Multi-Dimensional Tensor Structure of the Reverse Collatz Tree: An Ontological Proposal via a Fundamental Multidimensional Matrix

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Abstract

We present a multi-dimensional tensor reformulation of the reverse tree of the Collatz $3n+1$ problem, organized around a constructive 8-dimensional matrix whose diagonal is the sequence of increasing positive odd integers. Each odd integer is mapped to a unique point in $\mathbb{Z}^8$ via algebraic and dynamical coordinates: $(n, \sigma, q, \nu_3, \nu_2(3n+1), \text{freq}, P_{\max}, p_{\min})$, all generated by arithmetic principles over integers. **The central contribution is structural**: we show that the reverse Collatz tree is reducible to a system of simple arithmetic generators over integers, in number sufficient to capture the specific arithmetic of the problem. By *simple arithmetic generator* we mean an elementary operation defined directly on integers (addition, multiplication, division, p-adic valuation, iteration count), without need for extension to auxiliary structures. Generators of this type are accepted as definitional in the foundations of mathematics: $\mathbb{N}$ is generated by the rule $+1$ (Peano), the positive odd integers by the rule $+2$, without requiring any formal proof of coverage. We show that the reverse Collatz tree articulates into 8 generators of this same nature, organized into the 8-dimensional tensor matrix. Individual identities connecting the dimensions appear in equivalent forms in the classical Collatz literature [9, 6, 13] —— the contribution is not to rediscover them, but to reorganize them so that the structural reduction becomes visible. We present the matrix with its generators, demonstrate structural identities connecting the dimensions, systematically compare with universally accepted generating principles for $\mathbb{N}$ and $\mathbb{N}^+_{\text{odd}}$, and formulate an *ontological proposal* (Principle 16) that expresses the structural rationale shared between our formulation and accepted generators such as Peano. We argue from symmetry of treatment: generic objections such as the trivial counterexample apply uniformly to all generating formulations, and the acceptance of Peano as tautological reflects foundational convention, not formal demonstration of non-existence of exclusions. Computational verification of the identities in 17,501 odd integers of magnitude $\sim 10^9$ is presented together with the historical verification of the conjecture by Barina (2020) for $n \le 2^{68}$. We do not claim a proof: we offer a structural framework whose evaluation rests with the community. In an illustrative appendix, we apply the methodology to three analogous dynamical systems over $\mathbb{N}^+$ —— one convergent in 6D demonstrable by induction, another in 6D with closed form via binary structure, and a third in 7D where coverage demonstrably fails —— evidencing that the formulation is dimension-adaptive and does not constitute an *ad hoc* construction for the Collatz case.

Keywords: Collatz Conjecture, $3n+1$ problem, reverse tree, multi-dimensional tensor structure, arithmetic foliation, dyadic distribution, generating principles, mathematical ontology.

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1 Introduction

1.1 Historical context

The conjecture known as the "$3n+1$ problem" states that for every positive integer n, the iterated sequence defined by

$$T(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n+1)/2 & \text{if } n \text{ is odd,} \end{cases} \tag{1}$$

eventually reaches the value 1.

Historical attribution is debated [8]. Although frequently associated with Lothar Collatz from 1937, the problem also appears in correspondence attributed to Kakutani, Ulam, Hasse, and Thwaites in subsequent decades. We acknowledge this multiplicity of origins; we use "Collatz Conjecture" by convention, without implying exclusive attribution.

1.2 State of the art

Computational verifications by Barina [11] established the conjecture for all $n \leq 2^{68} \approx 2.95 \times 10^{20}$, finding no counterexample. Partial theoretical results include:

- **Terras [9]:** the density of integers whose trajectory eventually descends below the initial value equals 1;
- **Krasikov [12]:** quantitative estimates on the fraction of odd integers satisfying the conjecture;
- **Eliahou [18]:** non-trivial lower bounds for hypothetical cycle lengths;
- **Tao [10]:** almost every integer has a logarithmically bounded trajectory;
- **Lagarias [7]:** p-adic structure of the orbits;
- **Wirsching [13]:** systematic treatment as a discrete dynamical system;
- **Garner [17]:** classical heuristic analysis of the operation T and statistical properties of trajectories;
- **Sinai [15]:** ergodic and statistical approach.

Each partial result establishes convergence *in measure* or *in density*; the pointwise conjecture remains open.

1.3 Existing multi-dimensional reformulations

Several structural reformulations have been proposed:

- **Conway [14]:** FRACTRAN machines, demonstrating undecidability of generalizations;
- **Wirsching [13]:** Collatz operator as a dynamical system on $\mathbb{Z}_2$;
- **Sinai [15]:** multi-dimensional ergodic analysis via invariant measures;
- **Kontorovich-Lagarias [16]:** stopping time statistics;
- **Tao [10]:** harmonic analysis on $\mathbb{Z}_2$.

1.4 Contribution of this work

Our proposal is articulated around four contributions:

1. **Structural reduction** of the reverse Collatz tree to a system of 8 simple arithmetic generators over integers —— elementary operations (+, ×, /, p-adic valuation, iteration counting) of the same nature as those that generate $\mathbb{N}$ via Peano or the odd integers via +2. This reduction is articulated in the 8-dimensional tensor matrix;
2. **Structural identities** connecting the 8 dimensions and enabling articulated multi-dimensional analysis (several appear in equivalent forms in the classical literature; the contribution is the *organization*, not the discovery);
3. **Systematic comparison** with accepted generators of $\mathbb{N}$ and $\mathbb{N}^+_{\text{odd}}$ (Peano, +2 succession, factorization), identifying the shared structural rationale;
4. **Ontological proposal** (Principle 16) formalizing the underlying generating principle, offered for community evaluation.

Important: this work does not claim a formal proof of the Collatz Conjecture. We offer a structural framework and the formalization of a principle whose adoption as an axiom would resolve the conjecture by construction. The formal status —— whether *proof* or *proposal* —— is for the scientific community to confer.

1.5 Structure of the article

The article is organized as follows. Section 2 introduces the definitions and notation. Section 3 explicitly constructs the 8D tensor matrix, with the interpretation as a canonical factorization of the problem. Section 4 proves the structural identities connecting the dimensions and discusses their historical positioning. Section 5 articulates the systematic comparison with universally accepted generating principles (Peano, +2, factorization), introducing the ontological criterion of acceptance. Section 6 reports the computational verification of the identities. Section 7 formulates Principle 16, central to this work, with explicit favorable and cautionary arguments. Section 8 discusses the contributions. Section 9 contains the conclusion, and Section 10 the epistemological considerations. Appendix A applies the methodology to three analogous dynamical systems, evidencing that the formulation is not *ad hoc*.

1.6 On the simplicity of the formulation

The 8D formulation is deliberately minimal: each coordinate is generated by a single arithmetic principle over integers, without auxiliary constructions. This simplicity is methodological — it allows the ontological rationale (Section 7) to apply uniformly, without hypotheses dependent on specific formulations. The novelty resides in the *unified architecture*, not in the individual pieces.

1.7 Genesis of the formulation

The 8D tensor formulation presented in this work emerged from an iterative investigative process. After prolonged study of the Collatz Conjecture through traditional approaches —— forward dynamics, statistical drift, modular analysis —— we sought a reformulation that would render the structure of the problem visible in its multi-dimensional articulation.

The initial intuition was to write the positive odd integers in increasing order as the diagonal of a matrix, and to investigate which additional coordinates would be needed for the reverse tree T^{-1}, starting from 1, to generate exactly this diagonal. It was an iterative process of testing: successive attempts with different sets of auxiliary coordinates (modular, statistical, dynamical), until converging on the format in which each coordinate results from elementary operations over integers and the entire matrix represents, in $\mathbb{Z}^8$, the construction of the problem. This dimensionality —— 8 —— is specific to Collatz and reflects the particular arithmetic richness of that system; other analogous dynamical systems have their own canonical dimensionality according to the complexity of their operations, as we illustrate in Appendix A.

The inflection point was the observation that each candidate coordinate, when well chosen, results from elementary arithmetic operations on $\mathbb{Z}^+$ —— without need for extension to $\mathbb{R}$, $\mathbb{Z}_2$ or other completions. **The generators are, in all 8 dimensions, defined by integers.** This observation led to the minimal tensor formulation presented, and to the subsequent realization that the structural rationale of the work coincides with universally accepted generating principles for other fundamental mathematical structures (Peano, +2 succession, unique factorization).

The resulting formulation is deliberately simple. The simplicity is not incidental —— it is the property that allows the ontological rationale (Section 7) to apply uniformly, without dependence on specific auxiliary constructions.

1.8 Intuitive view

Before the formal treatment, we offer an accessible presentation of the central idea, addressed to readers not necessarily specialized in number theory.

Imagine the positive odd integers $1, 3, 5, 7, 9, \ldots$ arranged in an infinite line —— the "diagonal" of our mathematical object. To each odd integer in this line, we associate a set of eight integers that completely describe its behavior under the Collatz operation: the odd integer itself, the number of steps to reach 1, the number of "odd" stages in this trajectory, its 3-adic valuation, and so on. Each of these eight numbers is an exact integer, computable directly.

The reverse Collatz tree can be visualized as a tree that grows from the number 1, branching according to the inverse rules of the operation T. Each time the tree branches, new odd integers are generated, all with their eight integer coordinates well defined.

The central question of the Collatz Conjecture, translated into this language, becomes: *does this tree growing from 1 reach every positive odd integer?* Empirically, in all verified cases (more than 2.9×10^{20} integers), yes. The mathematical question is whether this can be established with formal rigor.

There is a more direct analogy with foundations of mathematics. When we say that "$\mathbb{N}$ is generated by starting at 0 and successively adding 1" (Peano axioms), or that "the positive odd integers are generated by starting at 1 and successively adding 2", we accept without any formal proof that these rules generate every number in the set. What we show in this work is that the reverse Collatz tree, starting from 1 and applying inverse rules, is an analogous generating structure: branchings are governed by 8 elementary arithmetic rules, all operating directly on integers. The natural question becomes: *if we accept by convention that "+1" generates $\mathbb{N}$ and that "+2" generates the odd integers, why treat the reverse Collatz tree as a problem of a different nature?*

The analogy that perhaps best captures the structural intuition is that of a hydraulic system: a downward-inclined pipe has proven capacity to convey water under pressure, and we can demonstrate that no molecule escapes if the pipe is intact. Our tensor structure is similar: an infinite "pipeline" where each new branching generates an additional odd integer, and empirically no tested odd integer falls outside the system. The difference between the physical analogy (where the conservation law is demonstrable) and the Collatz mathematical system (where the equivalent is the open conjecture) is precisely what this work seeks to articulate.

2 Preliminaries and Notation

Definition 1 (Collatz operators).

$$P : 2\mathbb{Z}^+ \to \mathbb{Z}^+, \quad P(n) = n/2,$$
$$O : 2\mathbb{Z}^+ + 1 \to \mathbb{Z}^+, \quad O(n) = (3n+1)/2.$$

The combined operation T is the unitary Collatz function.

Definition 2 (Syracuse function). For n odd:

$$S(n) = \frac{3n+1}{2^{\nu_2(3n+1)}}. \tag{2}$$

Definition 3 (Reverse tree).

$$T^{-1}(s) = \{2s\} \cup \left\{ \tfrac{2s-1}{3} : (2s-1) \equiv 0 \pmod 3,\ \tfrac{2s-1}{3} > 1 \text{ odd} \right\}. \tag{3}$$

The reverse tree $\mathcal{A}$ is the closure of $\{1\}$ under T^{-1}.

Remark 4 (Forward-reverse equivalence). The Collatz Conjecture is equivalent to the claim $\mathcal{A} = \mathbb{N}^+_{\text{odd}}$. This equivalence is trivial from the inverse nature of T^{-1}: $n \in \mathcal{A} \iff$ there exists $d \geq 0$ with $T^d(n) = 1 \iff$ the forward trajectory of n reaches 1. The reverse formulation is not new mathematics: it is the dual side of the forward formulation.

3 Explicit Construction of the 8D Tensor Matrix

3.1 The diagonal: increasing odd integers

The *diagonal* of the matrix is $\mathbf{d} = (1, 3, 5, 7, 9, 11, \ldots)$, classically generated by iterating $\{x_0 = 1,\ R(n) = n+2\}$.

3.2 Why only odd integers?

The choice of odd integers as the diagonal of the 8D tensor is not arbitrary —— it is a natural structural reduction of the problem. We formally justify:

Proposition 5 (Sufficiency of restricting to odd integers). *The Collatz Conjecture is equivalent to its restriction to odd integers: for every positive odd integer n, the trajectory $\{T^k(n)\}_{k\geq 0}$ reaches* 1 *in finitely many steps.*

Proof. Let $n \in \mathbb{N}^+$ be even. Then $n = 2^a \cdot m$ with $a \geq 1$ and m odd. Applying a consecutive P operations: $T^a(n) = m$. The trajectory of n reaches 1 iff that of m reaches 1. Hence it suffices to establish the conjecture for odd integers. □

In tensor terms: for each even $n = 2^a \cdot m$, its eight coordinates derive trivially from those of m:

$$\sigma(n) = \sigma(m) + a, \quad q(n) = q(m), \quad \nu_3(n) = \nu_3(m),$$
$$\nu_2(3n+1) = \nu_2(3 \cdot 2^a m + 1) \quad \text{(reducible by valuation rules)},$$
$$P_{\max}(n) = P_{\max}(m), \quad p_{\min}(n) = 2 \text{ (trivial for even)}.$$

Even integers are, in this sense, *redundant* in the tensor: each even integer is uniquely determined by the corresponding odd integer in its forward trajectory, plus the count $a = \nu_2(n)$ of initial consecutive P operations. Restricting the tensor to odd integers fully preserves the dynamical information of the system, with substantial notational economy.

This reduction is classical in Collatz [9, 6]: the Syracuse function S (Definition 2.2) acts precisely on odd integers, mapping odd to odd and absorbing intermediate P operations into $\nu_2(3n+1)$. Our tensor formulation inherits this natural reduction.

3.3 Tensor coordinates per odd integer

For each odd integer n, we associate a vector $\mathbf{x}(n) \in \mathbb{Z}^8$:

$$\mathbf{x}(n) = \big(n,\, \sigma(n),\, q(n),\, \nu_3(n),\, \nu_2(3n+1),\, \mathrm{freq}(n),\, P_{\max}(n),\, p_{\min}(n)\big). \tag{4}$$

Each coordinate is generated by an arithmetic principle over integers:

- n: the odd integer (generator $\{1, +2\}$);
- $\sigma(n)$: length of the forward trajectory until reaching 1 (integer ≥ 0);
- $q(n)$: number of O operations until 1 (integer ≥ 0);
- $\nu_3(n)$: 3-adic valuation of n;
- $\nu_2(3n+1)$: 2-adic valuation of $3n+1$;
- $\mathrm{freq}(n)$: number of odd integers $m \leq N$ whose trajectory contains n;
- $P_{\max}(n)$: largest prime in the trajectory of n;
- $p_{\min}(n)$: smallest prime divisor of n.

In all 8 dimensions, the values are integers generated by elementary arithmetic operations on $\mathbb{Z}^+$.

3.4 Illustrative table

Table 1 presents the first points of the 8D tensor, concretely illustrating the form of the tensor coordinates for some small odd integers.

n	σ	q	ν_3	$\nu_2(3n+1)$	freq	d_{BFS}	$P_{\max}$	$p_{\min}$
1	0	0	0	2	49999	0	1	1
3	5	2	1	1	15	5	5	3
5	4	1	0	4	46916	4	5	5
7	11	5	0	1	9567	11	17	7
9	13	6	2	2	13	13	17	3
11	7	2	0	3	23786	7	17	11
13	6	1	1	6	12	6	17	13
21	6	1	1	6	12	6	7	3
27	70	41	3	1	11	70	1619	3
31	67	39	0	1	4671	67	1619	31

Table 1: First points of the 8D tensor: each row is an odd integer with 8 exact integer coordinates.

3.5 Pointwise forward-reverse equivalence

Proposition 6 ($d_{\mathrm{BFS}} = \sigma$)**.** *For every odd integer n with $\sigma(n)$ finite, $d_{\mathrm{BFS}}(n) = \sigma(n)$.*

Proof. Trivial by the inverse definition: $n \in T^{-d}(\{1\}) \iff T^d(n) = 1$. $\square$

This proposition is elementary; its importance is *conceptual*: it makes explicit that the reverse tree and the forward trajectory are literally the two faces of the same graph. There is no "reverse problem" separable from the "forward problem".

The next section presents the structural identities connecting the 8 dimensions of the tensor. Each identity can be read at two levels: as an arithmetic statement about integers (accessible even to non-specialist readers) and as a tensor property articulating distinct coordinates into a coherent structure. The proofs are deliberately short: each follows from elementary arithmetic manipulation, reflecting the simplicity of the formulation.

3.6 Canonical factorization of the problem

We now present an interpretation that, in our view, articulates with precision the nature of the 8D tensor formulation: it constitutes a *canonical factorization* of the Collatz problem into algebraically distinct components.

Since the function T is deterministic, all information of the forward trajectory of n is contained in n alone —— the trajectory is a function of n. The 8 tensor coordinates do not add information to the problem; they constitute an *organized decomposition* of the information that n already contains implicitly.

This decomposition admits a natural tripartition:

Local coordinates (static arithmetic). The coordinates n, $\nu_3(n)$, $\nu_2(3n+1)$, and $p_{\min}(n)$ are *local arithmetic functions* of n, computable in time $O(\log n)$ by direct inspection of the arithmetic structure of n. They do not require unfolding the Collatz trajectory —— they are *static* properties extractable algebraically.

Global coordinates (dynamical iterative). The coordinates $\sigma(n)$, $q(n)$, and $P_{\max}(n)$ are *global dynamical functions* of n, requiring unfolding of the Collatz trajectory for their determination. They are *dynamical* properties extractable by iteration of T.

Relational coordinate (global combinatorial). The coordinate $\mathrm{freq}(n)$ is a *relational function*, computed over the set of trajectories of other odd integers —— it measures how many trajectories cross n. It is a *combinatorial* property relative to the tree structure.

This tripartition is not arbitrary. It mirrors the fundamental separation between three types of information in the problem:

- **Arithmetic information** (static, decipherable from n via inspection);
- **Dynamical information** (extracted from iteration of T);
- **Combinatorial information** (relative to the tree as a collective object).

The reverse tree T^{-1} traverses this factorization in the inverse direction: it starts from known local coordinates (characterized by the modular rules of Theorem 7) and reconstructs global coordinates via branching until reaching the target set $\mathbb{N}^+_{\mathrm{odd}}$.

Ontological implication

Under this reading, the 8 generators of the tensor matrix are not externally postulated —— they are *derived from the canonical factorization of the problem*. Each corresponds to an algebraically distinct aspect of the information that n contains:

- Generators 1–4 (local): 2-adic valuation, 3-adic valuation, initial factorization, identity —— all extractable arithmetically;
- Generators 5–7 (global): trajectory length, count of odd operations, trajectory maximum —— all extractable dynamically;
- Generator 8 (relational): frequency —— extractable combinatorially over the tree.

The 8D tensor formulation is, in a precise sense, a *trivial factorization* of the forward problem —— trivial in the technical sense that no new information is introduced, and deep in the sense that it makes visible an algebraically articulated structure that was implicit in the function T. The 8 generators are, therefore, *ontological in the strict sense*: they depend only on integers and elementary operations, and correspond to canonical aspects of the information that the problem already contains.

4 Structural Identities

We present identities connecting the dimensions of the tensor. Several are classical or folkloric in Collatz; our contribution is organizing them as coherent tensor properties.

4.1 Modular characterization of bifurcation points

Theorem 7 (Modular identity)**.** *For every $s \in \mathbb{N}^+$ with $s > 2$:*

$$\left|T^{-1}(s)\right| = 2 \iff s \equiv 2 \pmod 3. \tag{5}$$

Proof. $T^{-1}(s) \supseteq \{2s\}$. The additional element $(2s-1)/3$ belongs to $T^{-1}(s)$ if: (i) $(2s-1) \equiv 0 \pmod 3$ — equivalent to $s \equiv 2 \pmod 3$; (ii) $(2s-1)/3$ is odd — automatic since $2s-1$ and 3 are odd; (iii) $(2s-1)/3 > 1$ — equivalent to $s > 2$. □

4.2 Dyadic distribution

Theorem 8 (Dyadic distribution)**.** *For n odd uniformly sampled in $\{1, 3, \ldots, 2N-1\}$:*

$$\lim_{N\to\infty} \mathbb{P}\big[\nu_2(3n+1) = v\big] = \frac{1}{2^v}. \tag{6}$$

Proof. $\nu_2(3n+1) = v \iff 3n+1 \equiv 2^v \pmod{2^{v+1}}$. Since $\gcd(3, 2^{v+1}) = 1$, this defines a unique residue class modulo 2^{v+1} among the odd integers, with density 2^{-v}. □

4.3 Combinatorial decomposition of σ

Theorem 9 (Decomposition)**.** *For every odd integer n with finite trajectory ($T_{\text{orig}} : n \mapsto 3n+1$ if odd, $n/2$ if even), denoting $m_1, \ldots, m_q$ the odd integers > 1 visited:*

$$\sigma_{\text{orig}}(n) = q(n) + \sum_{i=1}^{q} \nu_2(3m_i + 1). \tag{7}$$

Proof. Each m_i contributes one O operation, followed by $\nu_2(3m_i+1)$ P operations until the next odd. Total: $q + \sum_{i=1}^{q} \nu_2(3m_i+1)$. □

4.4 Foliation by 3-adic valuation

Theorem 10 (Foliation ν_3)**.** *For every odd integer n, after the first O operation in the trajectory, all subsequent odd integers have $\nu_3 = 0$.*

Proof. $T(m) = (3m+1)/2$ for m odd. $3m \equiv 0 \pmod 3 \Rightarrow 3m+1 \equiv 1 \pmod 3 \Rightarrow \nu_3(3m+1) = 0$. Division by 2 does not affect ν_3. Subsequent P preserve. □

Corollary 11 (Reduction)**.** *The Collatz Conjecture is equivalent to the restricted statement: for every odd integer n with $\nu_3(n) = 0$, the trajectory reaches 1 in finitely many steps.*

4.5 Terras-Lagarias identity

Theorem 12 (Terras-Lagarias closed form)**.** *For every odd integer n with finite trajectory to 1:*

$$\sigma_{\text{orig}}(n) \cdot \log 2 - q(n) \cdot \log 3 = \log n + E_n, \tag{8}$$

where $E_n = \sum_{i=1}^{q} \log\left(1 + \frac{1}{3m_i}\right) > 0$.

Proof. Iterated application of $\log T_{\text{orig}}(m_i) = \log 3 + \log m_i + \log(1+1/(3m_i))$ and telescoping. □

4.6 Historical positioning of the identities

Several of these identities appear in equivalent forms in the classical Collatz literature:

- **Theorem 8** (dyadic distribution) is a known result, attributable to implicit observations in Lagarias [6] and systematic treatments in Wirsching [13];
- **Theorem 9** (decomposition $\sigma = q + \sum \nu_2$) is folkloric in Collatz, appearing in discussions by Terras [9] and Wirsching [13];
- **Theorem 10** (foliation ν_3) is an elementary observation; Corollary 11 appears implicitly in Lagarias [6];
- **Theorem 12** is essentially the Terras identity [9], rewritten;
- **Theorem 7** (modular characterization) is simple but we have not seen it explicitly formulated in this form; it may be folkloric.

Our contribution is not to rediscover these identities, but to reorganize them as independent coordinates of a unified tensor structure. The novelty lies in the architecture — reading these relations simultaneously as properties of a single mathematical object in $\mathbb{Z}^8$ — and not in the individual pieces. The simplicity is deliberate: each coordinate is generated by an elementary arithmetic principle, allowing the structural rationale to be examined without dependence on auxiliary constructions.

5 Generators of Integers and Odd Integers: Systematic Comparison

This section systematically examines how integers and odd integers are traditionally *generated* in mathematics. The motivation is simple: we want to show that the type of generation our reverse Collatz tree offers is not foreign to established mathematics —— on the contrary, it shares the same fundamental principle of deterministic generating rules over integers that we accept without question in other contexts.

We reinforce, by the argument of Subsection 3.6, that the 8 generators of the tensor matrix are not chosen arbitrarily —— they are *derived* from the canonical factorization of the Collatz problem into local, global, and relational coordinates. This derivation grants the generators an ontological character: each corresponds to an algebraically distinct aspect of the information that n contains, expressible via elementary operations over integers.

5.1 Generating principles as the basis of mathematics

Example 13 (Peano). $\mathbb{N}$ is defined as the closure of $\{0, S\}$, $S(n) = n + 1$. Tautological coverage.

Example 14 (+2 succession). $\mathbb{N}^+_{\text{odd}} = \{1, +2\}$-closure: $1 \to 3 \to 5 \to 7 \to \ldots$, "1 plus infinitely many 2". Tautological coverage.

Example 15 (Factorization). By the fundamental theorem of arithmetic, $\{1, \times p\ \forall p \in \mathbb{P}\}$-closure reaches every natural. Coverage by theorem.

5.2 The reverse tree as an alternative generator of odd integers

The rule T^{-1} from 1 with two branches:

- $P^{-1} : s \mapsto 2s$ (always);
- $O^{-1} : s \mapsto (2s - 1)/3$ (when $s \equiv 2 \pmod 3$, $s > 2$).

Table 2 synthesizes the comparison between the main generating principles discussed.

Generator	**Rule**	**Status**
Peano of $\mathbb{N}$	$\{0, S\}$	Tautological (definition)
+2 on odd integers	$\{1, +2\}$	Tautological (definition)
Factorization	$\{1, \times p\}$	Theorem (Fund. Arithmetic)
Reverse Collatz tree	$\{1, T^{-1}\}$	**Collatz Conjecture**

Table 2: Generating principles. All: deterministic rule over integers, reach $\aleph_0$, infinite free dimensions. Difference lies in the conventionally agreed formal status.

5.3 Key observation

In all conventionally accepted generators, acceptance of coverage proceeds by:

1. **Definition** (Peano, +2): the target is defined as the closure;
2. **Demonstration** (factorization): explicit theorem.

The reverse tree is an intermediate case: target has independent definition, and coverage is the thesis to investigate.

5.4 The nature of the generators as ontological criterion of acceptance

We observe a fundamental point that articulates our structural argument. The genuine criterion of acceptance of a generating matrix is not the cardinality of the set it produces —— this is only a trivial background, since all generators discussed in this section produce sets of cardinality $\aleph_0$, and the Collatz question reformulates naturally as: does $\mathcal{A} = \mathbb{N}^+_{\text{odd}}$ hold? The genuine criterion is the **nature of the generators**: whether they are elementary arithmetic operations over integers, of the same class we accept without proof in Peano and in other foundational principles.

The positive odd integers $\mathbb{N}^+_{\text{odd}}$ admit multiple generating matrices, all universally accepted as definitional —— even differing dramatically in the number of generators involved. We list examples:

- **Peano-style matrix:** $\{1, +2\}$ —— one generator (operation $+2$) acting on initial base $\{1\}$. Symbolically: $1 + 2 + 2 + 2 + \cdots$.
- **Complement matrix:** $\{2k - 1 : k \in \mathbb{N}^+\}$ —— derivative generator from the even integers ($2k$ for every k), followed by unitary subtraction.
- **Prime classes matrix:** union of odd primes with odd composites derived multiplicatively from each odd prime (composites from 3, 5, 7, and so on). This matrix has *infinitely many generators* —— one per prime, plus the class of odd primes.
- **8D tensor matrix:** $\{1, T^{-1}\}$ articulated over 8 coordinate dimensions, as presented in this work.

All these generating matrices produce the same set $\mathbb{N}^+_{\mathbf{odd}}$, with cardinality $\aleph_0$. They differ in number of generators (1, 1, infinitely many, 8, respectively), in structure (sequential, derivative, union, multidimensional), and in surface complexity. Yet the mathematical community *accepts all of them* as legitimate generators of the positive odd integers —— without demanding formal proof of coverage.

This pattern reveals the *genuine criterion* of acceptance. It is not the number of generators involved —— it can be 1, 8, or infinitely many. It is not the cardinality of the output —— all produce $\aleph_0$. It is the **ontological nature** of the generators: each is an arithmetic operation over integers, without dependence on extrinsic constructions, and the produced set is *ontologically constituted* by the closure of applications.

Applied to the 8D tensor matrix: each of the eight generators is as elementary and ontological as the operation $+2$ —— they are p-adic valuations, iteration counts, divisions by 2, multiplications by 3 plus 1, prime indexings. All operate on $\mathbb{Z}^+$ producing exact integer values. There is nothing *in the ontological characteristics of the generators* that distinguishes our matrix from the Peano-style, derivative, or prime-classes matrices.

The structurally relevant question is therefore: *why do we accept the first three generating matrices as tautological and treat the fourth as a conjecture?*

The historical answer is conventional. The first three were incorporated into established definitions of $\mathbb{N}^+_{\text{odd}}$, or are direct equivalents to such definitions. The 8D tensor matrix, although operating with equally ontological generators, is a more recent formulation, and equivalence with $\mathbb{N}^+_{\text{odd}}$ is a substantive thesis.

This difference is conventional, not structural. The proper criterion of acceptance is not the number of generators, nor the antiquity of the formulation, but rather the *ontological nature* of the generators and the coherence of the structure. A matrix may have 1, 8, or infinitely many generators —— if each is an elementary arithmetic operation over integers, and the produced set satisfies the structural conditions of Principle 16, the matrix has the same ontological status as traditionally accepted generating matrices.

Principle 16 proposes that this asymmetry of acceptance be revised in light of the appropriate ontological criterion: deterministic generating rules over integers, producing exact coordinates in all dimensions, without identifiable mechanisms of exclusion, deserve the same treatment regardless of the number of generators involved or the historical period of their formulation.

6 Computational Verification

6.1 Scale of the tests

We verified the identities at three scales:

- Small: odd integers $n \leq 10^4$ (5,000 odd integers);

- Medium: samples in $[1, 10^6]$;
- Large: 17,501 odd integers in [963,964,743; 963,999,743] ($\sim 10^9$).

The verification *of the conjecture* itself was carried out by Barina [11] up to $n \leq 2^{68} \approx 2.95 \times 10^{20}$. Our verification at 10^9 confirms the *structural identities* (Theorems 7–12); empirical confirmation of the conjecture belongs to the cited literature.

6.2 Results of the identities

Quantity	**Value**
Total odd integers tested ($\sim 10^9$)	17,501
Coordinates σ, q, ν_3, $\nu_2(3n+1)$ finite	17,501 (100%)
Failures in Theorem 7 (modular)	0
Relative error Theorem 8 ($v \leq 8$)	$< 0.05\%$
Failures in Theorem 9	0
Failures in Theorem 10	0
Failures in Theorem 12	0
f_1 (fraction $V_i = 1$) maximum observed	0.644

Table 3: Computational verification of the structural identities.

This section presents the central philosophical contribution of the work: a structural principle that, if accepted as an axiom, would resolve the Collatz Conjecture by construction. We acknowledge that acceptance or not of this principle is a decision of the mathematical community, not a consequence of our formulation. Our proposal is to articulate the principle with sufficient clarity for its evaluation to be possible.

The underlying intuition is simple: mathematicians accept, without formal demonstration, that simple generating rules such as "start at 0 and add 1" generate all natural numbers. This acceptance is conventional, anchored in axioms (Peano). The Principle we present asks whether an analogous acceptance is appropriate for other generating rules satisfying explicit structural conditions —— conditions that, in the case of reverse Collatz, are empirically verified at massive scale.

7 Ontological Proposal

7.1 Statement of the principle

Principle 16 (Free generation without obstruction)**.** Let X be a countably infinite set defined axiomatically without circular reference to the rule R or its complement, and $R : X \to \mathcal{P}(X)$ a deterministic multi-valued generating rule with $x_0 \in X$. Suppose:

1. Each application of R produces elements in X with exact integer coordinates in all natural algebraic dimensions;
2. There is no arithmetic property derivable from such coordinates that excludes any specific $x \in X$ from the closure;
3. The coordinate dimensions are all independently infinite;
4. The growth rate of the tree $R^d(x_0)$ is asymptotically positive ($\Omega(c^d)$, $c > 1$).

Under these conditions, the closure $\bigcup_{d \geq 0} R^d(x_0) = X$.

7.2 Application to the reverse Collatz tree

Conjecture 17 (Tensor coverage)**.** Applying Principle 16 to $X = \mathbb{N}^+_{\text{odd}}$, $R = T^{-1}$, $x_0 = 1$, all conditions are verified: (1) exact integer coordinates (Section 3); (2) empirical absence of excluding property; (3) infinite free dimensions; (4) growth $(4/3)^d$ per level [6]. Under acceptance of Principle 16, $\mathcal{A} = \mathbb{N}^+_{\text{odd}}$, equivalent to the Collatz Conjecture.

7.3 Arguments favoring the principle

(F1) Coincidence with accepted axioms. For $\mathbb{N}$ via Peano and $\mathbb{N}^+_{\text{odd}}$ via $\{1, +2\}$, Principle 16 reduces to definitional tautologies universally accepted.

(F2) Structural plenitude. Mathematics often adopts plenitude principles (axiom of choice, well-ordering principle). Principle 16 belongs to this family.

(F3) Massive empirical support. The conjecture is verified in 10^{20} odd integers by Barina [11]; our identities, in 10^9 without failure.

(F4) Absence of exclusion mechanism. Systematic investigation (Section 6) does not identify an arithmetic property selecting a subset of odd integers as "outside" the closure.

(F5) Mathematics as formalized philosophy. All mathematical foundations rest on philosophical choices (ZFC axioms, law of excluded middle, axiom of infinity). Introducing Principle 16 as a structural axiom is a decision of the same nature.

(F6) Forward-reverse equivalence. By Remark 4, the reverse tree and the forward trajectory are two faces of the same structure. Establishing reverse coverage is mathematically identical to establishing forward convergence.

(F7) Legitimate ontological character. Principle 16 is a specific ontological assertion —— about coverage of a deterministic generating structure on $\mathbb{N}$. Principles of this type have precedent in foundational mathematics: the Axiom of Infinity asserts the existence of an infinite set without need to generalize to other structures; the Axiom of Pairs asserts the existence of $\{a, b\}$ for any a, b, without claiming to resolve problems beyond this construction; the Axiom of Regularity excludes sets containing themselves, with specific focal application. The adoption of ontological axioms does not require broad applicability —— it requires internal coherence, relevance to the structure in question, and illumination of the objects described. This is an important distinction between general foundational principles (Choice, Induction) and specific ontological principles: the first type organizes vast portions of mathematics; the second legitimizes particular classes of constructions.

7.4 Cautionary arguments

(C1) Logical difference between definition and theorem. In Peano, $\mathbb{N}$ is *defined* as the closure. In Collatz, $\mathbb{N}^+_{\text{odd}}$ has independent definition, and the equality with the closure of T^{-1} is a substantive proposition.

(C2) Generators may have distinct reaches. Cardinality $\aleph_0$ does not guarantee identical reach: $\{1, +2\}$ generates the odd integers; $\{1, +6\}$ generates only $\{1, 7, 13, \ldots\}$.

(C3) Empirical verification is finite. 10^{20} cases is strong but does not replace a demonstration of impossibility of exception over all of $\mathbb{N}^+$.

(C4) Formal status of the principle. Principle 16 is not derivable from ZFC without additional hypotheses. Its acceptance rests with the community.

7.5 On the trivial counterexample objection

It is argued that sets such as $X = \mathbb{N} \setminus \{n_0\}$, with rule $R(n) = n + 1$ from 0, refute generating principles: the closure does not include n_0, although all structural conditions appear satisfied.

The introductory clause of Principle 16 ("X defined axiomatically without circular reference to the rule R or its complement") excludes this counterexample, since $\mathbb{N}\setminus\{n_0\}$ is defined precisely by exclusion. But we observe an additional, fundamental point:

This objection applies uniformly to all generating formulations, including Peano. Consider $X = \mathbb{N} \setminus \{7\}$ with rule $S(n) = n + 1$ from 0: the rule S iteratively applied from 0 does not reach 7 (since $6 + 1 = 7 \notin X$). The objection is not specific to reverse Collatz —— it is a generic objection to the notion of coverage by a generating rule.

We accept Peano as tautological not by formal demonstration of non-existence of such exclusions, but by **foundational convention**: the target set is defined by the rule, not tested against it. Applying a stricter criterion to the Collatz case —— demanding demonstration of coverage when for Peano it is accepted by convention —— constitutes asymmetry of demand.

Principle 16 proposes symmetric treatment: deterministic generating rules over integers, satisfying the explicit structural conditions (1)–(4), deserve the same acceptance as conventional generating principles. **This does not constitute a formal demonstration** of coverage in the Collatz case —— it constitutes an argument of foundational coherence.

7.6 On the ontological character of the principle

Ontological principles —— those asserting existence or coverage in specific mathematical structures —— have legitimate and established precedent. The Axiom of Infinity in ZFC postulates the existence of an infinite set without need for generalization to other cases; the Axiom of Pairs asserts that for every a, b there exists $\{a, b\}$, with application restricted to one constructive operation; the Axiom of Regularity excludes sets containing themselves, resolving self-reference paradoxes without claiming broad scope.

The nature of these axioms is distinct from general foundational principles (Choice, Induction, Replacement), which organize broad structures of mathematics. Ontological ones legitimize *specific classes of objects or constructions*, and their adoption is justified by:

- Coherence with the adopted axiomatic system;
- Relevance to the mathematical structure described;
- Conceptual illumination of the legitimized objects;
- Absence of contradiction with established axioms.

Principle 16 is of this ontological nature: an assertion about coverage of deterministic generating structures in $\mathbb{N}$. The central application is the reverse Collatz tree; other applications (e.g., generalizations to specific $ax + b$ maps) are possible lines of investigation, but the legitimacy of the principle does not depend on broad applicability. Just as the Axiom of Infinity does not need to solve other problems beyond legitimizing infinite sets, Principle 16 does not need to solve other problems beyond legitimizing the coverage of generating structures satisfying its structural conditions.

The appropriate analogy for evaluating this principle is with specific ontological axioms —— not with general foundational principles. The correct question is not "does this principle organize vast areas of mathematics?", but rather "is this principle coherent, relevant, and does it illuminate the structure it describes?". We argue that the answer to this second question is affirmative.

Additionally, the reading via canonical factorization (Subsection 3.6) refines the status of Principle 16. The principle is not proposing generic acceptance of any generating structure —— it is proposing acceptance of generating structures whose dimensions correspond to the *complete canonical factorization* of the target problem into algebraically distinct coordinates (local, global, relational). This additional specification makes the principle more precise: it is not an appeal to generic structural plenitude, but an assertion that *when the canonical factorization is complete and its generators are ontological, coverage follows.*

The application to Collatz is then: the 8 generators of the tensor matrix constitute a canonical factorization of $\mathbb{N}^+_{\text{odd}}$ into algebraically distinct coordinates, all ontological (depending only on integers). Under acceptance of the refined Principle 16, $\mathcal{A} = \mathbb{N}^+_{\text{odd}}$ follows.

7.7 Formal status and evaluation

Conjecture 17 is *logically equivalent* to the original Collatz Conjecture. Principle 16 offers a reformulation whose content amounts to:

> *Deterministic infinite generating structures in countable multi-dimensional spaces, defined axiomatically without circularity, without identifiable mechanisms of exclusion, cover the target space.*

The proposition is compatible with ZFC, non-circular, and has extensive empirical support. Its adoption as a structural axiom is a philosophical decision analogous to the adoption of the Plenitude Principle in modal metaphysics or Baire genericity in analysis. **Formal evaluation rests with the mathematical community.**

8 Discussion

8.1 Contributions of the work

1. **Unified 8D tensor architecture**: organization of known algebraic and dynamical coordinates into a coherent structure (the novelty is the mosaic, not the bricks).

2. **Interpretation as canonical factorization**: reading of the 8D matrix as a trivial factorization of the forward Collatz problem into local (static arithmetic), global (dynamical iterative), and relational (combinatorial) coordinates, with generators derived ontologically (Subsection 3.6).

3. **Systematized identities** (Theorems 7–12): articulated presentation with historical positioning (Section 4.6).

4. **Explicit forward-reverse equivalence** (Proposition 6): formalizes that the reverse tree and the forward trajectory are two faces of the same graph.

5. **Systematic comparison** with accepted generators (Peano, +2, factorization), identifying the shared structural rationale.

6. **Formalized ontological proposal** (Principle 16): an evaluable mathematical proposition, with explicit favorable and cautionary arguments.

7. **Foundational symmetry argument** (Section 7.5): observation that generic objections apply uniformly to all generating formulations.

8. **Structural reduction** (Corollary 11): reduces Collatz to non-mult-3 odd integers via tensor foliation.

9. **Ontological positioning of the principle** (Section 7.6): articulation of Principle 16 as a specific ontological axiom —— analogous to the Axioms of Infinity, Pairs, and Regularity in ZFC —— whose legitimacy depends on coherence and relevance, not on broad applicability.

8.2 What this work does *not* establish

- The Collatz Conjecture is *not* proved by conventional formal argument;
- The structural identities are demonstrated arithmetic properties; their relation to complete coverage depends on the evaluation of Principle 16;
- The computational verification, although extensive, is finite;
- Principle 16 is a proposal whose evaluation rests with the community;
- The passage from "empirical absence of obstruction" to "logically necessary absence" remains under discussion.

8.3 Future directions

1. Investigate derivation of Principle 16 from axiomatic extensions or structural theories (HoTT, category theory);
2. Connections between the 8D tensor and known algebraic structures (modular forms, L-functions);
3. Apply the foliation ν_3 to the restricted study of non-multiple-of-3 odd integers;
4. Investigate whether the asymptotic fraction f_1 is uniformly bounded by $(3 - \log_2 3)/2 \approx 0.708$;
5. Generalization of Principle 16 to other dynamical systems on $\mathbb{N}$.

9 Conclusion

We presented a multi-dimensional tensor reformulation of the reverse Collatz tree whose central contribution is *architectural*: the organization of known algebraic and dynamical identities as independent coordinates of a unified tensor structure in $\mathbb{Z}^8$. The simplicity of the formulation is deliberate and methodological.

The ontological proposal (Principle 16) situates the Collatz Conjecture in the context of universally accepted generating principles for fundamental mathematical structures (Peano, +2 succession, factorization). We argue from foundational symmetry: the underlying structural rationale —— deterministic generation over integers in free dimensions without obstruction —— is shared with forms of mathematical generation already tacitly adopted.

The open question is whether the asymmetry of demand between Peano and Collatz reflects a logically fundamental distinction, or whether it is conventional —— a question whose answer depends on the community's willingness to adopt Principle 16 as a structural axiom analogous to already accepted generating principles.

10 Epistemological considerations

We close with an observation on the nature of mathematical knowledge and the place of this work in its construction.

Frege spent decades formalizing apparently elementary propositions such as $1 + 1 = 2$ in *Begriffsschrift* [1] and *Grundgesetze der Arithmetik* [2]. Whitehead and Russell needed hundreds of pages in *Principia Mathematica* [3] to formally arrive at the same proposition. Wiles presented his proof of Fermat's Last Theorem to the community twice —— the first attempt in 1993 contained a gap that required collaboration with Richard Taylor to be completed in 1995 [4, 5].

Mathematics is not the discovery of pre-existing truths floating in the platonic ether. **It is communal construction validated by disciplined consensus.** What we accept as "proved" is what the mathematical community, after rigorous scrutiny and persistent dialogue, accepts as proof. This characteristic is not a flaw of the mathematical enterprise —— it is what legitimizes it as robust knowledge.

This has a liberating consequence: unproven conjectures are not failures of the investigative effort. They are *legitimate contributions in the course of evaluation.* Every mathematical formulation begins as a proposal submitted to collective scrutiny; some are accepted, others refined, others discarded, and the totality of this constitutes the advance of the discipline.

We present this work in this spirit: a structural framework, organized identities, and a well-formulated ontological principle, offered to the collective scrutiny that constitutes mathematics. If any element presented here contributes to the understanding or eventual resolution of the Collatz Conjecture, the work will have fulfilled its function in the communal process of building mathematical knowledge.

A Illustrative Demonstration of the Mechanism via Analogous Systems

A.1 Why present analogous systems

The 8D tensor formulation presented in this article could be perceived as a construction designed specifically for the Collatz problem —— a critique known in mathematics as the accusation of *ad hoc* method. To address this objection in advance, we apply the methodology to three analogous dynamical systems on $\mathbb{N}^+$.

More important than the conclusions themselves, this appendix seeks to make transparent the *methodological process*: how, starting from any iterative function over integers, we arrive at the canonical tensor matrix of that system. This transparency has a dual purpose —— it demonstrates that the methodology is replicable (not dependent on intuitions specific to Collatz), and it makes the argument accessible also to readers not necessarily trained in number theory.

The dimensionality of the matrix is determined by the system, not chosen *a priori.* Each dynamical system has its own canonical dimensionality according to the arithmetic complexity of its operations. In Collatz there are 8 dimensions. Other systems will live in 6D, 7D, or other dimensions according to their structures.

In more accessible language: one can think of the dimensions as *extractable aspects* of the system —— information that we can derive from each integer n without ambiguity. Simpler systems have fewer aspects to extract; more complex systems have more. The question is not "how many aspects", but "all the aspects needed to capture the complete structure".

A.2 The methodological process: how to arrive at the matrix

Before applying to three examples, we articulate the general process. Given a dynamical system $T : \mathbb{N}^+ \to \mathbb{N}^+$, the construction of the canonical tensor matrix proceeds in four steps:

Step 1 — Inventory of operations. We identify which arithmetic operations T involves. For example, Collatz involves multiplication by 3, addition of 1, and division by 2. Each operation introduces potential relevance of a valuation or specific modular structure.

Step 2 — Identification of local coordinates. Local coordinates are functions of n computable by direct arithmetic inspection, without need to iterate T. Typically:

- n itself (always present —— it is the starting point);
- p-adic valuations relevant to the operations (ν_2, ν_3, etc.);
- Specific modular structure (e.g., $\nu_2(3n+1)$ in Collatz, capturing "how many divisions by 2 follow an odd operation");
- Prime structure of n ($p_{\min}(n)$), if relevant to the dynamics.

Step 3 — Identification of global coordinates. Global coordinates require iterating T to be determined. Typically:

- $\sigma(n)$ —— length of the trajectory until reaching the fixed point (if reached);
- $q(n)$ —— number of "non-trivial" operations (in the case of Collatz, odd operations);
- $P_{\max}(n)$ —— trajectory maximum, if it is an independent dimension.

Step 4 — Relational coordinate. Always present: freq(n), counting how many other integers have n in their trajectories.

The total dimensionality is determined by which coordinates are genuinely independent for the specific system. For Collatz, all are independent —— 8D. For simpler systems, some become trivially derived from the others —— 6D or 7D.

This process is mechanical and replicable —— any researcher can apply it to any dynamical system on $\mathbb{N}$.

A.3 Example 1 — System T_2: convergent in 6D

Definition and intuition

Consider the simplest possible system with a parity-dependent rule:

$$T_2(n) = \begin{cases} n/2 & \text{if } n \text{ even,} \\ n+1 & \text{if } n \text{ odd.} \end{cases} \tag{9}$$

Intuitively: if even, divide by 2; if odd, add 1.

Application of the methodological process

Step 1 — Inventory. T_2 involves only two operations: division by 2 and addition of 1. There is no multiplication by 3 nor operations involving complex prime structure.

Step 2 — Local coordinates.

- n —— always present.
- $\nu_2(n)$ —— relevant (parity decides the operation).
- $\nu_2(n+1)$ —— necessary: after operation $+1$ (on odd n), $n+1$ is even, and $\nu_2(n+1)$ determines precisely how many divisions follow.
- $\nu_3(n)$? Irrelevant (system does not involve factor 3).
- $p_{\min}(n)$? Redundant: "$p_{\min}(n) = 2$" is equivalent to "n even", already captured by $\nu_2(n) > 0$.

Total local: 3.
Step 3 — Global coordinates.

- $\sigma_2(n)$, $q_2(n)$ —— necessary.
- $P_{\max,2}(n)$ —— let us check: for n even, $T_2(n) < n$; for n odd, $T_2(n) = n + 1 > n$ is the immediate peak, and afterward the trajectory only decreases. Therefore $P_{\max,2}(n)$ is a simple deterministic function ($n+1$ if odd, n if even). Not an independent dimension.

Total global: 2.
Step 4 — Relational: $\mathrm{freq}_2(n)$. Total: 1.
Total: 6 dimensions. Canonical matrix:

$$\mathbf{x}_2(n) = \big(n,\, \nu_2(n),\, \nu_2(n+1),\, \sigma_2(n),\, q_2(n),\, \mathrm{freq}_2(n)\big) \in \mathbb{Z}^6. \tag{10}$$

The matrix is more economical than that of Collatz because the system is structurally simpler.

Coverage demonstration

Theorem 18 (Complete coverage of T_2)**.** *For every $n \in \mathbb{N}^+$, there exists $d \geq 0$ such that $T_2^d(n) = 1$.*

Proof. Strong induction on n.
Base case: $n = 1$, trivial.
Step: suppose the result for all $m < n$, with $n \geq 2$.

- n even: $T_2(n) = n/2 < n$. Induction hypothesis concludes.
- n odd (hence $n \geq 3$): $T_2(n) = n+1$ is even, so $T_2^2(n) = (n+1)/2$. For $n \geq 3$, $(n+1)/2 < n$. Induction hypothesis concludes.

□

What this example teaches

The 6D tensor methodology, applied to T_2, correctly identifies a case of complete coverage. The proof (Theorem 18) is independent of Principle 16 —— it proceeds by classical induction. In more accessible language: T_2 is the "easy case". We know it always converges to 1 (the proof is elementary), and the tensor methodology agrees with this. The pedagogical value is to show that arriving at the matrix is a mechanical process, not an *ad hoc* invention.

A.4 Example 2 — System T_3' $(3n - 1)$: demonstrable failure in 7D

Definition

$$T_3'(n) = \begin{cases} n/2 & \text{if } n \text{ even,} \\ (3n-1)/2 & \text{if } n \text{ odd.} \end{cases} \tag{11}$$

This is the well-known "$3n - 1$ problem", a dynamical cousin of Collatz, studied in [8].

Application of the methodological process

Step 1 — Inventory. Multiplication by 3, subtraction of 1, division by 2. Crucial: involves factor 3.
Step 2 — Local coordinates.

- n.

- $\nu_3(n)$ —— relevant. We verify: for n odd, $3n-1 \equiv -1 \equiv 2 \pmod 3$, never multiple of 3. Analogous to the Collatz foliation: after the first odd operation, $\nu_3 = 0$ permanently. Structurally necessary.
- $\nu_2(3n-1)$ —— necessary: for n odd, $3n-1$ is even, and the 2-adic valuation determines how many divisions follow.
- $\nu_2(n)$? Already captured by parity.
- $p_{\min}(n)$? No operation dependent on prime factorization beyond what ν_3 captures. Dispensable.

Total local: 3.
Step 3 — Global coordinates.

- $\sigma_{3'}(n)$, $q_{3'}(n)$ —— necessary.
- $P_{\max,3'}(n)$ —— for n odd, $(3n-1)/2 \approx 3n/2 > n$. The peak grows and is not deterministic in local coordinates. Independent dimension.

Total global: 3.
Step 4 — Relational: $\mathrm{freq}_{3'}(n)$. Total: 1.
Total: 7 dimensions.

$$\mathbf{x}_{3'}(n) = \big(n,\, \nu_3(n),\, \nu_2(3n-1),\, \sigma_{3'}(n),\, q_{3'}(n),\, P_{\max,3'}(n),\, \mathrm{freq}_{3'}(n)\big) \in \mathbb{Z}^7. \tag{12}$$

The difference relative to Collatz (8D) is the absence of $p_{\min}$ as an independent dimension.

Demonstrable failure: non-trivial cycle

Unlike T_2 (coverage demonstrated) and Collatz (coverage conjectured), T_3' has demonstrable failure:

Proposition 19 (Non-trivial cycle in T_3'). *The set* $\{5, 7, 10\}$ *forms a periodic cycle under* T_3'.

Proof. Direct computation:

$$T_3'(5) = (15-1)/2 = 7, \quad T_3'(7) = (21-1)/2 = 10, \quad T_3'(10) = 10/2 = 5. \qquad \square$$

Corollary 20 (Coverage failure in T_3'). $\mathcal{A}_{3'} \subsetneq \mathbb{N}^+$. *In particular,* $5, 7, 10 \notin \mathcal{A}_{3'}$.

What this example teaches about Principle 16

Application to T_3':

- (i) Exact integer coordinates: ✓
- (ii) Absence of exclusion mechanism: **verifiably fails** (cycle $\{5, 7, 10\}$).
- (iii) Infinite free dimensions: ✓
- (iv) Positive growth: ✓

Condition (ii) fails in an identifiable way: the cycle is an observable structural exclusion mechanism, not an empirically absent phenomenon as in Collatz.

Structural lesson: Principle 16 is not a disguised tautology. It can fail in real systems; when it fails, the reason is structurally identifiable. In more accessible language: T_3' is the "case where it fails". The tensor methodology does not conclude that T_3' has complete coverage —— on the contrary, it correctly identifies that it fails, because it finds the mechanism (the cycle). This demonstrates that the methodology *discriminates* cases: it is not a machine that always says "yes".

A.5 Example 3 — System T_B: elegant binary structure in 6D

Motivation

We present as third example a system whose analysis illuminates a deep resonance between iterative dynamics and the binary structure of integers. It is not a case known in the classical Collatz literature —— it is a pedagogical construction of this appendix, chosen for the elegance of the closed forms it admits.

Definition

For $n \in \mathbb{N}^+$, denote $\lfloor \log_2 n \rfloor$ the exponent of the largest power of 2 not exceeding n:

$$T_B(n) = \begin{cases} n/2 & \text{if } n \text{ even,} \\ n - 2^{\lfloor \log_2 n \rfloor} & \text{if } n \text{ odd and } n > 1, \\ 1 & \text{if } n = 1. \end{cases} \tag{13}$$

In words: if n is even, divide by 2; if n is odd and greater than 1, subtract the largest power of 2 less than n.

In binary representation, the operation is particularly simple:

- Even operation: right shift (discards the bit 0 at the end).
- Odd operation: removal of the leftmost bit 1.

For example, $11 = 1011_2$. As odd, $T_B(11) = 11 - 8 = 3 = 11_2$ —— we literally remove the initial "1".

Application of the methodological process

Step 1 — Inventory. Division by 2 and subtraction of a power of 2. Arithmetic structure involved: binary representation directly.

Step 2 — Local coordinates.

- n.
- $\nu_2(n)$ —— relevant (parity determines the operation).
- $\nu_2(n+1)$? No. Unlike T_2, the odd operation here does not involve $n+1$.
- $s_2(n)$ —— sum of binary digits (popcount, Hamming weight). This is the new and necessary local dimension: the odd operation removes a "1" bit, so the count of "1" bits is fundamental structural information.
- $\nu_3(n)$? Irrelevant.
- $p_{\min}(n)$? Irrelevant.

Total local: 3.

Step 3 — Global coordinates.

- $\sigma_B(n)$, $q_B(n)$ —— necessary.
- $P_{\max,B}(n)$ —— the trajectory of T_B is monotonically decreasing: each operation strictly reduces n. Therefore $P_{\max,B}(n) = n$ trivially. Not an independent dimension.

Total global: 2.
Step 4 — Relational: $\mathrm{freq}_B(n)$. Total: 1.
Total: 6 dimensions.

$$\mathbf{x}_B(n) = \big(n,\, \nu_2(n),\, s_2(n),\, \sigma_B(n),\, q_B(n),\, \mathrm{freq}_B(n)\big) \in \mathbb{Z}^6. \tag{14}$$

Note how this 6D differs from the 6D of T_2: both have 6 dimensions, but the "intermediate" local coordinates are distinct ($\nu_2(n+1)$ in T_2, $s_2(n)$ in T_B). The mathematics is exact: each system has its own canonical dimensions.

Coverage with closed forms

Theorem 21 (Coverage of T_B with closed form)**.** *For every $n \in \mathbb{N}^+$:*

1. $T_B^d(n) = 1$ *for some finite* $d \geq 0$*;*
2. $\sigma_B(n) = \lfloor \log_2 n \rfloor$*;*
3. $q_B(n) = s_2(n) - 1$*;*
4. $P_{\max,B}(n) = n$.

Proof. Let $n = (b_k b_{k-1} \cdots b_0)_2$ be the binary representation with $b_k = 1$. Each operation removes exactly one bit:

- Operation /2 (when $b_0 = 0$): removes b_0.
- Odd operation (when $b_0 = 1$): removes b_k.

Starting from n with $k+1$ bits, we reach 1 (representation 1_2, a single bit) after exactly k removals. Therefore $\sigma_B(n) = k = \lfloor \log_2 n \rfloor$.

Of the k removals, exactly $s_2(n) - 1$ are odd operations (one for each "1" bit except the final one that remains as $n = 1$). Therefore $q_B(n) = s_2(n) - 1$.

The monotonicity of the trajectory guarantees $P_{\max,B}(n) = n$. □

Mathematical elegance

The reverse tree T_B^{-1} has equally beautiful structure. Starting from 1, each inverse operation adds one bit:

- Inverse of /2: doubles n, adding "0" to the right.
- Inverse of odd operation: adds "1" to the left of the highest bit.

After k levels, the tree contains exactly all integers with $k+1$ bits or fewer, that is, $\{1, 2, 3, \ldots, 2^{k+1} - 1\}$.

In more accessible language: the reverse tree of T_B is, literally, a binary tree that generates all numbers bit by bit, in the natural order of binary representation. At each level, the number of integers covered doubles. This exact correspondence between tree depth and binary magnitude of integers is a geometric manifestation of the unity between iterative dynamics and arithmetic structure of integers.

What this example teaches

T_B illustrates three properties of the method:

1. The dimensionality reflects the specific structure of the system —— even two 6D systems can have distinct local coordinates depending on the arithmetic involved.

2. Dynamical systems can admit exact closed forms —— T_B is a case where all dynamical information reduces to formulas in local coordinates.

3. There exists a deep resonance between dynamics and arithmetic —— the binary structure of integers and the reverse tree of T_B are literally the same object, in distinct representations.

A.6 Comparative synthesis

System	Odd operation	Dim.	Coverage	Why
T_B (binary)	$n - 2^{\lfloor \log_2 n \rfloor}$	6D	Demonstrated	Closed form
T_2 $(n+1)$	$n+1$	6D	Demonstrated	Simple induction
T_3' $(3n-1)$	$(3n-1)/2$	7D	**Failure**	Cycle $\{5, 7, 10\}$
Collatz $(3n+1)$	$(3n+1)/2$	8D	Conjectured	No known non-trivial cycles

Table 4: Comparison of the systems treated in this appendix. The tensor methodology is dimension-adaptive and structurally discriminating.

A.7 Implications for the Collatz case

Positioning Collatz in the picture:

- **As in T_2 and T_B:** conditions (i), (iii), (iv) are directly verified; empirical coverage is massive ($n \leq 2.95 \times 10^{20}$ [11]).

- **Unlike T_2 and T_B:** formal proof of coverage remains open.

- **Unlike T_3':** no identifiable exclusion mechanism has been found in decades of investigation.

Condition (ii) —— absence of exclusion mechanism —— is the point where Collatz separates from the other cases. In T_2 and T_B, it is demonstrated. In T_3', it demonstrably fails. In Collatz, it is empirically verified at massive scale but not formally established.

This gap is precisely the content of Principle 16 when applied to Collatz: it proposes that massively verified empirical absence constitutes sufficient structural evidence, without formal proof of impossibility. The philosophical question is whether this type of evidence —— which we routinely accept for generating principles such as Peano —— should also be accepted for Collatz, given that the system satisfies the remaining structural conditions.

A.8 Conclusion of the appendix

The three examples confirm:

1. The methodology is dimension-adaptive (6D, 6D, 7D, 8D depending on the system);

2. The process of arriving at the matrix is mechanical and replicable —— any researcher can apply it;

3. The generators are always ontological (elementary arithmetic operations over integers);

4. Principle 16 verifiably distinguishes coverage cases from failure cases;
5. Collatz occupies a structurally articulable intermediate position.

The methodology is therefore not an *ad hoc* construction for Collatz —— it produces demonstrable theorems in T_2 and T_B, correctly identifies the failure in T_3', and situates Collatz in a structurally coherent picture in which the only condition not formally established is precisely the one that constitutes the Conjecture.

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