A Continuation of Gauss's "Dioptrische Untersuchungen." By Professor R. A. Sampson. Read April 8th, 1897. Received, in recast form, September 18th, 1897.

Of all methods by which we can discuss the most general case of repeated asymmetrical refraction of a beam of light, that which Gauss invented and applied to the symmetrical instrument appears to be most appropriate; for it rests upon the correspondence of ray to ray—the simplest correspondence which no refraction can disturb. The deeper physical significance of the method of the characteristic function appears to have diverted attention from what may be done by direct consideration of the rays.

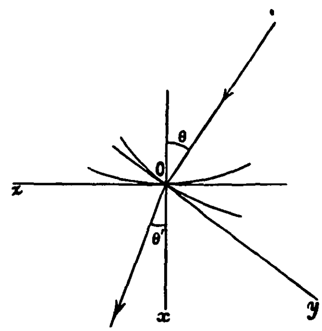
How complete a solution Gauss's method can furnish will be seen below, where a simple and elegant algebraical correspondence expresses the limits of the modification which any narrow beam can experience in passing in any manner through any singly refracting media which vary continuously or discontinuously.

In this scheme sixteen coefficients express the correspondence of ray to ray: six of these are absorbed in satisfaction of six identical relations; four more relate to an irrelevant choice of axes, and six constants are left to express the properties peculiar to any, the most general, given optical system. To exhibit these constants free from irrelevant quantities and identical relations requires a somewhat elaborate theory; which, however, yields an immediate test of the equivalence of two given systems, and, among other things, proves that direct incidence on a thick lens whose curvatures are unequal in each face and in different planes in the two faces is equivalent to any, the most general, given system.

The method of the characteristic function has been made by Larmor to yield the same result—in fact, to furnish an implicit, though unelaborated, solution of the whole problem (*Proceedings Lond. Math. Soc.*, Vol. XX., p. 192, and Vol. XXIII., p. 172). But a question of so much generality and importance will justify a second, radically different, discussion; and I trust that the new results of the following pages, which include an account of the geometrical properties and a sketch classification of the most general systems, will show that Gauss's method, applied freely and according to the spirit of it, is a capable rival of even so powerful, elegant, and well tried a method as the characteristic function.

I.

Let a narrow beam of light be composed of discrete rays, not related to one another in any manner, except that the squares of their mutual inclinations are negligible. Let this beam be refracted at any surfaces of discontinuity, arbitrary in number, in situation, and in curvatures. Let an arbitrary ray be called the axis of the beam, and at each surface let the origin of coordinates be placed where this ray impinges on the surface, with the axis of x along the normal and the axis of y along the tangent line which is perpendicular to the axis of the beam. Then the equation of the surface may be written



$$2x = A_0 y^2 + 2H_0 yz + B_0 z^2,$$

and those of the ray,

before refraction :

$$\frac{x}{\cos\theta} = \frac{y}{0} = \frac{z}{\sin\theta},$$

after refraction :

$$\frac{x}{\cos\theta'} = \frac{y}{0} = \frac{z}{\sin\theta'},$$

where $$\mu\sin\theta - \mu'\sin\theta' = 0.$$

If we write the equations of any other ray of the beam,

before refraction : $$\frac{x}{\cos(\theta+\delta)} = \frac{y-b_0}{\epsilon} = \frac{z-c_0}{\sin(\theta+\delta)},$$

after refraction : $$\frac{x}{\cos(\theta'+\delta')} = \frac{y-b_0}{\epsilon'} = \frac{z-c_0}{\sin(\theta'+\delta')},$$

where the quantities b_0, c_0, δ, ϵ, δ', ϵ' are all small, we may derive the relations between these quantities from the formula

$$\mu l - \mu' l' = (\mu\cos\Theta - \mu'\cos\Theta')\,p,$$

where l, l', p are direction cosines with respect to any axis of original and refracted ray and normal to the surface respectively, and Θ, Θ' are the inclinations of the rays to the normal. We get thus

$$\mu\epsilon - \mu'\epsilon' = (A_0 b_0 + H_0 c_0)(\mu\cos\theta - \mu'\cos\theta'),$$

$$\mu\cos\theta\,\delta - \mu'\cos\theta'\,\delta' = (H_0 b_0 + B_0 c_0)(\mu\cos\theta - \mu'\cos\theta').$$

Now refer the equations of the rays to new systems of coordinates in which their own axes are the axes of x. Rotate the axes of x, z

about that of y through an angle $-\theta$; the equation of the original ray becomes

$$\frac{x-c_0\sin\theta}{\cos\delta}=\frac{y-b_0}{\epsilon}=\frac{z-c_0\cos\theta}{\sin\delta},$$

or, ignoring squares of small quantities,

$$x=\frac{y-b_0}{\epsilon}=\frac{z-c_0\cos\theta}{\delta},$$

which we shall write in the form

$$y=\frac{\beta}{\mu}x+b,\quad z=\frac{\gamma}{\mu}x+c,$$

where $\qquad b=b_0,\quad \beta=\mu\epsilon,\quad c=c_0\cos\theta,\quad \gamma=\mu\delta.$

To find corresponding equations for the emergent ray we must shift the origin to that arbitrary point where the axis of the beam next impinges upon a refracting surface and rotate the axes of y, z until the axis of y is a tangent line to that surface. The equations of the ray then become

$$y'=\frac{\beta'}{\mu'}x'+b',\quad z'=\frac{\gamma'}{\mu'}x'+c',$$

where

$$\begin{aligned}\beta' &= \mu'\epsilon'\cos\phi'+\mu'\delta'\sin\phi',\\ \gamma' &= -\mu'\epsilon'\sin\phi'+\mu'\delta'\cos\phi',\\ b' &= b_0\cos\phi'+c_0\cos\theta'\sin\phi'+a'\beta',\\ c' &= -b_0\sin\phi'+c_0\cos\theta'\cos\phi'+a'\gamma',\end{aligned}$$

where $\mu'a'$ is the shift of the origin along the ray, and ϕ' is the angle of rotation. Eliminate ϵ, ϵ', δ, δ', b_0, c_0, and we get

$$b=(b'-a'\beta')\cos\phi'-(c'-a'\gamma')\sin\phi',$$

$$\beta=\beta'\cos\phi'-\gamma'\sin\phi'+Ab+Hc\frac{\cos\theta'}{\cos\theta},$$

$$c=\left[(b'-a'\beta')\sin\phi'+(c'-a'\gamma')\cos\phi'\right]\frac{\cos\theta}{\cos\theta'},$$

$$\gamma=\left[\beta'\sin\phi'+\gamma'\cos\phi'+Hb+Bc\frac{\cos\theta'}{\cos\theta}\right]\frac{\cos\theta'}{\cos\theta}$$

where we have written

$$\frac{A_0}{A}=\frac{H_0}{H\cos\theta'}=\frac{B_0}{B\cos^2\theta'}=\frac{1}{\mu\cos\theta-\mu'\cos\theta'}.$$

If we write
$$\begin{aligned} b &= g_1 b' + h_1 \beta' + p_1 c' + q_1 \gamma', \\ \beta &= k_1 b' + l_1 \beta' + r_1 c' + s_1 \gamma', \\ c &= p_2 b' + q_2 \beta' + g_2 c' + h_2 \gamma', \\ \gamma &= r_2 b' + s_2 \beta' + k_2 c' + l_2 \gamma', \end{aligned}$$

we get the following scheme of values, in which ϖ has been written for $\cos\theta/\cos\theta'$:—

$g_1 = \cos\phi'$	$h_1 = -a'\cos\phi'$	$p_1 = -\sin\phi'$	$q_1 = a'\sin\phi'$
$k_1 = A\cos\phi' + H\sin\phi'$	$l_1 = -Aa'\cos\phi' - Ha'\sin\phi' + \cos\phi'$	$r_1 = -A\sin\phi' + H\cos\phi'$	$s_1 = Aa'\sin\phi' - Ha'\cos\phi' - \sin\phi'$
$p_2 = \varpi\sin\phi'$	$q_2 = -\varpi a'\sin\phi'$	$g_2 = \varpi\cos\phi'$	$h_2 = -\varpi a'\cos\phi'$
$r_2 = \varpi^{-1}[H\cos\phi' + B\sin\phi']$	$s_2 = \varpi^{-1}[-Ha'\cos\phi' - Ba'\sin\phi' + \sin\phi']$	$k_2 = \varpi^{-1}[-H\sin\phi' + B\cos\phi']$	$l_2 = \varpi^{-1}[Ha'\sin\phi' - Ba'\cos\phi' + \cos\phi']$

Regarding this scheme of coefficients it may be verified by inspection that they obey the following twelve relations, namely :—

$$\begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix} + \begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix} = 1, \tag{1}$$

$$\begin{vmatrix} g_1 & p_1 \\ k_1 & r_1 \end{vmatrix} + \begin{vmatrix} p_2 & g_2 \\ r_2 & k_2 \end{vmatrix} = 0, \tag{2}$$

$$\begin{vmatrix} g_1 & q_1 \\ k_1 & s_1 \end{vmatrix} + \begin{vmatrix} p_2 & h_2 \\ r_2 & l_2 \end{vmatrix} = 0, \tag{3}$$

$$\begin{vmatrix} h_1 & p_1 \\ l_1 & r_1 \end{vmatrix} + \begin{vmatrix} q_2 & g_2 \\ s_2 & k_2 \end{vmatrix} = 0, \tag{4}$$

$$\begin{vmatrix} h_1 & q_1 \\ l_1 & s_1 \end{vmatrix} + \begin{vmatrix} q_2 & h_2 \\ s_2 & l_2 \end{vmatrix} = 0, \tag{5}$$

$$\begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix} + \begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix} = 1, \tag{6}$$

and

$$\begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix} = 1, \qquad (1')$$

$$\begin{vmatrix} g_1 & h_1 \\ p_2 & q_2 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ g_2 & h_2 \end{vmatrix} = 0, \qquad (2')$$

$$\begin{vmatrix} g_1 & h_1 \\ r_2 & s_2 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ k_2 & l_2 \end{vmatrix} = 0, \qquad (3')$$

$$\begin{vmatrix} k_1 & l_1 \\ p_2 & q_2 \end{vmatrix} + \begin{vmatrix} r_1 & s_1 \\ g_2 & h_2 \end{vmatrix} = 0, \qquad (4')$$

$$\begin{vmatrix} k_1 & l_1 \\ r_2 & s_2 \end{vmatrix} + \begin{vmatrix} r_1 & s_1 \\ k_2 & l_2 \end{vmatrix} = 0, \qquad (5')$$

$$\begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix} + \begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix} = 1. \qquad (6')$$

The structure of these equations is readily seen; each consists of two minors of the determinant formed with the sixteen coefficients as elements. The first six are composed of one minor made out of the first and second rows, together with a minor formed similarly from the third and fourth rows, and the second six are formed in the same way from the first and second, third and fourth columns.

Now these same relations hold when any number of successive refractions separate in the most general manner the rays (b, β, c, γ), $(b', \beta', c', \gamma')$. For, if we have the three schemes of relation

$$\begin{aligned} b &= g_1 b' + h_1 \beta' + p_1 c' + q_1 \gamma', & b' &= \gamma_1 b'' + \eta_1 \beta'' + \pi_1 c'' + \chi_1 \gamma'', \\ \beta &= k_1 b' + l_1 \beta' + r_1 c' + s_1 \gamma', & \beta' &= \kappa_1 b'' + \lambda_1 \beta'' + \rho_1 c'' + \sigma_1 \gamma'', \\ c &= p_2 b' + q_2 \beta' + g_2 c' + h_2 \gamma', & c' &= \pi_2 b'' + \chi_2 \beta'' + \gamma_2 c'' + \eta_2 \gamma'', \\ \gamma &= r_2 b' + s_2 \beta' + k_2 c' + l_2 \gamma', & \gamma &= \rho_2 b'' + \sigma_2 \beta'' + \kappa_2 c'' + \lambda_2 \gamma'', \end{aligned}$$

$$\begin{aligned} b &= G_1 b'' + H_1 \beta'' + P_1 c'' + Q_1 \gamma'', \\ \beta &= K_1 b'' + L_1 \beta'' + R_1 c'' + S_1 \gamma'', \\ c &= P_2 b'' + Q_2 \beta'' + G_2 c'' + H_2 \gamma'', \\ \gamma &= R_2 b'' + S_2 \beta'' + K_2 c'' + L_2 \gamma'', \end{aligned}$$

and if the first two obey the twelve relations just written, then the third also obeys the same relations. For we have

$$\begin{aligned}
G_1 &= g_1\gamma_1+h_1\kappa_1+p_1\pi_2+q_1\rho_2,\\
H_1 &= g_1\eta_1+h_1\lambda_1+p_1\chi_2+q_1\sigma_2,\\
P_1 &= g_1\pi_1+h_1\rho_1+p_1\gamma_2+q_1\kappa_2,\\
Q_1 &= g_1\chi_1+h_1\sigma_1+p_1\eta_2+q_1\lambda_2,\\
K_1 &= k_1\gamma_1+l_1\kappa_1+r_1\pi_2+s_1\rho_2,\\
&\&c., \qquad \&c.
\end{aligned}$$

Now form, for example, the minors which make up (3′):—

$$\begin{vmatrix} G_1 & H_1 \\ R_2 & S_2 \end{vmatrix} = \begin{vmatrix} g_1\gamma_1+h_1\kappa_1+p_1\pi_2+q_1\rho_2, & g_1\eta_1+h_1\lambda_1+p_1\chi_2+q_1\sigma_2 \\ r_2\gamma_1+s_2\kappa_1+k_2\pi_2+l_2\rho_2, & r_2\eta_1+s_2\lambda_1+k_2\chi_2+l_2\sigma_2 \end{vmatrix}$$

$$= \begin{vmatrix} g_1 & h_1 \\ r_2 & s_2 \end{vmatrix}\begin{vmatrix} \gamma_1 & \kappa_1 \\ \eta_1 & \lambda_1 \end{vmatrix} + \begin{vmatrix} g_1 & p_1 \\ r_2 & k_2 \end{vmatrix}\begin{vmatrix} \gamma_1 & \pi_2 \\ \eta_1 & \chi_2 \end{vmatrix} + \begin{vmatrix} g_1 & q_1 \\ r_2 & l_2 \end{vmatrix}\begin{vmatrix} \gamma_1 & \rho_2 \\ \eta_1 & \sigma_2 \end{vmatrix}$$

$$+ \begin{vmatrix} h_1 & p_1 \\ s_2 & k_2 \end{vmatrix}\begin{vmatrix} \kappa_1 & \pi_2 \\ \lambda_1 & \chi_2 \end{vmatrix} + \begin{vmatrix} h_1 & q_1 \\ s_2 & l_2 \end{vmatrix}\begin{vmatrix} \kappa_1 & \rho_2 \\ \lambda_1 & \sigma_2 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ k_2 & l_2 \end{vmatrix}\begin{vmatrix} \pi_2 & \rho_2 \\ \chi_2 & \sigma_2 \end{vmatrix}.$$

It will be seen that the minors of the (gh) system are all formed out of the first and fourth rows, while those of the $(\gamma\eta)$ system are formed out of the first and second columns; in the same way, we have

$$\begin{vmatrix} P_1 & Q_1 \\ K_2 & L_2 \end{vmatrix} = \begin{vmatrix} g_1 & h_1 \\ r_2 & s_2 \end{vmatrix}\begin{vmatrix} \pi_1 & \rho_1 \\ \chi_1 & \sigma_1 \end{vmatrix} + \begin{vmatrix} g_1 & p_1 \\ r_2 & k_2 \end{vmatrix}\begin{vmatrix} \pi_1 & \gamma_2 \\ \chi_1 & \eta_2 \end{vmatrix} + \begin{vmatrix} g_1 & q_1 \\ r_2 & l_2 \end{vmatrix}\begin{vmatrix} \pi_1 & \kappa_2 \\ \chi_1 & \lambda_2 \end{vmatrix}$$

$$+ \begin{vmatrix} h_1 & p_1 \\ s_2 & k_2 \end{vmatrix}\begin{vmatrix} \rho_1 & \gamma_2 \\ \sigma_1 & \eta_2 \end{vmatrix} + \begin{vmatrix} h_1 & q_1 \\ s_2 & l_2 \end{vmatrix}\begin{vmatrix} \rho_1 & \kappa_2 \\ \sigma_1 & \lambda_2 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ k_2 & l_2 \end{vmatrix}\begin{vmatrix} \gamma_2 & \kappa_2 \\ \eta_2 & \lambda_2 \end{vmatrix}.$$

Thus, in virtue of the relations

$$\begin{vmatrix} \gamma_1 & \kappa_1 \\ \eta_1 & \lambda_1 \end{vmatrix} + \begin{vmatrix} \pi_1 & \rho_1 \\ \chi_1 & \sigma_1 \end{vmatrix} = 1, \tag{1′}$$

$$\begin{vmatrix} \gamma_1 & \pi_2 \\ \eta_1 & \chi_2 \end{vmatrix} + \begin{vmatrix} \pi_1 & \gamma_2 \\ \chi_1 & \eta_2 \end{vmatrix} = 0, \tag{2′}$$

$$\begin{vmatrix} \gamma_1 & \rho_2 \\ \eta_1 & \sigma_2 \end{vmatrix} + \begin{vmatrix} \pi_1 & \kappa_2 \\ \chi_1 & \lambda_2 \end{vmatrix} = 0, \tag{3′}$$

$$\begin{vmatrix} \kappa_1 & \pi_2 \\ \lambda_1 & \chi_2 \end{vmatrix} + \begin{vmatrix} \rho_1 & \gamma_2 \\ \sigma_1 & \eta_2 \end{vmatrix} = 0, \tag{4'}$$

$$\begin{vmatrix} \kappa_1 & \rho_2 \\ \lambda_1 & \sigma_2 \end{vmatrix} + \begin{vmatrix} \rho_1 & \kappa_2 \\ \sigma_1 & \lambda_2 \end{vmatrix} = 0, \tag{5'}$$

$$\begin{vmatrix} \pi_2 & \rho_2 \\ \chi_2 & \sigma_2 \end{vmatrix} + \begin{vmatrix} \gamma_2 & \kappa_2 \\ \eta_2 & \lambda_2 \end{vmatrix} = 1, \tag{6'}$$

we have

$$\begin{vmatrix} G_1 & H_1 \\ R_2 & S_2 \end{vmatrix} + \begin{vmatrix} P_1 & Q_1 \\ K_2 & L_2 \end{vmatrix} = \begin{vmatrix} g_1 & h_1 \\ r_2 & s_2 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ k_2 & l_2 \end{vmatrix} = 0. \tag{3'}$$

In the same way, we get

$$\begin{vmatrix} P_1 & Q_1 \\ R_1 & S_1 \end{vmatrix} + \begin{vmatrix} G_2 & H_2 \\ K_2 & L_2 \end{vmatrix} = \begin{vmatrix} \pi_1 & \chi_1 \\ \rho_1 & \sigma_1 \end{vmatrix} + \begin{vmatrix} \gamma_2 & \eta_2 \\ \kappa_2 & \lambda_2 \end{vmatrix} = 1, \tag{6}$$

and the remainder follow on exactly the same lines.

Let us now remove two unnecessary restrictions. We have supposed the origin to lie in the first surface of discontinuity, and the axis of y to be a tangent line. But introduce at an arbitrary point of the beam before the first surface is reached a hypothetical surface with zero change of refractive index, and the origin will be transferred to an arbitrary point, and the axis of y to an arbitrary direction. And the same applies to the emergent beam.

Secondly, we have supposed density to vary discontinuously; but, since there is no restriction upon the number of surfaces of discontinuity, the results apply equally to continuous variation.

Thus we have proved the following theorem:—

If the equations of an original ray and of the same ray emergent after passing through any singly refracting continuous or discontinuous media whatever in any manner be respectively

$$\left.\begin{aligned} y &= \frac{\beta}{\mu}x+b \\ z &= \frac{\gamma}{\mu}x+c \end{aligned}\right\} \quad \textit{and} \quad \left.\begin{aligned} y' &= \frac{\beta'}{\mu'}x'+b', \\ z' &= \frac{\gamma'}{\mu'}x'+c', \end{aligned}\right\}$$

where b, β, c, γ, b', β', c', γ' *are all small, so that the relations between them are linear, viz.,*

$$\begin{aligned} b &= G_1 b' + H_1 \beta' + P_1 c' + Q_1 \gamma', \\ \beta &= K_1 b' + L_1 \beta' + R_1 c' + S_1 \gamma', \\ c &= P_2 b' + Q_2 \beta' + G_2 c' + H_2 \gamma', \\ \gamma &= R_2 b' + S_2 \beta' + K_2 c' + L_2 \gamma', \end{aligned}$$

then the sixteen coefficients obey the relations

$$G_1 L_1 - H_1 K_1 + P_2 S_2 - Q_2 R_2 = 1, \quad (1)$$
$$G_1 R_1 - P_1 K_1 + P_2 K_2 - G_2 R_2 = 0, \quad (2)$$
$$G_1 S_1 - Q_1 K_1 + P_2 L_2 - H_2 R_2 = 0, \quad (3)$$
$$H_1 R_1 - P_1 L_1 + Q_2 K_2 - G_2 S_2 = 0, \quad (4)$$
$$H_1 S_1 - Q_1 L_1 + Q_2 L_2 - H_2 S_2 = 0, \quad (5)$$
$$P_1 S_1 - Q_1 R_1 + G_2 L_2 - H_2 K_2 = 1, \quad (6)$$

$$G_1 L_1 - H_1 K_1 + P_1 S_1 - Q_1 R_1 = 1, \quad (1')$$
$$G_1 Q_2 - H_1 P_2 + P_1 H_2 - Q_1 G_2 = 0, \quad (2')$$
$$G_1 S_2 - H_1 R_2 + P_1 L_2 - Q_1 K_2 = 0, \quad (3')$$
$$K_1 Q_2 - L_1 P_2 + R_1 H_2 - S_1 G_2 = 0, \quad (4')$$
$$K_1 S_2 - L_1 R_2 + R_1 L_2 - S_1 K_2 = 0, \quad (5')$$
$$P_2 S_2 - Q_2 R_2 + G_2 L_2 - H_2 K_2 = 1. \quad (6')$$

Let us now consider further the passage from one scheme of coefficients to another, as written at length on p. 37, or, as we shall write it hereafter,

$$(b\beta c\gamma) = \left\{ \begin{matrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{matrix} \right\} (b'\beta'c'\gamma'),$$

$$(b'\beta'c'\gamma') = \left\{ \begin{matrix} \gamma_1 & \eta_1 & \pi_1 & \chi_1 \\ \kappa_1 & \lambda_1 & \rho_1 & \sigma_1 \\ \pi_2 & \chi_2 & \gamma_2 & \eta_2 \\ \rho_2 & \sigma_2 & \kappa_2 & \lambda_2 \end{matrix} \right\} (b''\beta''c''\gamma''),$$

$$(b\beta c\gamma)=\begin{Bmatrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{Bmatrix}(b''\beta''c''\gamma''),$$

where it is postulated that each group of coefficients satisfies the relations (1) ... (6), (1′) ... (6′).

First let us prove the following theorem:—

If we have

$$(b\beta c\gamma)=\begin{Bmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{Bmatrix}(b'\beta'c'\gamma'),$$

then inversely

$$(b'\beta'c'\gamma')=\begin{Bmatrix} l_1 & -h_1 & s_2 & -q_2 \\ -k_1 & g_1 & -r_2 & p_2 \\ s_1 & -q_1 & l_2 & -h_2 \\ -r_1 & p_1 & -k_2 & g_2 \end{Bmatrix}(b\beta c\gamma).$$

For $$l_1b-h_1\beta=\begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix}b'+0\,.\,\beta'+\begin{vmatrix} p_1 & h_1 \\ r_1 & l_1 \end{vmatrix}c'+\begin{vmatrix} q_1 & h_1 \\ s_1 & l_1 \end{vmatrix}\gamma',$$

$$s_2c-q_2\gamma=\begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix}b'+0\,.\,\beta'+\begin{vmatrix} g_2 & q_2 \\ k_2 & s_2 \end{vmatrix}c'+\begin{vmatrix} h_2 & q_2 \\ l_2 & s_2 \end{vmatrix}\gamma'.$$

Add these; then in virtue of (1), (4), (5), we get the first of the above relations, and the others follow similarly.

Now it is physically and also algebraically evident that the ray is reversible. Hence the scheme $\{l_1, -h_1, \text{\&c.}\}$ must satisfy the twelve fundamental relations. Make the necessary substitutions, l_1 for g_1, $-h_1$ for h_1, s_2 for p_1, $-q_2$ for q_1, &c., and we find that (1) becomes (1′), (2) becomes (5′), (3) becomes (4′), (4) becomes (3′), (5) becomes (2′), (6) becomes (6′).

Hence, from the reversibility of the ray, *if the relations* (1)...(6) *are satisfied, it follows that the relations* (1′)...(6′) *are also satisfied, and conversely.*

This may be proved without introducing the principle of reversibility. For we have

$$\begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix} \begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix} - \begin{vmatrix} g_1 & p_1 \\ k_1 & r_1 \end{vmatrix} \begin{vmatrix} g_2 & h_2 \\ s_2 & l_2 \end{vmatrix} + \begin{vmatrix} g_1 & q_1 \\ k_1 & s_1 \end{vmatrix} \begin{vmatrix} q_2 & g_2 \\ s_2 & k_2 \end{vmatrix}$$

$$+ \begin{vmatrix} h_1 & p_1 \\ l_1 & r_1 \end{vmatrix} \begin{vmatrix} p_2 & h_2 \\ r_2 & l_2 \end{vmatrix} - \begin{vmatrix} h_1 & q_1 \\ l_1 & s_1 \end{vmatrix} \begin{vmatrix} p_2 & g_2 \\ r_2 & k_2 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix} \begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix}$$

$$\equiv \begin{vmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{vmatrix} \equiv \Delta, \text{ say,}$$

and

$$\begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix} \begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix} - \begin{vmatrix} p_2 & g_2 \\ r_2 & k_2 \end{vmatrix} \begin{vmatrix} q_2 & h_2 \\ s_2 & l_2 \end{vmatrix} + \begin{vmatrix} p_2 & h_2 \\ r_2 & l_2 \end{vmatrix} \begin{vmatrix} q_2 & g_2 \\ s_2 & k_2 \end{vmatrix}$$

$$+ \begin{vmatrix} q_2 & g_2 \\ s_2 & k_2 \end{vmatrix} \begin{vmatrix} p_2 & h_2 \\ r_2 & l_2 \end{vmatrix} - \begin{vmatrix} q_2 & h_2 \\ s_2 & l_2 \end{vmatrix} \begin{vmatrix} p_2 & g_2 \\ r_2 & k_2 \end{vmatrix} + \begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix} \begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix}$$

$$\equiv \begin{vmatrix} p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{vmatrix} \equiv 0.$$

Add these:

$$(1) \begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix} - (2) \begin{vmatrix} q_2 & h_2 \\ s_2 & l_2 \end{vmatrix} + (3) \begin{vmatrix} q_2 & g_2 \\ s_2 & k_2 \end{vmatrix} + (4) \begin{vmatrix} p_2 & h_2 \\ r_2 & l_2 \end{vmatrix} - (5) \begin{vmatrix} p_2 & g_2 \\ r_2 & k_2 \end{vmatrix}$$

$$+ (6) \begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix} \equiv \Delta,$$

where (1), (2), ... denote the left-hand members of the equations so named.

In the same way,

$$(1) \begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix} - (2) \begin{vmatrix} h_1 & q_1 \\ l_1 & s_1 \end{vmatrix} + (3) \begin{vmatrix} h_1 & p_1 \\ l_1 & r_1 \end{vmatrix} + (4) \begin{vmatrix} g_1 & q_1 \\ k_1 & s_1 \end{vmatrix} - (5) \begin{vmatrix} g_1 & p_1 \\ k_1 & r_1 \end{vmatrix}$$

$$+ (6) \begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix} \equiv \Delta.$$

Add these: we get

$$\Delta \equiv (1)(6)-(2)(5)+(3)(4)$$
$$= 1,$$

in virtue of the values of the expressions (1)...(6); hence, substituting the values of Δ, (1) ... (6) above, we get

$$\begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix} + \begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix} = 1, \qquad (6')$$

$$\begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix} + \begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix} = 1. \qquad (1')$$

Again, in like manner, we have

$$0 \equiv \begin{vmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ g_1 & h_1 & p_1 & q_1 \\ p_2 & q_2 & g_2 & h_2 \end{vmatrix} + \begin{vmatrix} p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \\ g_1 & h_1 & p_1 & q_1 \\ p_2 & q_2 & g_2 & h_2 \end{vmatrix}$$

$$\equiv (1)\begin{vmatrix} p_1 & q_1 \\ g_2 & h_2 \end{vmatrix} - (2)\begin{vmatrix} h_1 & q_1 \\ q_2 & h_2 \end{vmatrix} + (3)\begin{vmatrix} h_1 & p_1 \\ q_2 & g_2 \end{vmatrix} + (4)\begin{vmatrix} g_1 & q_1 \\ p_2 & h_2 \end{vmatrix}$$
$$- (5)\begin{vmatrix} g_1 & p_1 \\ p_2 & g_2 \end{vmatrix} + (6)\begin{vmatrix} g_1 & h_1 \\ p_2 & q_2 \end{vmatrix},$$

$$0 \equiv (1)\begin{vmatrix} p_1 & q_1 \\ k_2 & l_2 \end{vmatrix} - (2)\begin{vmatrix} h_1 & q_1 \\ s_2 & l_2 \end{vmatrix} + (3)\begin{vmatrix} h_1 & p_1 \\ s_2 & k_2 \end{vmatrix} + (4)\begin{vmatrix} g_1 & q_1 \\ r_2 & l_2 \end{vmatrix}$$
$$- (5)\begin{vmatrix} g_1 & p_1 \\ r_2 & k_2 \end{vmatrix} + (6)\begin{vmatrix} g_1 & h_1 \\ r_2 & s_2 \end{vmatrix},$$

$$0 \equiv (1)\begin{vmatrix} r_1 & s_1 \\ g_2 & h_2 \end{vmatrix} - (2)\begin{vmatrix} h_1 & s_1 \\ q_2 & h_2 \end{vmatrix} + (3)\begin{vmatrix} l_1 & r_1 \\ q_2 & g_2 \end{vmatrix} + (4)\begin{vmatrix} k_1 & s_1 \\ p_2 & h_2 \end{vmatrix}$$
$$- (5)\begin{vmatrix} k_1 & r_1 \\ p_2 & g_2 \end{vmatrix} + (6)\begin{vmatrix} k_1 & l_1 \\ p_2 & q_2 \end{vmatrix},$$

$$0 \equiv (1)\begin{vmatrix} r_1 & s_1 \\ k_2 & l_2 \end{vmatrix} - (2)\begin{vmatrix} l_1 & s_1 \\ s_2 & l_2 \end{vmatrix} + (3)\begin{vmatrix} l_1 & r_1 \\ s_2 & k_2 \end{vmatrix} + (4)\begin{vmatrix} k_1 & s_1 \\ r_2 & l_2 \end{vmatrix}$$
$$- (5)\begin{vmatrix} k_1 & r_1 \\ r_2 & k_2 \end{vmatrix} + (6)\begin{vmatrix} k_1 & l_1 \\ r_2 & s_2 \end{vmatrix},$$

and these, in virtue of the values of the expressions (1) ... (6), give respectively

$$\begin{vmatrix} p_1 & q_1 \\ g_2 & h_2 \end{vmatrix} + \begin{vmatrix} g_1 & h_1 \\ p_2 & q_2 \end{vmatrix} = 0, \tag{2'}$$

$$\begin{vmatrix} p_1 & q_1 \\ k_2 & l_2 \end{vmatrix} + \begin{vmatrix} g_1 & h_1 \\ r_2 & s_2 \end{vmatrix} = 0, \tag{3'}$$

$$\begin{vmatrix} r_1 & s_1 \\ g_2 & h_2 \end{vmatrix} + \begin{vmatrix} k_1 & l_1 \\ p_2 & q_2 \end{vmatrix} = 0, \tag{4'}$$

$$\begin{vmatrix} r_1 & s_1 \\ k_2 & l_2 \end{vmatrix} + \begin{vmatrix} k_1 & l_1 \\ r_2 & s_2 \end{vmatrix} = 0. \tag{5'}$$

Hence the twelve fundamental relations contain only six independent statements.

Let us write the relation between the three schemes of p. 37, as follows:—

$$\left\{\begin{matrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{matrix}\right\} = \left\{\begin{matrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{matrix}\right\} \left\{\begin{matrix} \gamma_1 & \eta_1 & \pi_1 & \chi_1 \\ \kappa_1 & \lambda_1 & \rho_1 & \sigma_1 \\ \pi_2 & \chi_2 & \gamma_2 & \eta_2 \\ \rho_2 & \sigma_2 & \kappa_2 & \lambda_2 \end{matrix}\right\}$$

where in fact the value of any element on the left is found by treating the schemes on the right as if they were determinants, but multiplying the rows of the first into the columns of the second; *e.g.*,

$$Q_2 = p_2\eta_1 + q_2\lambda_1 + g_2\chi_2 + k_2\sigma_2,$$

that is, we take the third row of $\{g_1 h_1\}$ into the second column of $\{\gamma_1\, \eta_1 \ldots\}$, since Q_2 stands in the third row and second column on the left. It is clear that this multiplication is not in general commutative —a limitation that has a direct and obvious physical meaning. Inversely we get

$$\left\{\begin{matrix} L_1 & -H_1 & S_2 & -Q_2 \\ -K_1 & G_1 & -R_2 & P_2 \\ S_1 & -Q_1 & L_2 & -H_2 \\ -R_1 & P_1 & -K_2 & G_2 \end{matrix}\right\} = \left\{\begin{matrix} \lambda_1 & -\eta_1 & \sigma_2 & -\chi_2 \\ -\kappa_1 & \gamma_1 & -\rho_2 & \pi_2 \\ \sigma_1 & -\chi_1 & \lambda_2 & -\eta_2 \\ -\rho_1 & \pi_1 & -\kappa_2 & \gamma_2 \end{matrix}\right\} \left\{\begin{matrix} l_1 & -h_1 & s_2 & -q_2 \\ -k_1 & g_1 & -r_2 & p_2 \\ s_1 & -q_1 & l_2 & -h_2 \\ -r_1 & p_1 & -k_2 & g_2 \end{matrix}\right\},$$

and also

$$\left\{\begin{matrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{matrix}\right\} = \left\{\begin{matrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{matrix}\right\} \left\{\begin{matrix} \lambda_1 & -\eta_1 & \sigma_2 & -\chi_2 \\ -\kappa_1 & \gamma_1 & -\rho_2 & \pi_2 \\ \sigma_1 & -\chi_1 & \lambda_2 & -\eta_2 \\ -\rho_1 & \pi_1 & -\kappa_2 & \gamma_2 \end{matrix}\right\},$$

$$\left\{\begin{matrix} \gamma_1 & \eta_1 & \pi_1 & \chi_1 \\ \kappa_1 & \lambda_1 & \rho_1 & \sigma_1 \\ \pi_2 & \chi_2 & \gamma_2 & \eta_2 \\ \rho_2 & \sigma_2 & \kappa_2 & \lambda_2 \end{matrix}\right\} = \left\{\begin{matrix} l_1 & -h_1 & s_2 & -q_2 \\ -k_1 & g_1 & -r_2 & p_2 \\ s_1 & -q_1 & l_2 & -h_2 \\ -r_1 & p_1 & -k_2 & g_2 \end{matrix}\right\} \left\{\begin{matrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{matrix}\right\}.$$

It is to be observed that a scheme $\{g_1\, h_1 \dots\}$ obeying the fundamental relations may represent a mere shift of origin; thus, if

$$(b\beta c\gamma) = \left\{\begin{matrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 1 \end{matrix}\right\}(b'\beta'c'\gamma'),$$

we have

$$b = b' + \lambda\beta', \quad c = c' + \lambda\gamma',$$

$$\beta = \beta', \qquad \gamma = \gamma',$$

or the origin of the (b, c) system has been shifted a distance $+\mu\lambda$ along the axis of x from the origin of the (b', c') system.

Or, again, a mere rotation; thus, if

$$(b\beta c\gamma) = \left\{\begin{matrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{matrix}\right\}(b'\beta'c'\gamma'),$$

we have

$$b = b'\cos\phi + c'\sin\varphi, \quad c = -b'\sin\phi + c'\cos\phi,$$

$$\beta = \beta'\cos\phi + c'\sin\phi, \quad \gamma = -\beta'\sin\phi + \gamma'\cos\phi,$$

or the axes of (b, c) have been rotated through an angle $+\phi$ from the positions for (b', c').

It is possible that, if the fundamental relations are taken as definition of a scheme of sixteen coefficients, g_1, h_1, &c., then the scheme so defined includes other systems than can be produced by pure optical means; but I cannot adduce an example.

The foregoing results permit us to construct with mechanical facility the scheme of coefficients that belong to any given optical system.*

Example I.—Multiply together the schemes

$$\begin{Bmatrix} 1 & 0 & 0 & 0 \\ A & 1 & H & 0 \\ 0 & 0 & \varpi & 0 \\ H/\varpi & 0 & B/\varpi & 1/\varpi \end{Bmatrix} \begin{Bmatrix} 1 & -a' & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -a' \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

$$\begin{Bmatrix} \cos\phi' & 0 & -\sin\phi' & 0 \\ 0 & \cos\phi' & 0 & -\sin\phi' \\ \sin\phi' & 0 & \cos\phi' & 0 \\ 0 & \sin\phi' & 0 & \cos\phi' \end{Bmatrix},$$

which correspond respectively to refraction at a surface of discontinuity, shift of the origin of the emergent ray, a distance $+\mu' a'$ along the axis of the beam, and rotation of the axes for the same through an angle $+\phi'$. We obtain the original scheme of p. 36.

Example II. Thick Lens.—Let a beam impinge directly upon a lens of finite thickness whose faces are of arbitrary curvatures. Let the thickness be $-\mu c$, the equation of the first surface

$$2x = a_1 y^2 + b_1 z^2,$$

and that of the second with parallel axes

$$2x = A_2 y^2 + 2H_2 yz + B_2 z^2,$$

or with axes rotated through an angle $+\theta$ from these,

$$2x = a_2 y^2 + b_2 z^2.$$

* It is worth remembering that the same method applies to surfaces symmetrically disposed along an axis when each scheme contains only two rows.

First multiply together the schemes

$$\begin{Bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b_1 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & 0 & 0 & 0 \\ A_2 & 1 & H_2 & 0 \\ 0 & 0 & 1 & 0 \\ H_2 & 0 & B_2 & 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 & c & 0 & 0 \\ a_1 & 1+a_1c & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & b_1 & 1+b_1c \end{Bmatrix} \begin{Bmatrix} 1 & 0 & 0 & 0 \\ A_2 & 1 & H_2 & 0 \\ 0 & 0 & 1 & 0 \\ H_2 & 0 & B_2 & 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1+A_2c & c & cH_2 & 0 \\ a_1+A_2+a_1A_2c & 1+a_1c & H_2(1+a_1c) & 0 \\ cH_2 & 0 & 1+B_2c & c \\ H_2(1+b_1c) & 0 & b_1+B_2+b_1B_2c & 1+b_1c \end{Bmatrix},$$

or we may form this scheme otherwise by rotating the axes between the two refractions until the equation of the second surface assumes its simplest form. Thus

$$\begin{Bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b_1 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{Bmatrix} \begin{Bmatrix} 1 & 0 & 0 & 0 \\ a_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b_2 & 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} \cos\theta & c\cos\theta & -\sin\theta & -c\sin\theta \\ a_1\cos\theta & (1+a_1c)\cos\theta & -a_1\sin\theta & -(1+a_1c)\sin\theta \\ \sin\theta & c\sin\theta & \cos\theta & c\cos\theta \\ b_1\sin\theta & (1+b_1c)\sin\theta & b_1\cos\theta & (1+b_1c)\cos\theta \end{Bmatrix} \begin{Bmatrix} 1 & 0 & 0 & 0 \\ a_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b_2 & 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} (1+a_2c)\cos\theta & c\cos\theta & -(1+b_2c)\sin\theta & -c\sin\theta \\ (a_1+a_2+a_1a_2c)\cos\theta & (1+a_1c)\cos\theta & -(a_1+b_2+a_1b_2c)\sin\theta & -(1+a_1c)\sin\theta \\ (1+a_2c)\sin\theta & c\sin\theta & (1+b_2c)\cos\theta & c\cos\theta \\ (b_1+a_2+b_1a_2c)\sin\theta & (1+b_1c)\sin\theta & (b_1+b_2+b_1b_2c)\cos\theta & (1+b_1c)\cos\theta \end{Bmatrix}.$$

This scheme becomes identical with the one already obtained if we further multiply by

$$\left\{\begin{array}{cccc} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{array}\right\},$$

that is, if we rotate the axes back to their original positions.

Example III. Curved Prism.—Let the two faces of a prism of any angle be of arbitrary curvatures, the edge being formed by a common tangent line. Let a beam impinge at the edge, perpendicular to this line. Let this line be taken as axis of y and the equations of the surfaces be

$$2x = ay^2 + 2hyz + bz^2, \quad 2x = Ay^2 + 2Hyz + Bz^2,$$

respectively. The resulting coefficients are given by

$$\left\{\begin{array}{cccc} 1 & 0 & 0 & 0 \\ a & 1 & h & 0 \\ 0 & 0 & \varpi & 0 \\ \frac{h}{\varpi} & 0 & \frac{b}{\varpi} & \frac{1}{\varpi} \end{array}\right\} \left\{\begin{array}{cccc} 1 & 0 & 0 & 0 \\ A & 1 & H & 0 \\ 0 & 0 & \Pi & 0 \\ \frac{H}{\Pi} & 0 & \frac{B}{\Pi} & \frac{1}{\Pi} \end{array}\right\}$$

$$= \left\{\begin{array}{cccc} 1 & 0 & 0 & 0 \\ a+A & 1 & H+h\Pi & 0 \\ 0 & 0 & \varpi\Pi & 0 \\ \frac{H+h\Pi}{\varpi\Pi} & 0 & \frac{B+b\Pi^2}{\varpi\Pi} & \frac{1}{\varpi\Pi} \end{array}\right\};$$

that is to say, the prism is equivalent to a single surface.

II.

1. It is evident that, if there are two systems of rays in correspondence, so that the relations between the parameters that define individual pairs are expressed by linear equations, then a scheme of sixteen coefficients suffices to define the correspondence, and that irrespective of the means, whether optical or other, by which one beam is related to the other.

To introduce these sixteen coefficients the axes of x, x' are identified with one corresponding pair of rays while the situations of the origins and the directions of the axes of y, y' are undefined. But we must consider that two systems are equivalent which differ merely in respect to these undefined quantities, because the beams in two such systems might be identified ray with ray by mere screw shifts along the axes of x, x', and such shifts are not significant in relation to the intrinsic representation of geometrical relationship afforded by the sixteen coefficients.

Thus, out of sixteen quantities contained in the coefficients, four (called λ, ρ, ϕ, ψ below) must be allotted to describe irrelevant features, and this suggests a conception of invariant forms which are independent of irrelevant quantities, and are related only to the geometrical modification produced in the beam. When we fully understand how far a scheme of coefficients may be modified by changes of axes, we shall find it easy to detect expressions which are invariant in form and value in spite of such changes. These are the quantities we require; it will be evident that their number is limited only by their degree, that their degree is always even, and that in the most general case not more than twelve can be independent.

Associated with the invariants we shall observe forms which vary with one only of the quantities λ, ρ, ϕ, or ψ, which permit us to read off immediately the change of axes which will reduce a scheme of coefficients from one given shape to any other, its equivalent. These may be called seminvariants.

The following discussion, up to the introduction of the optical equations, applies to the modification of a narrow beam of rays by any means, *e.g.*, by repeated reciprocation at quadric surfaces, though its plan and scope have been determined by the optical problem.

2. Let λ, ρ define shifts of the origins, so that

$$\begin{Bmatrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{Bmatrix} \begin{Bmatrix} 1 & \rho & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \rho \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} g_1+\lambda k_1 & h_1+\rho g_1+\lambda l_1+\lambda\rho k_1 & p_1+\lambda r_1 & q_1+\rho p_1+\lambda s_1+\lambda\rho r_1 \\ k_1 & l_1+\rho k_1 & r_1 & s_1+\rho r_1 \\ p_2+\lambda r_2 & q_2+\rho p_2+\lambda s_2+\lambda\rho r_2 & g_2+\lambda k_2 & h_2+\rho g_2+\lambda l_2+\lambda\rho k_2 \\ r_2 & s_2+\rho r_2 & k_2 & l_2+\rho k_2 \end{Bmatrix}.$$

The forms which are independent of λ and ρ and not higher than the second degree are

$$k_1,\quad r_1,\quad r_2,\quad k_2,$$

$$\begin{vmatrix} g_1 & p_1 \\ k_1 & r_1 \end{vmatrix},\quad \begin{vmatrix} g_1 & p_2 \\ k_1 & r_2 \end{vmatrix},\quad \begin{vmatrix} g_1 & g_2 \\ k_1 & k_2 \end{vmatrix},\quad \begin{vmatrix} p_1 & p_2 \\ r_1 & r_2 \end{vmatrix},\quad \begin{vmatrix} p_1 & g_2 \\ r_1 & k_2 \end{vmatrix},\quad \begin{vmatrix} p_2 & g_2 \\ r_2 & k_2 \end{vmatrix},$$

$$\begin{vmatrix} k_1 & r_2 \\ l_1 & s_2 \end{vmatrix},\quad \begin{vmatrix} k_1 & r_1 \\ l_2 & s_1 \end{vmatrix},\quad \begin{vmatrix} k_1 & k_2 \\ l_1 & l_2 \end{vmatrix},\quad \begin{vmatrix} r_1 & r_2 \\ s_1 & s_2 \end{vmatrix},\quad \begin{vmatrix} r_1 & k_2 \\ s_1 & l_2 \end{vmatrix},\quad \begin{vmatrix} r_2 & k_2 \\ s_2 & l_2 \end{vmatrix},$$

$$\begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix},\quad \begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix},\quad \begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix},\quad \begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix}.$$

These are each equal to the similarly formed quantity with letters G_1, H_1, &c.

Next consider a rotation of the axis of y through an angle ϕ,

$$\begin{Bmatrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{Bmatrix} \begin{Bmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} g_1\cos\phi+p_2\sin\phi & h_1\cos\phi+q_2\sin\phi & p_1\cos\phi+g_2\sin\phi & q_1\cos\phi+h_2\sin\phi \\ k_1\cos\phi+r_2\sin\phi & l_1\cos\phi+s_2\sin\phi & r_1\cos\phi+k_2\sin\phi & s_1\cos\phi+l_2\sin\phi \\ -g_1\sin\phi+p_2\cos\phi & -h_1\sin\phi+q_2\cos\phi & -p_1\sin\phi+g_2\cos\phi & -q_1\sin\phi+h_2\cos\phi \\ -k_1\sin\phi+r_2\cos\phi & -l_1\sin\phi+s_2\cos\phi & -r_1\sin\phi+k_2\cos\phi & -s_1\sin\phi+l_2\cos\phi \end{Bmatrix}$$

Confine attention to the terms in k_1, r_1, r_2, k_2. We have

$$K_1^2+R_2^2 = k_1^2+r_2^2,$$

$$R_1^2+K_2^2 = r_1^2+k_2^2,$$

$$K_1K_2-R_1R_2 = k_1k_2-r_1r_2,$$

$$K_1R_1+K_2R_2 = k_1r_1+k_2r_2,$$

$$K_1^2-K_2^2+R_1^2-R_2^2 = (k_1^2-k_2^2+r_1^2-r_2^2)\cos 2\phi+2(k_1r_2+k_2r_1)\sin 2\phi,$$

$$2(K_1R_2+K_2R_1) = -(k_1^2-k_2^2+r_1^2-r_2^2)\sin 2\phi+2(k_1r_2+k_2r_1)\cos 2\phi.$$

Now consider a rotation (ψ) of the axis of y',

$$\left\{\begin{matrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{matrix}\right\}$$

$$=\left\{\begin{matrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{matrix}\right\}\left\{\begin{matrix} \cos\psi & 0 & -\sin\psi & 0 \\ 0 & \cos\psi & 0 & -\sin\psi \\ \sin\psi & 0 & \cos\psi & 0 \\ 0 & \sin\psi & 0 & \cos\psi \end{matrix}\right\}$$

$$=\left\{\begin{matrix} g_1\cos\psi+p_1\sin\psi & h_1\cos\psi+q_1\sin\psi & -g_1\sin\psi+p_1\cos\psi & -h_1\sin\psi+q_1\cos\psi \\ k_1\cos\psi+r_1\sin\psi & l_1\cos\psi+s_1\sin\psi & -k_1\sin\psi+r_1\cos\psi & -l_1\sin\psi+s_1\cos\psi \\ p_2\cos\psi+g_2\sin\psi & q_2\cos\psi+h_2\sin\psi & -p_2\sin\psi+g_2\cos\psi & -q_2\sin\psi+h_2\cos\psi \\ r_2\cos\psi+k_2\sin\psi & s_2\cos\psi+l_2\sin\psi & -r_2\sin\psi+k_2\cos\psi & -s_2\sin\psi+l_2\cos\psi \end{matrix}\right\}$$

When we observe

$$K_1^2+R_1^2 = k_1^2+r_1^2,$$

$$K_2^2+R_2^2 = k_2^2+r_2^2,$$

$$K_1K_2-R_1R_2 = k_1k_2-r_1r_2,$$

$$K_1R_2+K_2R_1 = k_1r_2+k_2r_1,$$

$$K_1^2-K_2^2-R_1^2+R_2^2 = (k_1^2-k^2-r_1^2+r_2^2)\cos 2\psi+2(k_1r_1+k_2r_2)\sin 2\psi,$$

$$2(K_1R_1+K_2R_2) = -(k_1^2-k_2^2-r_1^2+r_2^2)\sin 2\psi+2(k_1r_1+k_2r_2)\cos 2\psi.$$

3. Compare these with the former, and we find the following equations true in spite of all changes in axes, namely:—

$$(a)\quad K_1^2+K_2^2+R_1^2+R_2^2 = k_1^2+k^2+r_1^2+r_2^2,$$

$$(b)\quad K_1K_2-R_1R_2 = k_1k_2-r_1r_2,$$

$$(\alpha)\quad K_1^2-K_2^2+R_1^2-R_2^2 = (k_1^2-k_2^2+r_1^2-r_2^2)\cos 2\phi+2(k_1r_2+k_2r_1)\sin 2\phi,$$

$$(\alpha')\quad 2(K_1R_2+K_2R_1) = -(k_1^2-k_2^2+r_1^2-r_2^2)\sin 2\phi+2(k_1r_2+k_2r_1)\cos 2\phi,$$

$$(\beta)\quad K^2-K\ -R_1^2+R_2^2 = (k_1^2-k_2^2-r_1^2+r_2^2)\cos 2\psi+2(k_1r_1+k_2r_2)\sin 2\psi,$$

$$(\beta')\quad 2(K_1R_1+K_2R_2) = -(k_1^2-k_2^2-r_1^2+r_2^2)\sin 2\psi+2(k_1r_1+k_2r_2)\cos 2\psi.$$

The forms (a), (b) are two of the invariants we are seeking; (α), (α') are seminvariants with respect to ϕ; (β), (β') are seminvariants with respect to ψ.

It would be tedious to follow the details in all subsequent cases; it may be verified without difficulty that we arrive at the following results, true in spite of all changes of axes:—

$$(c)\quad G_1R_1-K_1P_1-G_2R_2+K_2P_2 = g_1r_1-k_1p_1-g_2r_2+k_2p_2,$$

$$(d)\quad G_1R_2-K_1P_2-G_2R_1+K_2P_1 = g_1r_2-k_1p_2-g_2r_1+k_2p_1,$$

$$(\gamma)\quad G_1R_1-K_1P_1+G_2R_2-K_2P_2 = (g_1r_1-k_1p_1+g_2r_2-k_2p_2)\cos 2\phi + (g_1k_2-g_2k_1-p_1r_2+p_2r_1)\sin 2\phi,$$

$$(\gamma')\quad G_1K_2-G_2K_1-P_1R_2+P_2R_1 = -(g_1r_1-k_1p_1+g_2r_2-k_2p_2)\sin 2\phi + (g_1k_2-g_2k_1-p_1r_2+p_2r_1)\cos 2\phi,$$

$$(\delta)\quad G_1R_2-K_1P_2+G_2R_1-K_2P_1 = (g_1r_2-k_1p_2+g_2r_1-k_2p_1)\cos 2\psi + (g_1k_2-g_2k_1+p_1r_2-p_2r_1)\sin 2\psi,$$

$$(\delta')\quad G_1K_2-G_2K_1+P_1R_2-P_2R_1 = -(g_1r_2-k_1p_2+g_2r_1-k_2p_1)\sin 2\psi + (g_1k_2-g_2k_1+p_1r_2-p_2r_1)\cos 2\psi;$$

and also

$$(e)\quad K_1S_2-L_1R_2-K_2S_1+L_2R_1 = k_1s_2-l_1r_2-k_2s_1+l_2r_1,$$

$$(f)\quad K_1S_1-L_1R_1-K_2S_2+L_2R_2 = k_1s_1-l_1r_1-k_2s_2+l_2r_2,$$

$$(\epsilon)\quad K_1S_2-L_1R_2+K_2S_1-L_2R_1 = (k_1s_2-l_1r_2+k_2s_1-l_2r_1)\cos 2\psi + (k_1l_2-k_2l_1+r_1s_2-r_2s_1)\sin 2\psi,$$

$$(\epsilon')\quad K_1L_2-K_2L_1+R_1S_2-R_2S_1 = -(k_1s_2-l_1r_2+k_2s_1-l_2r_1)\sin 2\psi + (k_1l_2-k_2l_1+r_1s_2-r_2s_1)\cos 2\psi,

$$(\phi)\quad K_1S_1-L_1R_1+K_2S_2-L_2R_2=\ (k_1s_1-l_1r_1+k_2s_2-l_2r_2)\cos 2\phi$$
$$+(k_1l_2-k_2l_1-r_1s_2+r_2s_1)\sin 2\phi,$$

$$(\phi')\quad K_1L_2-K_2L_1-R_1S_2+R_2S_1=-(k_1s_1-l_1r_1+k_2s_2-l_2r_2)\sin 2\phi$$
$$+(k_1l_2-k_2l_1-r_1s_2+r_2s_1)\cos 2\phi\ ;$$

and, finally,

$$(g)\quad G_1L_1-H_1K_1+P_1S_1-Q_1R_1+P_2S_2-Q_2R_2+G_2L_2-H_2K_2$$
$$=g_1l_1-h_1k_1+p_1s_1-q_1r_1+p_2s_2-q_2r_2+g_2l_2-h_2k_2.$$

There are also seminvariants with respect to λ and ρ, namely,

$$(\alpha_1)\quad G_1K_1+G_2K_2+P_1R_1+P_2R_2$$
$$=g_1k_1+g_2k_2+p_1r_1+p_2r_2+\lambda\,(k_1^2+k_2^2+r_1^2+r_2^2),$$

$$(\alpha_2)\quad L_1K_1+L_2K_2+S_1R_1+S_2R_2$$
$$=l_1k_1+l_2k_2+s_1r_1+s_2r_2+\rho\,(k_1^2+k_2^2+r_1^2+r_2^2),$$

$$(\beta_1)\quad G_1K_2+G_2K_1-P_1R_2-P_2R_1$$
$$=g_1k_2+g_2k_1-p_1r_2-p_2r_1+2\lambda\,(k_1k_2-r_1r_2),$$

$$(\beta_2)\quad L_1K_2+L_2K_1-S_1R_2-S_2R_1$$
$$=l_1k_2+l_2k_1-s_1r_2-s_2r_1+2\rho\,(k_1k_2-r_1r_2),$$

$$(\phi_1)\quad G_1S_1-L_1P_1-G_2S_2+L_2P_2$$
$$=g_1s_1-l_1p_1-g_2s_2+l_2p_2+\lambda\,(k_1s_1-l_1r_1-k_2s_2+l_2r_2)$$
$$+\rho\,(g_1r_1-k_1p_1-g_2r_2+k_2p_2),$$

$$(\phi_2)\quad K_1Q_1-R_1H_1-K_2Q_2+R_2H_2$$
$$=k_1q_1-r_1h_1-k_2q_2+r_2h_2+\lambda\,(k_1s_1-l_1r_1-k_2s_2+l_2r_2)$$
$$-\rho\,(g_1r_1-k_1p_1-g_2r_2+k_2p_2),$$

$$(\delta_1)\quad G_1S_2-L_1P_2-G_2S_1+L_2P_1$$
$$=g_1s_2-l_1p_2-g_2s_1+l_2p_1+\lambda\,(k_1s_2-l_1r_2-k_2s_1+l_2r_1)$$
$$+\rho\,(g_1r_2-k_1p_2-g_2r_1+k_2p_1),$$

$$(\delta_2)\quad K_1Q_2-H_1R_2-K_2Q_1+H_2R_1$$
$$=k_1q_2-h_1r_2-k_2q_1+h_2r_1+\lambda\,(k_1s_2-l_1r_2-k_2s_1+l_2r_1)$$
$$-\rho\,(g_1r_2-k_1p_2-g_2r_1+k_2p_1),$$

$$(\omega_1)\quad G_1Q_1-H_1P_1-G_2Q_2+H_2P_2$$
$$=g_1q_1-h_1p_1-g_2q_2+h_2p_2$$
$$+\lambda\,(g_1s_1-l_1p_1-g_2s_2+l_2p_2+k_1q_1-h_1r_1-k_2q_2+h_2r_2)$$
$$+\lambda^2(k_1s_1-l_1r_1-k_2s_2+l_2r_2),$$

$$(\omega_2)\quad H_1S_1-L_1Q_1-H_2S_2+L_2Q_2$$
$$=h_1s_1-l_1q_1-h_2s_2+l_2q_2$$
$$+\rho\;(g_1s_1-l_1p_1-g_2s_2+l_2p_2-k_1q_1+h_1r_1+k_2q_2-h_2r_2)$$
$$+\rho^2(g_1r_1-k_1p_1-g_2r_2+k_2p_2),$$

$$(\zeta_1)\quad G_1Q_2-H_1P_2-G_2Q_1+H_2P_1$$
$$=g_1q_2-h_1p_2-g_2q_1+h_2p_1$$
$$+\lambda\,(g_1s_2-l_1p_2-g_2s_1+l_2p_1+k_1q_2-h_1r_2-k_2q_1+h_2r_1)$$
$$+\lambda^2(k_1s_2-l_1r_2-k_2s_1+l_2r_1),$$

$$(\zeta_2)\quad H_1S_2-L_1Q_2-H_2S_1+L_2Q_1$$
$$=h_1s_2-l_1q_2-h_2s_1+l_2q_1$$
$$+\rho\;(g_1s_2-l_1p_2-g_2s_1+l_2p_1-k_1q_2+h_1r_2+k_2q_1-h_2r_1)$$
$$+\rho^2(g_1r_2-k_1p_2-g_2r_1+k_2p_1).$$

4. If two schemes are equivalent, all the above relations hold; inversely, let us investigate what conditions must hold in order that λ, ρ, ϕ, ψ determined from any of the above should be consistent.

First, consider the equations in φ, ψ. If there are four simultaneous equations

$$\begin{aligned}
a_1 &= a_2\cos\theta+b_2\sin\theta,\\
b_1 &= -a_2\sin\theta+b_2\cos\theta,\\
c_1 &= c_2\cos\theta+d_2\sin\theta,\\
d_1 &= -c_2\sin\theta+d_2\cos\theta,
\end{aligned}$$

then we must have

$$a_1^2+b_1^2=a_2^2+b_2^2,\qquad c_1^2+d_1^2=c_2^2+d_2^2,$$
$$a_1c_1+b_1d_1=a_2c_2+b_2d_2,\quad a_1d_1-b_1c_1=a_2d_2-b_2c_2,$$

between which an identity subsists

$$(a_1^2+b_1^2)(c_1^2+d_1^2) = (a_1c_1+b_1d_1)^2+(a_1d_1-b_1c_1)^2.$$

We must find the corresponding forms for the equations in ϕ, ψ.

We shall make repeated use of the identity

$$\begin{aligned}(a_1b_1c_3d_4) \equiv\ & [(a_1b_2)+(a_3b_4)][(c_1d_2)+(c_3d_4)] \\ & -[(a_1c_2)+(a_3c_4)][(b_1d_2)+(b_3d_4)] \\ & +[(a_1d_2)+(a_3d_4)][(b_1c_2)+(b_3c_4)] \\ \equiv - & [(a_1b_2)-(a_3b_4)][(c_1d_2)-(c_3d_4)] \\ & +[(a_1c_2)-(a_3c_4)][(b_1d_2)-(b_3d_4)] \\ & -[(a_1d_2)-(a_3d_4)][(b_1c_2)-(b_3c_4)],\end{aligned}$$

where (a_1b_2) stands for $a_1b_2-a_2b_1$, and $(a_1b_2c_3d_4)$ is any determinant of four rows and columns. This has already been proved (p. 43 *ante*) under the form

$$\Delta \equiv (1)(6)-(2)(5)+(3)(4).$$

Now for the consistency of the equations (α), (α'), we must have the following form invariant, viz.,

$$\begin{aligned}(A) &\equiv (K_1^2-K_2^2+R_1^2-R_2^2)^2+4(K_1R_2+K_2R_1)^2 \\ &\equiv (a)^2-4(b)^2 \\ &\equiv \begin{vmatrix} K_1 & K_2 & R_1 & R_2 \\ R_1 & -R_2 & -K_1 & K_2 \\ R_2 & -R_1 & K_2 & -K_1 \\ K_2 & K_1 & -R_2 & -R_1 \end{vmatrix}.\end{aligned}$$

Again, (γ), (γ') give

$$\begin{aligned}(C) &\equiv (G_1R_1-K_1P_1+G_2R_2-K_2P_2)^2+(G_1K_2-G_2K_1-P_1R_2+P_2R_1)^2 \\ &\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ -P_2 & P_1 & -G_2 & G_1 \\ K_1 & K_2 & R_1 & R_2 \\ R_2 & -R_1 & K_2 & -K_1 \end{vmatrix},\end{aligned}$$

and, taking (a), (a'), (γ), (γ') together,

$$\begin{aligned}(ac) \equiv\ & (K_1^2-K_2^2+R_1^2-R_2^2)(G_1R_1-K_1P_1+G_2R_2-K_2P_2)\\ & +2\,(K_1R_2+K_2R_1)(G_1K_2-G_2K_1-P_1R_2+P_2R_1)\end{aligned}$$

$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ K_1 & K_2 & R_1 & R_2 \\ K_2 & K_1 & -R_2 & -R_1 \\ R_2 & -R_1 & K_2 & -K_1 \end{vmatrix} \equiv (a)(c)+2\,(b)(d),$$

$$\begin{aligned}(ac') \equiv\ & (K_1^2-K_2^2+R_1^2-R_2^2)(G_1K_2-G_2K_1-P_1R_2+P_2R_1)\\ & -2\,(K_1R_2+K_2R_1)(G_1R_1-K_1P_1+G_2R_2-K_2P_2)\end{aligned}$$

$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_1 & -R_2 & -K_1 & K_2 \\ R_2 & -R_1 & K_2 & -K_1 \end{vmatrix},$$

and among these four forms we have the relation

$$(A)(C) \equiv (ac)^2+(ac')^2,$$

so that these four forms actually introduce only one new one.

Next, take (a), (a'), (ϕ), (ϕ'). We get

$$(F) \equiv (K_1S_1-L_1R_1+K_2S_2-L_2R_2)^2+(K_1L_2-K_2L_1-R_1S_2+R_2S_1)^2$$

$$\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2 \\ -S_2 & S_1 & -L_2 & L_1 \\ K_1 & K_2 & R_1 & R_2 \\ R_2 & -R_1 & K_2 & -K_1 \end{vmatrix}.$$

Also

$$\begin{aligned}(af) \equiv\ & (K_1^2-K_2^2+R_1^2-R_2^2)(K_1S_1-L_1R_1+K_2S_2-L_2R_2)\\ & +2\,(K_1R_2+K_2R_1)(K_1L_2-K_2L_1-R_1S_2+R_2S_1)\end{aligned}$$

$$\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2 \\ K_1 & K_2 & R_1 & R_2 \\ K_2 & K_1 & -R_2 & -R_1 \\ -R_2 & R_1 & -K_2 & K_1 \end{vmatrix} \equiv (a)(f)+2\,(b)(e),$$

and

$$(af') \equiv (K_1^2-K_2^2+R_1^2-R_2^2)(K_1L_2-K_2L_1-R_1S_2+R_2S_1)$$
$$-2\,(K_1R_2+K_2R_1)(K_1S_1-L_1R_1+K_2S_2-L_2R_2)$$
$$\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_1 & -R_2 & -K_1 & K_2 \\ -R_2 & R_1 & -K_2 & K_1 \end{vmatrix},$$

and we have the relation

$$(A)(F) \equiv (af)^2+(af')^2;$$

these three, then, introduce one new form.

Lastly, take (γ), (γ'), (ϕ), (ϕ'), which, of course, can introduce no new form.

$$(cf) \equiv (G_1R_1-K_1P_1+G_2R_2-K_2P_2)(K_1S_1-L_1R_1+K_2S_2-L_2R_2)$$
$$-(G_1K_2-G_2K_1-P_1R_2+P_2R_1)(K_1L_2-K_2L_1-R_1S_2+R_2S_1)$$
$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ -S_2 & S_1 & -L_2 & L_1 \\ K_1 & K_2 & R_1 & R_2 \\ -R_2 & R_1 & -K_2 & K_1 \end{vmatrix},$$

$$(cf') \equiv (G_1R_1-K_1P_1+G_2R_2-K_2P_2)(K_1L_2-K_2L_1-R_1S_2+R_2S_1)$$
$$-(G_1K_2-G_2K_1-P_1R_2+P_2R_1)(K_1S_1-L_1R_1+K_2S_2-L_2R_2)$$
$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ L_1 & L_2 & S_1 & S_2 \\ K_1 & K_2 & R_1 & R_2 \\ -R_2 & R_1 & -K_2 & K_1 \end{vmatrix},$$

and we have the relations

$$(A)(cf) \equiv (ac)(af)+(ac')(af'),$$
$$(A)(cf') \equiv (ac)(af')-(ac')(af),$$

leading to $$(C)(F) \equiv (cf)^2+(cf')^2.$$

5. Next, take the equations in ψ beginning with (β), (β'), (δ), (δ'); we get

$$(B) \equiv (K_1^2-K_2^2-R_1^2+R_2^2)^2+4\,(K_1R_1+K_2R_2)^2 \equiv (A),$$

$$(D) \equiv (G_1R_2-K_1P_2+G_2R_1-K_2P_1)^2+(G_1K_2-G_2K_1+P_1R_2-P_2R_1)^2.$$

$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ P_1 & -P_2 & -G_1 & G_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_1 & -R_2 & -K_1 & K_2 \end{vmatrix},$$

$$(ad) \equiv (K_1^2-K_2^2-R_1^2+R_2^2)(G_1R_2-K_1P_2+G_2R_1-K_2P_1)$$
$$+2\,(K_1R_1+K_2R_2)(G_1K_2-G_2K_1+P_1R_2-P_2R_1)$$

$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ K_1 & K_2 & R_1 & R_2 \\ K_2 & K_1 & -R_2 & -R_1 \\ -R_1 & R_2 & K_1 & -K_2 \end{vmatrix} \equiv (a)(d)+2\,(b)(c),$$

$$(ad') \equiv (K_1^2-K_2^2-R_1^2+R_2^2)(G_1K_2-G_2K_1+P_1R_2-P_2R_1)$$
$$-2\,(K_1R_1+K_2R_2)(G_1R_2-K_1P_2+G_2R_1-K_2P_1)$$

$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_2 & -R_1 & K_2 & -K_1 \\ -R_1 & R_2 & K_1 & -K_2 \end{vmatrix} \equiv (ac'),$$

and among these $\quad (A)(D) \equiv (ad)^2+(ad')^2.$

And we also have the identity, easily proved,

$$(C)-(c)^2 \equiv (D)-(d)^2.$$

These, then, introduce no new form.

In the same way

$$(E) \equiv (K_1S_2-L_1R_2+K_2S_1-L_2R_1)^2+(K_1L_2-K_2L_1+R_1S_2-R_2S_1)^2$$

$$\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2 \\ S_1 & -S_2 & -L_1 & L_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_1 & -R_2 & -K_1 & K_2 \end{vmatrix},$$

$$(ae) \equiv (K_1^2 - K_2^2 - R_1^2 + R_2^2)(K_1S_2 - L_1R_2 + K_2S_1 - L_2R_1)$$
$$+2\,(K_1R_1 + K_2R_2)(K_1L_2 - K_2L_1 + R_1S_2 - R_2S_1)$$
$$\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2 \\ K_1 & K_2 & R_1 & R_2 \\ K_2 & K_1 & -R_2 & -R_1 \\ R_1 & -R_2 & -K_1 & K_2 \end{vmatrix} \equiv (a)(e) + 2\,(b)(f),$$

$$(ae') \equiv (K_1^2 - K_2^2 - R_1^2 + R_2^2)(K_1L_2 - K_2L_1 + R_1S_2 - R_2S_1)$$
$$-2\,(K_1R_1 + K_2R_2)(K_1S_2 - L_1R_2 + K_2S_1 - L_2R_1)$$
$$\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_2 & -R_1 & K_2 & -K_1 \\ R_1 & -R_2 & -K_1 & K_2 \end{vmatrix} \equiv (af'),$$

and we have $$(A)(E) \equiv (ae)^2 + (ae')^2,$$

and also $$(E) - (e)^2 \equiv (F) - (f)^2.$$

No new form is introduced.

Again take (δ), (δ'); (ϵ), (ϵ'); we have

$$(de) \equiv (G_1R_2 - K_1P_2 + G_2R_1 - K_2P_1)(K_1S_2 - L_1R_2 + K_2S_1 - L_2R_1)$$
$$+ (G_1K_2 - G_2K_1 + P_1R_2 - P_2R_1)(K_1L_2 - K_2L_1 + R_1S_2 - R_2S_1)$$
$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ S_1 & -S_2 & -L_1 & L_2 \\ K_1 & K_2 & R_1 & R_2 \\ -R_1 & R_2 & K_1 & -K_2 \end{vmatrix},$$

$$(de') \equiv (G_1R_2 - K_1P_2 + G_2R_1 - K_2P_1)(K_1L_2 - K_2L_1 + R_1S_2 - R_2S_1)$$
$$-(G_1K_2 - G_2K_1 + P_1R_2 - P_2R_1)(K_1S_2 - L_1R_2 + K_2S_1 - L_2R_1)$$
$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ L_1 & L_2 & S_1 & S_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_1 & -R_2 & -K_1 & K_2 \end{vmatrix}.$$

These are related to the earlier forms by the equations

$$(A)(de) \equiv (ad)(ae)+(ad')(ae'),$$
$$(A)(de') \equiv (ad)(ae')-(ad')(ae),$$
$$(D)(E) \equiv (de)^2+(de')^2,$$

and also

$$(cf)-(c)(f) \equiv (de)-(d)(e),$$
$$(a)(cf')-2\,(b)(de')+(f)(ac')+(c)(af') \equiv 0.$$

6. Thus the consistency of all these equations requires the existence of only two new invariant forms. Further the same forms imply the consistency of several of the equations in λ, ρ. Thus take the equations (α_1), (β_1); (α_2), (β_2),

$$\begin{aligned}(ab_1) &\equiv (G_1K_1+G_2K_2+P_1R_1+P_2R_2)\,2\,(K_1K_2-R_1R_2)\\ &\qquad +(G_1K_2+G_2K_1-P_1R_2-P_2R_1)(K_1^2+K^2+R_1^2+R_2^2)\\ &\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2\\ K_1 & K_2 & R_1 & R_2\\ R_1 & -R_2 & -K_1 & K_2\\ R_2 & -R_1 & K_2 & -K_1\end{vmatrix} \equiv (ac') \equiv (ad'),\end{aligned}$$

$$\begin{aligned}(ab_2) &\equiv (L_1K_1+L_2K_2+S_1R_1+S_2R_2)\,2(K_1K_2-R_1R_2)\\ &\qquad +(L_1K_2+L_2K_1-S_1R_2-S_2R_1)(K_1^2+K_2^2+R_1^2+R_2^2)\\ &\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2\\ K_1 & K_2 & R_1 & R_2\\ R_1 & -R_2 & -K_1 & K_2\\ R_2 & -R_1 & K_2 & -K_1\end{vmatrix} \equiv (af') \equiv (ae').\end{aligned}$$

Again eliminate λ, ρ from (α_1), (α_2), (ϕ_1); we get

$$\begin{aligned}&-(G_1S_1-L_1P_1-G_2S_2+L_2P_2)(K^2+K_2^2+R_1^2+R_2^2)\\ &\quad +(G_1K_1+G_2K_2+P_1R_1+P_2R_2)(K_1S_1-L_1R_1-K_2S_2+L_1R_2)\\ &\quad +(L_1K_1+L_2K_2+S_1R_1+S_2R_2)(G_1R_1-K_1P_1-G_2R_2+K_2P_2) \equiv (de').\end{aligned}$$

In the same way, from (β_1), (β_2), (ϕ_1),

$$\begin{aligned}&-(G_1S_1-L_1P_1-G_2S_2+L_2P_2)\,2\,(K_1K_2-R_1R_2)\\ &\quad +(K_1S_1-L_1R_1-K_2S_2+L_2R_2)(G_1K_2+G_2K_1-P_1R_2-P_2R_1)\\ &\quad +(G_1R_1-G_1K_1-G_2R_2+P_2K_2)(K_1L_2+K_2L_1-S_1R_2-S_2R_1) \equiv (cf'),\end{aligned}$$

and again (α_1), (α_2), (δ_1), lead to (cf') and (β_1), (β_2), (δ_1) lead to (de'), with the equations :

$$\begin{aligned}-(G_1S_2-L_1P_2-G_2S_1+L_2P_1)(K_1^2+K_2^2+R_1^2+R_2^2)\\ +(K_1S_2-L_1R_2-K_2S_1+L_2R_1)(G_1K_1+G_2K_2+P_1R_1+P_2R_2)\\ +(G_1R_2-P_1K_2-G_2R_1+P_2K_1)(L_1K_1+L_2K_2+S_1R_1+S_2R_2)\equiv(cf'),\end{aligned}$$

and

$$\begin{aligned}-(G_1S_2-L_1P_2-G_2S_1+L_2P_1)\,2\,(K_1K_2-R_1R_2)\\ +(K_1S_2-L_1R_2-K_2S_1+L_2R_1)(G_1K_2+G_2K_1-P_1R_2-P_2R_1)\\ +(G_1R_2-P_1K_2-G_2R_1+P_2K_1)(L_1K_2+L_2K_1-S_1R_2-S_2R_1)\equiv(de').\end{aligned}$$

Now consider the forms (ω_1), (ω_2) in conjunction with (ϕ_1), (ϕ_2). Eliminate λ from ω_1. We get the invariant

$$\begin{aligned}(G_1Q_1-H_1P_1-G_2Q_2+H_2P_2)(K_1S_1-L_1R_1-K_2S_2+L_2R_2)\\ -\tfrac{1}{4}(G_1S_1-L_1P_1-G_2S_2+L_2P_2+K_1Q_1-H_1R_1-K_2Q_2+H_2R_2)^2.\end{aligned}$$

Eliminate ρ from ω_2. We get

$$\begin{aligned}(G_1R_1-K_1P_1-G_2R_2+K_2P_2)(H_1S_1-L_1Q_1-H_2S_2+L_2Q_2)\\ -\tfrac{1}{4}(G_1S_1-L_1P_1-G_2S_2+L_2P_2-K_1Q_1+H_1R_1+K_2Q_2-H_2R_2)^2.\end{aligned}$$

The difference of these forms gives

$$\begin{vmatrix} G_1 & H_1 & K_1 & L_1 \\ P_1 & Q_1 & R_1 & S_1 \\ P_2 & Q_2 & R_2 & S_2 \\ G_2 & H_2 & K_2 & L_2 \end{vmatrix}\equiv(K).$$

The form (K) is an addition to those already found. It may be found in the same way, using (ζ_1), (ζ_2), in place of (ω_1) (ω_2). We have not yet exhausted the forms that may be derived from these equations, but we shall stop here, because the discussion has already grown somewhat elaborate, and all that remain become ciphers in the optical problem.

7. Let us now introduce the six fundamental optical equations.

These render nugatory (c), (e), (g), (ω_2), (ζ_1), and make (ϕ_2), (δ_2) repetitions of (ϕ_1), (δ_1). Moreover, they reduce the number of independent quantities contained in the coefficients, other than λ, ρ, ϕ, ψ, to six, so that only six of the above-found invariant forms can be independent. We have already seen that two forms, in addition to (a), (b), (d), (f), will reconcile all the above equations, excepting (ω_1), (ζ_2), and these are reconciled by a seventh form (K). We must then express (K) in terms of the other six.

We have

$$(K)(A) \equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ H_1 & H_2 & Q_1 & Q_2 \\ K_1 & K_2 & R_1 & R_2 \\ L_1 & L_2 & S_1 & S_2 \end{vmatrix} \begin{vmatrix} R_1 & -R_2 & -K_1 & K_2 \\ R_2 & -R_1 & K_2 & -K_1 \\ K_1 & K_2 & R_1 & R_2 \\ K_2 & K_1 & -R_2 & -R_1 \end{vmatrix}$$

$$\equiv \begin{vmatrix} 0 & (d) & (G_1K_1) & (G_1K_2) \\ -(K_1Q_1) & -(K_1Q_2) & (H_1K_1) & (H_1K_2) \\ 0 & 0 & (a) & 2(b) \\ -(f) & 0 & (L_1K_1) & (L_1K_2) \end{vmatrix},$$

where we have written

$$(K_1Q_1) \quad \text{for} \quad K_1Q_1 - H_1R_1 - K_2Q_2 + H_2R_2,$$

$$(K_1Q_2) \quad \text{for} \quad K_1Q_2 - H_1R_2 - K_2Q_1 + H_2R_1,$$

&c.;

but
$$(K_1Q_1) = G_1S_1 - L_1P_1 - G_2S_2 + L_2P_2, \qquad (3),\ (4)$$

$$-(K_1Q_2) = G_1S_2 - L_1P_2 - G_2S_1 + L_2P_1. \qquad (3'),\ (4')$$

Hence
$$(A)(K) = (G_1S_1)(d)(ae') + (G_1S_2)(f)(ac') + (d)(f)\{2(b)(H_1K_1) - (a)(H_1K_2)\}.$$

But
$$(cf') = -2(b)(G_1S_1) + (f)(G_1K_2),$$

$$(de') = -2(b)(G_1S_2) + (d)(L_1K_2).$$

Therefore

$$2(b)(A)(K) + (d)(ae')(cf') + (f)(ac')(de')$$
$$= (d)(f)\{(ae')(G_1K_2) + (ac')(L_1K_2) + 4(b)^2(H_1K_1) - 2(a)(b)(H_1K_2)\}$$
$$= (d)(f)\{(G_1K_2)[(a)(L_1K_2) - 2(b)(L_1K_1)]$$
$$+ (L_1K_2)[(a)(G_1K_2) - 2(b)(G_1K_1)]$$
$$+ 4(b)^2(H_1K_1) - 2(a)(b)(H_1K_2)\}.$$

Consider the terms in $\{\ \}$.

The term

$$-2(a)(b)(H_1K_2+H_2K_1-Q_1R_2-Q_2R_1)$$
$$=-2(a)\{(K_2^2+R_1^2)(H_1K_1+Q_2R_2)+(K_1^2+R_2^2)(H_2K_2+Q_1R_1)\}$$
$$+2(a)(K_1R_1+K_2R_2)(G_1S_1+L_1P_1+G_2S_2+L_2P_2),$$

using (3) and (4).

The term

$$2(a)(G_1K_2+G_2K_1-P_1R_2-P_2R_1)(L_1K_2+L_2K_1-S_1R_2-S_2R_1)$$
$$=\ 2(a)\{(K_2^2+R_1^2)(G_1L_1+P_2S_2)+(K_1^2+R_2^2)(G_2L_2+P_1S_1)\}$$
$$-2(a)(K_1R_1+K_2R_2)(G_1S_1+L_1P_1+G_2S_2+L_2P_2)$$
$$+2(a)(b)(G_1L_2+G_2L_1-P_1S_2-P_2S_1).$$

Together these terms give

$$2(a)^2+2(a)(b)(G_1L_2+G_2L_1-P_1S_2-P_2S_1).$$

To the terms

$$-2(b)\big[(G_1K_2+G_2K_1-P_1R_2-P_2R_1)(L_1K_1+L_2K_2+S_1R_1+S_2R_2)$$
$$+(L_1K_2+L_2K_1-S_1R_2-S_2R_1)(G_1K_1+G_2K_2+P_1R_1+P_2R_2)\big]$$

add the term

$$-2(b)\big[(G_1R_1-K_1P_1-G_2R_2+K_2P_2)(K_1S_2-L_1R_2-K_2S_1+L_2R_1)$$
$$+(G_1R_2-P_1K_2-G_2R_1+P_2K_1)(K_1S_1-L_1R_1-K_2S_2+L_2R_2)\big]$$
$$=-2(b)(d)(f).$$

The sum is

$$-2(b)\{2(b)\left[G_1L_1+G_2L_2+P_1S_1+P_2S_2\right]$$
$$+(a)\left[G_1L_2+G_2L_1-P_1S_2-P_2S_1\right]\}.$$

Add in the terms already considered, and also the remaining term

$$2(b)\,.\,2(b)\left[H_1K_1+H_2K_2+Q_1R_1+Q_2R_2\right].$$

Altogether we get $\quad 2(a)^2-8(b)^2=2(A).$

Thus the required expression for (K) is

$$2(b)(A)(K)+(d)(ae')(cf')+(f)(ac')(de')$$
$$=2(d)(f)(A)+2(b)(d)^2(f)^2.$$

8. Let us adopt the following definition:—

Two optical systems are called equivalent when the original beams in each and also the corresponding emergent beams may be identified ray with ray by mere screw displacements.

Thus, if the systems are defined by means of schemes of coefficients $G_1H_1, \ldots, g_1'h_1', \ldots$, these schemes must differ only in respect to the quantities $\lambda, \rho, \phi, \psi$. We may then consider it almost self-evident that *two optical systems are equivalent when they give the same values to the invariants* (a), (b), (d), (f), (ac'), (af'), *or any other six implying these;* but, on account of the importance of the result, we shall add a direct proof of it.

Determine two angles ϕ, ψ by means of the equations

$$K_1^2-K_2^2+R_1^2-R_2^2 = (k_1'^2-k_2'^2+r_1'^2-r_2'^2)\cos 2\phi+2(k_1'r_2'+k_2'r_1')\sin 2\phi,$$

$$\text{or } 2(K_1R_2+K_2R_1) = -(k_1'^2-k_2'^2+r_1'^2-r_2'^2)\sin 2\phi+2(k_1'r_2'+k_2'r_1')\cos 2\phi,$$

and

$$K_1^2-K_2^2-R_1^2+R_2^2 = (k_1'^2-k_2'^2-r_1'^2+r_2'^2)\cos 2\psi+2(k_1'r_1'+k_2'r_2')\sin 2\psi,$$

$$\text{or } 2(K_1R_1+K_2R_2) = -(k_1'^2-k_2'^2-r_1'^2+r_2'^2)\sin 2\psi+2(k_1'r_1'+k_2'r_2')\cos 2\psi.$$

It is immaterial which of these we use, because, (a), (b), and therefore (A), having the same value for each, their consistency is implied. Then, by rotations of the axes (ϕ, ψ) in the scheme $G_1H_1\ldots$, transform it to a scheme $g_1h_1\ldots$. The equations that effect this transformation are (α), (α'), (β), (β'), &c., above. Compare these with the equations above in k_1', &c., and we see that

$$k_1 = k_1', \quad r = r_1', \quad r_2 = r_2', \quad k_2 = k_2',$$

with alternatives that are of no importance.

In exactly the same way,

$$g_1r_1-k_1p_1-g_2r_2+k_2p_2 = g_1'r_1'-k_1'p_1'-g_2'r_2'+k_2'p_2',$$

$$g_1r_2-k_1p_2-g_2r_1+k_2p_1 = g_1'r_2'-k_1'p_2'-g_2'r_1'+k_2'p_1',$$

and, since (ac') has the same value in the two systems,

$$g_1r_1-k_1p_1+g_2r_2-k_2p_2 = g_1'r_1'-k_1'p_1'+g_2'r_2'-k_2'p_2',$$

$$g_1k_2-g_2k_1-p_1r_2+p_2r_1 = g_1'k_2'-g_2'k_1'-p_1'r_2'+p_2'r_1';$$

whence
$$\frac{g_1-g_1'}{k_1} = \frac{p_1-p_1'}{r_1} = \frac{p_2-p_2'}{r_2} = \frac{g_2-g_2'}{k_2},$$

or $g_1' = g_1+\lambda k_1,\quad p_1' = p_1+\lambda r_1,\quad p_2' = p_2+\lambda r_2,\quad g_2' = g_2+\lambda k_2,$

and, in the same way,

$$l_1' = l_1+\rho k_1,\quad s_1' = s_1+\rho r_1,\quad s_2' = s_2+\rho r_2,\quad l_2' = l_2+\rho k_2.$$

Therefore shifts of the origins complete the identification, since the equality of $h_1q_1q_2h_2$ with $h_1'q_1'q_2'h_2'$ is implied by the fundamental equations (1), (6), (1′), (6′).

9. *Any optical transformation is equivalent to direct passage through a thick lens whose curvatures are unequal in each face and in different planes in the two faces.* This is equivalent to a theorem given by Larmor (*Proceedings*, Vol. xx.). We shall show by actual determination of them that the six features that determine such a lens appear in such a manner as to give independent values to six of the invariants. We shall use the invariants (b), (d), (f), (D), (F), (K). In the scheme of Part I., p. 47, write

$$1+a_1c = cX,\quad 1+b_1c = cY,\quad 1+a_2c = cx,\quad 1+b_2c = cy,$$

and we have

$$\left\{\begin{matrix} Xc\cos\theta & c\cos\theta & Yc\sin\theta & c\sin\theta \\ \left(cXx-\dfrac{1}{c}\right)\cos\theta & cx\cos\theta & \left(cYx-\dfrac{1}{c}\right)\sin\theta & cx\sin\theta \\ -Xc\sin\theta & -c\sin\theta & Yc\cos\theta & c\cos\theta \\ -\left(cXy-\dfrac{1}{c}\right)\sin\theta & -cy\sin\theta & \left(cYy-\dfrac{1}{c}\right)\cos\theta & cy\cos\theta \end{matrix}\right\};$$

whence

$$(b) = c^2XYxy-\tfrac{1}{2}(X+Y)(x+y)+\frac{1}{c^2}-\tfrac{1}{2}\cos 2\theta\,(X-Y)(x-y),$$

$$(d) = \tfrac{1}{2}c^2\sin 2\theta\,(X-Y)(x-y)(X+Y),$$

$$(f) = \tfrac{1}{2}c^2\sin 2\theta\,(X-Y)(x-y)(x+y),$$

$$(D) = (X-Y)^2+2c^2\cos 2\theta\,(X-Y)(x-y)\,XY+c^4\,(x-y)^2\,X^2Y^2,$$

$$(F) = (x-y)^2+2c^2\cos 2\theta\,(X-Y)(x-y)\,xy+c^4\,(X-Y)^2\,x^2y^2,$$

$$(K) = -\tfrac{1}{4}c^4\sin^2 2\theta\,(X-Y)^2\,(x-y)^2.$$

These equations admit of algebraic solution, for we have

$$X+Y = (d)/\sqrt{-(K)},$$

$$x+y = (f)/\sqrt{-(K)},$$

$$(b)-\tfrac{1}{2}\frac{(d)(f)}{(K)} = c^2XYxy+\frac{1}{c^2}-\frac{1}{c^2}\cot 2\theta\sqrt{-(K)},$$

$$(D) = -\frac{(d)^2}{(K)}-4c^2XY\left\{\frac{1}{c^2}-\frac{1}{c^2}\cot 2\theta\sqrt{-(K)}\right\}+c^4(x-y)^2X^2Y^2.$$

Eliminate $\cot 2\theta$ from between these;

$$(f)^2c^4X^2Y^2+2c^2XY\{2(b)(K)-(d)(f)\}+(d)^2+(K)(D) = 0.$$

This gives c^2XY; in the same way, c^2xy is found from

$$(d)^2c^4x^2y^2+2c^2xy\{2(b)(K)-(d)(f)\}+(f)^2+(K)(F) = 0,$$

and, eliminating $\cos 2\theta$ between (D) and (F), we get, to determine c,

$$\frac{4}{c^2}(\xi-\Xi) = (D)\xi-(F)\Xi-\frac{1}{(K)}\{(d)^2(\xi^2\Xi-1)-(f)^2(\xi\Xi^2-1)\},$$

where, for clearness, ξ, Ξ are written in place of c^2xy, c^2XY. θ is then easily found.

It appears that there are four such lenses, all real if any are;* but under what circumstances they are real we cannot usefully discuss without knowing what, besides real optical systems, is contained by a scheme of coefficients defined only by obedience to the fundamental equations; it is the less important to discover, because the equivalence is illusory in a very important class of cases. Namely, if (d), (f) are finite, while $(K) = 0$, all the curvatures are infinite, and this is the class which possesses a single pair of planes whereon point corresponds to point, as we shall immediately prove.

10. Introduce the scheme

$$b = (gb'+h\beta'+c')\frac{\sigma}{\varpi_1},$$

$$\beta = (kb'+l\beta'+\gamma')\,\varpi_1,$$

$$c = (b'-lc'+h\gamma')\frac{\sigma}{\varpi_2},$$

$$\gamma = (\beta'+kc'-g\gamma')\,\varpi_2,$$

where $\sigma^{-1} = gl-hk+1.$

* Generally one only is equivalent to the given system; (D) and (F) above would give $\pm(ac')$, $\pm(af')$ in § 8.

This scheme satisfies the fundamental equations, involves six symbols only, and is capable of representing any system whatever; as we may see by noticing that the forms

$$
\begin{aligned}
(a) &= k^2 (\varpi_1^2+\varpi_2^2), \\
(b) &= k^2 \varpi_1 \varpi_2, \\
(d) &= k\sigma (\varpi_2^2-\varpi_1^2)/\varpi_1^2 \varpi_2, \\
(f) &= k (\varpi_1^2-\varpi_2^2), \\
(ac') &= k^3 \sigma (\varpi_1-\varpi_2^2) \left(g \frac{\varpi_2}{\varpi_1} + l \frac{\varpi_1}{\varpi_2}\right), \\
(af') &= k^3 \varpi_1 \varpi_2 (\varpi_2^2-\varpi_1^2)(g+l),
\end{aligned}
$$

are independent; in such a scheme every letter is an invariant; if a name is wanted, it may be called canonical. Now here

$$(K) = hk\sigma^2 (\varpi_1^2-\varpi_2^2)^2/\varpi_1^2 \varpi_2^2,$$

and, if $(K) = 0$, while (d), (f) are finite, we must have $h = 0$, which implies

$$
\begin{aligned}
b &= (gb'+c')\, \sigma/\varpi_1, \\
c &= (b'-lc')\, \sigma/\varpi_2,
\end{aligned}
$$

or points in the planes Oyz, $O'y'z'$ correspond one with another. When such a pair of planes in correspondence exist, we shall call them principal planes, because, as we shall see in Part III., they form the extreme generalization of principal planes of Gauss.

III.

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1. It is our next task to elicit the properties of an optical system from the foregoing mass of algebra.

In a system symmetrical about an axis there exists a complete one-to-one correspondence of point to point, and from this fact alone all properties of such a system, excepting the relation of linear and angular magnification, can, and I should say ought to, be deduced; the excepted relation, generally known as Helmholtz's theorem, is the interpretation of Gauss's equation

$$gl - hk = 1.$$

The want of a one-to-one correspondence has been so far the main and effectual bar to any detailed theory of an unsymmetrical system, and therefore it is of the first importance to remark that such a correspondence not only exists completely, but exists twice over. To enunciate this statement formally I shall take a slight liberty with established language.

In any two narrow beams related linearly ray to ray, there exists a system of lines perpendicular to the original axis, corresponding one to one with a system perpendicular to the emergent axis, and such that the triply infinite complex of rays which meet one line emerge and meet its conjugate also. Let such a pair of lines be called conjugate focal lines. It follows that, out of the doubly infinite complex issuing from any point upon a focal line, a singly infinite complex passes through each point upon its conjugate line; this singly infinite complex lies in a plane in the original and also in the emergent system. Let such planes be called "planes of correspondence." Through any point two focal lines pass in directions which the position of the point and the constants of the system determine.*

In particular in an optical system every plane of correspondence is perpendicular to the associated focal line, and every two focal lines that intersect are conjugate to two which are at right angles.

2. To prove these statements let us begin by considering the conditions under which two rays which meet once will meet again. Let one of the rays be the axis, and let the other meet it at the origin; then

$$0 = b = G_1 b' + H_1 \beta' + P_1 c' + Q_1 \gamma',$$

$$0 = c = P_2 b' + Q_2 \beta' + G_2 c' + H_2 \gamma'.$$

If the emergent ray meets the axis at a distance $\mu' a'$ from the origin, then these are consistent with

$$b' + a'\beta' = 0, \quad c' + a'\gamma' = 0;$$

or

$$a'(G_1\beta' + P_1\gamma') - H_1\beta' - Q_1\gamma' = 0,$$

$$a'(P_2\beta' + G_2\gamma') - Q_2\beta' - H_2\gamma' = 0.$$

* The existence of such planes has been remarked by Larmor.

Eliminate a'; then

$$\beta'^2 (G_1 Q_2 - H_1 P_2) + \beta'\gamma' (G_1 H_2 - G_2 H_1 + P_1 Q_2 - P_2 Q_1) \\ + \gamma'^2 (P_1 H_2 - G_2 Q_1) = 0.$$

The two solutions of this ($\beta' : \gamma' = c_1', c_2'$) give two emergent planes of correspondence, associated, as we shall presently see, with the two focal lines that meet at O. They are at right angles in virtue of (2′); associated with each is a single point ($\mu' a_1'$, 0, 0; $\mu' a_2'$, 0, 0) and a single original plane of correspondence ($\beta : \gamma = c_1, c_2$).

3. Now, let us simplify, by a rotation of axes (ψ) such as to make

$$G_1 Q_2 - H_1 P_2 = G_2 Q_1 - H_2 P_1 = 0,$$

so that the coordinate planes are the emergent planes of correspondence, and

$$\frac{G_1}{H_1} = \frac{P_2}{Q_2} = \frac{1}{a_2'}, \qquad \frac{G_2}{H_2} = \frac{P_1}{Q_1} = \frac{1}{a_1'};$$

then the equations connecting any two rays $(B\beta C\gamma)$, $(B'\beta' C'\gamma')$ are

$$\begin{aligned} B' &= \quad L_1 B + S_2 C - a_2' (G_1 \beta + P_2 \gamma), \\ \beta' &= -K_1 B - R_2 C + \quad G_1 \beta + P_2 \gamma, \\ C' &= \quad S_1 B + L_2 C - a_1' (P_1 \beta + G_2 \gamma), \\ \gamma' &= -R_1 B - K_2 C + \quad P_1 \beta + G_2 \gamma; \end{aligned}$$

and the planes of correspondence in the original system are

(1) associated with $\beta' = 0$, $\mu' a_1'$, $0 = G_1 \beta + P_2 \gamma$,

(2) " " $\gamma' = 0$, $\mu' a_2'$, $0 = P_1 \beta + G_2 \gamma$.

Now, eliminate β, γ from between C', γ', and we see that, if any ray meets the line

$$x' = \mu' a_1', \quad z' = C_1' = C' + a_1' \gamma',$$

then it also meets the line

$$\begin{aligned} x = 0, \quad C_1' &= \quad (S_1 - a_1' R_1)\, B + \quad (L_2 - a_1' K_2)\, C \\ &= (P_1 S_1 - Q_1 R_1) \frac{B}{P_1} + (G_2 L_2 - H_2 K_2) \frac{C}{G_2}. \end{aligned}$$

But consider $\quad G_1 Q_2 - H_1 P_2 = 0, \quad G_2 Q_1 - H_2 P_1 = 0, \qquad (2')$

$$G_1 S_2 - H_1 R_2 + P_1 L_2 - Q_1 K_2 = 0; \qquad (3')$$

from these $\quad \dfrac{G_1}{P_2} (P_2 S_2 - Q_2 R_2) + \dfrac{P_1}{G_2} (G_2 L_2 - H_2 K_2) = 0;$

hence, in virtue of (1), (6), (1′), (6′),

$$\frac{G_2L_2-H_2K_2}{G_1G_2} = \frac{P_1S_1-Q_1R_1}{-P_1P_2} = \frac{1}{G_1G_2-P_1P_2},$$

or

$$C_1' = \frac{-P_2B+G_1C}{G_1G_2-P_1P_2},$$

and in the same way, if an emergent ray meets

$$x' = \mu' a_2', \quad y' = B_2' = B' + a_2'\beta'$$

then it follows that the original ray meets

$$x = 0, \quad B_2' = \frac{G_2B-P_1C}{G_1G_2-P_1P_2}.$$

These results contain the statements of p. 68; indeed, they contain more, namely, referring to the figure:—

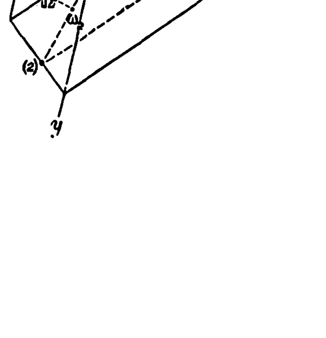
If $\Omega\omega_1$ be a focal line passing through Ω, and $\Omega_1'\omega_1'$ its conjugate, and similarly a pair of conjugates parallel to these pass through O, A_1', then the distance $O\omega_1$ between the lines at O, Ω is related to the distance $A_1'\Omega_1'$ between their conjugates by the equation

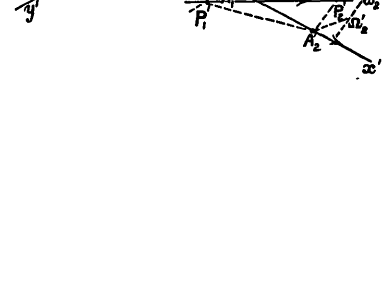
$$A_1'\Omega_1' = \frac{(G_1^2+P_2^2)^{\frac{1}{2}}}{G_1G_2-P_1P_2}\, O\omega_1,$$

and, in the same way, if $\Omega\omega_2$, $\Omega_2'\omega_2'$ be conjugates of the other system,

$$A_2'\Omega_2' = \frac{(G_2^2+P_1^2)^{\frac{1}{2}}}{G_1G_2-P_1P_2}\, O\omega_2.$$

These two multipliers shall be called the two linear magnifications for the point O; they are determined by the position of O and the constants of the instrument, and their values for any point, together with the position and directions of the focal lines, are given below.





4. Now consider the magnification of angles for any ray through O. If B, C vanish,

$$\beta' = G_1\beta + P_2\gamma,$$
$$\gamma' = P_1\beta + G_2\gamma;$$

these quantities vanish respectively when the ray OP is in the first or second plane of correspondence through O, and when this is not the case it may be seen with little calculation that

$$P_1\beta + G_2\gamma = \mu \frac{G_1G_2 - P_1P_2}{(G_1+P_2^2)^{\frac{1}{2}}} \angle p_1Ox,$$

where Pp_1 is parallel to O (2); also

$$\gamma' = \mu \angle P_2'A_1'A_2',$$

if $P_1'P_2'$ be the emergent ray corresponding to OP; hence we have a generalization of Helmholtz's theorem

$$\mu \,.\, O\omega_1 \angle p_1Ox = \mu' \,.\, A_1'\Omega_1' \angle P_2'A_1'A_2',$$
$$\mu \,.\, O\omega_2 \angle p_2Ox = \mu' \,.\, A_1'\Omega_2' \angle P_1'A_2'A_1'.$$

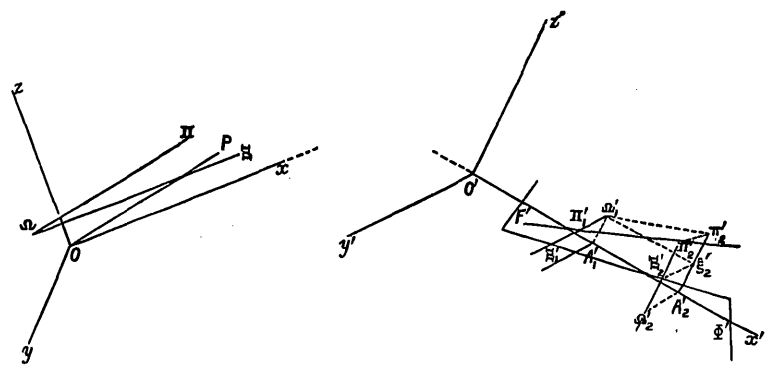
We notice that, if a ray be projected thus upon either plane of correspondence through any point of itself and upon the associated emergent plane, then its projections are optically related like original and emergent rays of the beam.

5. The foregoing results supply several alternative constructions for the emergent ray corresponding to any incident ray.

i. *Determination of Emergent Ray by means of two Points on Original Ray.*—Let Ω, Z, be the two points; find the focal lines Ω_1', Ω_2', Z_1', Z_2' whose conjugates pass through Ω, Z; the emergent ray is the ray meeting these four lines.*

ii. *Determination by means of Point and Direction.*—Let the ray be $\Omega\Pi$ parallel to OP. First find the emergent ray corresponding to $\Omega\Xi$, a parallel to Ox; this is the ray which meets the focal lines at Ω_1', Ω_2', and also the principal focal lines at F', Φ' whose conjugates are at infinity. This is the ray (B, 0, C, 0), and

* Observe that two lines meet any four, but in this case only one is eligible; for the lines are parallel to a plane, so that three are generators of a paraboloid, and the fourth parallel to a generator; hence one intersection with this paraboloid is at infinity and must be rejected.

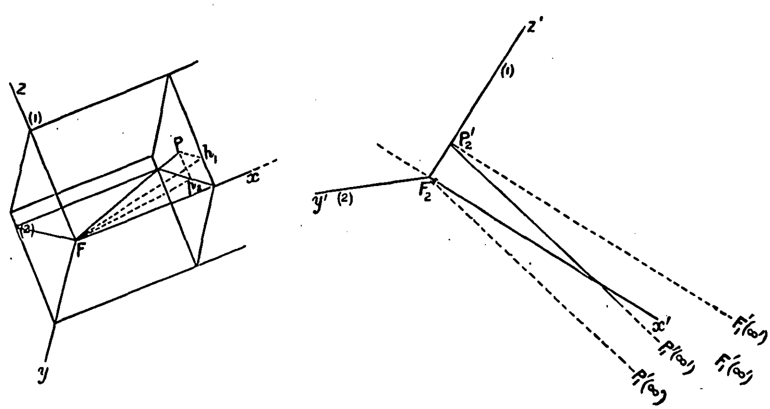


we see, from the equations of p. 69, that the projections of the emergent ray $\Pi_1'\Pi_2'$ make with the projections of $\Xi_1'\Xi_2'$ angles equal to $\angle P_2'A_1'A_2'$, $\angle P_1'A_2'A_1'$ respectively; evidently this determines the emergent ray.

6. We must now consider the problem :—*Given the position of a point O, to express the directions of the focal lines that meet in O, the positions and directions of their conjugates, and the magnification associated with each pair in terms of fixed geometrical features or algebraic invariants of the system.*

The chief fixed features of a system are the principal focal lines at F, Φ whose conjugates are at ∞', and those at F', Φ' whose conjugates are at ∞.

Consider F as an origin of rays. The focal lines conjugate to those at F are at ∞' and another point F_2'. All rays from F in the first plane of correspondence emerge parallel to $O'x'$ in the plane



$z'O'x'$; all rays in the second plane of correspondence lie in the plane $y'O'x'$, and meet the foot of this line (F_2); all other rays meet the line at other points and are parallel to the plane $y'O'x'$. Let us form the scheme of coefficients with F as origin, Ox as the first plane of correspondence, and F_2' as emergent origin. Then, if the scheme is

$$(b'c'\beta'\gamma') = \left\{\begin{matrix} l_1 & -h_1 & s_2 & -q_2 \\ -k_1 & g_1 & -r_2 & p_2 \\ s_1 & -q_1 & l_2 & -h_2 \\ -r_1 & p_1 & -k_2 & g_2 \end{matrix}\right\} (bc\beta\gamma),$$

we must have, when b, c vanish, $\gamma' = 0$, $b' = 0$, and $\beta' = 0$, implying $\beta = 0$; hence

$$p_1 = g_2 = h_1 = q_2 = p_2 = 0;$$

hence, from (2), $r_1 = 0$; from (3′), $q_1 = ng_1$, $s_2 = nk_2$; from (1), $g_1 l_1 = 1$; from (6), $h_2 k_2 = -1$; from (3), $g_1 s_1 - q_1 k_1 - h_2 r_2 = 0$; hence the scheme may be written

$$(bc\beta\gamma) = \left\{\begin{matrix} g_1 & 0 & 0 & ng_1 \\ k_1 & \dfrac{1}{g_1} & 0 & s_1 \\ 0 & 0 & 0 & -\dfrac{1}{k_2} \\ r_2 & nk_2 & k_2 & l_2 \end{matrix}\right\} (b'c'\beta'\gamma'),$$

where $$s_1 = nk_1 - r_2/g_1 k_2.$$

This is a reduction of the general scheme to expression by six independent letters—what we have called a canonical reduction on p. 67.

Now let us change the origin and axes until they occupy the corresponding position with respect to Φ which they now occupy with respect to F. Let these changes be $\lambda = f$, $\rho = d$, $\phi = \dfrac{\pi}{2}$, $\psi = a$; ϕ is a right angle because the planes of correspondence in question are both associated with a single point ∞'; the distance $F\Phi = \mu f$, $F_2'\Phi_2' = \mu' d$; let Greek letters denote the coefficients arrived at for Φ. Then, in the first place, $\rho_1 = 0$, which gives

$$0 = -r_2 \sin a + k_2 \cos a,$$

while $$\kappa_1 = \quad r_2 \cos a + k_2 \sin a;$$

hence $$r_2 = \kappa_1 \cos a, \quad k_2 = \kappa_1 \sin a;$$
in future we shall employ a, κ, and drop subscript marks. Then the result of the change $\phi = \frac{\pi}{2}$ is

$$\left\{\begin{matrix} 0 & 0 & 0 & -\dfrac{1}{\kappa \sin a} \\ \kappa \cos a & n\kappa \sin a & \kappa \sin a & l \\ -g & 0 & 0 & -ng \\ -k & -\dfrac{1}{g} & 0 & -s \end{matrix}\right\},$$

and the further change $\psi = a$ gives

$$\left\{\begin{matrix} 0 & -\dfrac{1}{\kappa} & 0 & -\dfrac{1}{\kappa}\cot a \\ \kappa & n\kappa \sin a \cos a + l \sin a & 0 & -n\kappa \sin^2 a + l \cos a \\ -g \cos a & -ng \sin a & g \sin a & -ng \cos a \\ -k \cos a & -\dfrac{1}{g}\cos a - s \sin a & k \sin a & \dfrac{1}{g}\sin a - s \cos a \end{matrix}\right\}$$

Further changes $\lambda = f$, $\rho = d$ make

$$h_1 = p_1 = p_2 = q_2 = g_2 = 0,$$

and $$g_1 = \gamma, \quad q_1 = \nu\gamma, \quad l_1 = \frac{1}{\gamma}, \quad s_1 = \nu\kappa + \frac{1}{\gamma}\cot a,$$

&c., &c.

Hence $$\gamma = f\kappa,$$

$$0 = p_2 = -\cos a\,(g + fk);$$

therefore $$g = -fk.$$

$$l_1 = \frac{1}{\gamma} = n\kappa \sin a \cos a + l \sin a + \kappa d,$$

$$s_1 = \nu\kappa + \frac{1}{\gamma}\cot a = -n\kappa \sin^2 a + l \cos a.$$

Eliminate l: $$-\nu\kappa \sin a = n\kappa \sin a + \kappa d \cos a,$$

or $$n + \nu + d \cot a = 0.$$

Solve for l:
$$l = \nu\kappa\cos a + \frac{1}{\gamma\sin a} - d\kappa\sin a$$
$$= (n\sin^2 a + \nu)\frac{\kappa}{\cos a} + \frac{1}{f\kappa\sin a}.$$

It may be verified that these results are got over and over again by taking other of the coefficients $g_1 h_1$, &c. ... . Hence, if the system is specified by the six constants k, κ, a, f, n, ν, the scheme of coefficients reads :—F as origin:

$$\left\{\begin{array}{cccc} -kf & 0 & 0 & nkf \\ k & -1/kf & 0 & nk+\cot a/kf \\ 0 & 0 & 0 & -1/\kappa\sin a \\ \kappa\cos a & n\kappa\sin a & \kappa\sin a & (n\sin^2 a+\nu)\,\kappa/\cos a + 1/f\kappa\sin a \end{array}\right\},$$

Φ as origin:

$$\left\{\begin{array}{cccc} \kappa f & 0 & 0 & -\nu\kappa f \\ \kappa & 1/\kappa f & 0 & \nu\kappa+\cot a/\kappa f \\ 0 & 0 & 0 & -1/k\sin a \\ -k\cos a & \nu k\sin a & k\sin a & -(\nu\sin^2 a+n)\,k/\cos a - 1/fk\sin a \end{array}\right\}.$$

7. The quantities we have employed are easily expressed in terms of algebraic invariants of the system

$$\begin{aligned}
(a) &= k^2+\kappa^2,\\
(b) &= k\kappa\sin a,\\
(d) &= -k\kappa f\cos a,\\
(f) &= nk^2+\nu\kappa^2,\\
(ac') &= -k^3\kappa f\sin a\,(k^2+\kappa^2) = -(a)(b)\,k^2 f,\\
(cf') &= -k\kappa f\sin a\left[\nu\kappa^2 - nk^2 + \frac{2}{f}\cot a\right],\\
(K) &= -n\nu k^2\kappa^2 f^2 - \cot^2 a\\
&\text{\&c.,} \qquad \text{\&c.}
\end{aligned}$$

The solution of these is easy; we shall employ them to find the schemes of coefficients when the instrument is reversed, and F' or Φ' is origin; let k', κ', &c., denote quantities corresponding to k, κ, &c.

Now when the instrument is reversed, out of the above invariants, (a), (b), (K), (cf') are unchanged in value, while $(d') = (f)$ and $(f') = (d)$. Hence we must solve

$$k'^2+\kappa'^2 = k^2+\kappa^2, \quad k'\kappa' \sin\alpha' = k\kappa\sin\alpha,$$

$$n'k'^2+\nu'\kappa'^2 = -k\kappa f\cos\alpha, \quad -k'\kappa' f'\cos\alpha' = nk^2+\nu\kappa^2,$$

$$n'\nu' k'^2\kappa'^2 f'^2+\cot^2\alpha' = n\nu k^2\kappa^2 f^2+\cot^2\alpha,$$

$$k'\kappa' f'\sin\alpha'\left[\nu'\kappa'^2-nk^2+\frac{2}{f'}\cot\alpha'\right] = k\kappa f\sin\alpha\left[\nu\kappa^2-nk^2+\frac{2}{f}\cot\alpha\right].$$

These are satisfied by the six results

$$k'=k, \quad \kappa'=\kappa, \quad \alpha'=\alpha,$$

$$f'n'=fn, \quad f'\nu'=f\nu,$$

$$k^2fn+\kappa^2f\nu+k\kappa ff'\cos\alpha = 0,$$

which also imply $$f'd' = fd.$$

We may use f', and d or d', to express n, ν, and then the four quantities f, f', α, d have clear geometrical meanings, and may be found for any system by the simplest direct observations. The quantities k, κ may also be found from observation in a variety of simple ways by help of the results of p. 78 below.

8. We shall have occasion to use the angle between a plane of correspondence at F' and one at F_2'. Let us say, the first plane of correspondence for each; it may be found as follows :—

The scheme for F' is equivalent to that for F_2', and this latter is simply the scheme for F, inverted; consider the terms in k_1, r_1, r_2, k_2 in each. They are respectively,

for F':	k	0	$\kappa\cos\alpha$	$\kappa\sin\alpha$,
for F_2':	$-k$	$-\kappa\cos\alpha$	0	$-\kappa\sin\alpha$.

The rotation (ϕ) which is required to reconcile these equivalent schemes is given by equation (α) or (α') of p. 52;

$$(\alpha) \quad k^2+\kappa^2\cos 2\alpha = \quad (k^2-\kappa^2)\cos 2\phi+2k\kappa\cos\alpha\sin 2\phi,$$

$$(\alpha') \quad 2\kappa^2\sin\alpha\cos\alpha = -(k^2-\kappa^2)\sin 2\phi+2k\kappa\cos\alpha\cos 2\phi,$$

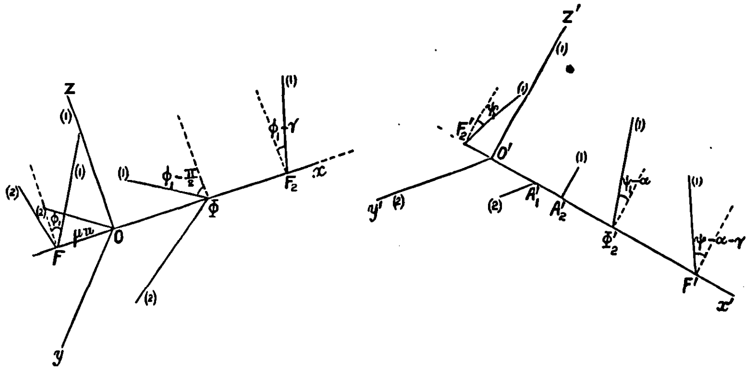
which are consistent. If we write

$$(k^4+\kappa^4+2k^2\kappa^2\cos 2\alpha)^{\frac{1}{2}} = (k^2-\kappa^2)/\cos\delta = (k^2+\kappa^2\cos 2\alpha)/\cos\epsilon,$$

then $$2\phi = \delta+\epsilon;$$

denote this value of ϕ by a single symbol γ. It may be easily shown that the corresponding angle upon the other side of the system is also equal to γ.

9. We are now in a position to answer the questions of p. 72; but first let us simplify the expressions by choosing the plane of correspondence of which we are speaking as the plane of zOx. Thus, for



the first plane of correspondence,

$$P_2 = 0,$$

$$A_1'\Omega_1' = \frac{1}{G_2} O\omega_1;$$

for the second plane of correspondence,

$$G_2 = 0,$$

$$A_2'\Omega_2' = -\frac{1}{P_2} O\omega_2.$$

" Now suppose displacements from the origin F, $\lambda = u$, $\phi = \phi_1$, $\psi = \psi_1$, convert the scheme for F into the scheme for another point O, distant μu from F.

Then, for the point O,

$$\begin{aligned} G_2 = &-\sin\psi_1 \left[-\sin\phi_1 (-kf + ku) + \cos\phi_1 \,.\, u\kappa\cos\alpha\right] \\ &+\cos\psi_1 \left[\qquad\qquad\qquad\qquad \cos\phi_1 \,.\, u\kappa\sin\alpha\right], \end{aligned}$$

$$\begin{aligned} 0 = P_2 = &\cos\psi_1 \left[-\sin\phi_1 (\;ku - kf) + \cos\phi_1 \,.\, u\kappa\cos\alpha\right] \\ &+\sin\psi_1 \left[\qquad\qquad\qquad\qquad \cos\phi_1 \,.\, u\kappa\sin\alpha\right]. \end{aligned}$$

Hence $$G_2 = u\kappa \sin a \cos \phi_1 / \cos \psi_1,$$

and $$\tan \phi_1 = \frac{\kappa}{k} \frac{u}{u-f} \frac{\cos(\psi_1 - a)}{\cos \psi_1}.$$

It may be verified that the same equations are found by starting from Φ as origin in place of F.

Again, if we take F' as origin and proceed to A_1', the angles ϕ, ψ interchange their *rôles*, and each is diminished by the angle γ. Hence if $\mu' u_1'$ be the distance $F'A_1'$, we have

$$\frac{1}{G_2} = u_1' \kappa \sin a \cos(\psi_1 - \gamma) / \cos(\phi_1 - \gamma),$$

$$\tan(\psi_1 - \gamma) = \frac{\kappa}{k} \frac{u_1'}{u_1' - f'} \frac{\cos(\phi_1 - \gamma - a)}{\cos(\phi_1 - \gamma)}.$$

These equations give G_2, ϕ_1, ψ_1, u_1', in terms of u, which solves the problem proposed. In the same way ϕ_2, ψ_2, u_2', and P_2 when $G_2 = 0$, may be found. Including the theorem on angular magnification already proved, we have thus a complete outline of the geometrical theory of a general optical system. It will be observed that most of its properties, with slight modifications, obtain in all systems, whether optical or not, where there is a linear relation between the rays.

10. Let us now compare our results with the corresponding relations in a symmetrical instrument, and, observing the successive steps by which the properties of the latter become generalized or disappear, and the associated values of the invariants, let us attempt a classification of optical systems in general.

I. When six invariants, say (b), (d), (f), (D), (F), (K), are all independent and finite, the system is completely general.

II. If $(K) = 0$, while (d) and (f) are finite, the system possesses one pair of planes whereon point corresponds to point.

III. If, in addition to $(K) = 0$, (d), (f) finite, we have a further relation, namely,

$$(cf')(de') = (a)(d)(f) + (b)\{(d)^2 + (f)^2\},$$

the system is equivalent to oblique refraction at a single surface. This may be seen by forming the invariants of the scheme which belongs to a single surface, and effecting the elimination (other than

$(K) = 0$) of the four quantities on which they depend; if we write the scheme

$$\left\{\begin{matrix} 1 & 0 & 0 & 0 \\ x & 1 & \varpi z & 0 \\ 0 & 0 & \varpi & 0 \\ z & 0 & y & 1/\varpi \end{matrix}\right\},$$

we find $(a) = x^2+y^2+z^2(1+\varpi^2)$, $(b) = xy-\varpi z^2$,

$$(d) = z(1-\varpi^2), \qquad (f) = z(1-\varpi^2)/\varpi,$$

$$(cf') = (f)(\varpi x+y), \qquad (de') = (d)(x/\varpi+y),$$

$$(D) = (x\varpi-y)^2+z^2(1+\varpi^2)^2,$$

$$(F) = (x/\varpi-y)^2+z^2(\varpi+1/\varpi)^2.$$

Eliminating, we get the relation above. The limitation which this imposes upon the general scheme of p. 67 is worth notice; there

$$(cf') = k^2\sigma\,(\varpi_1^2-\varpi_2^2)\left(g\,\frac{\varpi_2}{\varpi_1}-l\,\frac{\varpi_1}{\varpi_2}\right),$$

$$(de') = k^2\sigma\,(\varpi_1^2-\varpi_2^2)(g-l);$$

hence, removing $k^4(\varpi_1^2-\varpi_2^2)^2$, the relation is

$$(g-l)\left(g\,\frac{\varpi_2}{\varpi_1}-l\,\frac{\varpi_1}{\varpi_2}\right) = \left(\frac{1}{\varpi_1}-\frac{\varpi_1}{\sigma}\right)\left(\frac{1}{\varpi_2}-\frac{\varpi_2}{\sigma}\right),$$

which may be written

$$(g\varpi_2^2-l\varpi_1^2)_2 = \left(g^2\varpi_2^2+\varpi_1^2-\frac{\varpi_1^2\varpi_2^2}{\sigma^2}\right)\left(l\varpi + \varpi - \frac{\varpi_1^2\varpi_2^2}{\sigma^2}\right),$$

and this is the condition that the circle $x^2+y^2 = c$ and the ellipse

$$(gx+y)^2\,\sigma^2/\varpi_1^2+(x-ly)^2\,\sigma^2/\varpi_2^2 = c$$

have double contact. Now in a single surface the planes where point corresponds to point are evidently situated at the surface itself, and corresponding points are the projections of any, the same, point of the surface on planes perpendicular to the original and emergent axes respectively; and the traces of such points are evidently related as above, namely, one may be a circle and the other an ellipse, which, if in the same plane, would be concentric and in double contact.

IV. (d) cannot vanish without (f) in ordinary systems produced by finite curvatures. If (d), (f) both vanish, we see that, in the equivalent lens of p. 65, $\theta = 0$, or the principal curvatures of the two faces of the lens are parallel; also, from p. 75, $\alpha = \frac{\pi}{2}$, and this makes n, ν vanish, and hence (K). Turning to the expressions for magnification, we see that

$$\gamma = 0, \quad \phi_1 = \psi_1 = 0,$$

$$\kappa^2 u u_1' = 1;$$

and, in the same way, $k^2(u-f)(u_2'-f') = 1$.

That is to say, all the first planes of correspondence coincide, and all the second planes coincide, and are at right angles to the first. Thus the system treats the projections of any ray upon these planes exactly as two separate symmetrical instruments would treat them. Such a system possesses two pairs of planes whereon point corresponds to point. For we see by p. 47 that the relations are

$$\left.\begin{aligned} b &= g_1 b' + h_1 \beta', \\ \beta &= k_1 b' + l_1 \beta', \end{aligned}\right\} \qquad \left.\begin{aligned} c &= g_2 c' + h_2 \gamma', \\ \gamma &= k_2 c' + l_2 \gamma', \end{aligned}\right\}$$

and, if $$b + u\beta = 0, \qquad c + u\gamma = 0,$$

then $$(g_1 + uk_1)\, b' + (h_1 + ul_1)\, \beta' = 0,$$

$$(g_2 + uk_2)\, b' + (h_2 + ul_2)\, \beta' = 0,$$

which give $$b' + u'\beta' = 0, \qquad c' + u'\gamma' = 0,$$

when u, u' have the two pairs of values satisfying

$$u' = \frac{h_1 + ul_1}{g_1 + uk_1} = \frac{h_2 + ul_2}{g_2 + uk_2}.$$

The existence of systems possessing two pairs of planes of conjugate foci has been noticed by Larmor.

V. A sub-class of IV. is given by the scheme on p. 79, in which $z = 0$, that is to say, the equivalent of oblique refraction at a surface of double curvature, one of whose lines of curvature is perpendicular to the axis of the beam. We see that

$$(d) = (f) = 0,$$

and, in addition,

$$\sqrt{(D)(F)} + \{\sqrt{(D)} + \sqrt{(F)}\}\{\sqrt{(a) + 2(b)} \pm \sqrt{(a) - 2(b)}\} - \sqrt{(A)} = 0.$$

This class possesses two pairs of planes where point corresponds to point: one of these coincides with the surface ($u = 0$); the other is at distance

$$u = \frac{\varpi^2 - 1}{x - \varpi y} = \frac{\{(D)(F) - 2(a)\sqrt{(D)(F)} + (A)\}^{\frac{1}{2}}}{(b)\sqrt{(F)}},$$

which also coincides with the surface in the Case VII. below.

VI. The two pairs of planes in IV. coincide, and give a single pair of planes of conjugate foci, provided that, in addition to $(d) = (f) = 0$, giving $\theta = 0$ in the equivalent lens, we have $cx = cy$, which, expressed in invariants, requires

$$\{(a) - 2(b)\}^2 = (D)(F).$$

VII. If, in addition,

$$(a) - 2(b) = (D) = (F),$$

then, disregarding irrelevant alternatives,

$$c^2 xy = c^2 XY = 1,$$

and this implies $c = 0$, or the equivalent lens is a thin lens; the single pair of planes of conjugate foci coincide with the surface of the lens. Such a system is equivalent to direct incidence upon a single surface; for then, on p. 79, $\varpi = 1$, and this makes

$$(d) = (f) = 0, \qquad (a) - 2(b) = (D) = (F).$$

VIII. Lastly, if

$$(a) - 2(b) = (d) = (f) = (D) = (F) = 0,$$

the system reduces to the ordinary symmetrical instrument which Gauss discussed. It has but one invariant, the focal length, and the value of (a) or (b) defines it.

11. It is interesting to observe the gradual disappearance of the features of a symmetrical instrument.

The ordinary symmetrical instrument is, simply, a single spherical surface with direct incidence, duplicated, and the duplicates separated. The duplicates of the surface itself are Gauss's principal planes; the duplicates of its centres are the nodal points. Such a system has an infinite number of pairs of planes of conjugate foci. (VIII.)

Preserve direct incidence, but make the curvatures unequal; all planes of conjugate foci disappear except the principal planes, and we are left with a single pair of planes of conjugate foci, coinciding with the duplicates of the surface, and possessing unit magnification in all directions. (VII.)

Introduce a spherical surface perpendicular to the rays at a finite distance from the other surface; a pair of planes of conjugate foci remain, but do not coincide with either surface; magnification there is equal in all directions, but is no longer unity. (VI.)

Replace the spherical surface by a surface of double curvature whose principal curvatures are parallel to those of the first surface; the single pair of planes of conjugate foci separates into two, in which the magnification is unequal in different directions, but in two fixed directions is directly proportional to the distance of the point from the two foci corresponding to infinity. (IV.)

Rotate either of the surfaces with respect to the other, and the system becomes completely general; both pairs of planes of conjugate foci disappear together. (I.)

Or, again, take oblique incidence. Oblique incidence upon a single surface one of whose lines of curvature is perpendicular to the axes of the beam gives rise to two pairs of planes where point and point correspond; one of these always coincides with the surface, and the other degenerates into coincidence with the first if the incidence be made direct; the magnification at these planes is not the same in all directions, and at that one which coincides with the surface its maximum or minimum is unity. All the planes of correspondence of each series coincide with one or other of two planes at right angles, and generally the instrument behaves towards rays in these two planes exactly as two different symmetrical intruments would do. (V.)

If we introduce a spherical surface normal to the rays as they emerge, the two pairs of planes of conjugate foci remain, but the condition of unit magnification as a maximum or minimum disappears. (IV.)

If the beam falls upon any single surface whose lines of curvature are oblique to it, we have one and only one pair of planes of conjugate foci, situated at the surface itself; magnification in this plane is unequal in different directions, but its maximum or minimum is unity. (III.)

If we introduce a spherical surface normal to the rays as they emerge, the pair of planes of conjugate foci remain, though no longer coinciding with either surface, and the maximum or minimum magnification is no longer unity. (II.)

Any other change makes the planes of conjugate foci disappear, and the system becomes completely general. (I.)

12. Again, consider the relations between the positions of conjugate focal lines. All systems of the type IV., and onwards, are, to put it briefly, resolvable into two symmetrical systems acting in two planes at right angles, both as regards position of conjugate lines and magnification; systems of the types I., II., III. are not so resolvable, and the simplest expressions for the relations between conjugate focal lines or for the magnification are involved with the directions of those lines. Nevertheless, the change of form is not very great, and we can even see that, were the systems not related optically at all, it would not be very much greater.

13. This classification of optical systems is probably very imperfect, for the few properties which I have examined and used as a base for it must have omitted many notable cases; but they seem sufficient to show that no analytical obstacle remains against examining every corner of the problem, whether in its broadest or its finest features.

On the Poncelet Polygons of a Limaçon. By F. Morley.

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1. *The Limaçon.*

If we take n elements, points or lines, in a determinate order, say they are numbered $1, 2, \ldots, n, 1$, and also the elements, lines or points $12, 23, \ldots, n1$, we have the polygon. The problem of determining the condition under which the points are elements of one conic and the lines are elements of a second conic is a famous one, and it is sufficient to make the single reference to Halphen's *Fonctions Elliptiques*, t. II. The two conics constitute a curve of degree and class 4, any element of which determines two elements of the other kind, for a line of the one conic determines two points of the other,