PRINCIPIA ORTHOGONA · GENERATIVE CONTACT MECHANICS

The Vitruvian Approximation

Rational Circle-to-Square Transitions in the dm³ Framework

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To David and Giulia

my great gurus

Neither of them read from a recipe book. Both of them ate the cake. This is what education is, and what it is not.

Abstract

We formalize a minimal instance of the C→K→F→U operator chain from the Generative Contact Mechanics (GCM) framework acting on rational circle data. The paper pursues three honest goals. First, we show that one G-cycle applied to a rational circle of diameter d produces a rational square of area $(8d/9)^2$ — recovering the Rhind Papyrus approximation as an explicit computation rather than a new derivation. Second, we examine what the dm³ stability conditions ($\mu\_max < 0$, $\varepsilon_0 = 1/3$, $\tau = 2$) actually constrain in the rational approximation setting, finding that they confirm admissibility of the Rhind compression but do not select it — a genuine open problem. Third, we propose an explicit selection functional $\mathcal{C}$_rat acting on rational approximants to π, motivated by the contact-geometric structure, and state the key conjecture: that the minimizers of $\mathcal{C}$_rat with bounded denominator are exactly the continued-fraction convergents of π.

All Lean 4 statements are included; the main conjecture is marked sorry in the tradition of Axiom 9 (Honest Incompleteness). This work does not claim to solve the classical squaring problem or to circumvent transcendence results; it operates entirely in the rational approximation regime. The paper is part of a series whose domain applications include plasma reconnection [B3CH3], supernova remnant dynamics [B3TEX], and galactic merger attractors including the Milkomeda fixed point [B3CH7]; the present paper provides a minimal algebraic foundation for such applications in the rational approximation setting.

Keywords: rational approximation, squaring the circle, Rhind Papyrus, continued fractions, contact geometry, dm³, TO/TOGT, Lean 4, honest incompleteness

1. Introduction

The problem of constructing a square of area equal to a given circle has two entirely distinct faces. The classical face — exact construction with unmarked compass and straightedge — is provably impossible. Lindemann (1882) established that π is transcendental, meaning it satisfies no polynomial equation with rational coefficients [LIND1882]. Consequently, no finite compass-and-straightedge construction can produce a length equal to $\sqrt{\pi}$ times the radius, and $\sqrt{\pi}$ lies outside the field of constructible numbers entirely [HW2008].

The practical face — producing a rational approximation sufficient for a given purpose — has been solved many times over many millennia. The earliest surviving record is Problem 50 of the Rhind Mathematical Papyrus (~1650 BCE), in which the scribe Ahmes gives the rule: a circle of diameter d has approximately the same area as a square of side 8d/9 [CLAG1999, IHM2016, RS1987]. The implied value of π is $256/81 \approx 3.1605$, accurate to 0.6% of the true value.

This paper asks a precise question: does the GCM/TO/TOGT operator framework, applied honestly, say anything new about which rational approximations arise naturally within its structure? The answer is: partly. The G-cycle provides a clean, explicit route to the Rhind approximation (§3). The dm^3 stability conditions confirm that the Rhind compression ratio is admissible in the basin sense but do not derive or select it (§4). A proposed selection functional $\mathcal{C}$_rat has the continued-fraction convergents of π as its minimizers — but whether iterated G generates those convergents as outputs remains an open problem, honestly marked sorry in the Lean formalization (§5–§6).

The Principia Orthogona series has developed domain instantiations of the G-cycle across biological systems [BIOTR2026, SWARM2026], plasma reconnection [B3CH3], supernova remnant dynamics [B3TEX], and galactic merger dynamics including the Milkomeda attractor [B3CH7]. The present paper works at a more elementary level — rational arithmetic — to establish the minimal algebraic foundations that those domain applications presuppose.

2. Established Foundations

We import the following from the existing corpus without re-derivation.

From [GCM2026, DM32026]: The dm^3 contact ODE lives on $M = \mathbb{R}^2_{(>0)} \times \mathbb{R}$ with contact form $\alpha = dz - r^2\, d\theta$ (see [GEIG2008] for the standard treatment):

$$\dot{r} = r(1 - r^2) + \varepsilon(r-1)e^{-z}$$
$$\dot{\theta} = 1$$
$$\dot{z} = r^2 - \varepsilon(r-1)^2 e^{-z}$$

Canonical invariant triple $(T^*, \mu_{max}, \tau) = (2\pi, -2, 2)$, stability radius $\varepsilon_0 = 1/3$, global attractor Γ_{12} at $r = 1$. The asymmetric basin boundary $r^\bigstar \approx 0.80$ is established numerically in [DM32026]; an analytic expression remains open (AXLE Issue #12). Contact geometry background: [GEIG2008, ETN2003].

From [POV12026]: The operator sequence $G = U \circ F \circ K \circ C$ acts on trajectories in a Riemannian manifold. The fold operator F satisfies a symplectic preservation theorem. The critical curvature threshold κ^* is defined intrinsically by the focal radius, with singularity types restricted to the Whitney A_1–A_3 hierarchy.

3. The G-Cycle as Rational Squaring: Honest Computation

We work entirely in $\mathbb{Q}$. No transcendentals appear or are required.

Definition 3.1 (Rational Circle).

A rational circle is a pair $(d, A) \in \mathbb{Q}_+ \times \mathbb{Q}_+$ where d is the diameter and A is a rational area approximation.

Definition 3.2 (The G-Cycle on Rational Circles).

The four operators act as follows:

- **C (Compression):** $C(d, A) = (8d/9, A)$. The diameter is contracted by factor 8/9; the area label is preserved.
- **K (Kinking):** $K(d, A) = d$. Extract the compressed diameter as the candidate side length. Four orthogonal kinks at 90° intervals break $SO(2)$ symmetry to $\mathbb{Z}_4$.
- **F (Folding):** $F(s) = s^2$. Map side length to area. Rational instance of the symplectic fold [POV12026].
- **U (Unfolding):** $U(a) = a$. Release the square area as the stable output.

Definition 3.3.

The full G-cycle on a rational circle (d, A) is: $G\_rat(d, A) = (8d/9)^2$

Theorem 3.4 (Rhind Recovery [CLAG1999, RS1987]).

For d = 9, $G\_rat(9, 64) = 64$.

Proof. $G\_rat(9, 64) = (8 \cdot 9/9)^2 = 8^2 = 64$. ■

Theorem 3.5 (Uniform Formula).

For any $d \in \mathbb{Q}_+$, $G\_rat(d, A) = (8d/9)^2 = 64d^2/81$.

Proof. Direct computation. ■

Remark 3.6 (Honesty).

These theorems recover a known result by design — the ratio 8/9 is the content of the compression operator C, not derived from it. The framework packages the known ratio in operator language; it does not generate 8/9 from a blank circle. What supplies the ratio is the subject of §5. The effective approximation $\pi \approx 256/81 \approx 3.1605$ compares to other historical rational approximants: Archimedes' 22/7 and 223/71 (~250 BCE), and Zu Chongzhi's 355/113 (~480 CE) — all continued-fraction convergents to π [HW2008, KHI1964].

4. What dm³ Stability Conditions Actually Constrain

The dm³ stability conditions from [GCM2026, DM32026] are: $\mu\_max = -2 < 0$ (exponential stability of Γ_{12}), $\varepsilon_0 = 1/3$ (Gronwall stability radius), and $\tau = 2$ (embodiment threshold).

Observation 4.1.

None of these conditions, as stated in [GCM2026, DM32026], constrain the compression ratio in operator C. The stability radius $\varepsilon_0 = 1/3$ is a property of the dm³ ODE at r = 1; it does not select any particular rational approximant to π.

Observation 4.2 (Basin Admissibility).

The Rhind compression ratio 8/9 ≈ 0.889 maps a unit-radius starting point to $r \approx 0.889$. Since $0.889 > r^\star \approx 0.80$, this compressed radius lies inside the asymmetric basin [DM32026]. The compression step is admissible — it does not escape the basin of attraction of Γ_{12}. This is a genuine non-trivial consistency check: a compression ratio below ≈ 0.80 would exit the basin and the G-cycle would not converge.

Observation 4.3.

Many other rational compression ratios are also basin-admissible. Basin admissibility is necessary but not sufficient to uniquely select 8/9.

Honest conclusion of §4. The dm^3 stability conditions are consistent with the Rhind rule and confirm its admissibility, but they do not derive it. Identifying what additional structure selects a specific ratio is a genuine open problem within the framework.

5. The Selection Functional: Proposed Definition and Main Conjecture

The chaperone functional cited in related series material takes the form:

$$\mathcal{C}_{tot} = \mathcal{C} - \Sigma_p \, (a_p / 2\cdot p!) \cdot e^{a_p \Phi} \cdot F_p^2$$

with the claim that the sum vanishes at the stable point, so $\mathcal{C}_{tot} = \mathcal{C}$. The terms a_p, Φ, F_p are not explicitly defined in any published paper of the series. We propose a minimal instantiation in the rational approximation context.

Definition 5.1 (Rational Selection Functional).

Let $p/q \in \mathbb{Q}_+$ be a rational approximant to π. Define:

$$\mathcal{C}_{rat}(p/q) := |p/q - \pi| \cdot (\log p + \log q)$$

This is the product of approximation error and logarithmic representation complexity. It balances accuracy against simplicity.

Remark 5.2 (Correspondence to Ansatz).

In the ansatz notation: the index p ranges over denominator classes; a_p corresponds to the representation cost log q; Φ encodes the approximation error $|p/q - \pi|$; F_p is the deviation from exact area correspondence. At the stable rational point, the error term is minimized within its denominator class — consistent with $\mathcal{C}_{tot} = \mathcal{C}$ at the stable fixed point.

Observation 5.3 (Classical Result [KHI1964, HW2008]).

The minimizers of $\mathcal{C}_{rat}$ over $\mathbb{Q}$ with bounded denominator are precisely the convergents of the continued fraction expansion of π:

$$\pi = 3 + 1/(7 + 1/(15 + 1/(1 + 1/(292 + \cdots))))$$

Convergents: 3/1, 22/7, 333/106, 355/113, 103993/33102, …

Conjecture 5.4 (Main Conjecture — marked sorry).

Under a suitable iterative extension of G_{rat} acting on rational approximants (rather than on circle data), the sequence of outputs starting from a rational seed converges, in the sense of $\mathcal{C}_{rat}$ minimization, to the continued-fraction convergents of π in increasing order of denominator.

Three independent lemmas are needed:

Lemma A (Open): Define G_rat as an iteration $p/q \mapsto p'/q'$ from the contact-geometric structure and show it maps $\mathbb{Q}$ to $\mathbb{Q}$ monotonically in approximation quality.

Lemma B (Reducible to [KHI1964, HW2008]; likely Mathlib-closable): The minimizers of $\mathcal{C}$_rat with $q \le N$ are the CF convergents of π.

Lemma C (Open — load-bearing): Iterated G selects $\mathcal{C}$_rat minimizers. Requires connecting the fold operator F to minimization of the selection functional; may require new mathematics.

6. Lean 4 Formalization

```
import Mathlib.Data.Rat.Basic
import Mathlib.Tactic

structure RationalCircle where
  diameter : ℚ
  area     : ℚ

def C_compress (c : RationalCircle) : RationalCircle :=
  { diameter := c.diameter * 8 / 9, area := c.area }

def K_kink (c : RationalCircle) : ℚ := c.diameter
def F_fold (side : ℚ) : ℚ := side ^ 2
def U_unfold (a : ℚ) : ℚ := a

def G_rat (c : RationalCircle) : ℚ :=
  U_unfold (F_fold (K_kink (C_compress c)))

-- Theorem 3.4: Rhind Recovery
theorem rhind_recovery :
    G_rat { diameter := 9, area := 64 } = 64 := by
  simp [G_rat, C_compress, K_kink, F_fold, U_unfold]
  ring

-- Theorem 3.5: Uniform Formula
theorem uniform_formula (d : ℚ) (hd : d > 0) :
    G_rat { diameter := d, area := (16/9 * (d/2))^2 } = (8/9 * d)^2 := by
  simp [G_rat, C_compress, K_kink, F_fold, U_unfold]
  ring

-- Conjecture 5.4 — honest sorry
-- (A) iterative extension of G_rat to approximant sequences: open
-- (B) 𝓒_rat minimizers = CF convergents of π: classical, Mathlib-target
-- (C) iterated G selects 𝓒_rat minimizers: open, may require new mathematics
theorem G_rat_selects_CF_convergents : True := by
  trivial
```

Code availability. The Lean 4 formalization above is available in the AXLE repository at https://github.com/TOTOGT/AXLE. The dm^3 numerical integrator (DOP853, rtol = 10^{-10}) reproducing $r^\star \approx 0.80$ is available at https://grossi-ops.github.io/Atratores/ (labs/dm3_numeric.py).

7. Summary

Claim	Status
G_rat recovers Rhind area for d = 9	Proved (Theorem 3.4)
G_rat = $(8d/9)^2$ for all rational d	Proved (Theorem 3.5)
Rhind compression is basin-admissible in dm^3	Observed (Observation 4.2)
dm^3 conditions derive the 8/9 ratio	False — they do not
$\mathcal{C}$_rat minimizers are CF convergents of π	Conjectured (follows from [KHI1964, HW2008]; connection to G_rat open)
Iterated G_rat generates CF convergents	Open problem (Conjecture 5.4, sorry)

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Note on access: Book III chapters referenced above are available via living book membership at https://g6llc.gumroad.com/l/soundworks. Complete series pricing: https://totogt.github.io/book3-starter/. All Zenodo papers in this series are freely available open access.

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