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THE METHOD OF 'RIGHT AND WRONG CASES' ('CONSTANT STIMULI') WITHOUT GAUSS'S FORMULAE.

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1. *Need of dispensing with the Gaussian formulae in the method of 'right and wrong cases.'*
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1. *Need of dispensing with the Gaussian formulae.*

GAUSS, as is well known, discovered that the fluctuations of 'chance' errors of measurement are regulated by a mathematical law. His formulae were welcomed by scientists, met with remarkably close empirical corroboration and contributed not a little to the exactness of modern physical and psychical research. But of late years opinion would seem to be veering round; the use of the formulae encounters a continually increasing opposition, especially in psychology.

This opposition is, the present writer believes, sound enough in general principle, but defective in execution. It contents itself too easily with reiterating that deviations from the formulae are conceivable and actually occur, instead of determining the particular circumstances under which such deviations must and do take place; it is too eager to cast aside the formulae altogether because they are occasionally misleading, instead of first seeing if there are not also occasions where these formulae render right good service. Now, Gauss's law deals—by the terms of its demonstration—only with such variations as are the additive result of very many independently varying and very small factors[1]. And accordingly, it has shown itself to fit admirably just those variations which may not unreasonably be regarded as the accumulative effect of factors of such a kind (notably, the fluctuations of the spatial measurements of physical science). But the law ought never to have been expected to apply accurately to any variations where the above two conditions, the independent additive effect and the relative smallness of the factors, are known not to be fulfilled.

The second of these conditions is of more interest for psychology; beyond all question, the factors influencing the observed variations are in psychology often relatively *large*[2]. The measurements taken of a person are perturbed by practice, fatigue, 'absolute impression,' attention, suggestion, confidence, etc. Those of a class are differentially affected by natural selection, artificial elimination, mixture of species, external limits and a host more. These relatively large factors inevitably cause more or less disagreement between Gauss's 'normal' distribution of values and that actually observed; and such disagreements are daily becoming less tolerable, as psychology is growing into a more and more exact science.

Now, the question of these disagreements is in a peculiarly acute stage as concerns the admittedly most scientific method of psychophysical measurement, that of 'right and wrong cases' or, as it has recently been termed, of 'constant stimuli.' For it is beginning to be realized that this method is applicable to a very extensive field over and beyond the determination of limina, to which it has hitherto almost exclusively been confined[3]. Further, these liminal determinations, formerly cultivated with such zeal and then suffered to fall into neglect, are now beginning to be taken up again with something of the old interest but with a new purpose.

[1] It is sometimes but incorrectly said, that these factors also need to be all equal in amount and all equally probable.

[2] As regards the first condition, to postulate that the factors should be independent and additive is to postulate the independence of their respective effects upon the total variations—expressed more mathematically, the independence of the partial differential coefficients of the total variable, taken with respect to the part variables. But it should be remembered that, even should this independence really exist when choosing the 'argument' rightly, it will no longer hold good when using any other argument not a linear function of the right one. This fact is not taken into consideration by those who think to refute Gauss's law simply by pointing out, as they believe, its discrepancy from observation. To take one instance among many, whenever absolute measurements are wrongly substituted for relative ones, such a wrong argument is not a linear but a logarithmic function of the right one.

[3] As far as I can see, it is applicable to the whole range of judgments that depend on stimuli admitting of fine gradation.

For it has lately been discovered, that just those anomalies of experimental results, which so grievously obstructed all attempts at accurate verification of the law of Weber, will bring in a rich psychological harvest when investigated on their own account. But this involves work of the utmost delicacy and exactness; the disagreements between the 'normal' and the actual distribution of values, far from being considered too small to be taken into account, must form one of the chief objects of investigation. Hence, we are now in great need of a method whereby to measure this actual distribution.

This does not at all mean that the 'normal' distribution is to be thrown away like a worn-out garment. On the contrary, it still applies to a vast number of cases with sufficient approximation to do great service in research (this service, so precious to the researcher, is sometimes ignored by the mathematician 'at the green table'). And even when the 'normal' distribution ceases to indicate closely the factual one, it at any rate remains an invaluable standard for estimating the latter; the very discrepancy between the two furnishes the principal means of investigation.

2. *Proposed general procedure.*

As it is required to measure the actual distribution—whatever this may turn out to be—of the values investigated, it is of course necessary that the experimental data should suffice to determine this actual distribution. This means that the 'stimuli of comparison' must be at small intervals throughout the whole region within which any of the observer's judgments are likely to be undecided; but it is not necessary, although convenient, that these intervals should form a regular series (arithmetical, geometrical, etc.).

To illustrate our procedure, we will take a series of experimental observations by Merkel[1], both as having been very carefully executed and as having been searchingly discussed by G. E. Müller[2]. In these experiments, the observer had to judge which of two noises appeared to be the louder[3]. First came the 'standard' noise, which we will call S, and whose intensity was calculated by Merkel at the value 1772. Then

[1] Wundt's *Philosoph. Studien*, Bd. IV. S. 141.

[2] *Die Gesichtspunkte und die Tatsachen der psychophysischen Methodik*, 1904, S. 52, and later.

[3] The noises were caused by dropping brass balls on to hard blocks of wood. Following Müller, Merkel's ascending and descending series have here been added together, as they show no appreciable discrepancy from one another.

followed the 'comparative' noise, which we will call C, and which had 11 different intensities varying not quite regularly from 1078 to 3011; with each of these intensities 100 judgments were made. Table I. shows each value of $C-S$, which we will call D, together with the ensuing amounts of the three different kinds of judgments, 'greater,' 'undecided,' and 'less.'

TABLE I. *The data given by Merkel.*

D $(=C-S)$	Judgments as to intensity of C 'Greater' $(=g)$	'Undecided' $(=u)$	'Less' $(=l)$
-694		2	98
-538		11	89
-370	11	14	73[1]
-195	28	20	52
0	48	24	28
$+200$	71	18	11
$+397$	83	15	2
$+603$	91	8[1]	
$+807$	95	5	
$+1021$	98	2	
$+1239$	100		
	625 $(=\Sigma g)$	119 $(=\Sigma u)$	353 $(=\Sigma l)$

We will now consider how these data may be made to furnish the measurement, say, of the upper limen; by this will be understood, as usual, the exact value of D necessary, in order that the observer should just balance between judging C to be equal to or greater than S. Our considerations however, as will readily be seen, are equally applicable to any other sort of limen or overlimen, quantitative or qualitative, absolute or differential; further, to the wide range of other psychological problems amenable to the method of constant stimuli, as the measurement of an optical illusion, the determination of colours in peripheral vision, or that of the most pleasing musical intervals, etc.

It must be borne in mind that the experimenter has only to deal with the actual judgments and the actual physical stimuli; also that the judgments elicited by any given stimulus are not constant but *variable*. Any observer is liable at one moment to judge C greater than S when it is objectively nearly equal, and at the next moment to judge it equal when it is objectively far greater. The conception of an underlying constant limen that would occur on elimination of all variable disturbances is (when not a mere auxiliary hypothesis, for use at a

[1] In these cases, the number of judgments would appear not to have been exactly 100, but this slight difference will here—for convenience of calculation—be neglected.

pinch) highly speculative; it is a luxury to be reserved—together with the question, as to whether the causes of variation lie in the stimuli, in the sensations, or in the apprehension of their difference—for the theoretical discussion at the close of proceedings. So long as any investigation is still at the experimental stage, 'the' limen must be taken as only an abbreviated expression for the *mean* limen.

Now, it is evident (being implied in the meaning of the words), that on every occasion when the judgment is a g, the (upper) limen is some value less than the D then used[1]. To take an instance from Table I., when D amounted to + 200, the percentage of g's was 71; from this we conclude that 71 °/ₒ of the limina were less than + 200. Hence, under the given experimental conditions generally, 71 °/ₒ must be taken as the most probable percentage of limina less than + 200[2]. Similarly, 83 °/ₒ must be taken as the percentage of limina below + 397. But if 71 °/ₒ occur below + 200 and 83 °/ₒ below + 397, then 12 °/ₒ must occur between + 200 and + 397; in other words, the most probable percentage of limina falling into the interval between + 200 and + 397 is 12 °/ₒ. By extending the same reckoning process to the whole series of observed data, we get a complete distribution of the percentages of the upper limina[3]; these will be found in the 2nd column of Table II.

TABLE II. *Distribution of upper limina (deduced from Table I.).*

Between the limits	p = percentage of limina	f = measure of average relative frequency	No. of column in graph
− 694 and − 538	0	0	
− 538 „ − 370	11	12·9	1
− 370 „ − 195	17	19·1	2
− 195 „ 0	20	20·2	3
0 „ + 200	23	22·7	4
+ 200 „ + 397	12	12·0	5
+ 397 „ + 603	8	7·6	6
+ 603 „ + 807	4	3·9	7
+ 807 „ + 1021	3	2·8	8
+ 1021 „ + 1239	2	1·8	9

If the D's are equidistant, evidently the percentage for each interval may be taken as proportional to the average relative frequency of such limina; the percentage will therefore without further ado serve as a measure of the average relative frequency. But if (as here) the intervals

[1] In this paper, D_k will be termed less than D_{k+1} whenever $D_k - D_{k+1}$ is negative.

[2] Obviously, the limen is some value greater than D whenever the observer answers either l or even u.

[3] Any *minus* percentage shows that the experimental series was too short, or that some illegitimate influence was present and affected the D's unequally.

between the D's are appreciably unequal, a correction is required: each percentage must be divided by its interval. And then—to make the results comparable with those in the cases where no such correction is needed—each such quotient may be multiplied by the general average of all the intervals. Thus in our example a few lines above, 12 must first be divided by 197 (the interval being in this case $397 - 200$); it is then, to facilitate comparison, multiplied by the average interval, $\frac{1239 - (-438)}{9} = 197{\cdot}4$. In this way we get the values in the 3rd column of Table II.

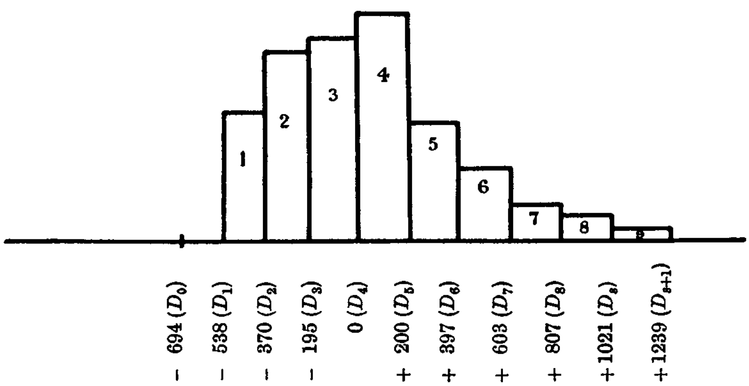

FIG. 1. Actual distribution of Limina.

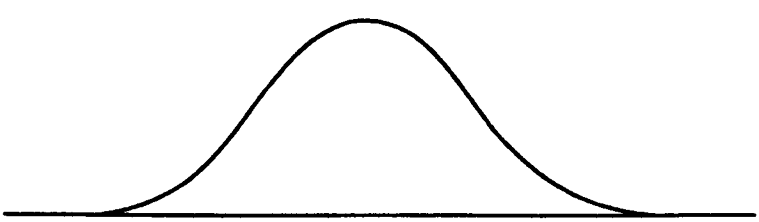

FIG. 2. Distribution calculated from the same data as fig. 1, but by means of Gauss's formula.

Having thus easily arrived at a complete list of average relative frequencies, the next question is how to represent these data summarily. In the opinion of the present writer, there is one and only one procedure, by which any given series of values can be summarized clearly, completely, and without questionable assumptions: this is by means of a *graph*. Once enlightened generally by such a lucid picture of the entire state of affairs, the researcher is in a position to choose any specially interesting characteristics for exact measurement. When he comes to make this particular measurement—and not before—the graphic procedure ought to be replaced by that of analysis.

For this graph, we will take the D's as abscissas and the f's as ordinates. In this way we get figure 1 (the numbering of the D's will be explained later on). Clearly, it differs from the 'normal' distribution (fig. 2) to a remarkable extent. That the asymmetry is no mere chance, is proved at once by its magnitude, its regularity, and by a quite similar shape resulting from Merkel's parallel series, where S followed C. It hardly needs saying, that the study of such abnormal distributions is likely to be very fruitful for psychology.

The chief point for the present is that our proposed method has brought this abnormality so clearly to view.

3. *First method of measuring the average limen.*

Having obtained a general survey of the distribution of the limina, we come to our second task, the exact measurement of the chief particular characteristics. Foremost, but still only one of several, of these is the mean limen. And even as regards this mean, there are various different values to be considered. First, there is the 'mode' or limen occurring most frequently[1]; it is represented by the highest point in the frequency distribution and evidently lies somewhere between 0 and + 200 (see fig. 1); its position could be found definitely by interpolation, but this is unfortunately a precarious proceeding, depending upon only a small part of the experimental data and upon a more or less arbitrary method of calculation. Secondly, there is the 'median' or the value having just as many limina above as below it; this seems to be the only sort of limen measured previously (without assuming the Gaussian formulae); but again recourse must be had to a precarious interpolation[2]. And lastly, there is the 'average,' or the sum of the liminal values divided by the number of them; of the three means this will be found much the most accurate. In the present paper, the calculation of this one only will be discussed.

For this purpose, let us turn to fig. 1 again. We see that the total distribution is made up of the part distributions belonging to the

[1] Conceivably, of course, the mode may assume an extreme value instead of a mean one. But it does this so extremely rarely and it presents so many analogies with the true means, that it may conveniently be classed with them.

[2] G. E. Müller, to whom the idea of measuring the mode and the median without assuming the Gaussian formulae is originally due, points out the necessity and the disadvantages of using interpolation (*loc. cit.* S. 42). Formulae for interpolating are given by Lehmann, *Archiv f. d. g. Psychologie*, Bd. VIII. S. 484 ff.

separate columns respectively (here, nine); the area of each column represents the number of limina between the two magnitudes denoted by the extremities of the column's base. Therefore, using a known rule, the required total average magnitude of all the individual limina is equal to the average of the part averages for the separate columns; 'weight' must of course be allowed for the columns' varying areas. But the part average belonging to any single column is represented with fair approximation by the centre of that column's base; strictly speaking, it must lie somewhat more towards the highest column, but the errors thus engendered in the columns on the left are approximately compensated by errors of similar amount and opposite direction in the columns on the right. Now, let

c_k = the centre of the base of any, say the kth, column from the left; if we call the two extremities of this base D_k and D_{k+1}, then $c_k = \frac{1}{2}(D_k + D_{k+1})$*;

p_k = the area of the kth column (2nd column, Table II.);

Σcp denote the sum obtained by multiplying c by p for each column and adding together all the products;

Σp denote the sum of all the p's (= the number of limina);

and L denote the required average limen.

Then $$L = \frac{\Sigma cp}{\Sigma p} = \frac{\Sigma cp}{100} \qquad (a).$$

The meaning of this will be found very simple on working it out, as shown in the following table.

TABLE III. *Calculation of the average upper limen. First method.*

Col.	c			p	cp	
		+	−		+	−
1	$\frac{1}{2}(D_1+D_2) = \frac{1}{2}(-538-370) =$		454	11		4994
2	$\frac{1}{2}(D_2+D_3) = \frac{1}{2}(-370-195) =$		282·5	17		4802·5
3	$\frac{1}{2}(D_3+D_4) = \frac{1}{2}(-195+0) =$		97·5	20		1950
4	$\frac{1}{2}(D_4+D_5) = \frac{1}{2}(0+200) =$	100		23	2300	
5	$\frac{1}{2}(D_5+D_6) = \frac{1}{2}(200+397) =$	298·5		12	3582	
6	$\frac{1}{2}(D_6+D_7) = \frac{1}{2}(397+603) =$	500		8	4000	
7	$\frac{1}{2}(D_7+D_8) = \frac{1}{2}(603+807) =$	705		4	2820	
8	$\frac{1}{2}(D_8+D_9) = \frac{1}{2}(807+1021) =$	914		3	2742	
9	$\frac{1}{2}(D_9+D_{10}) = \frac{1}{2}(1021+1239) =$	1130		2	2260	
				$100 = \Sigma p$	17704	11746·5
					11746·5	
					$5957{\cdot}5 = \Sigma cp$ †	

* The numbering of the D's in fig. 1 is taken from this definition.

† The largeness of these numbers is, of course, not due to the present method, but to the unusually large numbers in Merkel's D's (see Table I.).

Hence, the average limen,

$$L = \frac{\Sigma cp}{100} = + \frac{5957 \cdot 5}{100} = + 60, \text{ rounded off.}$$

4. *The case of equal intervals.*

Much trouble is saved when, as will usually be the case, all the D's are equidistant. Then the f's, the ordinates for the graph, become identical with the p's. Also our formula (a) can be modified so as to make the calculation much shorter.

To measure the upper limen,

let i = the constant interval between the D's,

D_{s+1} = the smallest D furnishing no u's or l's (D_s = the D next smallest, etc.),

Σg = the sum of the g's for all D's smaller than D_{s+1},

n = the number of judgments for each D,

and L_{up} = the required average upper limen.

Then $$L_{up} = \tfrac{1}{2}(D_{s+1} + D_s) - \frac{i \, . \, \Sigma g}{n} \qquad \text{(b)}[1].$$

Thus in our previous example, *if* all the intervals had been equal to the average of the intervals in question (*i.e.* those between D_1 and D_{s+1}), this common interval would have been 197·4, while $\Sigma g = 625$ (Table I.); consequently,

$$L_{up} = \tfrac{1}{2}(1239 + 1021) - \frac{197 \cdot 4 \times 625}{100} = -104.$$

The large discrepancy of this from the previous result (+ 60) shows that formula (b) cannot be used unless the D's really have a common interval. We may not substitute the average interval, even when, as here, the differences between the intervals is slight (except in the particular case, that these differences can be shown to tend to compensate one another).

For a lower limen, it will be found similarly that

$$L_{lo} = \tfrac{1}{2}({}_{t+1}D + {}_{t}D) + \frac{i \, . \, \Sigma l}{n} \qquad \text{(c)},$$

where ${}_{t+1}D$ = the largest D furnishing no u's or g's (${}_{t}D$ = the D next largest, etc.),

[1] For proof, see Appendix 1.

Σl = the sum of the l's for all D's larger than $_{t+1}D$,

i and n = the same as before,

and L_{lo} = the required average lower limen.

It may be remarked that

$$L_{up} - L_{lo} \text{ will be found equal to } \frac{i \,.\, \Sigma u}{n},\text{[1]}$$

which latter expression is that given by Müller as measuring the 'ideal region of u-judgments.'[2] In fact, the present paper may be regarded generally as an attempt to develop some ideas of Müller a little further[3].

5. *Second and more accurate method of measuring the average limen.*

With but little extra trouble, we can substitute for formula (a) the following more accurate one:

$$L' = \frac{\Sigma ap}{\Sigma p} = \frac{\Sigma ap}{100} \text{ ....}(d),$$

where $a_k = D_k + \dfrac{D_{k+1} - D_k}{3} \,.\, \dfrac{2f_{k+1} + 3f_k + f_{k-1}}{f_{k+1} + 2f_k + f_{k-1}}$ and is a more approximate value of the average limen for the kth column[4],

f_k = the average relative frequency for the kth column (see Table II., 3rd column),

p_k is as before,

Σ again denotes summation over all values of k,

and L' = the required more approximate limen.

Applying this formula to the same example as before, we get, for instance:

$$\begin{aligned} a_1 &= D_1 + \frac{D_2 - D_1}{3} \,.\, \frac{2f_2 + 3f_1 + f_0}{f_2 + 2f_1 + f_0}, \\ &= -538 + \frac{-370 - (-538)}{3} \,.\, \frac{(2 \times 19) + (3 \times 13) + 0}{19 + (2 \times 13) + 0}, \\ &= -442, \end{aligned}$$

[1] For proof, see Appendix 2.

[2] *Die Gesichtspunkte und Tatsachen der psychophysischen Methodik*, S. 148.

[3] Anyone acquainted with the *Gesichtspunkte etc.* will readily see this for himself. Prof. Müller has, moreover, had the kindness to look critically through the MS. of this paper.

[4] For proof, see Appendix 3.

$$a_2 = D_2 + \frac{D_3 - D_2}{3} \cdot \frac{2f_3 + 3f_2 + f_0}{f_3 + 2f_2 + f_0},$$
$$= -370 + \frac{-195 - (-370)}{3} \cdot \frac{(2 \times 20) + (3 \times 19) + 13}{20 + (2 \times 19) + 13},$$
$$= -280.$$

Having in this manner calculated all the values of a, we can construct a table as before.

TABLE IV. *Calculation of the average upper limen. Second method.*

Col.	a +	a –	p	ap +	ap –
1		442	11		4862
2		280	17		4760
3		96	20		1920
4	97		23	2231	
5	289		12	3468	
6	490		8	3920	
7	696		4	2784	
8	907		3	2721	
9	1114		2	2228	
			100	17352	11542
				11542	
				$5810 = \Sigma ap$	

Hence $$L' = \frac{\Sigma ap}{100} = \frac{5810}{100} = 58{\cdot}1.$$

Thus, although the distribution investigated is very asymmetrical, and although the value for each column by this method differs sensibly from that by the former method (see Table III.), yet the total results by the two methods very closely coincide with one another. Still, the additional work involved in the second method is so insignificant in comparison with the labour of originally making the whole experimental series, that it seems worth undertaking even for such a slight gain of accuracy.

6. *Mean deviation.*

Next to the mean, the most important characteristic of the distribution of the limina is the mean deviation from the mean. By our method we are able to measure the upper and lower mean deviations separately. Here the advantage of the a (used in the 2nd method) over the c (used in the 1st method) is much more marked than before, as the errors of c on the right and left sides respectively of the distribution do not compensate one another, but all tend in the same direction, namely, to make the calculated value too big.

The average deviation from the average will be given by the formula

$$\theta = \frac{\Sigma pd}{\Sigma p} \qquad \text{.............................}(e),$$

where $d_k = | a_k - L' |$[1],

p_k = the same as before,

Σ indicates summation over all values of k,

and θ is the required average deviation.

The only difficulty concerns the particular column into which the average limen itself falls. This column must evidently furnish two p's and two d's, one on each side of L'. Very fair approximations to these values are given by the following formulae:

$$p_r = p_a \frac{D_r - L'}{D_r - D_l} \text{.........................}(f),$$

where D_r and D_l are the D's on the right and left respectively of L', p_a is the percentage for the whole column in which L' falls, and p_r is the required percentage for the portion of this column on the right of L'. To find p_l, we have similarly

$$p_l = p_a \cdot \frac{L' - D_l}{D_r - D_l} \text{..........................}(g).$$

Further, $$a_r = \frac{D_r - L'}{2} \text{..............................}(h),$$

where D_r means the same as before and a_r is the required a on the right of L'. And similarly,

$$a_l = \frac{L' - D_l}{2} \text{..............................}(i).$$

We get, then, the following table.

TABLE V. *Average lower deviation (from average upper limen).*

Col.	$d\,(=58{\cdot}10 - a)$	p	pd
1	500·10	11	5501
2	338·10	17	5748
3	154·10	20	3082
4	29·25	6·7 $(=p_l)$	194
		54·7 $(=\Sigma p)$	14425 $(=\Sigma pd)$

Hence $$\theta^{lo} = \frac{\Sigma pd}{\Sigma p} = \frac{14425}{54{\cdot}7} = 261.$$

[1] The brackets denote that this difference is always treated as positive. L' was found in the preceding section, see equation (d).

For the average upper deviation we get similarly:

TABLE VI. *Average upper deviation (from average upper limen).*

Col.	$d\ (=a-58{\cdot}10)$	p	pd
4	59·45	16·3 $(=p_r)$	969
5	230·90	12	2771
6	431·90	8	3455
7	637·90	4	2552
8	848·90	3	2547
9	1055·90	2	2112
		45·3 $(=\Sigma p)$	14406 $(=\Sigma pd)$

Hence $$\theta^{up}=\frac{\Sigma pd}{\Sigma p}=\frac{14405}{45{\cdot}3}=319.$$

We can further determine the 'measure of precision' h from the usual equation, $h=\frac{1}{\theta\sqrt{\pi}}$. And actually I have found this value of h to coincide very closely with that obtained by means of Müller's method. For instance, we find in our above example the average deviation (upper and lower) $=\theta=288{\cdot}31$. As $\sqrt{\pi}=1{\cdot}772$, we get

$$h=\frac{1}{288{\cdot}31\times 1{\cdot}772}=0{\cdot}00196,$$

whereas Müller, using a unit of measurement 10 times larger than ours, gives $h_{oI}=0{\cdot}0180$. The coincidence becomes still closer when the distribution of values is more approximately Gaussian, as occurs for instance in Merkel's lower limen with C preceding S.

Here our method gives

$$h=\frac{1}{274{\cdot}781\times 1{\cdot}772}=0{\cdot}00205,$$

whereas Müller's h_{uII} is 0·020 (*loc. cit.* p. 71).

7. *Conclusion.*

To sum up, the advantages claimed by the proposed method are:

(1) It dispenses with all questionable assumptions (as of Gauss's law).

(2) It utilizes the experimental data exhaustively. Hereby the exactness and, above all, the fulness of the results are greatly increased.

(3) Nevertheless, the calculations are less difficult than those required for the best methods now in use.

Appendix.

(1) Proof that

$$\frac{\Sigma cp}{100} = \tfrac{1}{2}(D_{s+1} + D_s) - i/n \, . \, \Sigma g.$$

By definition of c,

$$2\Sigma cp = p_s(D_{s+1} + D_s) + p_{s-1}(D_s + D_{s-1}) + \ldots + p_2(D_3 + D_2) + p_1(D_2 + D_1),$$

and therefore, by definition of p,

$$= \frac{100}{n}[(n - g_s)(D_{s+1} + D_s) + (g_s - g_{s-1})(D_s + D_{s-1}) + \ldots$$
$$+ (g_3 - g_2)(D_3 + D_2) + g_2(D_2 + D_1)],$$
$$= 100(D_{s+1} + D_s) - \frac{100}{n}[g_s(D_{s+1} - D_{s-1}) + g_{s-1}(D_s - D_{s-2}) + \ldots$$
$$+ g_2(D_3 - D_1)],$$
$$= 100(D_{s+1} + D_s) - \frac{100}{n} \, . \, 2i\Sigma g,$$

from which $$\frac{\Sigma cp}{100} = \tfrac{1}{2}(D_{s+1} + D_s) - i/n \, . \, \Sigma g.$$

(2) Proof that $$L'_{up} - L'_{lo} = i/n \, . \, \Sigma u.$$

Let N denote the number of intervals between D_{p+1} and ${}_tD$,

= the number of D's producing a mixture of different judgments,

$= 1/n \, . \, (\Sigma g + \Sigma u + \Sigma 1)$.

Then $$D_{s+1} + D_s - ({}_{t+1}D + {}_tD) = 2(D_{s+1} - {}_tD) = 2iN.$$

Consequently,

$$L'_{up} - L'_{lo} = \tfrac{1}{2}(D_{s+1} + D_s) - i/n \, . \, \Sigma g - [\tfrac{1}{2}({}_{t+1}D + {}_tD) + i/n \, . \, \Sigma 1],$$
$$= iN - i/n \, . \, (\Sigma g + \Sigma l),$$
$$= i/n \, . \, \Sigma u.$$

(3) Proof of the formula:

$$a_k = D_k + \frac{D_{k+1} - D_k}{3} \, . \, \frac{2f_{k+1} + 3f_k + f_{k-1}}{f_{k+1} + 2f_k + f_{k-1}}.$$

The vertical and horizontal straight lines of fig. 3 have the same meaning as those of fig. 1; that is, the height of each column represents the *observed average* relative frequency (f in Table II.) of all the limina falling between the two D's at the extremities of the column's base. The curved line, on the other hand, represents the *unknown particular* relative frequency of the limina for every particular value of D; it is thus only limited by the equations: area α = area β, area γ = area δ, area ϵ = area ζ, etc.

Our problem is to find an approximation to the average size of all the limina falling between D_k and D_{k+1}. The true average size is evidently represented by the abscissa of the centroid of the area bounded by the ordinates f_k and f_{k+1}, the axis of x, and the curve. And a very fair approximation will be seen to be furnished by the abscissa of the centroid of the area bounded by the ordinates f_k and f_{k+1}, the axis of x, and the straight line $y_k y_{k+1}$, where $y_k = \frac{1}{2}(f_{k-1} + f_k)$ and $y_{k+1} = \frac{1}{2}(f_k + f_{k+1})$. This approximative value will, it is true, always lie slightly too near the ordinate on the concave side of the curve (in fig. 3, too near to D_k). But all such errors must almost exactly com-

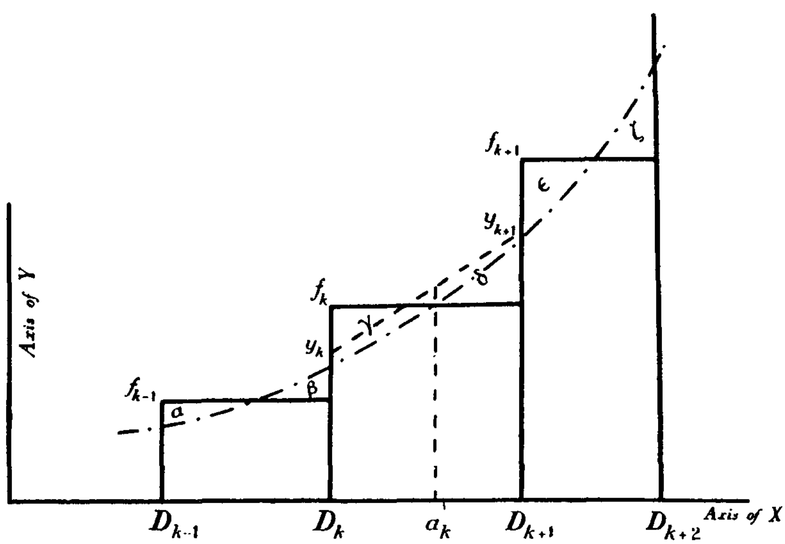

FIG. 3.

pensate one another, not only as regards the ascending compared with the descending branch of the curve, but even as regards the upper compared with the lower portion of each branch separately.

Now let our approximative abscissa be denoted as before by a_k. We get, by known rule, the equation

$$a_k = D_k + \frac{\int_0^\Delta xy \, . \, dx}{\int_0^\Delta y \, . \, dx},$$

where the equation to the straight line $y_k y_{k+1}$ is

$$y = \frac{x(y_{k+1} - y_k)}{D_{k+1} - D_k} + y_k,$$

and
$$\Delta = D_{k+1} - D_k.$$

Hence, on eliminating the variable y,

$$a_k = D_k + \frac{\int_0^\Delta 1/\Delta \, . \, x^2 y_{k+1} \, . \, dx - \int_0^\Delta 1/\Delta \, . \, x^2 y_k \, . \, dx + \int_0^\Delta x y_k \, . \, dx}{\int_0^\Delta 1/\Delta \, . \, x y_{k+1} \, . \, dx - \int_0^\Delta 1/\Delta \, . \, x y_k \, . \, dx + \int_0^\Delta y_k \, . \, dx}$$

$$= D_k + \frac{\Delta^2 (\frac{1}{3} y_{k+1} - \frac{1}{3} y_k + \frac{1}{2} y_k)}{\Delta (\frac{1}{2} y_{k+1} - \frac{1}{2} y_k + y_k)},$$

or, on replacing y_k, y_{k+1}, and Δ by values of f and D, and then simplifying,

$$= D_k + \frac{D_{k+1} - D_k}{3} \, . \, \frac{2f_{k+1} + 3f_k + f_{k-1}}{f_{k+1} + 2f_k + f_{k-1}},$$

which was required to be proved[1].

[1] When the D's are equidistant, there is another still closer approximation given by a_k', where

$$a_k' = D_k + \frac{i}{6} \, . \, \frac{5f_{k+1} + 18f_k + f_{k-1}}{f_{k+1} + 6f_k + f_{k-1}};$$

but the gain by this way is not sensible practically, and the formula becomes very complicated when the intervals are not equal.