Turán–Hankel determinants of Stieltjes sequences: a pushforward proof and strict positivity refinements

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Abstract

For a Stieltjes moment sequence $\{a_n\}_{n\geq 0}$ with representing measure μ on $[0,\infty)$, let $T_r(n) = \det[a_{n+i+j}]_{0\leq i,j\leq r}$ be the Turán–Hankel determinant of level r at shift n. We record two complementary results.

(1) A pushforward proof of the Stieltjes-moment property. $\{T_r(n)\}_{n\geq 0}$ is itself a Stieltjes moment sequence, with explicit representing measure $\mu_r = W_*\nu_r$, the pushforward under the product map $W(v_0,\ldots,v_r) = \prod_k v_k$ of the Heine measure $d\nu_r = \frac{1}{(r+1)!}\prod_{i<j}(v_j - v_i)^2\, d\mu^{\otimes(r+1)}$. The proof uses only Heine's integral formula (1881), the change-of-variables formula for measure pushforward, and the elementary identity $\prod_k v_k^n = (\prod_k v_k)^n$. This is the same conclusion as Wang–Zhu (2016) for $r=1$ and Zhu (2019) for general r (with an alternative proof by Park (2023)), reached without compound matrices, PSD characterisation, or lattice paths, and producing an explicit representing measure.

(2) Strict positivity refinements. Under $|\mathrm{supp}(\mu)| = \infty$, the pushforward measure μ_r has positive mass on $(0,\infty)$ and is not a single point mass; combined with Cauchy–Schwarz on $L^2(\mu_r)$, this yields strict positivity $T_r(n) > 0$ and strict log-convexity $T_r(n+1)^2 < T_r(n)T_r(n+2)$ for all $r, n\geq 0$, sharpening the non-strict $T_r(n)\geq 0$ conclusion. The classical Desnanot–Jacobi recurrence $T_{r+1}(n)\,T_{r-1}(n+2) = T_r(n)T_r(n+2) - T_r(n+1)^2$ provides an alternative inductive route to strict positivity.

We illustrate both results on two benchmark Stieltjes sequences: the Riemann xi-derivatives $a_n = \xi^{(2n)}(\frac{1}{2})$, verified at 40-digit precision, and the Hilbert moments $a_n = 1/(n+1)$, where the Barnes G-function asymptotic gives the analytic scaling $\ln T_r(0) \sim -2(r+1)^2 \ln 2$.

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1 Introduction

A sequence $\{a_n\}_{n\geq 0}$ of real numbers is a *Stieltjes moment* (SM) sequence if there exists a positive Borel measure μ on $[0,\infty)$ with

$$a_n \;=\; \int_0^\infty t^n\, d\mu(t), \qquad n\geq 0. \tag{1}$$

Equivalently (Stieltjes 1894), the Hankel matrices $H(\mathbf{a}) := [a_{i+j}]_{i,j\geq 0}$ and $H(\theta\mathbf{a}) := [a_{i+j+1}]_{i,j\geq 0}$ are both positive semidefinite. For integers $r\geq 0$ and $n\geq 0$ define the *Turán–Hankel determinant* of level r at shift n:

$$T_r(n) \;:=\; \det\big[a_{n+i+j}\big]_{0\leq i,j\leq r}. \tag{2}$$

For $r=0$, $T_0(n) = a_n$; for $r=1$, $T_1(n) = a_n a_{n+2} - a_{n+1}^2$ is the standard log-convex deficit.

The non-negativity $T_r(n)\geq 0$ for SM sequences is a classical consequence of the Heine integral formula (see Section 2). The stronger statement that the sequence $\{T_{k-1}(n)\}_{n\geq 0}$ is itself SM for every $k \geq 1$ — equivalently, that the operator

$$L_k \colon (a_n) \mapsto (T_{k-1}(n))$$

preserves the SM property — was established for $k = 2$ by Wang–Zhu [14] (giving infinite log-convexity of any SM sequence) and for general k by Zhu [15]. Both proofs use the PSD characterisation of SM combined with compound matrices: $H(L_k\mathbf{a})$ is exhibited as a principal submatrix of $C_k(H(\mathbf{a}))$, and total positivity of $H(\mathbf{a})$ implies the same for the compound matrix. An alternative proof, via lattice paths and the Lindström–Gessel–Viennot lemma applied to a weighted graph whose path-counting yields $T_r(n)$, was given by Park [10].

Result. We give a third proof, with the following features:

(1) It uses only Heine's integral formula and the change-of-variables formula for measure push-forward; no compound matrices, no PSD characterisation, no lattice paths.

(2) It produces an *explicit representing measure* $\mu_r = W_*\nu_r$ for $\{T_r(n)\}_n$, where ν_r is the Heine measure of order r and $W(v_0, \ldots, v_r) = \prod_k v_k$ is the product map.

(3) By the Cauchy–Schwarz equality criterion on ν_r, the strict refinements $T_r(n) > 0$ and $T_r(n+1)^2 < T_r(n)T_r(n+2)$ follow under $|\mathrm{supp}(\mu)| = \infty$, with the support condition cleanly traced to the distinctness of $(r+1)$-tuples in $\mathrm{supp}(\mu)$.

Comments on novelty. Each ingredient — Heine's formula (1881), the pushforward of a measure under a measurable map, the elementary identity $\prod_k v_k^n = (\prod_k v_k)^n$ — is classical. The combination yielding Theorem 3.1 appears not to be in the recent literature on Hankel total positivity [14, 15, 10, 4, 7], nor in the contemporary moment-problem expositions of Schmüdgen [11], Akhiezer's classical text [1][1], or Karlin's monograph on total positivity [5]. Given the elementary nature of the argument, it is plausible that it is implicit in classical pre-arXiv references; we discuss this in Remark 7.1.

Organisation. Section 2 recalls the Heine integral formula and the resulting non-strict positivity. Section 3 proves the main pushforward theorem and derives Wang–Zhu / Zhu / Park as a corollary. Section 4 adds the strict refinement via Cauchy–Schwarz. Section 5 works out two illustrations (Riemann xi, Hilbert). Section 6 records the Hamburger and Hausdorff variants. Section 7 compares with prior proofs and discusses scope of the contribution.

2 The Heine integral formula and the Heine measure

We begin by recalling the classical Heine integral formula for Hankel determinants of moments.

Lemma 2.1 (Heine 1881; Andréief; cf. Szegő **.** *Szego1939] Let $\{a_n\}_{n\geq 0}$ be a Stieltjes moment sequence with representing measure μ on $[0, \infty)$. For all $r \geq 0$ and $n \geq 0$,*

$$T_r(n) \;=\; \frac{1}{(r+1)!} \int_{[0,\infty)^{r+1}} \left(\prod_{k=0}^{r} v_k^n\right) \prod_{0\leq i<j\leq r} (v_j - v_i)^2 \, d\mu(v_0)\, d\mu(v_1) \cdots d\mu(v_r). \tag{3}$$

We define the *Heine measure of order r* on $[0, \infty)^{r+1}$:

$$d\nu_r(v_0, \ldots, v_r) \;:=\; \frac{1}{(r+1)!} \prod_{0\leq i<j\leq r} (v_j - v_i)^2 \, d\mu(v_0) \cdots d\mu(v_r). \tag{4}$$

[1] Although the present author has not been able to verify this in full text; see Remark 7.1 below.

Then ν_r is a non-negative Borel measure on $[0,\infty)^{r+1}$ (the Vandermonde square is non-negative and μ is non-negative), and (3) reads

$$T_r(n) \;=\; \int_{[0,\infty)^{r+1}} \Big(\prod_{k=0}^{r} v_k^n\Big)\, d\nu_r. \tag{5}$$

The qualitative structure of ν_r is determined by $\operatorname{supp}(\mu)$: since $\prod_{i<j}(v_j-v_i)^2$ vanishes on the diagonals where any two v_i coincide, ν_r is supported on tuples $(v_0,\dots,v_r)$ all of whose components are distinct points of $\operatorname{supp}(\mu)$. Hence:

- if $|\operatorname{supp}(\mu)| \le r+1$, then $\nu_r = 0$ and $T_r(n) = 0$;
- if $|\operatorname{supp}(\mu)| \ge r+2$, then $\nu_r \ne 0$ and $T_r(n) > 0$ for every $n \ge 0$.

3 The pushforward theorem

3.1 Main result

Theorem 3.1 (Heine pushforward $\Rightarrow$ Stieltjes-moment property)**.** *Let $\{a_n\}_{n\ge 0}$ be a Stieltjes moment sequence with representing measure μ on $[0,\infty)$. For each $r \ge 0$, the sequence $\{T_r(n)\}_{n\ge0}$ defined by* (2) *is itself a Stieltjes moment sequence, with representing measure*

$$\mu_r \;:=\; W_*\nu_r \qquad \text{on } [0,\infty), \tag{6}$$

where ν_r is the Heine measure of order r defined by (4) *and $W\colon [0,\infty)^{r+1} \to [0,\infty)$ is the product map $W(v_0,\dots,v_r) = v_0v_1\cdots v_r$.*

Proof. Both W and ν_r are well-defined. W is continuous (so measurable) and maps $[0,\infty)^{r+1}$ into $[0,\infty)$; the image lies in $[0,\infty)$ since the $v_k \ge 0$. ν_r is non-negative on $[0,\infty)^{r+1}$ since the Vandermonde square is non-negative and $\mu^{\otimes(r+1)}$ is non-negative. The pushforward $\mu_r := W_*\nu_r$ is therefore a non-negative Borel measure on $[0,\infty)$.

For the moment computation, observe that $\prod_{k=0}^r v_k^n = \big(\prod_{k=0}^r v_k\big)^n = W(v)^n$ for every $n \ge 0$. Therefore, by (5) and the change-of-variables formula for measure pushforward,

$$T_r(n) \;=\; \int W(v)^n\, d\nu_r(v) \;=\; \int_{[0,\infty)} t^n\, d(W_*\nu_r)(t) \;=\; \int_{[0,\infty)} t^n\, d\mu_r(t). \tag{7}$$

The integrals on the right exist (and are finite) because the moments a_n exist: ν_r has finite total mass $T_r(0) < \infty$ (by (5) with $n=0$), and the moments $\int t^n\, d\mu_r = T_r(n) < \infty$ are finite for every n. By (7), $\{T_r(n)\}_n$ is the moment sequence of μ_r, hence is SM. $\square$

Remark 3.2 (Concrete representation for finite-support μ)**.** *If $\mu = \sum_{i=1}^N c_i\,\delta_{p_i}$ is a finitely-supported non-negative measure with $N \ge r+1$ distinct support points $p_i \ge 0$ and weights $c_i > 0$, then*

$$\mu_r \;=\; \sum_{\substack{S\subset\{1,\dots,N\}\\ |S|=r+1}} \Big(\prod_{i\in S} c_i\Big)\Big(\prod_{\substack{i,j\in S\\ i<j}} (p_j-p_i)^2\Big)\, \delta_{\prod_{i\in S} p_i}\,. \tag{8}$$

That is, μ_r is concentrated on the set $\{\prod_{i\in S} p_i : |S| = r+1\}$ (with possible coincidences giving aggregated point masses), with weight on each point equal to the product of μ-masses at those points times the Vandermonde square of the support.

Justification of (8)*.* For finite-support μ, the Heine measure ν_r is the discrete sum

$$\nu_r \;=\; \frac{1}{(r+1)!} \sum_{(i_0,\dots,i_r)} \prod_{a<b} (p_{i_b}-p_{i_a})^2 \Big(\prod_{k=0}^{r} c_{i_k}\Big)\delta_{(p_{i_0},\dots,p_{i_r})},$$

where the sum is over all ordered $(r+1)$-tuples $(i_0, \ldots, i_r) \in \{1, \ldots, N\}^{r+1}$. Each unordered $(r+1)$-element subset $S = \{i_0, \ldots, i_r\}$ contributes $(r+1)!$ orderings, all with the same Vandermonde square (since the square is invariant under permutations). Combined, ν_r on the diagonal-free part is concentrated on unordered subsets, with mass per subset equal to $\prod_{i \in S} c_i \cdot \prod_{i<j \in S}(p_j - p_i)^2$. Pushing forward under W collects each subset's mass at the point $\prod_{i \in S} p_i$. □

Numerical verification. We verified (8) numerically on several finite-support examples at 30-digit precision:

- $\mu = \delta_2 + \delta_3$: $\mu_1 = (3-2)^2\, \delta_6 = \delta_6$, giving $T_1(n) = 6^n$. Direct: $T_1(0) = 1$, $T_1(1) = 6$, $T_1(2) = 36$. ✓
- $\mu = \delta_2 + \delta_3 + \delta_5$: $\mu_2 = (3-2)^2(5-2)^2(5-3)^2\, \delta_{30} = 36\, \delta_{30}$, giving $T_2(n) = 36 \cdot 30^n$. Verified for $n = 0, \ldots, 3$. ✓
- $\mu = \sum_{k=1}^{4} \delta_k$: $\mu_2 = 4\delta_6 + 36\delta_8 + 36\delta_{12} + 4\delta_{24}$, giving $T_2(0) = 80$, $T_2(1) = 840$, $T_2(2) = 9936$, $T_2(3) = 136800$ matching direct computation. ✓ $\mu_3 = 144\, \delta_{24}$, giving $T_3(n) = 144 \cdot 24^n$. ✓

3.2 Iterated pushforward and infinite log-convexity

Corollary 3.3 (Iterated SM-tower; recovers Wang–Zhu, Zhu, Park)**.** *The operator $L_k \colon \{a_n\} \mapsto \{T_{k-1}(n)\}$ preserves the SM property for every $k \geq 1$. By iteration, every Stieltjes moment sequence is* infinitely log-convex *in the sense of Wang–Zhu: the sequences $L_2\mathbf{a}$, $L_2L_2\mathbf{a}$, $L_2L_2L_2\mathbf{a}, \ldots$ are all SM, with representing measures obtained by iterating the pushforward construction: $\mu^{(1)} := \mu$, $\mu^{(j+1)} := W_*\nu_1[\mu^{(j)}]$.*

Proof. The SM-preservation by L_k is the content of Theorem 3.1: $\{L_k\mathbf{a}\} = \{T_{k-1}(n)\}_n$ is SM with representing measure $\mu_{k-1} = W_*\nu_{k-1}[\mu]$. Iteration: applying Theorem 3.1 to $L_k\mathbf{a}$ with measure μ_{k-1} gives that $L_kL_k\mathbf{a}$ is SM with measure $W_*\nu_{k-1}[\mu_{k-1}]$, and so on. The case $k = 2$ recovers Wang–Zhu's infinite log-convexity. □

4 Strict positivity via the support of μ_r

Theorem 3.1 provides the SM property and (via Lemma 2.1) the non-strict positivity $T_r(n) \geq 0$. The strict refinements $T_r(n) > 0$ and $T_r(n+1)^2 < T_r(n)T_r(n+2)$ follow directly from the structure of the pushforward measure μ_r under the infinite-support hypothesis.

Proposition 4.1 (Strict positivity from pushforward structure)**.** *Let $\{a_n\}$ be SM with representing measure μ on $[0, \infty)$ and $|\mathrm{supp}(\mu)| = \infty$ (so in particular $|\mathrm{supp}(\mu)| \geq r + 2$ for every $r \geq 0$). Then for all $r, n \geq 0$:*

(1) $T_r(n) > 0$ strictly*;*

(2) $T_r(n+1)^2 < T_r(n)T_r(n+2)$ *(strict log-convexity of $\{T_r(n)\}_{n \geq 0}$).*

Proof. We work with the pushforward measure $\mu_r = W_*\nu_r$ from Theorem 3.1.

Step 1. *μ_r has positive total mass.* The total mass $\mu_r([0, \infty)) = \nu_r([0, \infty)^{r+1}) = T_r(0)$. By Lemma 2.1, $T_r(0) > 0$ since $|\mathrm{supp}(\mu)| \geq r + 2$.

Step 2. *μ_r has positive mass on $(0, \infty)$.* Let $S \subset \mathrm{supp}(\mu)$ be any $(r+2)$-element subset of distinct support points; at most one of them equals 0, so S contains at least $r+1$ positive points $p_0, \ldots, p_r > 0$. Since $\mathrm{supp}(\mu) = \overline{\{t : \mu(U) > 0 \text{ for every nbhd } U \ni t\}}$, each p_i has positive μ-mass in any neighbourhood; hence the product set $\bigcap_i \{(v_0, \ldots, v_r) : v_i \in U_{p_i}\}$ has positive ν_r-mass for sufficiently small disjoint neighbourhoods $U_{p_i} \ni p_i$ in $(0, \infty)$. Pushed forward, this gives positive μ_r-mass on $(0, \infty) \setminus \{0\}$.

Step 3. *Strict positivity (1).* By Step 2, $\mu_r((0,\infty)) > 0$. Since $t^n > 0$ on $(0,\infty)$ for every $n \geq 0$,

$$T_r(n) = \int_0^\infty t^n \, d\mu_r(t) \;\geq\; \int_{(0,\infty)} t^n \, d\mu_r(t) \;>\; 0.$$

Step 4. *Strict log-convexity (2).* By the standard Cauchy–Schwarz inequality on $L^2(\mu_r)$, applied to $f(t) = t^{n/2}$ and $g(t) = t^{(n+2)/2}$,

$$T_r(n+1)^2 \;=\; \Big(\int t^{n+1}\, d\mu_r\Big)^2 \;=\; \Big(\int fg\, d\mu_r\Big)^2 \;\leq\; \Big(\int f^2\, d\mu_r\Big)\Big(\int g^2\, d\mu_r\Big) \;=\; T_r(n)\, T_r(n+2).$$

Equality holds iff f and g are proportional μ_r-a.e., i.e. $f/g = t^{-1}$ is constant on $\mathrm{supp}(\mu_r)$. This forces $\mathrm{supp}(\mu_r) \subset \{t_0\}$ for a single t_0, i.e. μ_r is a single point mass. We rule this out: by Remark 3.2 applied to any two $(r+1)$-element subsets of $\mathrm{supp}(\mu)$ that differ in exactly one component, their products $\prod p_i$ differ; hence $\mathrm{supp}(\mu_r)$ contains at least two distinct points, contradicting the equality criterion. $\square$

Remark 4.2 (The Desnanot–Jacobi recurrence)**.** *The classical Desnanot–Jacobi (Sylvester) identity for Hankel matrices,*

$$T_{r+1}(n)\, T_{r-1}(n+2) \;=\; T_r(n)\, T_r(n+2) - T_r(n+1)^2, \tag{9}$$

provides an alternative route to strict positivity: combined with the strict log-convexity of Proposition 4.1(2), the right-hand side of (9) *is strictly positive, so by induction on r the determinant $T_{r+1}(n)$ is strictly positive. This DJ + CS scheme (without the pushforward perspective) predates the present work and appears in classical combinatorial Hankel-determinant literature (Sylvester 1851; Eu–Wong–Yen [4]; Krattenthaler [7]). The pushforward proof of Proposition 4.1 gives the same conclusions without invoking* (9).

5 Two illustrations

5.1 The Riemann xi-sequence

The Taylor coefficients of the Riemann xi-function at the central point, $a_n = \xi^{(2n)}(\frac{1}{2})$, form an SM sequence (via the Pólya–de Bruijn density representation of $\xi(\frac{1}{2} + iz)$: see [3]). At 40-digit precision, the first values are

$$a_0 = 0.49712\,07781\,88314\,1099\ldots, \qquad a_1 = 0.02297\,19443\,15145\,4375\ldots,$$
$$a_2 = 0.00296\,28484\,33687\,6322\ldots, \qquad a_3 = 0.00059\,92959\,46597\,5795\ldots,$$
$$a_4 = 0.00016\,09665\,74550\,1956\ldots.$$

The first few Turán–Hankel determinants are $T_1(0) \approx 9.45 \times 10^{-4}$, $T_2(0) \approx 2.92 \times 10^{-8}$, $T_3(0) \approx 2.87 \times 10^{-14}$, $T_4(0) \approx 1.39 \times 10^{-21}$, $T_5(0) \approx 4.56 \times 10^{-30}$. The CS deficit $\delta_1(0) = T_1(0)T_1(2) - T_1(1)^2 \approx 8.64 \times 10^{-11} > 0$ gives the log-convex ratio $\theta_1(0) \approx 0.224$; the recurrence $T_2(0) = \delta_1(0)/a_2 \approx 2.917 \times 10^{-8}$ matches the direct 3×3-determinant evaluation to relative error $< 10^{-39}$. At higher levels $(r, n \leq 5)$, all 30 strict log-convexity inequalities $T_r(n+1)^2 < T_r(n)T_r(n+2)$ are verified.

The values $-\ln T_r(0)$ exhibit the empirical scaling $\ln T_r(0) \approx -C_r \cdot r(r+1)/2$ with $\{C_r\}_{r=1}^6 = \{6.96, 5.78, 5.20, 4.80, 4.50, 4.26\}$. The exponent $r(r+1)/2$ counts the number of factors $(v_j - v_i)^2$ in the Heine integrand; the constants C_r have not been determined analytically. The form is consistent with the $\sim r^2 \log r$ behaviour of analogous Hankel determinants of zeta values $H_n[\zeta] := \det[\zeta(i+j+r)]_{i,j}$ studied by Monien [9], who derived $\log H_n[\zeta] \sim -n^2(\log(2n) - 3/2)$ via a Coulomb-gas equilibrium-measure analysis.

5.2 The Hilbert moment sequence

For $a_n = 1/(n+1)$, the moments of Lebesgue measure on $[0,1]$ (Hausdorff sequence with infinite support, hence Stieltjes), the Turán–Hankel determinant has the Choi closed form

$$T_r(0) \;=\; \frac{c_{r+1}^4}{c_{2(r+1)}}, \qquad c_n := \prod_{k=0}^{n-1} k! \;=\; G(n+1), \tag{10}$$

where G is the Barnes G-function. Using the standard asymptotic $\ln c_n \sim \frac{n^2}{2}\ln n - \frac{3n^2}{4} + O(n\ln n)$,

$$-\ln T_r(0) \;=\; 2(r+1)^2 \ln 2 + O(r\ln r), \quad C_r \xrightarrow{r\to\infty} 4\ln 2 \;\approx\; 2.7726. \tag{11}$$

Numerical computation at 50-digit precision verifies the convergence:

r	$T_r(0)$	C_r	$C_r/(4\ln 2)$
1	8.333×10^{-2}	2.485	0.896
2	4.630×10^{-4}	2.559	0.923
3	1.653×10^{-7}	2.603	0.939
5	5.367×10^{-18}	2.651	0.956
8	9.720×10^{-43}	2.687	0.969
10	3.019×10^{-65}	2.701	0.974

The pushforward $\mu_r = W_*\nu_r$ for the Hilbert sequence is the absolutely continuous measure on $[0,1]$ with density $f_r(t)$ given (implicitly) by

$$\int_0^1 g(t) f_r(t)\,dt \;=\; \frac{1}{(r+1)!}\int_{[0,1]^{r+1}} g\Big(\prod_k v_k\Big) \prod_{i<j}(v_j - v_i)^2\, dv_0 \cdots dv_r,$$

i.e. the distribution of the product of $(r+1)$ independent Lebesgue points on $[0,1]$ weighted by Vandermonde-square.

6 Hamburger and Hausdorff variants

The pushforward construction extends naturally.

Proposition 6.1 (Hausdorff case)**.** *Let $\{a_n\}$ be a Hausdorff moment sequence with measure μ on $[0,1]$. Then $\{T_r(n)\}_n$ is itself a Hausdorff moment sequence with representing measure $\mu_r = W_*\nu_r$ supported on $[0,1]$, since $W([0,1]^{r+1}) \subset [0,1]$. The strict-positivity refinement of Proposition 4.1 holds under $|\mathrm{supp}(\mu)| = \infty$.*

Proposition 6.2 (Hamburger case)**.** *Let $\{a_n\}$ be a Hamburger moment sequence with measure μ on $\mathbb{R}$. Then $\{T_r(n)\}_n$ is itself a Hamburger moment sequence with representing measure $\mu_r = W_*\nu_r$ supported on $\mathbb{R}$. The strict-positivity conclusion does not hold in general: $T_r(n)$ may take both signs, depending on the parity of n and the sign structure of* $\mathrm{supp}(\mu)$.

Proof. For Hausdorff: μ is supported on $[0,1]$, so $v_k \in [0,1]$ forces $\prod v_k \in [0,1]$ and W maps into $[0,1]$. The pushforward μ_r is therefore supported on $[0,1]$, which is the Hausdorff condition. For Hamburger: μ is on $\mathbb{R}$, so W maps into $\mathbb{R}$ (positive products of even number of negatives, negative products of odd number). The integrand $\prod v_k^n$ has variable sign, so $T_r(n)$ is real but not sign-constrained. The pushforward μ_r on $\mathbb{R}$ is non-negative, and $T_r(n) = \int t^n\, d\mu_r$ recovers the Hamburger property. $\square$

7 Comparison with prior work and discussion

Theorem 3.1 (the SM-property of $\{T_r(n)\}$ via explicit pushforward) is the same conclusion as Wang–Zhu [14] for $r = 1$ and Zhu [15] for general r. The proofs differ structurally:

- **Wang–Zhu 2016, Zhu 2019.** Use the characterisation that $\{a_n\}$ is SM iff $H(\mathbf{a})$ and $H(\theta\mathbf{a})$ are PSD, then show that $H(L_k\mathbf{a})$ is a principal submatrix of the compound matrix $C_k(H(\mathbf{a}))$, whose total positivity follows from that of $H(\mathbf{a})$. The proof is non-constructive: no explicit representing measure is produced.
- **Park 2023.** Constructs a weighted graph whose path counts via Lindström–Gessel–Viennot give $T_r(n)$, then identifies a probabilistic structure on closed walks yielding the SM property. The representing measure is implicit in the path-counting data but not given in closed form.
- **Present pushforward proof.** Uses Heine's integral formula to write $T_r(n)$ as an integral of W^n against the Heine measure ν_r; the SM property follows from the push-forward $\mu_r = W_*\nu_r$ being a non-negative measure on $[0,\infty)$, with $T_r(n)$ being its n-th moment. The representing measure is given explicitly (by (6)), with concrete formulas in the finite-support case (8).

The pushforward proof has the advantage of producing an explicit representing measure with a transparent combinatorial interpretation in the finite-support case (Remark 3.2): μ_r assigns mass $\prod c_i \cdot V^2$ to each $(r+1)$-element subset's product of support points. This may be useful for explicit asymptotic analysis of $\{T_r(n)\}_n$ and its determinacy properties (whether the pushforward measure is determinate, etc.).

Remark 7.1 (On the absence of the pushforward argument in recent literature)**.** *The argument of Theorem 3.1 uses only Heine's formula and the change of variables for measure pushforward, both classical. The pushforward perspective on Hankel determinants of moments has been used implicitly in random-matrix theory: the joint eigenvalue density of a $\beta = 2$ random matrix is $\propto \prod(v_j - v_i)^2 d\mu^{\otimes n}$, and Heine integrals encode partition functions of these ensembles. However, the explicit observation that $\{T_r(n)\}_{n\geq 0}$ is itself a Stieltjes moment sequence* with explicit representing measure $W_*\nu_r$ *does not appear in:*

- *the recent Hankel-positivity literature (Wang–Zhu 2016, Zhu 2019, 2024, Park 2023, Krattenthaler 2021, Eu–Wong–Yen 2012, Berg–Christensen–Jensen 2014, Belton–Guillot–Khare–Putinar 2016, Bostan–Elvey-Price–Guttmann–Maillard 2020);*
- *Schmüdgen's monograph* The Moment Problem *[11] (Graduate Texts in Mathematics, 2017) or his expository* Ten Lectures *[12] (arXiv:2008.12698, 2020);*
- *the contemporary moment-problem expositions visible to the author.*

The author has been unable to access in full text the classical pre-arXiv references Akhiezer [1], Karlin [5], Karlin–Studden [6], Krein–Nudelman [8], all of which discuss aspects of moment problems and Hankel determinants that overlap with the present work. The pushforward argument is sufficiently elementary that it may be implicit in these references; we offer Theorem 3.1 as an explicit statement, with the priority caveat that any earlier explicit form in the classical literature should be acknowledged upon being identified.

Open questions.

(1) *Asymptotic of C_r on the xi-sequence.* Empirical $C_r \in \{6.96, \ldots, 4.26\}$ for $r = 1, \ldots, 6$; analytical determination of the leading large-r behaviour requires Coulomb-gas analysis of the Heine integrand with the Pólya–de Bruijn potential.

(2) *Determinacy of μ_r.* For determinate μ, is μ_r also determinate? Under what conditions does μ_r have a density (rather than singular components on the diagonals)?

(3) *Higher Turán–Hankel pushforwards.* The map $\{a_n\} \mapsto \{T_r(n)\}$ has explicit pushforward $\mu \mapsto W_*\nu_r$. Iterating gives a measure-theoretic dynamical system on the cone of SM measures; its long-term behaviour (does it converge in some sense, what are its fixed points?) is open.

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