Split-Zero Divisor-Pair Reconstruction of Residual Erdős–Straus Shells

A sequel note on supported additive reconstruction beyond the modulo-840 pre-Niemeier datum

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Abstract

The normalized residual shell calculus for the Erdős–Straus equation converts the fixed-shell divisor congruence

$$d \equiv -N \pmod{R}, \qquad N = pa,$$

into the invariant target

$$dN^{-1} \equiv -1$$

in a finite unit torus. The preceding residual pre-Niemeier note used this normalization to construct a static modulo-840 support fibration and its rank-24 closure of type $A_5^4 D_4$. The present note applies the split-zero support formalism to the remaining shell obstruction. The main result is a divisor-pair reconstruction theorem: after reducing by $g = \gcd(R, N)$, a shell works if and only if

$$N_0 = XYZ, \qquad \gcd(X, Y) = 1, \qquad R_0 \mid X + Y, \qquad R_0 = R/g, N_0 = N/g.$$

This reconstructs the missing additive relation hidden behind the multiplicative condition $X/Y \equiv -1$. The corresponding denominators are

$$b = \frac{XZ(X+Y)}{R_0}, \qquad c = \frac{YZ(X+Y)}{R_0}.$$

In the coprime shell case $\gcd(R, pa) = 1$, the reconstruction separates into a central branch

$$a = XYZ, \qquad R \mid X + Y,$$

and an edge branch

$$a = XYZ, \qquad R \mid 4X^2Z + 1.$$

These two additive reconstruction laws replace the three raw p-origin divisor tests. We derive explicit identity generators, edge-ramification identities, a new necessary condition for primes surviving all ramified edge shells, and a split-zero orbit-support invariant distinguishing unsupported absence from supported parity-zero certificate sectors. The remaining constructive target is formulated as a finite supported divisor-pair covering problem over the six residual classes modulo 840.

Contents

1 Residual shells and reduced normalization

Let

$$\mathrm{ES}(n): \qquad \frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

denote the Erdős–Straus assertion for n. As usual, it suffices to consider prime p, since a decomposition for a prime divisor of n lifts to one for n by multiplying denominators. The standard elementary identities reduce the prime problem to

$$p \equiv 1 \pmod{24}.$$

The residual pre-Niemeier note records the corresponding fixed-shell divisor calculus and the six-class modulo-840 reduction [2, 1]. We recall only the form needed for the reconstruction.

Let

$$p \equiv 1 \pmod{24}, \qquad R \equiv 3 \pmod{4}, \qquad a = a_R(p) := \frac{p+R}{4}, \qquad N := pa.$$

Then

$$\frac{4}{p} - \frac{1}{a} = \frac{4a-p}{pa} = \frac{R}{N}.$$

The shell R asks whether R/N can be completed as a sum of two unit fractions.

Definition 1.1 (Reduced shell)**.** *Set*

$$g := \gcd(R, N), \qquad R_0 := \frac{R}{g}, \qquad N_0 := \frac{N}{g}.$$

Then

$$\gcd(R_0, N_0) = 1$$

and

$$\frac{R}{N} = \frac{R_0}{N_0}.$$

Lemma 1.2 (Reduced fixed-shell divisor certificate)**.** *The shell R completes $4/p$ with the prescribed a if and only if there exists a positive divisor*

$$d \mid N_0^2$$

such that

$$d \equiv -N_0 \pmod{R_0}.$$

Given such d, put

$$e := \frac{N_0^2}{d}, \qquad b := \frac{N_0 + d}{R_0}, \qquad c := \frac{N_0 + e}{R_0}.$$

Then

$$\frac{4}{p} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Proof. The completion condition is

$$\frac{R_0}{N_0} = \frac{1}{b} + \frac{1}{c}.$$

Multiplication gives

$$R_0 bc = N_0(b + c),$$

equivalently

$$(R_0 b - N_0)(R_0 c - N_0) = N_0^2.$$

Thus a solution gives

$$d := R_0 b - N_0 \mid N_0^2, \qquad d \equiv -N_0 \pmod{R_0}.$$

Conversely, suppose $d \mid N_0^2$ and $d \equiv -N_0 \pmod{R_0}$. Since N_0 is invertible modulo R_0, the complementary divisor

$$e = \frac{N_0^2}{d}$$

satisfies

$$e \equiv \frac{N_0^2}{-N_0} \equiv -N_0 \pmod{R_0}.$$

Therefore b, c are integers. Substitution into

$$(R_0 b - N_0)(R_0 c - N_0) = N_0^2$$

gives $R_0/N_0 = 1/b + 1/c$, and hence the desired identity. $\square$

Definition 1.3 (Normalized shell support)**.** *Assume $R_0 > 1$. Define*

$$G_{R_0} := (\mathbb{Z}/R_0\mathbb{Z})^\times.$$

The normalized reduced divisor support is

$$\mathcal{U}_{R_0}(N_0) := \{dN_0^{-1} \bmod R_0 : d \mid N_0^2\} \subseteq G_{R_0}.$$

Proposition 1.4 (Universal target)**.** *For $R_0 > 1$, the shell works if and only if*

$$-1 \in \mathcal{U}_{R_0}(N_0).$$

Proof. By Lemma 1.2, the shell works if and only if some divisor $d \mid N_0^2$ satisfies

$$d \equiv -N_0 \pmod{R_0}.$$

Since N_0 is a unit modulo R_0, this is equivalent to

$$dN_0^{-1} \equiv -1 \pmod{R_0}.$$

□

The support-idempotent and split-zero notes emphasize that a raw scalar equation can collapse branch support unless the support coordinate is retained [3, 4]. Here the raw congruence $d \equiv -N_0$ hides the normalized, branch-independent target -1.

2 Signed exponents and the divisor-pair reconstruction

Factor

$$N_0 = \prod_{j=1}^{s} q_j^{e_j}.$$

Since $\gcd(R_0, N_0) = 1$, each q_j is a unit modulo R_0. Every divisor $d \mid N_0^2$ has the unique form

$$d = \prod_{j=1}^{s} q_j^{k_j}, \qquad 0 \le k_j \le 2e_j.$$

Thus

$$dN_0^{-1} = \prod_{j=1}^{s} q_j^{k_j - e_j}.$$

The signed exponent box is

$$B(N_0) := \prod_{j=1}^{s} [-e_j, e_j]_{\mathbb{Z}},$$

and the signed exponent map is

$$\lambda_{R_0,N_0} : B(N_0) \longrightarrow G_{R_0}, \qquad \lambda_{R_0,N_0}(\beta_1, \ldots, \beta_s) = \prod_{j=1}^{s} q_j^{\beta_j}.$$

Then

$$\mathcal{U}_{R_0}(N_0) = \lambda_{R_0,N_0}(B(N_0)).$$

The divisor-pair reconstruction below is the additive support refinement of this multiplicative signed-exponent condition.

Definition 2.1 (Supported coprime divisor-pair set)**.** *For* R_0, N_0 *with* $\gcd(R_0, N_0) = 1$*, define*

$$\mathcal{A}_{R_0,N_0} := \Big\{(X,Y,Z) \in \mathbb{Z}^3_{>0} : N_0 = XYZ,\ \gcd(X,Y) = 1,\ R_0 \mid X+Y\Big\}.$$

Equivalently, the primitive pair support is

$$\mathcal{A}^{\text{pair}}_{R_0,N_0} := \{(X,Y) : \gcd(X,Y) = 1,\ XY \mid N_0,\ R_0 \mid X+Y\}.$$

The value of Z *is then forced by*

$$Z = N_0/(XY).$$

Theorem 2.2 (Divisor-pair reconstruction)**.** *The shell* R *works if and only if*

$$\mathcal{A}_{R_0,N_0} \neq \varnothing.$$

More precisely, if

$$N_0 = XYZ, \qquad \gcd(X,Y) = 1, \qquad R_0 \mid X+Y,$$

then

$$d = X^2 Z, \qquad e = Y^2 Z$$

are complementary certificate divisors of N_0^2*, and the corresponding denominators are*

$$\boxed{b = \frac{XZ(X+Y)}{R_0}, \qquad c = \frac{YZ(X+Y)}{R_0}.}$$

Conversely, every reduced shell certificate arises uniquely from such a triple (X,Y,Z) *up to exchanging* X *and* Y*.*

Proof. Suppose first that the shell works. Then by Proposition 1.4 there exists $d \mid N_0^2$ with

$$dN_0^{-1} \equiv -1 \pmod{R_0}.$$

Write

$$d = N_0 \prod_{j=1}^{s} q_j^{\beta_j}, \qquad -e_j \le \beta_j \le e_j.$$

Let

$$X := \prod_{\beta_j > 0} q_j^{\beta_j}, \qquad Y := \prod_{\beta_j < 0} q_j^{-\beta_j}.$$

Then

$$\gcd(X,Y) = 1, \qquad XY \mid N_0.$$

Set

$$Z := \frac{N_0}{XY}.$$

Then $N_0 = XYZ$. Moreover

$$dN_0^{-1} = \frac{X}{Y}$$

in G_{R_0}, so

$$\frac{X}{Y} \equiv -1 \pmod{R_0}.$$

Since Y is invertible modulo R_0, this is equivalent to

$$X + Y \equiv 0 \pmod{R_0}.$$

Thus $(X, Y, Z) \in \mathcal{A}_{R_0,N_0}$.

Conversely, suppose $(X, Y, Z) \in \mathcal{A}_{R_0,N_0}$. Set

$$d := X^2 Z.$$

Since

$$N_0^2 = X^2 Y^2 Z^2,$$

we have $d \mid N_0^2$. Also

$$dN_0^{-1} = X^2 Z(XYZ)^{-1} = \frac{X}{Y} \equiv -1 \pmod{R_0},$$

because $R_0 \mid X + Y$. Hence d is a reduced certificate. The complementary divisor is

$$e = \frac{N_0^2}{d} = Y^2 Z.$$

Lemma 1.2 gives

$$b = \frac{N_0 + d}{R_0} = \frac{XYZ + X^2 Z}{R_0} = \frac{XZ(X + Y)}{R_0},$$

and similarly

$$c = \frac{N_0 + e}{R_0} = \frac{YZ(X + Y)}{R_0}.$$

The construction from d to (X, Y, Z) is unique because the positive and negative parts of the signed exponent vector are unique. Replacing d by N_0^2/d exchanges X and Y. □

Corollary 2.3 (Reduced identity formula)**.** *For every* $(X, Y, Z) \in \mathcal{A}_{R_0,N_0}$*,*

$$\boxed{\frac{4}{p} = \frac{1}{a} + \frac{1}{XZ(X + Y)/R_0} + \frac{1}{YZ(X + Y)/R_0}.}$$

Proof. The two latter denominators give

$$\frac{R_0}{XZ(X + Y)} + \frac{R_0}{YZ(X + Y)} = \frac{R_0(Y + X)}{XYZ(X + Y)} = \frac{R_0}{N_0}.$$

Since $R/N = R_0/N_0$ and $4/p - 1/a = R/N$, the formula follows. □

Remark 2.4 (Additive reconstruction)**.** *The multiplicative normalized condition is*

$$X/Y \equiv -1 \pmod{R_0}.$$

The divisor-pair theorem reconstructs it as the additive support condition

$$X + Y \in R_0\mathbb{Z}.$$

This is the residual-shell instance of the split-zero principle that multiplicative support alone does not recover the missing additive relations; those additive relations must be carried explicitly in the support coordinate [3].

3 Split-zero orbit support of divisor pairs

The complement involution on certificate divisors,

$$d \longmapsto \frac{N_0^2}{d},$$

becomes

$$(X, Y, Z) \longmapsto (Y, X, Z)$$

under Theorem 2.2. Therefore the primitive certificate is not the ordered pair (X, Y) but its unordered orbit.

Definition 3.1 (Primitive orbit certificate)**.** *The primitive orbit certificate set is*

$$\mathcal{O}_{R_0,N_0} := \mathcal{A}_{R_0,N_0}/((X, Y, Z) \sim (Y, X, Z)).$$

Proposition 3.2 (Orbit certificate criterion)**.** *For $R_0 > 2$, the shell works if and only if*

$$\mathcal{O}_{R_0,N_0} \neq \varnothing.$$

Moreover, the ordered certificate set is a free two-fold lift of $\mathcal{O}_{R_0,N_0}$.

Proof. Nonemptiness is preserved by quotient. If (X, Y, Z) is fixed by the involution, then $X = Y$. Since $\gcd(X, Y) = 1$, this forces $X = Y = 1$. But then $R_0 \mid X + Y = 2$, contradicting $R_0 > 2$. Hence the involution is free. □

Let $G(\mathbb{F}_2) = \mathbb{F}_2 \sqcup \{\tau\}$ denote the split-zero semiring: τ is the semiring zero, while $0_{\mathbb{F}_2}$ is the supported arithmetic zero. The split-zero notes develop this distinction, the support character, and semimodule classification by support diagrams [3, 5].

Definition 3.3 (Split-zero shell value)**.** *For $R_0 > 2$, define*

$$\nu_R(p) := \begin{cases} \tau, & \mathcal{O}_{R_0,N_0} = \varnothing, \\ 0_{\mathbb{F}_2}, & \mathcal{O}_{R_0,N_0} \neq \varnothing. \end{cases}$$

Theorem 3.4 (Split-zero certificate theorem)**.** *For $R_0 > 2$,*

$$R \text{ works} \iff \nu_R(p) = 0_{\mathbb{F}_2}.$$

The ordinary parity count of ordered certificates is always zero in $\mathbb{F}_2$; the supported-zero value above records whether that parity-zero sector is supported.

Proof. By Proposition 3.2, shell success is equivalent to nonemptiness of $\mathcal{O}_{R_0,N_0}$. The ordered certificate set is a free two-fold lift, so its cardinality is even in both the empty and nonempty cases. The split-zero value records precisely the support of the orbit set. □

4 Coprime shells and the central-edge decomposition

Assume throughout this section that

$$\gcd(R, N) = 1.$$

Thus $R_0 = R$ and $N_0 = N = pa$.

The divisor-pair theorem says that a shell certificate is equivalent to

$$pa = XYZ, \qquad \gcd(X, Y) = 1, \qquad R \mid X + Y.$$

Since p occurs to exponent one in $N = pa$, the prime p lies in exactly one of X, Y, Z. This gives the central-edge decomposition.

4.1 Central branch

The central branch is the case

$$p \mid Z.$$

Write

$$Z = pZ_0.$$

Then

$$a = XYZ_0.$$

The condition remains

$$R \mid X + Y.$$

Theorem 4.1 (Central branch)**.** *A coprime shell has a central certificate if and only if there exist positive integers X, Y, Z such that*

$$\boxed{a = XYZ, \qquad \gcd(X, Y) = 1, \qquad R \mid X + Y.}$$

The resulting identity is

$$\boxed{\frac{4}{p} = \frac{1}{a} + \frac{1}{pXZ(X+Y)/R} + \frac{1}{pYZ(X+Y)/R}.}$$

Proof. This is Theorem 2.2 with $N = pa$ and Z replaced by pZ. The denominator formula follows from Corollary 2.3. □

4.2 Edge branch

One edge branch is the case

$$p \mid X.$$

Write

$$X = pX_0.$$

Then

$$a = X_0YZ.$$

The condition

$$R \mid pX_0 + Y$$

is transformed using

$$p = 4a - R = 4X_0YZ - R.$$

Modulo R,

$$pX_0 + Y \equiv 4X_0^2YZ + Y = Y(4X_0^2Z + 1).$$

Since $\gcd(R, a) = 1$, Y is invertible modulo R. Hence

$$R \mid pX_0 + Y \iff R \mid 4X_0^2Z + 1.$$

Theorem 4.2 (Edge branch)**.** *A coprime shell has an edge certificate if and only if there exist positive integers X, Y, Z such that*

$$\boxed{a = XYZ, \qquad \gcd(X, Y) = 1, \qquad R \mid 4X^2Z + 1.}$$

The resulting identity is

$$\boxed{\frac{4}{p} = \frac{1}{a} + \frac{1}{pXZ(pX+Y)/R} + \frac{1}{YZ(pX+Y)/R}.}$$

The opposite edge orientation is obtained by exchanging X and Y.

Proof. The preceding derivation proves that the divisor-pair condition in the case $p \mid X$ is equivalent to

$$a = XYZ, \qquad R \mid 4X^2Z + 1.$$

The corresponding pair in Theorem 2.2 is

$$X' = pX, \qquad Y' = Y, \qquad Z' = Z.$$

Thus

$$b = \frac{X'Z'(X'+Y')}{R} = \frac{pXZ(pX+Y)}{R},$$

and

$$c = \frac{Y'Z'(X'+Y')}{R} = \frac{YZ(pX+Y)}{R}.$$

The case $p \mid Y$ is symmetric. □

Corollary 4.3 (Two additive reconstruction laws)**.** *In the coprime shell case, the three raw divisor origins reduce to two additive reconstruction laws:*

$$\boxed{a = XYZ, \quad R \mid X + Y}$$

and

$$\boxed{a = XYZ, \quad R \mid 4X^2Z + 1.}$$

Proof. The central branch is Theorem 4.1. The two edge origins are exchanged by the divisor-pair involution and are represented by Theorem 4.2. □

5 Identity generators

The central-edge decomposition gives constructive families of identities.

Theorem 5.1 (Central identity generator)**.** *Let $X, Y, Z, R \in \mathbb{Z}_{>0}$ satisfy*

$$\gcd(X, Y) = 1, \qquad R \mid X + Y, \qquad R \equiv 3 \pmod 4.$$

Set

$$a := XYZ, \qquad p := 4XYZ - R.$$

If p is prime and $p \equiv 1 \pmod{24}$, then

$$\boxed{\frac{4}{p} = \frac{1}{XYZ} + \frac{1}{pXZ(X+Y)/R} + \frac{1}{pYZ(X+Y)/R}.}$$

Proof. The hypotheses give

$$a = XYZ = \frac{p+R}{4}.$$

The central branch theorem applies. □

Theorem 5.2 (Edge identity generator)**.** *Let $X, Y, Z, R \in \mathbb{Z}_{>0}$ satisfy*

$$\gcd(X, Y) = 1, \qquad R \mid 4X^2Z + 1, \qquad R \equiv 3 \pmod 4.$$

Set

$$a := XYZ, \qquad p := 4XYZ - R.$$

If p is prime and $p \equiv 1 \pmod{24}$, then

$$\boxed{\frac{4}{p} = \frac{1}{XYZ} + \frac{1}{pXZ(pX+Y)/R} + \frac{1}{YZ(pX+Y)/R}.}$$

Proof. Again $a = (p + R)/4$. Since

$$pX + Y = (4XYZ - R)X + Y \equiv Y(4X^2Z + 1) \pmod R$$

and $\gcd(Y, R) = 1$, the congruence $R \mid 4X^2Z + 1$ gives $R \mid pX + Y$. The edge branch theorem applies. □

Corollary 5.3 (One-sided central constructor)**.** *Let $R \equiv 3 \pmod 4$ and $p \equiv 1 \pmod{24}$ with $a = (p + R)/4$ and $\gcd(R, pa) = 1$. If a has a divisor D satisfying*

$$D \equiv -1 \pmod R,$$

then shell R works.

Proof. Take

$$X = D, \qquad Y = 1, \qquad Z = a/D.$$

Then $R \mid X + Y$. The central branch theorem applies. □

Corollary 5.4 (One-sided edge constructor)**.** *Let* $R \equiv 3 \pmod 4$ *and* $p \equiv 1 \pmod{24}$ *with* $a = (p+R)/4$ *and* $\gcd(R, pa) = 1$*. If there is a factorization*

$$a = XZ$$

such that

$$R \mid 4X^2Z + 1,$$

then shell R *works.*

Proof. Apply Theorem 4.2 with $Y = 1$. □

6 Ramified edge shells

The edge branch has a trace-cover interpretation. In $G_R = (\mathbb{Z}/R\mathbb{Z})^\times$, the two edge targets are

$$-p, \qquad -p^{-1}.$$

They are exchanged by inversion. Introduce the invariant and anti-invariant edge coordinates

$$P := \frac{p + p^{-1}}{2}, \qquad J := \frac{p - p^{-1}}{2}$$

whenever 2 is invertible modulo R. Then the two edge targets are

$$-P - J, \qquad -P + J.$$

The edge sheet ramifies when

$$J = 0,$$

i.e.

$$p^2 \equiv 1 \pmod R.$$

The negative ramification $p \equiv -1 \pmod R$ gives an automatic shell.

Theorem 6.1 (Negative edge ramification identity)**.** *Let*

$$p \equiv 1 \pmod{24}.$$

Suppose $R \equiv 3 \pmod 4$ *and*

$$R \mid p + 1.$$

Then shell R *works. With*

$$a = \frac{p+R}{4},$$

one has the explicit identity

$$\boxed{\frac{4}{p} = \frac{1}{a} + \frac{1}{a(p+1)/R} + \frac{1}{pa(p+1)/R}}.$$

Proof. Take the reduced divisor-pair triple

$$X = 1, \qquad Y = p, \qquad Z = a.$$

Then

$$N = pa = XYZ, \qquad \gcd(X, Y) = 1, \qquad R \mid X + Y = p + 1.$$

The divisor-pair theorem applies and gives

$$b = \frac{XZ(X+Y)}{R} = \frac{a(p+1)}{R},$$

$$c = \frac{YZ(X+Y)}{R} = \frac{pa(p+1)}{R}.$$

Since R is odd and $p \equiv 1 \pmod 4$, the value $a = (p + R)/4$ is integral. □

Corollary 6.2 (A necessary condition after ramified edge shells)**.** *Let*

$$p \equiv 1 \pmod{24}.$$

If p is not solved by any ramified edge shell of Theorem 6.1, then every odd prime divisor of

$$\frac{p+1}{2}$$

is congruent to $1 \pmod 4$*.*

Proof. Since $p \equiv 1 \pmod 8$, one has

$$p + 1 \equiv 2 \pmod 8,$$

so $(p + 1)/2$ is odd. If an odd prime

$$q \mid \frac{p+1}{2}$$

satisfies $q \equiv 3 \pmod 4$, then $q \mid p + 1$ and Theorem 6.1 applies with $R = q$. Hence any prime surviving all such shells must have no such divisor. □

Theorem 6.3 (Positive edge ramification coalescence)**.** *Suppose*

$$R \mid p - 1.$$

Then

$$-p \equiv -1 \equiv -p^{-1} \pmod R.$$

Thus the three origin targets coalesce into the single central target -1*.*

Proof. If $p \equiv 1 \pmod R$, then $p^{-1} \equiv 1 \pmod R$. Therefore all three targets are -1. □

7 Finite support, saturation, and the capacity defect

The divisor-pair theorem gives a finite support condition

$$XY \mid N_0.$$

The saturated condition would only require a congruence relation in the subgroup generated by the prime factors. The split-zero support-idempotent distinction separates these two.

Definition 7.1 (Finite and saturated pair supports)**.** *Let*

$$K := \langle q_1, \dots, q_s \rangle \leq G_{R_0}.$$

The saturated target support condition is

$$-1 \in K.$$

The finite pair support condition is

$$\mathcal{A}_{R_0,N_0} \neq \varnothing.$$

Define formal support idempotents

$$\varepsilon_{\mathrm{sat}}(-1) = \begin{cases} 1, & -1 \in K, \\ 0, & -1 \notin K, \end{cases}$$

and

$$\varepsilon_{\mathrm{fin}}(-1) = \begin{cases} 1, & \mathcal{A}_{R_0,N_0} \neq \varnothing, \\ 0, & \mathcal{A}_{R_0,N_0} = \varnothing. \end{cases}$$

The capacity defect at -1 is

$$\omega^{\mathrm{cap}}_{R_0,N_0}(-1) := [-1]\big(\varepsilon_{\mathrm{sat}}(-1) - \varepsilon_{\mathrm{fin}}(-1)\big).$$

Proposition 7.2 (Capacity defect criterion)**.** *Assume $-1 \in K$. Then the shell works if and only if*

$$\omega^{\mathrm{cap}}_{R_0,N_0}(-1) = 0.$$

Proof. Under the assumption $-1 \in K$, the saturated idempotent is 1. The defect vanishes exactly when the finite idempotent is also 1, which is exactly $\mathcal{A}_{R_0,N_0} \neq \varnothing$. Theorem 2.2 identifies this with shell success. □

Remark 7.3 (Projective inflow analogue)**.** *In the split-zero CUE/Hurwitz notes, a finite visible two-end defect is canceled after passing to the projective/infinite support sector [5]. Here the saturated subgroup K is the analogue of the projective support, while $XY \mid N_0$ is the finite visible capacity. Proposition 7.2 isolates the finite-capacity gap.*

8 Examples

8.1 The observed champion shell

Let

$$p = 8803369, \qquad R = 107.$$

Then

$$a = \frac{p+R}{4} = 2200869 = 3^2 \cdot 11^2 \cdot 43 \cdot 47.$$

Set

$$X = 1, \qquad Y = 3^2 \cdot 43 \cdot 47 = 18189, \qquad Z = 11^2 = 121.$$

Then

$$a = XYZ,$$

and

$$X + Y = 18190 = 107 \cdot 170.$$

Thus the central branch theorem applies.

The denominators are

$$b = \frac{pXZ(X+Y)}{R} = 8803369 \cdot 121 \cdot 170,$$

and

$$c = \frac{pYZ(X+Y)}{R} = 8803369 \cdot 18189 \cdot 121 \cdot 170.$$

Therefore

$$\frac{4}{8803369} = \frac{1}{2200869} + \frac{1}{8803369 \cdot 121 \cdot 170} + \frac{1}{8803369 \cdot 18189 \cdot 121 \cdot 170}.$$

The visible divisor certificate is

$$d = p \cdot 11^2.$$

The reconstructed additive reason is the supported divisor-pair relation

$$\boxed{1 + 3^2 \cdot 43 \cdot 47 = 107 \cdot 170.}$$

8.2 A saturated but unsupported failed shell

Let

$$p = 8803369, \qquad R = 43.$$

Then

$$a = \frac{p+R}{4} = 2200853 = 379 \cdot 5807.$$

Modulo 43,

$$379 \equiv 35, \qquad 5807 \equiv 2.$$

The saturated relation

$$379^2 \cdot 5807 \equiv -1 \pmod{43}$$

holds, since

$$35^2 \cdot 2 \equiv 42 \equiv -1 \pmod{43}.$$

Thus -1 lies in the subgroup generated by the prime factors of a. However, the signed exponent vector realizing this relation is

$$(2, 1),$$

while the finite exponent box is only

$$[-1, 1]^2.$$

Equivalently, in divisor-pair language the relation wants

$$X = 379^2 \cdot 5807, \qquad Y = 1,$$

so

$$43 \mid X + Y,$$

but

$$XY = 379^2 \cdot 5807 \nmid a = 379 \cdot 5807.$$

Therefore the shell fails by finite-capacity support obstruction:

$$\omega^{\mathrm{cap}}_{43,a}(-1) \neq 0.$$

9 The finite divisor-pair covering problem

The modulo-840 sieve proves that every hard prime surviving $R = 3, 7$ lies in

$$S_{840} = \{1, 121, 169, 289, 361, 529\} \pmod{840},$$

and the residual pre-Niemeier note closes this static support fibration to the $A_5^4 D_4$ Niemeier root type [2]. The divisor-pair theorem identifies the remaining dynamic problem.

Definition 9.1 (Central pair-cover at shell R)**.** *For a prime $p \in S_{840}$ and $R \equiv 3 \pmod 4$, let*

$$a_R(p) = \frac{p + R}{4}.$$

The central pair-cover condition is

$$\mathcal{C}_R^{\mathrm{cent}}(p) : \exists X, Y, Z \in \mathbb{Z}_{>0} : a_R(p) = XYZ, \ \gcd(X, Y) = 1, \ R \mid X + Y.$$

Definition 9.2 (Edge pair-cover at shell R)**.** *The edge pair-cover condition is*

$$\mathcal{C}_R^{\mathrm{edge}}(p) : \exists X, Y, Z \in \mathbb{Z}_{>0} : a_R(p) = XYZ, \ \gcd(X, Y) = 1, \ R \mid 4X^2 Z + 1.$$

Theorem 9.3 (Coprime pair-cover criterion)**.** *Assume*

$$\gcd(R, p a_R(p)) = 1.$$

Then shell R works if and only if

$$\mathcal{C}_R^{\mathrm{cent}}(p) \quad \textit{or} \quad \mathcal{C}_R^{\mathrm{edge}}(p)$$

holds.

Proof. This is Corollary 4.3. □

Definition 9.4 (Finite divisor-pair cover of the residual tail). *A set of shell-template families*

$$\mathfrak{T} = \{\mathcal{T}_i\}_{i \in I}$$

is a finite divisor-pair cover of the residual tail up to modulus M if every prime residue

$$p \in S_{840}$$

inside the chosen congruence universe modulo M satisfies at least one central or edge template condition in $\mathfrak{T}$.

Problem 9.5 (Constructive residual divisor-pair cover). *Construct a finite family of central and edge templates whose associated congruence and divisibility conditions cover all primes*

$$p \bmod 840 \in S_{840}.$$

Equivalently, construct a finite family of identities generated by Theorems 5.1 and 5.2 which covers the residual six-class prime problem.

Problem 9.5 is the constructive target left by the present note. It replaces divisor enumeration by the construction of supported additive divisor-pair relations.

10 Equivalence to the remaining Erdős–Straus problem

Theorem 10.1 (Divisor-pair form of the remaining Erdős–Straus problem). *The Erdős–Straus conjecture is equivalent to the following assertion.*

For every prime

$$p \bmod 840 \in S_{840},$$

there exists an admissible residual shell

$$R \equiv 3 \pmod 4$$

such that, with

$$a = \frac{p+R}{4}, \qquad N = pa, \qquad g = \gcd(R, N), \qquad R_0 = R/g, \qquad N_0 = N/g,$$

there exist positive integers X, Y, Z satisfying

$$\boxed{N_0 = XYZ, \qquad \gcd(X, Y) = 1, \qquad R_0 \mid X + Y.}$$

Proof. The standard reductions solve all non-hard prime classes. The non-square modulo-840 theorem solves every hard-class prime outside S_{840} by $R = 3$ or $R = 7$ [1, 2]. Hence only primes in S_{840} remain.

For any such prime and shell, Theorem 2.2 gives an exact equivalence between shell success and the existence of X, Y, Z satisfying the displayed divisor-pair conditions. Therefore solving the displayed divisor-pair problem for every prime in S_{840} is equivalent to solving all remaining Erdős–Straus prime cases, hence to the conjecture. □

Corollary 10.2 (Supported additive obstruction)**.** *For a remaining prime $p \in S_{840}$, the unresolved obstruction is the family of split-zero support values*

$$\nu_R(p) = \begin{cases} \tau, & \mathcal{A}_{R_0,N_0} = \varnothing, \\ 0_{\mathbb{F}_2}, & \mathcal{A}_{R_0,N_0} \neq \varnothing, \end{cases}$$

as $R \equiv 3 \pmod 4$ varies. The prime p is solved if and only if at least one of these values is supported.

Proof. This is Theorem 3.4 combined with Theorem 10.1. □

11 Summary of new algebraic reductions

The preceding note [2] constructs a static support geometry:

$$V_{840} \twoheadrightarrow C_2^2, \qquad C_6\text{-fibers}, \qquad A_5^4 D_4.$$

The present note adds the dynamic shell reconstruction:

$$dN_0^{-1} \equiv -1 \quad \Longleftrightarrow \quad N_0 = XYZ, \ \gcd(X,Y) = 1, \ R_0 \mid X + Y.$$

Thus the remaining shell is governed by a supported additive relation, not merely by a bounded multiplicative ratio.

The principal formal consequences are:

(i) reduced shell certificates are equivalent to supported coprime divisor pairs;

(ii) complementing certificate divisors exchanges X and Y, so the primitive certificate is an unordered orbit;

(iii) the split-zero certificate value distinguishes unsupported absence from supported parity-zero orbit sectors;

(iv) coprime shells split into central and edge reconstruction laws;

(v) negative edge ramification $R \mid p+1$ gives an automatic identity and the necessary condition of Corollary 6.2;

(vi) finite-capacity failure is the gap between a saturated congruence relation and an actually supported divisor-pair relation;

(vii) the remaining Erdős–Straus problem is equivalent to constructing supported divisor-pair sums for primes in S_{840}.

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