear support recovery

Define

$$T_9(\tau) = \frac{\eta(\tau)^3}{\eta(9\tau)^3}.$$

Write

$$T_9 = q^{-1} P_9(q), \qquad P_9(q) = \frac{(q;q)_\infty^3}{(q^9;q^9)_\infty^3}.$$

Jacobi's eta-cube identity gives

$$P_9(q) = H(q^3) - 3q$$

for some $H(q^3) \in \mathbb{Z}[[q^3]]$. Hence

$$T_9 = q^{-1} H(q^3) - 3.$$

Theorem 10.1 (Raw A_8^3 vanishing)**.** *For every $n > 0$,*

$$[q^n]T_9 = 0$$

unless

$$n \equiv 2 \pmod 3.$$

In particular

$$[q^{744}]T_9 = [q^{747}]T_9 = [q^{750}]T_9 = 0.$$

Proof. All positive exponents of $q^{-1}H(q^3)$ are of the form $3m-1$, hence congruent to 2 (mod 3). The three indices $744, 747, 750$ are divisible by 3. □

Define

$$I_9 = \frac{27}{T_9}, \qquad J_9 = T_9 + 3 + \frac{27}{T_9}, \qquad Y_9 = T_9 - \frac{27}{T_9},$$

and

$$f_9 = -q\frac{d}{dq}\log T_9.$$

Since

$$T_9 = \eta(\tau)^3 \eta(9\tau)^{-3},$$

one has

$$f_9 = \frac{-3E_2(\tau) + 27E_2(9\tau)}{24} = 1 + \sum_{n\geq 1} C_9(n) q^n,$$

where

$$\boxed{C_9(n) = 3\sigma_1(n) - 27\sigma_1(n/9)}$$

with $\sigma_1(n/9) = 0$ if $9 \nmid n$.

Modulo 107, exact coefficient recurrence gives

n	n mod 107	T_9	I_9	J_9	Y_9	f_9
744	102	0	35	35	72	89
747	105	0	31	31	76	45
750	1	0	42	42	65	52.

The source note proves the endpoint cases by inverse-series recurrence and divisor summation [2, 3]; the verification script accompanying this note checks the hub as well.

Theorem 10.2 (A_8^3 completed-arm identities)**.** *Let $N_* = 35$ and $a_* = 93$ in $\mathbb{F}_{107}$. Then*

$$J_9(744) = N_*, \qquad Y_9(744) = -N_*, \qquad f_9(744) = a_*^2.$$

At the hub,

$$J_9(747)^2 = -2.$$

At the positive endpoint,

$$J_9(750)^2 = f_9(750), \qquad f_9(750)^{-1} = N_*.$$

Proof. From the table,

$$J_9(744) = 35 = N_*, \qquad Y_9(744) = 72 = -35 = -N_*.$$

Also

$$a_*^2 = 93^2 = 8649 \equiv 89 = f_9(744).$$

At 747,

$$J_9(747)^2 = 31^2 = 961 \equiv 105 \equiv -2.$$

At 750,

$$J_9(750)^2 = 42^2 = 1764 \equiv 52 = f_9(750).$$

Finally,

$$52 \cdot 35 = 1820 \equiv 1 \pmod{107}.$$

□

Because $T_9 = 0$ on the critical arm, the linear pair (J_9, Y_9) has rank one. The hub requires a nonlinear support coordinate. Define

$$\mathcal{R}_9(n) := (J_9(n), J_9(n)^2, f_9(n)).$$

Then

$$M_9 := \begin{pmatrix} 35 & 48 & 89 \\ 31 & 105 & 45 \\ 42 & 52 & 52 \end{pmatrix}.$$

Theorem 10.3 (A_8^3 nonlinear support determinant)**.**

$$\boxed{\det M_9 \equiv 103 \equiv -4 \pmod{107}.}$$

Proof. Expanding,

$$\begin{aligned} \det M_9 = {} & 35(105 \cdot 52 - 45 \cdot 52) - 48(31 \cdot 52 - 45 \cdot 42) \\ & + 89(31 \cdot 52 - 105 \cdot 42). \end{aligned}$$

Reduction modulo 107 gives $103 = -4$. □

Thus the raw A_8^3 sheet sees scalar zero, while the Fricke/Eisenstein support-resolvent recovers a determinant equal to the negative affine coefficient.

11 The $A_5^4 D_4$ lambency-six sheet and linear support recovery

Define

$$T_6(\tau) = \frac{\eta(\tau)^5 \eta(3\tau)}{\eta(2\tau)\eta(6\tau)^5}.$$

Its Fricke-even quotient, Fricke-odd sheet, umbral weight-two channel, and Eisenstein channel are

$$J_6 = T_6 + 5 + \frac{72}{T_6}, \qquad Y_6 = T_6 - \frac{72}{T_6},$$

$$F_6 = -2\eta(6\tau)^4 T_6, \qquad f_6 = -q\frac{d}{dq}\log T_6.$$

The eta-quotient computation identifies T_6 as the reciprocal Coxeter eta product of the residual $A_5^4 D_4$ frame, J_6 as the Monster $6B/6+6$ Fricke Hauptmodul, and F_6 as the corresponding umbral weight-two eta product [10]. The moonshine literature records the broader mechanism in which finite group representations and modular objects are related through graded traces and, in the umbral case, mock modular McKay–Thompson series [14, 15, 16].

Modulo 107, on the same arm:

n	n mod 107	T_6	F_6	J_6	Y_6	f_6
744	102	34	19	16	52	67
747	105	83	77	1	58	48
750	1	101	106	2	93	21.

Theorem 11.1 (Lambency-six hub and endpoint)**.** *At the supported hub,*

$$\boxed{J_6(747) = 1.}$$

At the positive endpoint,

$$\boxed{Y_6(750) = a_*, \qquad F_6(750) = -1, \qquad f_6(750) = S(3) = 21.}$$

Proof. The table gives $J_6(747) = 1$. At 750,

$$Y_6(750) = 93 = a_*, \qquad F_6(750) = 106 = -1, \qquad f_6(750) = 21.$$

The value $S(3) = 21$ is the classical three-state Busy-Beaver runtime used in the previous rigidity notes. ∎

Define the two lambency-six support-resolvent matrices

$$M_{6,1} := \begin{pmatrix} J_6(744) & Y_6(744) & F_6(744) \\ J_6(747) & Y_6(747) & F_6(747) \\ J_6(750) & Y_6(750) & F_6(750) \end{pmatrix} = \begin{pmatrix} 16 & 52 & 19 \\ 1 & 58 & 77 \\ 2 & 93 & 106 \end{pmatrix}$$

and

$$M_{6,2} := \begin{pmatrix} J_6(744) & F_6(744) & f_6(744) \\ J_6(747) & F_6(747) & f_6(747) \\ J_6(750) & F_6(750) & f_6(750) \end{pmatrix} = \begin{pmatrix} 16 & 19 & 67 \\ 1 & 77 & 48 \\ 2 & 106 & 21 \end{pmatrix}.$$

Theorem 11.2 (Lambency-six support determinants)**.**

$$\boxed{\det M_{6,1} = 82 = -25 \pmod{107},}$$

and

$$\boxed{\det M_{6,2} = 25 \pmod{107}.}$$

Proof. Direct determinant expansion over $\mathbb{F}_{107}$ gives

$$\det \begin{pmatrix} 16 & 52 & 19 \\ 1 & 58 & 77 \\ 2 & 93 & 106 \end{pmatrix} = 82 = -25,$$

and

$$\det \begin{pmatrix} 16 & 19 & 67 \\ 1 & 77 & 48 \\ 2 & 106 & 21 \end{pmatrix} = 25.$$

$\square$

Hence the interaction is:

$$\boxed{A_8^3 : \ \det(J_9, J_9^2, f_9) = -4,}$$

while

$$\boxed{A_5^4 D_4 : \ \det(J_6, Y_6, F_6) = -25, \quad \det(J_6, F_6, f_6) = 25.}$$

The A_8^3 completion resolves raw zero nonlinearly. The $A_5^4 D_4$ completion resolves the same arm linearly through the Fricke/umbral channels.

12 Transport from the A_8^3 resolvent to the lambency-six resolvents

Define

$$P^{(1)}_{9\to 6} := M_{6,1} M_9^{-1}, \qquad P^{(2)}_{9\to 6} := M_{6,2} M_9^{-1}.$$

All matrices are over $\mathbb{F}_{107}$.

Theorem 12.1 (First transport matrix)**.**

$$\boxed{P^{(1)}_{9\to 6} = \begin{pmatrix} 32 & 67 & 44 \\ 98 & 72 & 41 \\ 51 & 81 & 71 \end{pmatrix}.}$$

It satisfies

$$\det P^{(1)}_{9\to 6} = 33, \qquad \operatorname{tr} P^{(1)}_{9\to 6} = 68,$$

and

$$\chi_{P^{(1)}_{9\to 6}}(X) = X^3 + 39X^2 + 18X - 33.$$

This cubic is irreducible over $\mathbb{F}_{107}$*.*

Proof. Matrix multiplication gives the displayed matrix. The determinant is

$$\det P^{(1)}_{9\to 6} = \frac{\det M_{6,1}}{\det M_9} = \frac{-25}{-4} = 25 \cdot 4^{-1}.$$

Since $4^{-1} = 27$ in $\mathbb{F}_{107}$,

$$25 \cdot 27 = 675 \equiv 33.$$

The trace is $32+72+71 = 175 \equiv 68$. The characteristic polynomial is computed from $\det(XI - P)$; it has no root in $\mathbb{F}_{107}$ and is therefore irreducible. □

Theorem 12.2 (Second transport matrix)**.**

$$\boxed{P^{(2)}_{9\to 6} = \begin{pmatrix} 3 & 28 & 46 \\ 106 & 64 & 81 \\ 38 & 47 & 84 \end{pmatrix}.}$$

It satisfies

$$\det P^{(2)}_{9\to 6} = 74, \qquad \operatorname{tr} P^{(2)}_{9\to 6} = 44,$$

and

$$\chi_{P^{(2)}_{9\to 6}}(X) = X^3 - 44X^2 - 28X + 33.$$

Moreover

$$\chi_{P^{(2)}_{9\to 6}}(X) = (X+17)(X^2 + 46X + 46).$$

Proof. The determinant is

$$\det P^{(2)}_{9\to 6} = \frac{25}{-4} = -25 \cdot 27 \equiv 74.$$

The matrix product and characteristic polynomial are direct computations in $\mathbb{F}_{107}$. □

These transport matrices compute the interaction between the two completions. They are the finite bridge from raw-zero A_8^3 support recovery to moonshine/umbral $A_5^4 D_4$ support recovery.

13 Moonshine, glue, and defect completion

The residual Niemeier refinement constructs a finite equivariant quadratic-module datum

$$\mathfrak{N}_{\mathrm{ES}} = (V_{840} \twoheadrightarrow Q_{\mathrm{ES}}, S_{840}, R_{\mathrm{ES}}, H_{\mathrm{ES}}, N_{\mathrm{ES}}, A_{\mathrm{ES}}),$$

where

$$Q_{\mathrm{ES}} \cong C_2^2, \quad S_{840} \cong C_6, \quad R_{\mathrm{ES}} = A_5^4 D_4,$$

H_{ES} is a maximal isotropic subgroup of order 72, and the glue-preserving group is

$$A_{\mathrm{ES}} \cong GL_2(3).$$

The verification report confirms $|H_{\mathrm{ES}}| = 72$, isotropy, no new roots, glue symmetry order 48, and projection to 24 projective permutations with two lifts over each projective permutation [4, 5].

The present defect lives inside this structure as follows:

finite support object	role
$S_{840} = \langle 289 \rangle \cong C_6$	Coxeter fiber
C_3	gerbe arm
C_2	projective endpoint sign
$A_5 = \operatorname{Aug} \mathbb{Z}[C_6]$	support-completed exponent frame
$A_5^4 D_4$	four-fiber Niemeier root frame
$GL_2(3)$	non-split glue symmetry
T_6, J_6, Y_6, F_6, f_6	moonshine/umbral modular channels.

Moonshine supplies the modular representation side. Monstrous moonshine relates normalized Hauptmoduln to graded traces on the Monster module, and the proof of umbral moonshine constructs graded modules whose McKay–Thompson series are the prescribed mock modular forms for the umbral cases [14, 15]. In the present system, the $6+6$ Hauptmodul and its Fricke-odd and umbral channels provide explicit support coordinates on the residual defect arm.

The recent 3d modularity framework gives the structural template for defect completion: ordinary $\widehat{Z}$-invariants may span only a subspace, and adding supersymmetric defect invariants completes the vector-valued modular representation [13]. The finite theorem here is exactly analogous at the C_6 level:

$$\langle C_6 \cdot \mathfrak{d}_k \rangle = \mathbb{C}[C_6], \qquad \langle C_6 \cdot \delta_k \rangle = \operatorname{Aug} \mathbb{C}[C_6].$$

14 Binary-quadratic Jacobi and paramodular envelope

Let L be an even unimodular rank-24 lattice. Its genus-two theta series is

$$\vartheta_2(L)(Z) = \sum_{x,y \in L} \exp\left(\pi i \left(\langle x,x\rangle \tau_1 + 2\langle x,y\rangle z + \langle y,y\rangle \tau_2\right)\right),$$

where

$$Z = \begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix} \in \mathfrak{H}_2.$$

The Fourier coefficients are indexed by positive semidefinite half-integral binary quadratic forms

$$T = \frac{1}{2} \begin{pmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle x,y\rangle & \langle y,y\rangle \end{pmatrix}.$$

The Eisenstein/Leech note records this genus-two theta coefficient envelope and the Chenevier–Lannes Hecke-intertwining theta map from Niemeier lattice classes to Siegel modular forms [11]. Pitale gives a classical and representation-theoretic approach to Siegel modular forms [21]. Shahidi's Eisenstein-series framework supplies the automorphic constant-term and Fourier-coefficient context in which such coefficient systems connect to L-data [22].

Let

$$r_L(E_{r,\epsilon}; T)$$

count pairs $(x, y) \in L^2$ with binary quadratic index T and sector label $E_{r,\epsilon}$. Define the defect-weighted coefficient

$$\boxed{r_L(\mathfrak{D}_k; T) = r_L(E_{0,-}; T) + 2r_L(E_{-k,-(-1)^k}; T).}$$

Theorem 14.1 (Genus-two theta envelope preserves the total defect)**.** *For every positive semidefinite binary quadratic index T, the map*

$$\mathfrak{D}_k \longmapsto r_L(\mathfrak{D}_k; T)$$

is a linear functional on $\mathbb{Z}[C_3 \times C_2]$. In Fourier coordinates, the defect multiplier is

$$\widehat{\mathfrak{D}_k}(u, v) = (-1)^v \left(1 + 2(-1)^{kv}\omega^{-uk}\right),$$

which is nonzero for every (u, v). Hence the genus-two theta envelope kills no arm/sign mode of the total target defect.

Proof. Linearity follows from the definition. If

$$R_L(T) = \sum_{r,\epsilon} r_L(E_{r,\epsilon}; T) E^{\vee}_{r,\epsilon},$$

then

$$r_L(\mathfrak{D}_k; T) = \langle R_L(T), \mathfrak{D}_k \rangle.$$

Fourier inversion decomposes this pairing into products of the Fourier modes of $R_L(T)$ and of $\mathfrak{D}_k$. The latter are nonzero by the Fourier nondegeneracy theorem. □

For paramodular forms, the same binary-quadratic indexing persists with level conditions. Johnson-Leung–Roberts–Schmidt develop stable Klingen vectors and paramodular newforms, including stable Hecke operators, paramodularization, and Fourier expansions indexed by positive semidefinite symmetric matrices with level structure [20]. Their local recurrence results place radial coefficient families

$$\sum_{t \geq 0} a(p^t S) p^{-ts}$$

under local spin L-factors. The paramodular envelope of the defect is therefore

$$\mathfrak{D}_k \longmapsto r_L(\mathfrak{D}_k; T) \longmapsto \sum_{t \geq 0} r_L(\mathfrak{D}_k; p^t T) p^{-ts}.$$

The defect remains the same weighted object under this lift; only the coefficient domain changes from a shell divisor box to binary quadratic Fourier indices.

15 The total obstruction stack

The full obstruction is the following stack of interacting maps:

$$\begin{aligned}
&\textbf{Layer 0: raw shell} && d \equiv -N \pmod{R}, \\
&\textbf{Layer 1: projective normalization} && dN^{-1} \equiv -1 \pmod{R}, \\
&\textbf{Layer 2: signed divisor box} && \mathcal{B}_R(a) = \{\textstyle\prod q^{\beta_q} : -e_q \le \beta_q \le e_q\}, \\
&\textbf{Layer 3: weighted targets} && \Omega_R(p) = \tilde{c}_R(a; -1) + 2\tilde{c}_R(a; -p^{-1}), \\
&\textbf{Layer 4: sector defect} && \mathfrak{D}_k = E_{0,-} + 2E_{-k,-(-1)^k}, \\
&\textbf{Layer 5: split-zero support} && \tau, \quad 0_{\mathbb{Z}}, \quad \mathbb{Z}_{>0}, \\
&\textbf{Layer 6: Fourier frame} && \widehat{\mathfrak{D}_k}(u, v) \neq 0, \\
&\textbf{Layer 7: Fricke completion} && (T_9, J_9, Y_9, f_9), \quad (T_6, J_6, Y_6, F_6, f_6), \\
&\textbf{Layer 8: support determinants} && -4, \quad -25, \quad 25, \\
&\textbf{Layer 9: genus-two envelope} && \mathfrak{D}_k \mapsto r_L(\mathfrak{D}_k; T).
\end{aligned}$$

The interaction is directional but not one-way: the projective normalization fixes the target, the gerbe coordinate places the target in the arm/sign sector, the split-zero semiring decides positivity, the Fourier frame proves target nondegeneracy, the Fricke sheets compute finite support coordinates, and the genus-two envelope lifts the same weighted object into automorphic coefficient space.

Theorem 15.1 (Total obstruction theorem)**.** *Let $p \equiv 1 \pmod{24}$ be a hard prime surviving the $R = 3, 7$ sieve, and suppose*

$$p \equiv g^k \pmod{840}, \qquad g = 289.$$

For every residual shell $R \equiv 3 \pmod{4}$ with $a = (p + R)/4$, the shell obstruction is the split-zero evaluation of the full projective–gerbe defect

$$\mathfrak{D}_k = E_{0,-} + 2E_{-k,-(-1)^k}$$

against the bounded signed divisor box $\mathcal{B}_R(a)$. Explicitly,

$$\boxed{R \text{ succeeds} \iff \tilde{c}_R(a; -1) + 2\tilde{c}_R(a; -p^{-1}) \in \mathbb{Z}_{>0}.}$$

The target defect has full Fourier support, is invertible as a translated defect frame over $\mathbb{F}_{107}$, has the integral Smith types listed above, and is realized simultaneously as a snowflake edge, a gerbe-sector object, a split-zero support stalk, a Fricke/Eisenstein critical-arm completion, and a binary-quadratic theta coefficient functional.

Proof. Projective normalization converts the raw congruence $d \equiv -N$ to the invariant target -1. The signed-divisor analysis reduces the three p-origins to two targets -1 and $-p^{-1}$ with weights 1 and 2. The target-star/snowflake identification maps these to $E_{0,-}$ and $E_{-k,-(-1)^k}$, producing $\mathfrak{D}_k$. The split-zero semiring gives the positivity trichotomy. Therefore the shell succeeds exactly when the weighted split-zero count is positive.

The Fourier and determinant theorems prove nondegeneracy of $\mathfrak{D}_k$. The critical-arm theorems compute its modular completions. The genus-two theorem proves that the same weighted defect lifts linearly to binary-quadratic theta coefficients without losing any sector mode. □

16 Operational positivity record

For future computation, the scalar

$$\Omega_R(p)$$

should be stored together with its labelled stalk. Define

$$C_0(R,p) = \tilde{c}_R(a;-1), \qquad C_1(R,p) = \tilde{c}_R(a;-p^{-1}).$$

Then the complete support record is

$$\boxed{\mathfrak{S}_R(p) = C_0(R,p)E_{0,-} + 2C_1(R,p)E_{-k,-(-1)^k} \in G(\mathbb{Z}_{\geq 0})[C_3 \times C_2].}$$

The scalar projection is

$$\mathfrak{S}_R(p) \longmapsto \Omega_R(p) = C_0(R,p) + 2C_1(R,p).$$

The support character retains whether each component is τ, $0_{\mathbb{Z}}$, or positive. The finite Fourier transform tests arm/sign modes. The translated defect-response vector

$$(\langle M_R(p), \tau^r \mathfrak{D}_k \rangle)_{r=0}^{5}$$

recovers the field-valued sector mass distribution over $\mathbb{F}_{107}$, because the defect frame is invertible. The genus-two envelope maps the same labelled support mass into

$$r_L(\mathfrak{S}_R(p); T).$$

Thus the total computable obstruction record is

$$\boxed{\Big(C_0(R,p), C_1(R,p), k, \mathfrak{S}_R(p), \widehat{\mathfrak{S}_R(p)}, r_L(\mathfrak{S}_R(p);T)\Big).}$$

This is the point where positivity can emerge: a shell is positive precisely when one of the two labelled support coordinates becomes a positive integer. Every other layer records where that positivity lives, how it transforms, and which modes it occupies.

A Verification recurrences

For

$$T_6 = q^{-1}P_6(q),$$

we use

$$P_6(q) = \prod_{n\geq 1}(1-q^n)^5(1-q^{2n})^{-1}(1-q^{3n})(1-q^{6n})^{-5}.$$

For

$$T_9 = q^{-1}P_9(q),$$

we use

$$P_9(q) = \prod_{n\geq 1}(1-q^n)^3(1-q^{9n})^{-3}.$$

If

$$P(q) = 1 + \sum_{n\geq 1} p_n q^n, \qquad P(q)^{-1} = \sum_{n\geq 0} b_n q^n,$$

then

$$b_0 = 1, \qquad b_n = -\sum_{j=1}^{n} p_j b_{n-j}.$$

Thus

$$\frac{72}{T_6} = 72qP_6^{-1}, \qquad \frac{27}{T_9} = 27qP_9^{-1}.$$

The Eisenstein divisor channels are

$$[q^n]f_6 = 5\sigma_1(n) - 2\sigma_1(n/2) + 3\sigma_1(n/3) - 30\sigma_1(n/6),$$

where terms with nonintegral arguments are omitted, and

$$[q^n]f_9 = 3\sigma_1(n) - 27\sigma_1(n/9).$$

B Exact verification tables

The verification script produces:

n	n_{107}	n_9	T_6	F_6	J_6	Y_6	f_6	T_9	I_9	J_9	Y_9	f_9
744	102	6	34	19	16	52	67	0	35	35	72	89
747	105	0	83	77	1	58	48	0	31	31	76	45
750	1	3	101	106	2	93	21	0	42	42	65	52.

It also verifies

$$\det M_9 = 103, \quad \det M_{6,1} = 82, \quad \det M_{6,2} = 25,$$

$$P^{(1)}_{9\to 6} = \begin{pmatrix} 32 & 67 & 44 \\ 98 & 72 & 41 \\ 51 & 81 & 71 \end{pmatrix}, \qquad P^{(2)}_{9\to 6} = \begin{pmatrix} 3 & 28 & 46 \\ 106 & 64 & 81 \\ 38 & 47 & 84 \end{pmatrix},$$

and the defect determinant residues

$$20, 63, 26, 27, 26, 63.$$

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