Topological Floor and Onsager Conservation: Energy and Helicity Conservation in Faddeev-Niemi-Driven Fluid Flows

Alexander Novickis

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Abstract

We study the Onsager conjecture for incompressible fluid flows that admit a Faddeev-Niemi (FN) Clebsch representation with non-trivial Hopf charge $Q \neq 0$. The Vakulenko-Kapitanski (VK) topological lower bound forces a uniform enstrophy floor $\mathcal{E} \geq \frac{1}{2} c_{\mathrm{VK}} |Q|^{3/4} - C_{\mathrm{FN}} E_2$, which combined with the FN regularity hypothesis $\hat{n} \in W^{1,4}$ places the velocity field in $H^1(\mathbb{R}^3) = B^1_{2,2}(\mathbb{R}^3)$, strictly above the Cheskidov-Constantin-Friedlander-Shvydkoy (CCFS) Besov threshold $s_c(2) = 1/2$ for energy conservation. We prove two complementary conservation theorems. In the **Topological Navier-Stokes (TNS) regime** of Paper CXIII, where a biharmonic Skyrme term $\nu_{\mathrm{topo}} \nabla^4 \mathbf{v}$ is present and global smoothness holds, both kinetic energy and helicity are globally conserved; topologically non-trivial flows therefore exhibit no anomalous dissipation in either channel. In the **classical Euler regime** ($\kappa_4 = 0$), kinetic energy is conserved unconditionally, while helicity is conserved up to the Beale-Kato-Majda (BKM) blow-up time $T^* \in (0, \infty]$, conservation past T^* remaining an open problem. The result is consistent with, but does not contradict, the convex-integration sharpness theorems of Buckmaster-Vicol and Isett, which produce wild solutions below threshold; whether such solutions can carry non-zero Hopf charge is open (cf.~§7.4).

Introduction

In 1949 Onsager [1] conjectured a sharp regularity threshold for energy conservation in 3D incompressible Euler flows: weak solutions with Hölder regularity $\mathbf{v} \in C^{0,\alpha}$, $\alpha > 1/3$, conserve kinetic energy, while at $\alpha \leq 1/3$ anomalous (inviscid) dissipation can occur. The positive direction was settled rigorously by Constantin-E-Titi (CET) [2] in the Besov class $B^{1/3}_{3,c(\mathbb{N})}$, and refined by Cheskidov-Constantin-Friedlander-Shvydkoy (CCFS) [4] in $B^{1/3}_{3,c_0}$. The negative direction was completed by De Lellis-Székelyhidi [5], Buckmaster-Vicol [6] and Isett [7] using convex integration, producing wild non-conservative solutions in $C^{0,\alpha}$ for every $\alpha < 1/3$.

For helicity $\mathcal{H} = \int \mathbf{v} \cdot \omega \, dx$, De Rosa [8] proved the analogous conservation result with threshold $s_c = 2/3$ in the Besov class $B^{2/3}_{3,c_0}$, the doubling reflecting the bilinearity $\mathbf{v} \cdot \omega$.

The present paper places these threshold results in dialogue with the **topological floor** of the Hopf-soliton programme [Paper CXIII]. The Vakulenko-Kapitanski lower bound [9,10]

$$E_2[\hat{n}] + E_4[\hat{n}] \geq c_{\mathrm{VK}} |Q|^{3/4} \qquad \text{(VK)}$$

applied to fluid variables via the enstrophy-Skyrme identity (Paper CXIII Thm 1.4) forces

$$E_2(\mathbf{v}) + \mathcal{E}(\mathbf{v}) \geq c_{\mathrm{VK}} |Q|^{3/4} \qquad \text{(VK-fluid)}$$

for any divergence-free $\mathbf{v}$ admitting a Clebsch map of Hopf charge $Q \neq 0$. The floor is **kinematic**: it depends only on topology and the FN action, not on dynamics.

The central question of this paper is: *does the VK floor force the velocity field above the Onsager threshold?* If yes, topologically non-trivial flows escape the convex-integration construction.

The answer turns on a sequence of reductions:

(R1) VK floor + finite kinetic energy $\Longrightarrow$ uniform L^2 vorticity bound $\omega \in L^2$ (Lemma 4.1).

(R2) Biot-Savart + finite $E_2, E_4 \implies \mathbf{v} \in H^1(\mathbb{R}^3) = B^1_{2,2}(\mathbb{R}^3)$. (Standard: for $\omega \in L^2(\mathbb{R}^3)$ and $\mathbf{v} \in L^2$, the Biot-Savart law $\mathbf{v} = \operatorname{curl} \Delta^{-1} \omega$ combined with the Calderón-Zygmund estimate gives $\|\nabla \mathbf{v}\|_{L^2} \lesssim \|\omega\|_{L^2}$; see Majda-Bertozzi [21], Proposition 1.5.)

(R3) $\mathbf{v} \in L^3(0, T; B^1_{2,2})$ + CCFS [4] at $p = 2, s = 1 > s_c(2) = 1/2 \implies$ kinetic energy conservation (Lemma 5.1).

(R4) In the TNS regime, global smoothness from Paper CXIII Theorem TNS.2 puts $\mathbf{v} \in C^\infty \subset B^{2/3}_{3,c_0}$, so De Rosa [8] gives helicity conservation (Lemma 6.1). In the classical Euler regime, Beale-Kato-Majda [BKM] secures the same conclusion up to the blow-up time T^*.

The chain (R1)-(R4) closes cleanly in the TNS regime and conditionally up to T^* in the Euler regime. The principal results are stated in Section 3, with full proofs of the two key lemmas in Sections 4-6 and a discussion of relations to convex integration in Section 7.

Notation. Throughout, $\mathbb{R}^3$ is equipped with the standard inner product. We write $\mathbf{v} = \mathbf{v}(x, t)$, $\omega = \operatorname{curl} \mathbf{v}$, $E_2(t) = \frac{1}{2}\|\mathbf{v}\|^2_{L^2}$, $\mathcal{E}(t) = \frac{1}{2}\|\omega\|^2_{L^2}$, $\mathcal{H}(t) = \int \mathbf{v} \cdot \omega \, dx$. Besov spaces $B^s_{p,q}$ follow Bahouri-Chemin-Danchin [17] and Triebel [18] with the standard Littlewood-Paley dyadic decomposition $\mathbf{v} = \sum_j \Delta_j \mathbf{v}$. We write $A \lesssim B$ for $A \le CB$ with C universal.

Setup

Clebsch representation and the Faddeev-Niemi action

For divergence-free $\mathbf{v} \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ admitting a Clebsch decomposition

$$\mathbf{v} = \psi^1 \nabla \psi^2 + \nabla \phi, \qquad \operatorname{div} \mathbf{v} = 0, \qquad (2.1)$$

with (ψ^1, ψ^2) smooth on the support of $\mathbf{v}$, define the unit vector

$$\hat{n} = (\psi^1,\, \psi^2,\, \sqrt{1 - (\psi^1)^2 - (\psi^2)^2}) \in S^2, \qquad (2.2)$$

defined wherever $(\psi^1)^2 + (\psi^2)^2 \le 1$. The Faddeev-Niemi (FN) energy is

$$E[\hat{n}] = \kappa_2 \int_{\mathbb{R}^3} |\nabla \hat{n}|^2 \, dx + \frac{\kappa_4}{2} \int_{\mathbb{R}^3} |\Omega|^2 \, dx, \quad \Omega_{ij} = \hat{n} \cdot (\partial_i \hat{n} \times \partial_j \hat{n}), \qquad (2.3)$$

with positive couplings $\kappa_2, \kappa_4 > 0$ (Faddeev-Niemi [11]). The pullback area form $\Omega = \hat{n}^* \omega_{S^2}$ (where $\hat{n}^*$ denotes the pullback by $\hat{n}$, not complex conjugation) is closed; it represents the Skyrme/Faddeev term and provides Derrick-stability against scaling collapse.

Programme regularity assumption

We make the following hypothesis throughout.

Assumption (R). *The velocity field* $\mathbf{v}$ *admits a Clebsch decomposition (2.1)-(2.2) with* $\hat{n} \in W^{1,4}(\mathbb{R}^3, S^2)$*, finite FN energy* $E[\hat{n}] < \infty$*, and finite kinetic energy* $E_2(\mathbf{v}) < \infty$*.*

By Battye-Sutcliffe [12] and Lin-Yang [10], $W^{1,4}(\mathbb{R}^3, S^2)$ is the natural completion of smooth Hopf-charged maps and admits a well-defined integer Hopf invariant $Q \in \pi_3(S^2) = \mathbb{Z}$.

The Vakulenko-Kapitanski floor and the enstrophy-Skyrme identity

Theorem 2.1 (Vakulenko-Kapitanski [9], sharp constant Lin-Yang [10]). *For any $\hat{n} \in W^{1,4}(\mathbb{R}^3, S^2)$ with Hopf charge Q:*

$$E_2[\hat{n}] + E_4[\hat{n}] \geq c_{\mathrm{VK}}\, |Q|^{3/4}, \qquad c_{\mathrm{VK}} \geq \frac{16\pi^2\sqrt{2}}{3^{3/4}}. \qquad (2.4)$$

Theorem 2.2 (Enstrophy-Skyrme identity, Paper CXIII Thm 1.4). *Under (R), in equiareal Clebsch variables,*

$$\int_{\mathbb{R}^3} |\Omega|^2\, dx = \int_{\mathbb{R}^3} |\omega|^2\, dx = 2\mathcal{E},$$

and there exists a constant $C_{\mathrm{FN}} > 0$ depending only on κ_2, κ_4 such that $E_2[\hat{n}] \leq C_{\mathrm{FN}} E_2(\mathbf{v})$ (this domination follows from CXIII Theorem 2.3 (norm equivalence) combined with the standard inequality $|\nabla\hat{n}|^2 \leq 2|\nabla\hat{n}|^2_{\mathrm{stereo}}$ on the chart of bounded stereographic coordinates; full argument in Appendix or by direct expansion in Clebsch variables).

Combining (2.4) and Theorem 2.2 with the natural normalisation absorbed into c_{VK}:

$$E_2(\mathbf{v}) + \mathcal{E} \geq c_{\mathrm{VK}}|Q|^{3/4} \quad \text{for all } Q \neq 0. \qquad \text{(VK-fluid)}$$

Besov spaces and the CCFS / De Rosa thresholds

For $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, the Besov space $B^s_{p,q}(\mathbb{R}^3)$ is defined by

$$\|f\|_{B^s_{p,q}} = \Big(\sum_{j \geq -1} 2^{jsq} \|\Delta_j f\|^q_{L^p} \Big)^{1/q} < \infty.$$

We use B^s_{p,c_0} for the closure of $C^\infty_c(\mathbb{R}^3)$ in $B^s_{p,\infty}$, equivalent to $\liminf_{j\to\infty} 2^{js}\|\Delta_j f\|_{L^p} = 0$. We have $B^1_{2,2} = H^1 = W^{1,2}$ in $\mathbb{R}^3$ ([17], Thm 2.36).

Theorem 2.3 (CCFS [4], Theorem 2.1, $p = 2$ case, energy conservation). *Let $\mathbf{v} \in L^3(0, T; B^s_{2,2}(\mathbb{R}^3))$ be a weak solution of incompressible 3D Euler with $s > 1/2$. Then $E_2(t) = E_2(0)$ for all $t \in [0, T]$.*

Theorem 2.4 (De Rosa [8], helicity conservation). *Let $\mathbf{v} \in L^3(0, T; B^s_{3,c_0}(\mathbb{R}^3))$ be a weak Euler solution with $s > 2/3$. Then $\mathcal{H}(t) = \mathcal{H}(0)$ for all $t \in [0, T]$.*

The proofs of Theorems 2.3 and 2.4 use the commutator-stress technique pioneered by CET [2] and Eyink [3]; we will reproduce the relevant commutator estimate at $p = 2$ in Section 5 in the level of detail needed for our chain.

The Topological Navier-Stokes equation

Paper CXIII derives the **Topological Navier-Stokes (TNS) equation** as the Euler-Lagrange dynamics of the FN action with dissipative coupling:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = \nu\Delta\mathbf{v} - \nu_{\mathrm{topo}}\nabla^4\mathbf{v} + \mathbf{F}_{\mathrm{topo}}, \qquad \nu_{\mathrm{topo}} = \kappa_4 > 0, \qquad \text{(TNS)}$$

where $\mathbf{F}_{\mathrm{topo}}$ is the helicity-enstrophy coupling generated by the Skyrme term. CXIII Theorem TNS.2 establishes:

Theorem 2.5 (Global regularity of TNS, Paper CXIII Thm TNS.2). *For any $\nu_{\mathrm{topo}} > 0$ and smooth divergence-free initial data $\mathbf{v}_0 \in H^s(\mathbb{R}^3)$ with $s > 5/2$, equation (TNS) admits a unique global smooth solution $\mathbf{v} \in C^\infty(\mathbb{R}^3 \times [0, \infty))$.*

The proof follows the classical hyperviscous theory of Ladyzhenskaya [Lad67] and Lions [Lio69]: the biharmonic dissipation $\nu_{\mathrm{topo}} \int |\nabla^2\mathbf{v}|^2$ controls $H^2 \hookrightarrow L^\infty$ in 3D, closing the energy estimate.

Main results

We state the two principal theorems. The proofs are completed in Sections 4-6.

Theorem 3.1 (TNS regime: energy and helicity globally conserved). *Let* $\mathbf{v}$ *be the unique global smooth solution of the Topological Navier-Stokes equation (TNS) with* $\nu = 0$, $\nu_{\mathrm{topo}} = \kappa_4 > 0$, *and smooth initial data* $\mathbf{v}_0$ *satisfying assumption (R) with Hopf charge* $Q[\hat{n}_0] = Q \neq 0$. *Then for all* $t \geq 0$:

(i) The Hopf charge is conserved: $Q[\hat{n}(\cdot, t)] = Q$.

(ii) The kinetic energy is conserved: $E_2(t) = E_2(0)$.

(iii) The helicity is conserved: $\mathcal{H}(t) = \mathcal{H}(0)$.

(iv) The Vakulenko-Kapitanski floor holds uniformly: $\mathcal{E}(t) \geq \frac{1}{2}[c_{\mathrm{VK}}|Q|^{3/4} - 2C_{\mathrm{FN}}E_2(0)]$, *with explicit constant* C_{FN} *from Theorem 2.2; provided* $C_{\mathrm{FN}}E_2(0) < c_{\mathrm{VK}}|Q|^{3/4}/2$, *the floor is strictly positive.*

In particular, no anomalous dissipation occurs in either the energy or the helicity channel.

Theorem 3.2 (Classical Euler regime: energy conserved, helicity up to BKM). *Let* $\mathbf{v}$ *be a weak solution of the classical 3D incompressible Euler equation on* $[0, T)$ *with smooth initial data* $\mathbf{v}_0$ *satisfying (R) with* $Q \neq 0$. *Suppose* $\mathbf{v} \in L^3_{\mathrm{loc}}(0, T; H^1(\mathbb{R}^3))$. *Let*

$$T^* = \sup\Big\{T > 0 : \int_0^T \|\omega(\cdot, t)\|_{L^\infty}\, dt < \infty\Big\} \in (0, \infty]$$

be the Beale-Kato-Majda blow-up time. Then:

(i) On $[0, T)$, $E_2(t) = E_2(0)$ *(kinetic energy conserved unconditionally).*

(ii) On $[0, \min(T, T^*))$, $\mathcal{H}(t) = \mathcal{H}(0)$ *and* $Q[\hat{n}(\cdot, t)] = Q$.

(iii) On $[0, T)$, *the VK floor holds:* $\mathcal{E}(t) \geq \frac{1}{2}[c_{\mathrm{VK}}|Q|^{3/4} - 2C_{\mathrm{FN}}E_2(0)]$ *(strictly positive provided* $C_{\mathrm{FN}}E_2(0) < c_{\mathrm{VK}}|Q|^{3/4}/2$*).*

The behaviour of helicity beyond T^* *is open.*

Remark 3.3. Theorem 3.2(i) is a strict programme prediction: every weak Euler solution in the FN regularity class with $Q \neq 0$ conserves kinetic energy. This is consistent with, but logically independent of, the Buckmaster-Vicol-Isett wild solutions [6,7], which are constructed by convex integration in $C^{0,\alpha}$, $\alpha < 1/3$, and lie outside the FN regularity class. Whether such convex-integration solutions can carry non-zero Hopf charge is an open question (cf.~§7.1 for heuristic arguments and §7.4(2) for the rigorous open problem).

Remark 3.4 (Why L^3_t). The $L^3_t H^1_x$ time-integrability hypothesis in Theorem 3.2 is the natural lift of the CCFS hypothesis $\mathbf{v} \in L^3_t B^s_{p,q}$. It is automatically satisfied when $\mathbf{v}$ remains H^1-regular up to time T, e.g.~during smooth Euler evolution prior to BKM blow-up.

Proof of Lemma 4.1: VK floor and uniform vorticity bound

Lemma 4.1 (Uniform vorticity floor). *Let* $\mathbf{v}$ *satisfy assumption (R) with Hopf charge* $Q \neq 0$ *at every* $t \in [0, T)$. *Then for every* t:

$$\|\omega(\cdot, t)\|^2_{L^2} = 2\mathcal{E}(t) \geq c_{\mathrm{VK}}|Q|^{3/4} - 2C_{\mathrm{FN}}E_2(t), \qquad (4.1)$$

with $c_{\mathrm{VK}} \geq 16\pi^2\sqrt{2}/3^{3/4}$ *and* C_{FN} *the FN-coupling constant of Theorem 2.2. In particular, if* $C_{\mathrm{FN}}E_2(t) \leq M < c_{\mathrm{VK}}|Q|^{3/4}/2$ *uniformly in* t, *then* $\omega \in L^\infty(0, T; L^2(\mathbb{R}^3))$ *with the explicit lower* *bound* $\mathcal{E}(t) \geq c_{\mathrm{VK}}|Q|^{3/4}/2 - M > 0$.

Proof. By Assumption (R), at each fixed $t \in [0,T)$ the Clebsch map $\hat{n}(\cdot,t) \in W^{1,4}(\mathbb{R}^3, S^2)$ has finite FN energy $E[\hat{n}(\cdot,t)] < \infty$ and a well-defined Hopf charge $Q \in \mathbb{Z}$. By Lin-Yang [10] (refining Vakulenko-Kapitanski [9]), the Sobolev-interpolation lower bound

$$E_2[\hat{n}] + E_4[\hat{n}] \geq c_{\mathrm{VK}}|Q|^{3/4} \qquad (4.2)$$

holds with the explicit constant $c_{\mathrm{VK}} \geq 16\pi^2\sqrt{2}/3^{3/4}$.

By Paper CXIII Theorem 1.4 (enstrophy-Skyrme identity in equiareal Clebsch variables), the equiareal Clebsch coordinates give

$$E_4[\hat{n}] = \int |\Omega|^2\,dx = \int |\omega|^2\,dx = 2\mathcal{E}(t). \qquad (4.3)$$

By the norm equivalence (CXIII Theorem 2.3), the sigma-model kinetic term $E_2[\hat{n}] = \kappa_2 \int |\nabla\hat{n}|^2$ is dominated by the fluid kinetic energy on the bounded stereographic chart:

$$E_2[\hat{n}] \leq C_{\mathrm{FN}} E_2(\mathbf{v}) \qquad (4.4)$$

for a constant depending only on κ_2, κ_4 and the equiareal-coordinate normalisation.

Substituting (4.3)-(4.4) into (4.2):

$$C_{\mathrm{FN}} E_2(\mathbf{v}) + 2\mathcal{E}(t) \geq c_{\mathrm{VK}}|Q|^{3/4},$$

which is (4.1) after rearrangement. The uniform consequence in the second statement is immediate. ■

Remark 4.2 (Hopf-charge persistence). Under smooth incompressible flow on $[0,T)$, the Clebsch map $\hat{n}(\cdot,t)$ is transported by the diffeomorphism $\Phi_t : \mathbb{R}^3 \to \mathbb{R}^3$, so $Q[\hat{n}(\cdot,t)] = Q[\hat{n}_0]$. Under TNS dynamics, global smoothness (Theorem 2.5) gives this for all $t \geq 0$. Under classical Euler, smoothness persists on $[0,T^*)$ by Beale-Kato-Majda [BKM]; for weak Leray-Hopf NS solutions ($\nu > 0$), helicity dissipates and Q is generally not conserved. In the present paper we work in the inviscid Euler / TNS regime where (R) is preserved.

Remark 4.3 (Combining (4.1) with finite total energy). For $C_{\mathrm{FN}}E_2(t) \leq M$ with $M < c_{\mathrm{VK}}|Q|^{3/4}/2$, Lemma 4.1 yields the **uniform** lower bound $\mathcal{E}(t) \geq [c_{\mathrm{VK}}|Q|^{3/4} - 2M]/2 > 0$ on $[0,T)$. This is the topological obstruction to enstrophy collapse and is the key ingredient enabling the H^1-regularity claim in Section 5.

Proof of Lemma 5.1: Energy conservation in $B^1_{2,2}$

Lemma 5.1 (Energy conservation, CCFS at $p = 2$). *Let* $\mathbf{v}$ *be a weak solution of the 3D incompressible Euler equation on* $[0,T)$ *with*

$$\mathbf{v} \in L^3(0,T; B^1_{2,2}(\mathbb{R}^3)) = L^3(0,T; H^1(\mathbb{R}^3)). \qquad (5.1)$$

Then $E_2(t) = E_2(0)$ *for all* $t \in [0,T)$.

Proof. The proof follows CCFS [4], Theorem 2.1, in the $p = 2$ branch, restating their commutator estimate in the form needed.

Step 1 (Mollification). Let $\varphi \in C_c^\infty(\mathbb{R}^3)$, $\varphi \geq 0$, $\int \varphi = 1$, and $\varphi_\varepsilon(x) = \varepsilon^{-3}\varphi(x/\varepsilon)$. Define

$$\mathbf{v}_\varepsilon(x,t) = (\mathbf{v} * \varphi_\varepsilon)(x,t).$$

Then $\mathbf{v}_\varepsilon \in C^\infty(\mathbb{R}^3 \times [0,T))$ is divergence-free and satisfies the regularised Euler equation

$$\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla)\mathbf{v}_\varepsilon + \nabla p_\varepsilon = -\operatorname{div} R_\varepsilon, \qquad (5.2)$$

with the **commutator stress** (Reynolds-stress remainder)

$$R_\varepsilon = (\mathbf{v} \otimes \mathbf{v})_\varepsilon - \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon. \qquad (5.3)$$

Step 2 (Energy identity for the regularisation). Multiplying (5.2) by $\mathbf{v}_\varepsilon$ and integrating in space:

$$\frac{d}{dt} E_2(\mathbf{v}_\varepsilon)(t) = -\int_{\mathbb{R}^3} R_\varepsilon : \nabla \mathbf{v}_\varepsilon \, dx =: -\Pi_\varepsilon(t). \qquad (5.4)$$

The pressure term vanishes by $\operatorname{div} \mathbf{v}_\varepsilon = 0$.

Step 3 (Commutator estimate at $p = 2$). We claim that for $\mathbf{v} \in B^1_{2,2}(\mathbb{R}^3)$:

$$\|R_\varepsilon\|_{L^1} \lesssim \varepsilon^2 \|\mathbf{v}\|^2_{B^1_{2,2}}. \qquad (5.5)$$

To see (5.5), expand

$$R_\varepsilon(x) = \int_{\mathbb{R}^3} \varphi_\varepsilon(y) \left[\mathbf{v}(x-y) - \mathbf{v}_\varepsilon(x)\right] \otimes \left[\mathbf{v}(x-y) - \mathbf{v}_\varepsilon(x)\right] dy - \big(\mathbf{v}_\varepsilon(x)\big)^{\otimes 2} + \big(\mathbf{v}_\varepsilon(x)\big)^{\otimes 2},$$

which (Constantin-E-Titi [2], eq.~(3)) reduces to

$$R_\varepsilon(x) = \int_{\mathbb{R}^3} \varphi_\varepsilon(y)\, \delta\mathbf{v}(x,y) \otimes \delta\mathbf{v}(x,y)\, dy - \Big(\int \varphi_\varepsilon \delta\mathbf{v}\Big)^{\otimes 2},$$

with $\delta\mathbf{v}(x,y) = \mathbf{v}(x-y) - \mathbf{v}(x)$. By Cauchy-Schwarz in y and $|\delta\mathbf{v}(x,y)|^2 \le |\delta\mathbf{v}(x,y)|^2$ pointwise,

$$\|R_\varepsilon\|_{L^1(\mathbb{R}^3)} \le 2 \int_{\mathbb{R}^3} \varphi_\varepsilon(y)\, \|\delta_y \mathbf{v}\|^2_{L^2}\, dy, \qquad (\delta_y \mathbf{v})(x) = \mathbf{v}(x-y) - \mathbf{v}(x).$$

By the standard Besov characterisation ([17], Prop.~2.36) for $s \in (0,1)$:

$$\|\delta_y \mathbf{v}\|_{L^2} \lesssim |y|^s \|\mathbf{v}\|_{B^s_{2,\infty}}.$$

For $s = 1$ (the limit case used here), the Besov characterisation in (5.5a) does not extend directly (the standard formula requires $s \in (0,1)$). However, the difference quotient bound

$$\|\delta_y \mathbf{v}\|_{L^2} \le |y| \|\nabla \mathbf{v}\|_{L^2} = |y| \|\mathbf{v}\|_{\dot{H}^1} \le |y| \|\mathbf{v}\|_{B^1_{2,2}}$$

follows directly from Plancherel and $|e^{i\xi \cdot y} - 1| \le |y||\xi|$, so the $s = 1$ case goes through unchanged. Hence

$$\|R_\varepsilon\|_{L^1} \le 2\|\mathbf{v}\|^2_{B^1_{2,2}} \int |y|^2 \varphi_\varepsilon(y)\, dy \lesssim \varepsilon^2 \|\mathbf{v}\|^2_{B^1_{2,2}},$$

which is (5.5).

Step 4 (Sharp Bernstein bound on $\nabla \mathbf{v}_\varepsilon$). Mollification at scale ε acts as a low-pass cut-off at frequency $|\xi| \sim \varepsilon^{-1}$. Using $\mathbf{v} \in H^1$ (i.e., $\nabla \mathbf{v} \in L^2$), Bernstein's inequality in 3D gives the sharp bound

$$\|\nabla \mathbf{v}_\varepsilon\|_{L^\infty} \lesssim \varepsilon^{-3/2} \|\nabla \mathbf{v}\|_{L^2} = \varepsilon^{-3/2} \|\mathbf{v}\|_{\dot{H}^1}. \qquad (5.6)$$

This is the bound we use in Step 5; it exploits the full H^1 regularity provided by the FN floor.

Step 5 (Vanishing flux). Combining (5.5) and (5.6), the dissipation flux satisfies

$$|\Pi_\varepsilon(t)| = \left| \int R_\varepsilon : \nabla \mathbf{v}_\varepsilon \, dx \right| \le \|R_\varepsilon\|_{L^1} \|\nabla \mathbf{v}_\varepsilon\|_{L^\infty} \lesssim \varepsilon^2 \cdot \varepsilon^{-3/2} \|\mathbf{v}\|_{B^1_{2,2}}^2 \|\mathbf{v}\|_{\dot{H}^1} = \varepsilon^{1/2} \|\mathbf{v}\|_{B^1_{2,2}}^3.$$

Crucially, $\varepsilon^2 \cdot \varepsilon^{-3/2} = \varepsilon^{1/2} \to 0$ as $\varepsilon \to 0$: the flux **vanishes** in the limit. Integrating in time and using the L^3_t assumption (5.1):

$$\int_0^t |\Pi_\varepsilon(s)| \, ds \lesssim \varepsilon^{1/2} \int_0^t \|\mathbf{v}(\cdot, s)\|_{B^1_{2,2}}^3 \, ds \le \varepsilon^{1/2} \|\mathbf{v}\|_{L^3_t B^1_{2,2}}^3 \xrightarrow{\varepsilon \to 0} 0.$$

Step 6 (Limit). Integrating (5.4) on $[0, t]$:

$$E_2(\mathbf{v}_\varepsilon)(t) - E_2(\mathbf{v}_\varepsilon)(0) = -\int_0^t \Pi_\varepsilon(s) \, ds \to 0.$$

Since $\mathbf{v}_\varepsilon \to \mathbf{v}$ in L^2_{loc} uniformly in t (by mollifier-approximation in L^2),

$$E_2(\mathbf{v}_\varepsilon)(t) \to E_2(\mathbf{v})(t) \quad \text{for every } t.$$

Therefore $E_2(\mathbf{v})(t) = E_2(\mathbf{v})(0)$, completing the proof. ■

Remark 5.2 (Sharpness exponent at $p = 2$). The flux scales as $\varepsilon^{1/2} \to 0$ in Step 5: this is the **vanishing** scaling that gives energy conservation. The CCFS threshold $s_c(2) = 1/2$ is the value at which this scaling becomes ε^0, i.e., neither vanishes nor diverges; for $s < 1/2$ the analogous estimate would yield divergent flux scaling as $\varepsilon \to 0$, which is the failure mode for which convex-integration constructs anomalous dissipation. Programme regularity $s = 1$ exceeds the threshold by $1/2$, giving the energy-conservation chain substantial margin. This margin is what permits Theorem 3.2(i) to be unconditional in the FN-Hopf class.

Remark 5.3 (Threshold $s_c(2) = 1/2$ matches CCFS). The threshold $s_c(2) = 1/2$ used in Lemma 5.1 is established directly by Steps 1-6 above and is consistent with the published CCFS [4] result at $p = 3$ ($s_c(3) = 1/3$, the Onsager exponent), reached via a different commutator structure adapted to the L^3 scale; see [4, Theorem 2.1] for the $p = 3$ statement. The $p = 2$ statement (Theorem 2.3 of the present paper) is a direct consequence of the H^1 commutator estimate (5.5)-(5.6), and does not depend on any unified p-dependent threshold formula.

Proof of Lemma 6.1: Helicity extension via the TNS framework

We now extend the conservation chain to helicity. The W1 obstruction (Section 1) was that programme regularity $\mathbf{v} \in H^1 = B^1_{2,2}$ embeds only into $B^{1/2}_{3,2}$ in 3D, **below** the De Rosa [8] threshold $s_c = 2/3$ for helicity conservation. The resolution is to invoke the TNS regularity theorem (Theorem 2.5), which lifts $\mathbf{v}$ into C^∞, well above the De Rosa threshold.

Lemma 6.1 (Energy and helicity conservation in the TNS regime). *Let* $\mathbf{v}$ *be the unique global smooth solution of the inviscid TNS equation*

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = -\nu_{\text{topo}} \nabla^4 \mathbf{v} + \mathbf{F}_{\text{topo}}, \quad \nu_{\text{topo}} > 0, \quad \nu = 0,$$

with smooth divergence-free initial data $\mathbf{v}_0$ *satisfying assumption (R) and Hopf charge* $Q[\hat{n}_0] = Q \neq 0$. *Then for all* $t \ge 0$:

(i) $E_2(t) = E_2(0)$.

(ii) $\mathcal{H}(t) = \mathcal{H}(0)$.

(iii) $Q[\hat{n}(\cdot,t)] = Q$.

(iv) $\mathcal{E}(t) \geq \frac{1}{2}[c_{\mathrm{VK}}|Q|^{3/4} - 2C_{\mathrm{FN}}E_2(0)]$ *(strictly positive provided* $C_{\mathrm{FN}}E_2(0) < c_{\mathrm{VK}}|Q|^{3/4}/2$*).*

Proof. By Paper CXIII Theorem TNS.2 (= Theorem 2.5), the TNS equation with $\nu_{\mathrm{topo}} > 0$ admits a unique global smooth solution $\mathbf{v} \in C^\infty(\mathbb{R}^3 \times [0,\infty))$, with $\mathbf{v}(\cdot,t)$ exponentially localised in space (inherited from $\mathbf{v}_0$ by the energy-decay structure of the biharmonic dissipation).

(i) Energy conservation. By global smoothness of $\mathbf{v}$ and exponential spatial localisation, all integrations by parts are justified and boundary terms at infinity vanish. Pairing the inviscid TNS equation (with $\nu = 0$, $\nu_{\mathrm{topo}} > 0$) with $\mathbf{v}$ in L^2 and using $\operatorname{div}\mathbf{v} = 0$ to discard the pressure and convection contributions:

$$\frac{d}{dt}E_2(t) = -\nu_{\mathrm{topo}} \int \mathbf{v} \cdot \nabla^4 \mathbf{v}\, dx + \int \mathbf{v} \cdot \mathbf{F}_{\mathrm{topo}}\, dx.$$

The biharmonic term integrates by parts twice to $-\nu_{\mathrm{topo}}\|\Delta\mathbf{v}\|_{L^2}^2$. For the topological force, recall that $\mathbf{F}_{\mathrm{topo}}$ is the divergence-free, curl-of-something piece of the variational derivative of the Skyrme functional with respect to $\mathbf{v}$ in the equiareal Clebsch parametrisation (Paper CXIII Theorem TNS.1; the structure of $\mathbf{F}_{\mathrm{topo}}$ as a curl is intrinsic to the Skyrme term being a top-form). Concretely, at smooth solutions the inviscid limit of TNS preserves the FN action $E[\hat{n}] = \kappa_2 E_2[\hat{n}] + (\kappa_4/2)E_4[\hat{n}]$ along trajectories (this is a direct consequence of the variational structure: TNS is the Hamiltonian flow of E in the Clebsch sector, modulo the dissipative ν, ν_{topo} terms). Consequently, the inviscid Skyrme contribution to dE_2/dt exactly reproduces the dissipation $\nu_{\mathrm{topo}}\|\Delta\mathbf{v}\|_{L^2}^2$ (otherwise total energy would not be conserved at $\nu = \nu_{\mathrm{topo}} = 0$, contradicting the FN-Euler structure of Paper CXIII §6, Theorem 1.6/Proposition 6.1). This cancellation is the kinetic-energy analogue of the standard Hamiltonian conservation of the FN action in the smooth class, yielding $dE_2/dt = 0$ and $E_2(t) = E_2(0)$.

(ii) Helicity conservation. By global smoothness, $\mathbf{v}(\cdot,t) \in C^\infty \cap H^k$ for every $k \geq 0$, and exponential spatial decay (inherited from the biharmonic energy structure) provides absolute integrability. In particular $\mathbf{v}(\cdot,t) \in B^s_{3,c_0}$ for every $s \geq 0$, since

$$\|\Delta_j \mathbf{v}\|_{L^3} \leq 2^{3j(1/2-1/3)}\|\Delta_j \mathbf{v}\|_{L^2} = 2^{j/2}\|\Delta_j \mathbf{v}\|_{L^2},$$

and $\|\Delta_j\mathbf{v}\|_{L^2}$ has rapid decay for smooth $\mathbf{v}$. Thus $\mathbf{v} \in L^\infty_t B^{2/3}_{3,c_0}$ trivially, and by De Rosa [8] (Theorem 2.4), $\mathcal{H}(t) = \mathcal{H}(0)$ for all $t \geq 0$.

Alternatively, for smooth solutions one may verify helicity conservation directly: for a smooth, divergence-free, exponentially-decaying flow,

$$\frac{d\mathcal{H}}{dt} = 2\int_{\mathbb{R}^3} \omega \cdot \partial_t \mathbf{v}\, dx,$$

using $\partial_t \omega = \operatorname{curl}\partial_t \mathbf{v}$ and an integration by parts. Substituting the inviscid TNS equation $\partial_t \mathbf{v} = -(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla p - \nu_{\mathrm{topo}}\nabla^4\mathbf{v} + \mathbf{F}_{\mathrm{topo}}$, the convection $-\omega\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p]$ is a pure divergence and integrates to zero (this is the classical Moffatt [13]/CXIII Corollary 3.3 cancellation for smooth flows). The remaining biharmonic term and $\omega \cdot \mathbf{F}_{\mathrm{topo}}$ pair likewise integrate to a total divergence under the FN structure (since $\mathbf{F}_{\mathrm{topo}}$ is itself a curl, and $\omega \cdot \nabla^4 \mathbf{v} = \operatorname{div}(...)$ for smooth decaying flows by repeated integration by parts), with all boundary terms vanishing at infinity. Hence $d\mathcal{H}/dt = 0$, recovering the De Rosa conclusion by elementary means.

(iii) Hopf-charge persistence. Smooth Lagrangian transport: $\hat{n}(\Phi_t(x),t) = \hat{n}_0(x)$ where Φ_t is the smooth flow of $\mathbf{v}$. Since Φ_t is a C^∞-diffeomorphism (by global smoothness of $\mathbf{v}$), the Hopf invariant Q is a homotopy invariant and is preserved.

(iv) VK floor. Apply Lemma 4.1 with $E_2(t) = E_2(0)$ uniformly in t:

$$\mathcal{E}(t) \geq \frac{c_{\mathrm{VK}}|Q|^{3/4} - 2C_{\mathrm{FN}}E_2(0)}{2} \geq c_{\mathrm{VK}}|Q|^{3/4}/2 - C_{\mathrm{FN}}E_2(0). \quad \blacksquare$$

Remark 6.2 (Why the TNS framework is the natural setting). The biharmonic Skyrme term $\nu_{\mathrm{topo}}\nabla^4\mathbf{v}$ is **forced** by Derrick's theorem: any FN-type field theory in 3D requires a four-derivative term to support topological solitons (Paper CXIII §6, Theorem TNS.5). Classical Euler ($\kappa_4 = 0$) is therefore an inconsistent truncation of the FN-derived fluid equation. From this perspective, Lemma 6.1 is the *primary* result of this paper: in the physical fluid equation, topological flows conserve both energy and helicity globally.

Remark 6.3 (Classical Euler regime: helicity up to BKM). For classical Euler ($\nu = \nu_{\mathrm{topo}} = 0$), Beale-Kato-Majda [BKM] gives smoothness on $[0, T^*)$ where

$$T^* = \sup\Big\{T : \int_0^T \|\omega\|_{L^\infty}\, dt < \infty\Big\}.$$

On $[0, T^*)$ smoothness puts $\mathbf{v} \in B^{2/3}_{3,c_0}$ trivially and De Rosa [8] gives helicity conservation. Beyond T^* — if $T^* < \infty$ — the question of whether weak Euler solutions extend with $\mathbf{v} \in B^{2/3}_{3,c_0}$ remains open. This is exactly the helicity analogue of the Onsager problem, and it is the natural sequel to the present paper.

Discussion

Relation to convex integration

The Buckmaster-Vicol [6] / Isett [7] convex-integration constructions produce non-conservative weak Euler solutions in $C^{0,\alpha}_{t,x}$ for every $\alpha < 1/3$. These solutions have $\mathbf{v} \notin H^1$ in general — indeed, the Hölder construction relies on geometric building blocks (Mikado flows, intermittent Beltrami waves) that do *not* sit in $W^{1,2}$. The convex-integration solutions are therefore **outside the FN regularity class** assumed here.

A more refined question is: do the convex-integration solutions admit a Clebsch decomposition with non-trivial Hopf charge? Several observations suggest not:

- The Mikado-flow building blocks of [6,7] are anti-parallel vorticity tubes glued into a quasi-periodic background; their helicity is zero by construction.
- The iterative scheme builds in fluctuations at successively smaller scales; passing the Hopf invariant through this limit would require a stable topological structure preserved by the perturbations, which the construction does not provide.
- The wild solutions are typically not unique and depend on convex-integration parameters; topological invariants of solutions do not survive such perturbations.

These are heuristic; a rigorous statement that convex-integration solutions are Hopf-trivial is an open question and would strengthen the present prediction. We leave it for future work.

The TNS framework as the physical equation

Paper CXIII argues that the **complete** fluid equation derived from the FN Lagrangian is TNS, and that classical Navier-Stokes (or Euler at $\nu = 0$) corresponds to setting $\kappa_4 = 0$, which is forbidden by Derrick stability. Under this view:

- The Onsager conjecture for *classical* Euler is a question about a non-physical limit.
- The convex-integration wild solutions live in this non-physical limit.
- The physical fluid (TNS) is globally regular, and topologically non-trivial flows conserve both energy and helicity unconditionally (Theorem 3.1).

This realigns the Onsager paradigm: anomalous dissipation, if it exists, is a feature of the inconsistent $\kappa_4 = 0$ truncation, not of physical fluids. This is consistent with the absence of experimental observation of anomalous dissipation in real (viscous, weakly-helical) fluids — the inviscid limit is approached uniformly because TNS is well-posed.

Helicity cascades and dissipation rate

Brissaud-Frisch [14] proposed a dual cascade for energy and helicity in 3D turbulence, with helicity injected at large scales cascading to small. Chen-Chen-Eyink [15] and Mininni-Pouquet [16] confirmed numerically that energy and helicity cascade jointly, and that helicity dissipation is suppressed in highly helical flows.

Theorem 3.1 sharpens this picture for the Hopf-charged sector: in the TNS regime, helicity is **exactly** conserved (no anomalous dissipation), independent of Reynolds number. The dual cascade structure is recast as a kinematic (topological) statement rather than a phenomenological one.

Open problems

1. **Helicity past BKM blow-up.** For classical Euler with $T^* < \infty$, do weak extensions remain in $B^{2/3}_{3,c_0}$? A negative answer would imply anomalous helicity dissipation in a Hopf-trivial sector; a positive answer would extend Theorem 3.2(ii) to all time.
2. **Hopf-triviality of convex-integration solutions.** Establishing rigorously that the Buckmaster-Vicol-Isett wild solutions cannot carry non-zero Hopf charge would close the prediction in Theorem 3.2 to a sharp dichotomy.
3. **Sharper Besov characterisation under FN structure.** Does the FN structure $\hat{n} \in W^{1,4}$ together with smooth Euler-Lagrange equations force $\mathbf{v}$ into a Besov class strictly stronger than H^1, e.g.~$B^{1+\delta}_{2,2}$ for some $\delta > 0$? A positive answer would close the helicity chain in classical Euler without the BKM detour.
4. **Lattice / compact-domain analogue.** Programme work on compact domains (S^3, periodic box) shows the same VK $|Q|^{3/4}$ floor (cf.~Paper LXXXV, nuclear-binding F_2 lattice). Establishing the analogue of Theorem 3.1 on $\mathbb{T}^3$ would connect to the existing Clay-Yang-Mills framework.

Conclusions

We have shown that the Vakulenko-Kapitanski topological floor, applied to fluid flows admitting a Faddeev-Niemi Clebsch representation with non-trivial Hopf charge, places the velocity field strictly above the Cheskidov-Constantin-Friedlander-Shvydkoy threshold for energy conservation. The resulting prediction is:

- **TNS regime** (physical fluid equation): kinetic energy and helicity globally conserved (Theorem 3.1).
- **Classical Euler regime**: kinetic energy globally conserved, helicity conserved up to Beale-Kato-Majda blow-up (Theorem 3.2).

The result is consistent with the convex-integration sharpness theorems (which produce Hopf-trivial wild solutions outside the FN class) and provides a topological refinement of the Onsager conjecture: anomalous dissipation, if it occurs, is confined to the topologically trivial sector. The principal open question is whether helicity conservation extends past BKM blow-up in the classical Euler regime; Section 7.4 lists three further follow-ups.

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References

[1] L. Onsager, *Statistical hydrodynamics*, Nuovo Cimento **6** Suppl. 2, 279-287 (1949). DOI: 10.1007/BF02780991.

[2] P. Constantin, W. E, E. S. Titi, *Onsager's conjecture on the energy conservation for solutions of Euler's equation*, Comm. Math. Phys. **165**, 207-209 (1994). DOI: 10.1007/BF02099744.

[3] G. L. Eyink, *Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer*, Physica D **78**, 222-240 (1994). DOI: 10.1016/0167-2789(94)90117-1.

[4] A. Cheskidov, P. Constantin, S. Friedlander, R. Shvydkoy, *Energy conservation and Onsager's conjecture for the Euler equations*, Nonlinearity **21**, 1233-1252 (2008). DOI: 10.1088/0951-7715/21/6/005. arXiv: 0801.2289.

[5] C. De Lellis, L. Székelyhidi Jr., *The Euler equations as a differential inclusion*, Ann. Math. **170**, 1417-1436 (2009). DOI: 10.4007/annals.2009.170.1417.

[6] T. Buckmaster, V. Vicol, *Convex integration and phenomenologies in turbulence*, EMS Surveys Math. Sci. **6**, 173-263 (2019). DOI: 10.4171/EMSS/34. arXiv: 1901.09023.

[7] P. Isett, *A proof of Onsager's conjecture*, Ann. Math. **188**, 871-963 (2018). DOI: 10.4007/annals.2018.188.3.4. arXiv: 1608.08301.

[8] L. De Rosa, *Onsager's conjecture for the helicity*, J. Differential Equations **271**, 968-995 (2021). DOI: 10.1016/j.jde.2020.09.002. arXiv: 1904.04313.

[9] A. F. Vakulenko, L. V. Kapitanski, *Stability of solitons in S^2 in the nonlinear σ-model*, Soviet Physics Doklady **24**, 433-434 (1979).

[10] F. H. Lin, Y. Yang, *Existence of energy minimizers as stable knotted solitons in the Faddeev model*, Comm. Math. Phys. **269**, 137-152 (2006). DOI: 10.1007/s00220-006-0123-0.

[11] L. Faddeev, A. J. Niemi, *Stable knot-like structures in classical field theory*, Nature **387**, 58-61 (1997). DOI: 10.1038/387058a0. Preprint: hep-th/9610193.

[12] R. A. Battye, P. M. Sutcliffe, *Knots as stable soliton solutions in a three-dimensional classical field theory*, Phys. Rev. Lett. **81**, 4798-4801 (1998). DOI: 10.1103/PhysRevLett.81.4798. arXiv: hep-th/9808129.

[13] H. K. Moffatt, *The degree of knottedness of tangled vortex lines*, J. Fluid Mech. **35**, 117-129 (1969). DOI: 10.1017/S0022112069000991.

[14] A. Brissaud, U. Frisch, *Helicity cascades in fully developed turbulence*, Phys. Fluids **16**, 1366-1367 (1973). DOI: 10.1063/1.1694520.

[15] Q. Chen, S. Chen, G. L. Eyink, *Joint cascade of energy and helicity in three-dimensional turbulence*, Phys. Fluids **15**, 361-374 (2003). DOI: 10.1063/1.1533070.

[16] P. D. Mininni, A. Pouquet, *Rotating helical turbulence. I. Global evolution and spectral behavior*, Phys. Fluids **22**, 035105 (2010). DOI: 10.1063/1.3358466.

[17] H. Bahouri, J.-Y. Chemin, R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften **343**, Springer (2011).

[18] H. Triebel, *Theory of Function Spaces II*, Monographs in Mathematics **84**, Birkhäuser (1992).

[19] J. T. Beale, T. Kato, A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Comm. Math. Phys. **94**, 61-66 (1984). DOI: 10.1007/BF01212349.

[20] A. Novickis, *Topological Navier-Stokes: A Hopf-Soliton Reformulation of Incompressible Fluid Dynamics*, Paper CXIII of the Hopf Soliton Programme (2026). [Companion paper.]

[21] A. J. Majda, A. L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge Texts in Applied Mathematics, Cambridge University Press (2002). [Standard reference for the Biot-Savart Calderón-Zygmund estimate, Proposition 1.5.]

[22] A. Novickis, *Yang-Mills Mass Gap via Faddeev-Niemi Decomposition and Interval Arithmetic*, Paper CIII of the Hopf Soliton Programme (2026). [Programme reference for the FN regularity theory.]