Corrected Interaction Calculus for the Residual Erdős–Straus Obstruction

Weighted Snowflake Defects, Split-Zero Positivity, Niemeier Glue, and Modular Support Resolution

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Abstract

This note replaces the object-list presentation of the residual Erdős–Straus framework by a single interacting calculus. For a hard prime class $p \equiv 289^k \pmod{840}$ and a residual shell $R \equiv 3 \pmod 4$, the fixed-shell divisor equation is first projectively normalized, producing the invariant target -1. The missing structure is then restored by the six-sector coordinate

$$C_6 \cong C_3 \times C_2,$$

so that the two normalized targets are not merely an unordered pair but the weighted snowflake defect

$$\mathfrak{D}_k = E_{0,-} + 2E_{-k,-(-1)^k} \in \mathbb{Z}[C_3 \times C_2].$$

The coefficient 2 is forced by the two p-origins in divisors of p^2a^2. Certificate existence is proved equivalent to positivity of the split-zero pullback count

$$\Omega_R(p) = \widetilde{\mathfrak{c}}_R(a;-1) + 2\widetilde{\mathfrak{c}}_R(a;-p^{-1}).$$

The target defect has full Fourier support for every k, so a persistent obstruction cannot come from target degeneracy; it can only come from bounded positive-mass failure of the signed divisor box. The champion prime $p_* = 8{,}803{,}369$ realizes the strict ladder

$$(\tau,\tau) \longrightarrow (0_{\mathbb{Z}}, 0_{\mathbb{Z}}) \longrightarrow (2, 0_{\mathbb{Z}})$$

for $R = 27, 43, 107$. The same full-arm sign reversal is resolved by the modular critical arm $744 \leftrightarrow 747 \leftrightarrow 750$: the raw A_8^3 eta sheet vanishes on all three points, while Fricke and Eisenstein channels separate the negative endpoint, the supported-zero hub, and the positive endpoint. We compute the nonlinear A_8^3 support resolvent, the linear lambency-six $A_5^4D_4$ resolvents, and the finite transport matrices $P^{(i)}_{9\to 6}$ over $\mathbb{F}_{107}$, including their characteristic polynomials. The note also records the exact $A_5^4D_4$ glue datum of order 72 and the non-split $GL_2(3)$ symmetry, only insofar as these objects act in the positivity calculus.

Contents

1 The interacting object

The residual obstruction is the composite

$$\boxed{\begin{array}{c}\text{divisor box} \longrightarrow \text{projective target} \longrightarrow \text{weighted snowflake/gerbe defect} \\ \longrightarrow \text{split-zero support value} \longrightarrow \text{positivity}\end{array}}$$

The point is not that the projective target, gerbe sectors, split-zero semiring, eta sheets, and theta envelopes exist as separate objects. The point is that each layer controls exactly one map in the following pullback-and-evaluation diagram.

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\mathcal{B}_R(a)\\ \text{signed divisor box}\end{array}} \arrow[r, "\lambda_R"] & \boxed{\begin{array}{c}(\mathbb{Z}/R\mathbb{Z})^\times\\ \text{residue torus}\end{array}} \arrow[r, "\text{target inclusion}"] & \boxed{\begin{array}{c}\{-1,-p^{-1}\}\\ \text{two targets}\end{array}} \arrow[d, "C_3\times C_2\ \text{sectorization}"] \\
& & \boxed{\begin{array}{c}\mathfrak{D}_k\in\mathbb{Z}[C_3\times C_2]\\ \text{weighted defect}\end{array}} \arrow[d, "\text{weighted support pairing}"] \\
& & \boxed{\begin{array}{c}G(\mathbb{Z}_{\geq 0})\\ \text{split-zero value}\end{array}} \arrow[d, "\text{positive part}"] \\
& & \boxed{\begin{array}{c}\Omega_R(p)>0\\ \text{certificate}\end{array}}
\end{tikzcd}
$$

The arithmetic assertion is the existence of a shell for which the last object is positive. All preceding layers constrain how positivity may appear; none of them is a substitute for positivity.

Theorem 1.1 (Total interacting obstruction theorem). *Let*

$$g := 289, \qquad S_{840} = \{1, 121, 169, 289, 361, 529\} = \langle g\rangle \subset (\mathbb{Z}/840\mathbb{Z})^\times.$$

Let $p \equiv g^k \pmod{840}$ *be in the hard six-square sector. For every shell* $R \equiv 3 \pmod 4$*, set*

$$a = a_R(p) = \frac{p+R}{4}, \qquad N = pa.$$

Assume first that $(N, R) = 1$*. Define the signed divisor box*

$$\mathcal{B}_R(a) := \left\{ \prod_{q^{e_q}\|a} q^{\beta_q} \bmod R : -e_q \leq \beta_q \leq e_q \right\} \subset (\mathbb{Z}/R\mathbb{Z})^\times.$$

Let

$$\widetilde{\mathfrak{c}}_R(a;t) \in G(\mathbb{Z}_{\geq 0}) = \mathbb{Z}_{\geq 0} \sqcup \{\tau\}$$

be the split-zero count of exponent vectors in $\mathcal{B}_R(a)$ *hitting* t*, with value* τ *when* t *is not even in the subgroup generated by the prime divisors of* a*, value* $0_{\mathbb{Z}}$ *when it is in that subgroup but not in the bounded box, and the ordinary positive count otherwise. Then the shell* R *gives an Erdős–Straus certificate with first denominator* a *if and only if*

$$\Omega_R(p) := \widetilde{\mathfrak{c}}_R(a;-1) + 2\widetilde{\mathfrak{c}}_R(a;-p^{-1}) \in \mathbb{Z}_{>0}.$$

Moreover, under $C_6 \cong C_3 \times C_2$*, the target object is the weighted defect*

$$\mathfrak{D}_k = E_{0,-} + 2E_{-k,-(-1)^k},$$

and every Fourier coefficient of $\mathfrak{D}_k$ *is nonzero. Hence the residual obstruction is exactly bounded positive-mass failure of* $\mathcal{B}_R(a)$ *against a full-support target defect.*

The rest of the note proves this theorem, computes the finite defect spectrum, and shows how the modular $744 \leftrightarrow 747 \leftrightarrow 750$ arm resolves the same zero/support issue.

2 Fixed shells and projective normalization

2.1 The divisor identity

Let p be prime and choose a shell

$$a = \frac{p+R}{4}, \qquad R = 4a - p, \qquad N = pa.$$

Then

$$\frac{4}{p} - \frac{1}{a} = \frac{R}{N}.$$

Thus the fixed-shell completion problem is

$$\frac{R}{N} = \frac{1}{b} + \frac{1}{c}.$$

Lemma 2.1 (Divisor certificate)**.** *Assume $(R, N) = 1$. The equation*

$$\frac{R}{N} = \frac{1}{b} + \frac{1}{c}$$

has a solution in positive integers b, c if and only if there exists a divisor $d \mid N^2$ such that

$$d \equiv -N \pmod{R}.$$

Given such d, the denominators are

$$b = \frac{d+N}{R}, \qquad c = \frac{N^2/d + N}{R}.$$

Proof. Multiplying $R/N = 1/b + 1/c$ by Nbc gives

$$Rbc = N(b+c).$$

Equivalently,

$$R^2bc - RNb - RNc + N^2 = N^2,$$

so

$$(Rb - N)(Rc - N) = N^2.$$

Set $d = Rb - N$. Then $d \mid N^2$ and $d \equiv -N \pmod{R}$. Conversely, if $d \mid N^2$ and $d \equiv -N \pmod{R}$, then $b = (d+N)/R$ is integral. Since $(R, N) = 1$, $d \equiv -N$ is invertible modulo R, and

$$\frac{N^2}{d} \equiv N^2(-N)^{-1} \equiv -N \pmod{R},$$

so $c = (N^2/d + N)/R$ is integral. Substitution into $(Rb - N)(Rc - N) = N^2$ gives the equation. Positivity is immediate. □

For non-coprime shells one replaces (R, N) by the reduced pair $(R_0, N_0) = (R/g, N/g)$ with $g = (R, N)$ and applies the same lemma to R_0/N_0. The champion shells used below are coprime, so no reduction is hidden in the numerical tables.

2.2 Why the invariant target is -1

The congruence in Theorem 2.1 is scale-dependent:

$$d \equiv -N \pmod{R}.$$

Since N is invertible modulo R, multiplying by N^{-1} gives

$$dN^{-1} \equiv -1 \pmod{R}.$$

Thus the projective target is -1. This is the same algebraic pattern as the projective Springborn reduction: a raw scale-dependent obstruction is divided by a common projective scale, leaving a smaller invariant obstruction. In the present problem, however, this removes only the sign coordinate. It does not remember the three-arm coordinate of the residual six-sector.

2.3 The three p-origins and the forced coefficient 2

Let

$$a = \prod_q q^{e_q}.$$

Every divisor $d \mid N^2 = p^2 a^2$ has the form

$$d = p^i u, \qquad i \in \{0, 1, 2\}, \qquad u \mid a^2.$$

Writing

$$ua^{-1} = \prod_{q^{e_q} \| a} q^{\beta_q}, \qquad -e_q \le \beta_q \le e_q,$$

the normalized condition $dN^{-1} \equiv -1$ becomes

$$p^{i-1} u a^{-1} \equiv -1.$$

Therefore the signed a-ratio must hit

i	ua^{-1} target
0	$-p$
1	-1
2	$-p^{-1}$.

The signed box is inversion-invariant: $x \in \mathcal{B}_R(a)$ if and only if $x^{-1} \in \mathcal{B}_R(a)$, with the same number of exponent vectors. Since $(-p)^{-1} = -p^{-1}$, the $i = 0$ and $i = 2$ channels combine and force multiplicity 2.

Proposition 2.2 (Two-channel shell coefficient)**.** *For a coprime shell,*

$$R \text{ succeeds} \iff \widetilde{\mathfrak{c}}_R(a; -1) + 2\widetilde{\mathfrak{c}}_R(a; -p^{-1}) \in \mathbb{Z}_{>0}.$$

Proof. The preceding table gives the three target channels. The two channels $-p$ and $-p^{-1}$ have equal bounded counts by inversion of exponent vectors. Therefore the ordinary certificate count equals

$$\mathfrak{c}_R(a; -1) + 2\mathfrak{c}_R(a; -p^{-1}).$$

Replacing ordinary counts by split-zero counts preserves positivity and separates unsupported absence from supported zero. Hence positivity of the displayed split-zero expression is equivalent to existence of a divisor certificate. □

3 Split-zero support values

The coefficient semiring is

$$G(\mathbb{Z}_{\geq 0}) = \mathbb{Z}_{\geq 0} \sqcup \{\tau\}.$$

Here τ is the unsupported external zero, while $0_{\mathbb{Z}}$ is the supported arithmetic zero. Addition uses τ as the additive identity:

$$\tau + n = n, \qquad n, m \in \mathbb{Z}_{\geq 0},$$

and multiplication uses τ as an absorber. The three states used in the obstruction are

$$\tau \quad < \quad 0_{\mathbb{Z}} \quad < \quad \mathbb{Z}_{>0}.$$

For a target t and shell (R, a), let

$$\Gamma_R(a) := \langle q \bmod R : q \mid a \rangle \leq (\mathbb{Z}/R\mathbb{Z})^{\times}.$$

Define

$$\widetilde{\mathfrak{c}}_R(a; t) = \begin{cases} \tau, & t \notin \Gamma_R(a), \\ 0_{\mathbb{Z}}, & t \in \Gamma_R(a) \text{ but } t \notin \mathcal{B}_R(a), \\ \#\left\{(\beta_q) : \prod q^{\beta_q} \equiv t \pmod{R}\right\}, & t \in \mathcal{B}_R(a). \end{cases}$$

This is the precise location of positivity: not at the projective target, not at the snowflake, not at the eta sheet, but at the split-zero evaluation of the arithmetic box against the weighted target.

4 The modulo-840 support fibration and the glue datum

The modulo-840 support set is

$$V_{840} := \{r \in (\mathbb{Z}/840\mathbb{Z})^{\times} : r \equiv 1 \pmod{24}\}.$$

By the Chinese remainder theorem,

$$V_{840} \cong (\mathbb{Z}/5\mathbb{Z})^{\times} \times (\mathbb{Z}/7\mathbb{Z})^{\times},$$

so $|V_{840}| = 4 \cdot 6 = 24$. The two active quadratic characters are

$$\chi_5(r) = \left(\frac{r}{5}\right), \qquad \chi_7(r) = \left(\frac{r}{7}\right).$$

The fibration

$$V_{840} \twoheadrightarrow C_2^2, \qquad r \mapsto (\chi_5(r), \chi_7(r))$$

has four fibers of size six. The principal square fiber is

$$S_{840} = \{1, 121, 169, 289, 361, 529\} = \langle 289 \rangle \cong C_6.$$

The cyclic order is

$$1, 289, 361, 169, 121, 529.$$

$$
\begin{tikzcd}
\boxed{\begin{array}{c} V_{840} = \{r \in (\mathbb{Z}/840\mathbb{Z})^{\times} : r \equiv 1 \pmod{24}\} \\ \text{24 residual classes} \end{array}} \arrow[d, "r \mapsto ((r/5), (r/7))"] \\
\boxed{\begin{array}{c} C_2^2 \text{ character base} \\ (\chi_5, \chi_7) = (+,+), (+,-), (-,+), (-,-) \end{array}} \arrow[d, "\text{select } (+,+)"] \\
\boxed{\begin{array}{c} S_{840} = \langle 289 \rangle \cong C_6 \\ \text{principal six-class fiber} \end{array}}
\end{tikzcd}
$$

Each six-point fiber has augmentation lattice A_5. The four fibers give A_5^4. The C_2^2 base contributes the marked D_4 discriminant component. The resulting root frame is

$$R_{\mathrm{ES}} = A_5^4 D_4.$$

The glue calculation is included because it is the exact reason the residual object sits in the lambency-six umbral row rather than in a decorative ADE analogy.

4.1 The explicit isotropic subgroup

For A_5,

$$D(A_5) \cong \mathbb{Z}/6\mathbb{Z}, \qquad q_{A_5}(k) = \frac{5k^2}{6} \pmod{2\mathbb{Z}}.$$

For D_4,

$$D(D_4) \cong \mathbb{F}_2^2, \qquad q_{D_4}(0) = 0, \qquad q_{D_4}(y) = 1 \quad (y \neq 0).$$

Thus

$$D(R_{\mathrm{ES}}) \cong (\mathbb{Z}/6\mathbb{Z})^4 \oplus \mathbb{F}_2^2$$

with

$$q(x_1, x_2, x_3, x_4; y) = \frac{5}{6} \sum_{i=1}^{4} x_i^2 + q_D(y) \pmod{2\mathbb{Z}}.$$

Its order is

$$6^4 \cdot 4 = 5184 = 72^2.$$

Let $Q = \mathbb{F}_2^2 = \{0, a, b, a+b\}$. Define $H_2 \subset \mathbb{F}_2^4 \oplus Q$ by

u	y
0000	0
1111	0
1100	a
0011	a
1010	b
0101	b
1001	$a+b$
0110	$a+b.$

Let

$$H_3 := \mathrm{Span}_{\mathbb{F}_3}\{(1,1,1,0),(1,2,0,1)\} \subset \mathbb{F}_3^4.$$

Then $|H_3| = 9$, and every nonzero word of H_3 has Hamming weight 3: indeed a word is

$$\alpha(1,1,1,0)+\beta(1,2,0,1) = (\alpha+\beta, \alpha+2\beta, \alpha, \beta),$$

and if $(\alpha,\beta) \neq (0,0)$ exactly one coordinate is zero.

Embed $\mathbb{F}_2 \oplus \mathbb{F}_3$ into $\mathbb{Z}/6\mathbb{Z}$ by CRT,

$$k_i = 3u_i + 2v_i \pmod 6,$$

and define

$$H_{\mathrm{ES}} := \{((3u_1+2v_1,\dots,3u_4+2v_4),y) : (u,y) \in H_2,\ v \in H_3\}.$$

Equivalently, H_{ES} is generated by

$$\begin{aligned}
g_1 &= (3,3,3,3;0,0),\\
g_2 &= (3,3,0,0;1,0),\\
g_3 &= (3,0,3,0;0,1),\\
g_4 &= (2,2,2,0;0,0),\\
g_5 &= (2,4,0,2;0,0).
\end{aligned}$$

Theorem 4.1 (Glue theorem)**.** *The subgroup H_{ES} is isotropic of order* 72*. The corresponding overlattice*

$$N_{\mathrm{ES}} := \{x \in R_{\mathrm{ES}}^\vee : x \bmod R_{\mathrm{ES}} \in H_{\mathrm{ES}}\}$$

is even unimodular of rank 24 *and has root system exactly $A_5^4 D_4$.*

Proof. The CRT map is a coordinatewise bijection $\mathbb{F}_2 \oplus \mathbb{F}_3 \to \mathbb{Z}/6\mathbb{Z}$, so

$$|H_{\mathrm{ES}}| = |H_2||H_3| = 8 \cdot 9 = 72.$$

For $k = 3u + 2v$ one has in $\mathbb{Q}/2\mathbb{Z}$

$$\frac{5k^2}{6} = \frac{5}{6}(3u+2v)^2 \equiv \frac{3}{2}u + \frac{4}{3}v^2,$$

since the cross-term is even. Therefore

$$q((3u+2v),y) = \frac{3}{2}\,\mathrm{wt}(u) + \frac{4}{3}\,\mathrm{wt}(v) + q_D(y).$$

For $(u,y) \in H_2$, the possibilities are $(0,0)$, $u = 1111, y = 0$, or $\mathrm{wt}(u) = 2, y \neq 0$. The binary-plus-$D_4$ contribution is respectively

$$0, \qquad \frac{3}{2}\cdot 4 = 6, \qquad 3+1 = 4,$$

all even. For $v \in H_3$, either $v = 0$ or $\mathrm{wt}(v) = 3$, so the ternary contribution is either 0 or $\frac{4}{3}\cdot 3 = 4$. Hence q vanishes on H_{ES}.

Even overlattices of an even lattice correspond to isotropic subgroups of its discriminant form. Since H_{ES} is isotropic, N_{ES} is even. Its discriminant group has order

$$|D(N_{\mathrm{ES}})| = \frac{|D(R_{\mathrm{ES}})|}{|H_{\mathrm{ES}}|^2} = \frac{72^2}{72^2} = 1,$$

so N_{ES} is unimodular.

It remains to check that no roots are introduced by nonzero glue cosets. In the discriminant coset $k \in \mathbb{Z}/6\mathbb{Z}$ of A_5, the minimum norm is

$$\mu_A(k) = \frac{k(6-k)}{6}.$$

Thus the coordinate possibilities are

u_i	v_i	k_i	$\mu_A(k_i)$
0	0	0	0
1	0	3	3/2
0	$\neq 0$	2 or 4	4/3
1	$\neq 0$	1 or 5	5/6.

The allowed cases from H_2 and H_3 give the lower bounds

$\mathrm{wt}(u)$	$\mathrm{wt}(v)$	$\mathrm{wt}(\operatorname{supp} u \cap \operatorname{supp} v)$	minimum
2	0	0	$2(3/2)+1=4$
4	0	0	$4(3/2)=6$
0	3	0	$3(4/3)=4$
2	3	1	$(3/2)+2(4/3)+(5/6)+1=6$
2	3	2	$(4/3)+2(5/6)+1=4$
4	3	3	$(3/2)+3(5/6)=4.$

Every nonzero glue coset has minimum at least 4, while roots have norm 2. Hence all norm-2 vectors lie in the zero glue coset, and the root system remains $A_5^4 D_4$. □

The glue-preserving finite group is the non-split double cover

$$1 \to \{\pm I\} \to GL_2(3) \to PGL_2(3) \cong S_4 \to 1.$$

It projects to the permutation action on the four A_5 fibers and retains the central sign/glue datum invisible in the projective quotient. This is the finite symmetry used by the weighted defect calculus below.

5 The six-class snowflake and the weighted target defect

The cyclic fiber admits the product decomposition

$$C_6 \cong C_3 \times C_2, \qquad g^k \longmapsto (k \bmod 3, (-1)^k).$$

The A_8^3 discriminant realization writes

$$v_k = (-1)^k 3e_{k \bmod 3} \in D(A_8^3),$$

so

$$S_{840} \cong \{\pm 3e_0, \pm 3e_1, \pm 3e_2\}.$$

The projective target -1 is the distinguished hub

$$h = v_3 = -3e_0.$$

If $p \equiv g^k \pmod{840}$, then the second normalized target is

$$-p^{-1} \longleftrightarrow v_{3-k}.$$

Thus the target edge is

$$E_k = \{v_3, v_{3-k}\}.$$

But the actual coefficient is not the unweighted edge. It is the weighted defect coming from Theorem 2.2.

Definition 5.1 (Weighted target defect)**.** *Let $E_{r,\epsilon}$ denote the basis element of $\mathbb{Z}[C_3 \times C_2]$ with arm $r \in \mathbb{Z}/3\mathbb{Z}$ and sign $\epsilon \in \{+,-\}$. Define*

$$\mathfrak{D}_k := E_{0,-} + 2E_{-k,-(-1)^k}.$$

The six explicit matrices, with rows $r = 0, 1, 2$ and columns $+, -$, are

k	$\mathfrak{D}_k$
0	$\begin{pmatrix} 0 & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
1	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 0 \end{pmatrix}$
2	$\begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
4	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 2 \end{pmatrix}$
5	$\begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}$.

The champion prime satisfies

$$p_* = 8{,}803{,}369 \equiv 169 \equiv g^3 \pmod{840},$$

so

$$\mathfrak{D}_3 = 2E_{0,+} + E_{0,-}.$$

It lies on one arm and flips endpoint sign.

5.1 Why neither projection is sufficient

Define the arm marginal

$$A_k(r) = \sum_{\epsilon=\pm} \mathfrak{D}_k(r, \epsilon)$$

and the sign marginal

$$B_k(\epsilon) = \sum_{r=0}^{2} \mathfrak{D}_k(r, \epsilon).$$

Then

k	A_k	B_k
0	$(3,0,0)$	$(0,3)$
1	$(1,0,2)$	$(2,1)$
2	$(1,2,0)$	$(0,3)$
3	$(3,0,0)$	$(2,1)$
4	$(1,0,2)$	$(0,3)$
5	$(1,2,0)$	$(2,1)$.

Theorem 5.2 (Both coordinates are necessary)**.** *The sign projection B_k does not determine $\mathfrak{D}_k$. The arm projection A_k does not determine $\mathfrak{D}_k$. The product grading $C_3 \times C_2$ is the minimal grading that separates the six weighted defects.*

Proof. The sign projection loses arm data because

$$B_0 = B_2 = B_4 = (0, 3),$$

while

$$A_0 = (3, 0, 0), \quad A_2 = (1, 2, 0), \quad A_4 = (1, 0, 2).$$

The arm projection loses sign data because

$$A_0 = A_3 = (3, 0, 0),$$

while

$$B_0 = (0, 3), \quad B_3 = (2, 1).$$

Together the ordered pair (A_k, B_k) separates the six displayed matrices. Hence neither the projective C_2 sheet nor the gerbe C_3 arm coordinate is sufficient alone. □

This is the formal reason the Springborn/projective reduction and the gerbe sectorization must be composed rather than listed separately.

5.2 Mixed arm-sign defect

The independent tensor predicted by the two marginals is

$$\mathfrak{D}_k^{\mathrm{ind}}(r, \epsilon) = \frac{A_k(r) B_k(\epsilon)}{3}.$$

Define the mixed defect

$$\mathfrak{M}_k := \mathfrak{D}_k - \mathfrak{D}_k^{\mathrm{ind}}.$$

Theorem 5.3 (Mixed defect classification)**.** *One has*

$$\mathfrak{M}_k = 0 \quad \Longleftrightarrow \quad k \notin \{1, 5\}.$$

For the genuinely mixed sectors,

$$\mathfrak{M}_1 = \begin{pmatrix} -2/3 & 2/3 \\ 0 & 0 \\ 2/3 & -2/3 \end{pmatrix}, \qquad \mathfrak{M}_5 = \begin{pmatrix} -2/3 & 2/3 \\ 2/3 & -2/3 \\ 0 & 0 \end{pmatrix}.$$

Proof. Substitute each of the six explicit matrices into

$$\mathfrak{M}_k = \mathfrak{D}_k - A_k \otimes B_k/3.$$

For $k = 0, 2, 3, 4$, the matrix is supported in a single row or single column, and the tensor reconstruction is exact. For $k = 1, 5$, direct subtraction gives the two displayed rank-one signed rectangles. $\square$

Thus the six classes split into loop defects ($k = 0$), cross-arm same-sign defects ($k = 2, 4$), same-arm opposite-sign defects ($k = 3$), and genuinely mixed arm-sign defects ($k = 1, 5$). The champion is the same-arm opposite-sign sector.

6 Fourier nondegeneracy and all characteristic polynomials

Let

$$\omega = e^{2\pi i/3}.$$

Characters of $C_3 \times C_2$ are indexed by

$$(u, v) \in \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

and evaluate

$$(r, \epsilon) \mapsto \omega^{ur}\epsilon^{v}.$$

Theorem 6.1 (Full Fourier support)**.** *For every k,*

$$\widehat{\mathfrak{D}_k}(u, v) = (-1)^v \left(1 + 2(-1)^{kv}\omega^{-uk}\right).$$

In particular,

$$\widehat{\mathfrak{D}_k}(u, v) \neq 0$$

for every character (u, v).

Proof. The hub endpoint $E_{0,-}$ contributes $(-1)^v$. The second endpoint $E_{-k,-(-1)^k}$ contributes

$$2\omega^{-uk}(-(-1)^k)^v = 2\omega^{-uk}(-1)^v(-1)^{kv}.$$

Adding gives the formula. The parenthetical factor cannot vanish because its two summands have absolute values 1 and 2. $\square$

Corollary 6.2 (No target-mode obstruction)**.** *The target defect never loses a Fourier mode. Therefore a shell failure is not caused by a missing target character. It is caused by the arithmetic box $\mathcal{B}_R(a)$ failing to deliver bounded positive mass to the full-support target defect.*

For completeness, we also record the characteristic polynomial of the six translation/circulant defect operators. Let $C(\mathfrak{D}_k)$ be multiplication by the group-ring element $e_3 + 2e_{3-k}$ on $\mathbb{Z}[C_6]$.

Theorem 6.3 (Characteristic polynomials of all six defect operators). *The determinants and characteristic polynomials over $\mathbb{Z}$ are*

k	$\det C(\mathfrak{D}_k)$	$\chi_k(X)$
0	-729	$(X-3)^3(X+3)^3$
1	63	$(X-3)(X-1)(X^2+3)(X^2+4X+7)$
2	-81	$(X-3)(X+3)(X^2+3)^2$
3	27	$(X-3)^3(X-1)^3$
4	-81	$(X-3)(X+3)(X^2+3)^2$
5	63	$(X-3)(X-1)(X^2+3)(X^2+4X+7)$.

In particular every determinant is nonzero modulo 107*.*

Proof. The eigenvalues of a circulant operator are the Fourier values

$$\zeta_6^{3m} + 2\zeta_6^{(3-k)m}, \qquad m = 0, \dots, 5.$$

Multiplying $(X - \lambda_m)$ over the six Fourier modes gives the displayed polynomials. The determinant is the product of the same eigenvalues. Each listed determinant is prime to 107. □

7 Champion positivity with all layers active

Let

$$p_* = 8{,}803{,}369.$$

Then

$$p_* \equiv 169 \equiv g^3 \pmod{840},$$

so the target defect is

$$\mathfrak{D}_3 = 2E_{0,+} + E_{0,-}.$$

For each shell below, the ordered support state is

$$\mathcal{C}_R(p_*) = \big(\widetilde{\mathfrak{c}}_R(a_R(p_*); -1), \widetilde{\mathfrak{c}}_R(a_R(p_*); -p_*^{-1})\big).$$

R	$a_R(p_*)$	support state	meaning
27	$7 \cdot 314407$	(τ, τ)	no subgroup descent
43	$379 \cdot 5807$	$(0_{\mathbb{Z}}, 0_{\mathbb{Z}})$	subgroup descent, bounded miss
107	$3^2 \cdot 11^2 \cdot 43 \cdot 47$	$(2, 0_{\mathbb{Z}})$	positive certificate

Theorem 7.1 (Champion support ladder). *The shells $R = 27, 43, 107$ for p_* realize*

$$(\tau, \tau) \longrightarrow (0_{\mathbb{Z}}, 0_{\mathbb{Z}}) \longrightarrow (2, 0_{\mathbb{Z}}).$$

Hence positivity first appears at $R = 107$, and it appears in the -1 channel.

Proof. For $R = 27$,

$$a = 2{,}200{,}849 = 7 \cdot 314407.$$

The subgroup generated by the two primes modulo 27 is

$$\{1, 4, 7, 10, 13, 16, 19, 22, 25\}.$$

The target pair is

$$-1 \equiv 26, \qquad -p_*^{-1} \equiv 17 \pmod{27},$$

and neither target lies in the subgroup. Thus the state is (τ, τ).

For $R = 43$,

$$a = 2{,}200{,}853 = 379 \cdot 5807.$$

The target pair is

$$-1 \equiv 42, \qquad -p_*^{-1} \equiv 41 \pmod{43}.$$

Both targets lie in the generated subgroup, but the bounded box is

$$\mathcal{B}_{43}(a) = \{1, 2, 8, 16, 22, 27, 32, 35, 39\},$$

so neither target is hit. Thus the state is $(0_{\mathbb{Z}}, 0_{\mathbb{Z}})$.

For $R = 107$,

$$a = 2{,}200{,}869 = 3^2 \cdot 11^2 \cdot 43 \cdot 47.$$

The targets are

$$-1 \equiv 106, \qquad -p_*^{-1} \equiv 86 \pmod{107}.$$

The signed exponent box has two exponent vectors hitting 106,

$$(-2, 0, -1, -1), \qquad (2, 0, 1, 1),$$

with coordinates ordered by $(3, 11, 43, 47)$, and has no exponent vector hitting 86. Hence the state is $(2, 0_{\mathbb{Z}})$. □

The strictness of this ladder is the operational meaning of split zero: $R = 27$ has no support descent; $R = 43$ has support but no bounded mass; $R = 107$ has bounded positive mass.

8 The critical arm $744 \leftrightarrow 747 \leftrightarrow 750$

The completed A_8 arm is

$$-3e_0, 0_0, +3e_0.$$

It is represented by

$$744, \qquad 747, \qquad 750.$$

Define

$$v(n) = \frac{n - 747}{3}.$$

Then

$$v(744) = -1, \qquad v(747) = 0, \qquad v(750) = 1.$$

The involution

$$n \mapsto 1494 - n$$

is exactly $v \mapsto -v$.

index	744	747	750
arm state	$-3e_0$	0_0	$+3e_0$
v	-1	0	1
role	negative endpoint	supported-zero hub	positive endpoint

The important point is that 747 is not absence. It is the supported-zero hub of the arm.

8.1 The A_8^3 raw-zero sheet

Set

$$T_9(\tau) = \frac{\eta(\tau)^3}{\eta(9\tau)^3} = q^{-1}P_9(q).$$

Jacobi's eta-cube dissection gives

$$P_9(q) = H(q^3) - 3q$$

for some $H(q^3) \in \mathbb{Z}[[q^3]]$. Therefore

$$T_9 = q^{-1}H(q^3) - 3.$$

All positive nonconstant exponents of T_9 are of the form $3m-1$, i.e. congruent to 2 (mod 3). Since $744, 747, 750$ are divisible by 3,

$$[q^{744}]T_9 = [q^{747}]T_9 = [q^{750}]T_9 = 0.$$

The raw A_8^3 eta sheet sees the whole completed arm as zero. The support states are recovered only after Fricke and Eisenstein completion.

Define

$$I_9 := \frac{27}{T_9}, \qquad J_9 := T_9 + 3 + \frac{27}{T_9}, \qquad Y_9 := T_9 - \frac{27}{T_9},$$

and

$$f_9 := -q\frac{d}{dq}\log T_9.$$

Then

$$f_9 = \frac{-3E_2(\tau) + 27E_2(9\tau)}{24},$$

so

$$[q^n]f_9 = 3\sigma_1(n) - 27\sigma_1(n/9),$$

with the second term omitted if $9 \nmid n$.

Modulo 107, the critical-arm values are

n	T_9	I_9	J_9	Y_9	f_9
744	0	35	35	72	89
747	0	31	31	76	45
750	0	42	42	65	52.

Thus

$$J_9(744) = 35 = N_*, \qquad Y_9(744) = -35 = -N_*,$$
$$f_9(744) = 89 = 93^2 = a_*^2,$$
$$J_9(747)^2 = 31^2 \equiv 105 \equiv -2,$$

and

$$J_9(750)^2 = 42^2 \equiv 52 = f_9(750), \qquad f_9(750)^{-1} = 35 = N_*.$$

The hub identity $J_9(747)^2 = -2$ records the two-edge support state at $747 \equiv -2 \pmod{107}$.

8.2 The nonlinear A_8^3 support resolvent

Since $Y_9 = -J_9$ on this arm, the pair (J_9, Y_9) is rank one. The nonlinear coordinate J_9^2 is required to separate the hub. Define

$$\mathcal{R}_9(n) = (J_9(n), J_9(n)^2, f_9(n)).$$

For $n = 744, 747, 750$,

$$M_9 = \begin{pmatrix} 35 & 48 & 89 \\ 31 & 105 & 45 \\ 42 & 52 & 52 \end{pmatrix}.$$

Theorem 8.1 (A_8^3 support determinant)**.**

$$\det M_9 \equiv 103 \equiv -4 \pmod{107}.$$

Proof. Expanding along the first row,

$$\det M_9 = 35(105 \cdot 52 - 45 \cdot 52) - 48(31 \cdot 52 - 45 \cdot 42) + 89(31 \cdot 52 - 105 \cdot 42).$$

Reduction modulo 107 gives $103 \equiv -4$. □

9 Lambency-six moonshine support resolution

The residual lambency-six eta quotient is

$$T_6(\tau) = \frac{\eta(\tau)^5 \eta(3\tau)}{\eta(2\tau)\eta(6\tau)^5}.$$

It has expansion

$$T_6 = q^{-1} - 5 + 6q + 4q^2 - 3q^3 - 12q^4 - 8q^5 + \cdots.$$

Define

$$J_6 := T_6 + 5 + \frac{72}{T_6}, \qquad Y_6 := T_6 - \frac{72}{T_6},$$

$$F_6 := -2\eta(6\tau)^4 T_6, \qquad f_6 := -q\frac{d}{dq}\log T_6.$$

The logarithmic derivative is

$$f_6 = \frac{-5E_2(\tau) + 2E_2(2\tau) - 3E_2(3\tau) + 30E_2(6\tau)}{24},$$

and the coefficient formula is

$$[q^n]f_6 = 5\sigma_1(n) - 2\sigma_1(n/2) + 3\sigma_1(n/3) - 30\sigma_1(n/6),$$

with terms omitted when the denominator does not divide n.

On the same critical arm, modulo 107,

n	T_6	F_6	J_6	Y_6	f_6
744	34	19	16	52	67
747	83	77	1	58	48
750	101	106	2	93	21.

Consequently

$$J_6(747) = 1, \qquad Y_6(750) = 93 = a_*,$$
$$F_6(750) = 106 = -1, \qquad f_6(750) = 21 = S(3).$$

The lambency-six sheet separates the hub and endpoint linearly.

Define

$$M_{6,1} := \begin{pmatrix} J_6(744) & Y_6(744) & F_6(744) \\ J_6(747) & Y_6(747) & F_6(747) \\ J_6(750) & Y_6(750) & F_6(750) \end{pmatrix} = \begin{pmatrix} 16 & 52 & 19 \\ 1 & 58 & 77 \\ 2 & 93 & 106 \end{pmatrix}$$

and

$$M_{6,2} := \begin{pmatrix} J_6(744) & F_6(744) & f_6(744) \\ J_6(747) & F_6(747) & f_6(747) \\ J_6(750) & F_6(750) & f_6(750) \end{pmatrix} = \begin{pmatrix} 16 & 19 & 67 \\ 1 & 77 & 48 \\ 2 & 106 & 21 \end{pmatrix}.$$

Theorem 9.1 (Lambency-six support determinants)**.**

$$\det M_{6,1} \equiv 82 \equiv -25 \pmod{107}, \qquad \det M_{6,2} \equiv 25 \pmod{107}.$$

Proof. Direct determinant expansion of the two displayed matrices gives 82 and 25 modulo 107 respectively. □

10 Finite transport from A_8^3 to $A_5^4 D_4$

The A_8^3 sheet is raw-zero and requires the nonlinear support coordinate J_9^2. The $A_5^4 D_4$ lambency-six sheet separates linearly. Their finite relation on the critical arm is encoded by transport matrices

$$P^{(1)}_{9\to 6} := M_{6,1} M_9^{-1}, \qquad P^{(2)}_{9\to 6} := M_{6,2} M_9^{-1}$$

over $\mathbb{F}_{107}$.

Theorem 10.1 (Transport matrices and characteristic polynomials)**.** *Modulo* 107*,*

$$P^{(1)}_{9\to 6} = \begin{pmatrix} 32 & 67 & 44 \\ 98 & 72 & 41 \\ 51 & 81 & 71 \end{pmatrix},$$

with

$$\det P^{(1)}_{9\to 6} \equiv 33,$$

and

$$\chi_{P^{(1)}_{9\to 6}}(X) = X^3 + 39X^2 + 18X - 33.$$

This polynomial is irreducible over $\mathbb{F}_{107}$*. Also*

$$P^{(2)}_{9\to 6} = \begin{pmatrix} 3 & 28 & 46 \\ 106 & 64 & 81 \\ 38 & 47 & 84 \end{pmatrix},$$

with

$$\det P^{(2)}_{9\to 6} \equiv 74,$$

and

$$\chi_{P^{(2)}_{9\to 6}}(X) = X^3 - 44X^2 - 28X + 33 = (X + 17)(X^2 + 46X + 46).$$

Proof. Since $\det M_9 = -4$ modulo 107, the inverse M_9^{-1} exists. Multiplying the displayed matrices gives the two stated transport matrices. The determinants follow from

$$\det P^{(1)}_{9\to 6} = \frac{-25}{-4} = 25 \cdot 4^{-1} \equiv 25 \cdot 27 \equiv 33$$

and

$$\det P^{(2)}_{9\to 6} = \frac{25}{-4} = -25 \cdot 4^{-1} \equiv 74.$$

The characteristic polynomials are obtained by evaluating $\det(XI - P)$ for the two matrices. The factorization of the second polynomial is direct. For the first, substituting all 107 field elements gives no root, and a cubic over a field is reducible if and only if it has a root; hence it is irreducible. □

This is the finite computation linking the A_8^3 raw-zero support completion to the $A_5^4 D_4$ lambency-six moonshine completion. It is not a symbolic arrow; it is an invertible $\mathbb{F}_{107}$-linear transport between the three resolved support states.

11 Gerbe and paramodular envelope as functorial lifts of the same defect

The gerbe construction is used here only for one purpose: it supplies a natural six-sector module basis indexed by

$$\mathbb{Z}_3 \times \mathbb{Z}_2.$$

Under

$$C_6 \cong C_3 \times C_2,$$

the snowflake sector $E_{r,\epsilon}$ is the gerbe sector $\Theta_{r,\epsilon}$. The actual object is therefore

$$\mathfrak{D}_k = \Theta_{0,-} \oplus 2\Theta_{-k,-(-1)^k}.$$

The full Fourier support theorem shows that this defect is representation-complete on the six-sector slice.

The genus-two theta/paramodular envelope lifts the same weighted defect from sectors to binary quadratic coefficient indices. If L is an even unimodular rank-24 lattice and T is a positive semidefinite half-integral binary quadratic form, write $r_L(E_{r,\epsilon}; T)$ for the number of lattice pairs in sector $E_{r,\epsilon}$ with Gram index T. Define

$$r_L(\mathfrak{D}_k; T) := r_L(E_{0,-}; T) + 2r_L(E_{-k,-(-1)^k}; T).$$

Proposition 11.1 (No mode loss under the binary-quadratic envelope)**.** *For each fixed T, the map*

$$\mathfrak{D} \mapsto r_L(\mathfrak{D}; T)$$

is a linear functional on $\mathbb{Z}[C_3 \times C_2]$. Since $\mathfrak{D}_k$ has nonzero Fourier coefficient in every character, the envelope does not kill any target mode on the defect side. Any vanishing in $r_L(\mathfrak{D}_k; T)$ is therefore a property of the lattice-pair coefficient at T, not a degeneracy of the target defect.

Proof. Linearity is immediate from the definition. Fourier inversion on $C_3 \times C_2$ decomposes the pairing into modewise products of coefficient modes and target modes. By Theorem 6.1, all target modes are nonzero. □

This is the exact sense in which the gerbe and paramodular objects interact with the arithmetic obstruction: both are lifts of the same weighted target defect. They are not independent analogies.

12 The remaining arithmetic theorem

The preceding results reduce the deep-tail problem to one positivity statement.

Theorem 12.1 (Equivalent positivity formulation)**.** *For hard primes in the six-square sector, the Erdős–Straus assertion is equivalent to the following reachability statement:*

For every prime $p \equiv g^k \pmod{840}$ in S_{840}, there exists a shell $R \equiv 3 \pmod 4$ such that

$$\Omega_R(p) = \widetilde{\mathfrak{c}}_R(a_R(p); -1) + 2\widetilde{\mathfrak{c}}_R(a_R(p); -p^{-1}) \in \mathbb{Z}_{>0}.$$

Equivalently, the signed divisor box $\mathcal{B}_R(a_R(p))$ intersects the weighted target defect

$$\mathfrak{D}_k = E_{0,-} + 2E_{-k,-(-1)^k}$$

with positive split-zero mass.

Proof. The standard congruence identities and the $R = 3, 7$ sieve reduce the remaining primes to the six-square sector. For a fixed shell, Theorem 2.2 proves equivalence of shell success and positivity of $\Omega_R(p)$. Existence of an Erdős–Straus decomposition is equivalent to success of at least one shell. The target-edge statement is the $C_3 \times C_2$ sectorization of the two normalized targets. $\square$

13 Verification recurrences

All coefficient calculations in this note are finite. The verification script accompanying the note implements the following recurrences.

Let

$$T_h = q^{-1}P_h(q), \qquad P_h(q) = \prod_{n\geq 1}(1-q^n)^{E_h(n)}.$$

For T_6,

$$E_6(n) = 5 - \mathbf{1}_{2|n} + \mathbf{1}_{3|n} - 5\mathbf{1}_{6|n}.$$

For T_9,

$$E_9(n) = 3 - 3\mathbf{1}_{9|n}.$$

If

$$P_h(q) = \sum_{n\geq 0} a_n q^n, \qquad a_0 = 1,$$

and

$$C_h(m) = \sum_{d|m} dE_h(d),$$

then

$$na_n = -\sum_{m=1}^{n} C_h(m)a_{n-m}.$$

The inverse series

$$P_h(q)^{-1} = \sum_{n\geq 0} b_n q^n, \qquad b_0 = 1,$$

is computed by

$$b_n = -\sum_{j=1}^{n} a_j b_{n-j}.$$

Then

$$\frac{72}{T_6} = 72qP_6(q)^{-1}, \qquad \frac{27}{T_9} = 27qP_9(q)^{-1}.$$

All congruences in the tables are taken in $\mathbb{F}_{107}$.

14 Conclusion

The corrected object is not a list of structures. It is the positivity evaluation

$$\mathcal{B}_R(a) \longrightarrow \{-1, -p^{-1}\} \longrightarrow \mathfrak{D}_k \longrightarrow G(\mathbb{Z}_{\geq 0}) \longrightarrow \mathbb{Z}_{>0}.$$

Projective normalization creates the invariant sign target -1. The $C_3 \times C_2$ snowflake restores the arm/sign coordinates. The coefficient 2 in $\mathfrak{D}_k$ is forced by divisor origins. Split-zero support distinguishes no subgroup descent, supported bounded miss, and genuine positive mass. The target defect has full Fourier support, so failure cannot be blamed on missing target modes. The champion prime demonstrates the entire ladder explicitly. The $744, 747, 750$ modular arm shows the same phenomenon in the eta sheets: raw zero is resolved only after Fricke/Eisenstein completion, and the transport matrices compute the finite passage from the A_8^3 support resolution to the lambency-six $A_5^4 D_4$ moonshine resolution.

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