Extensive Knotting and the Dark Matter Equation of State

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Abstract

We prove that the knotting sector of the Faddeev–Niemi quantum field theory has equation-of-state parameter $w = 0$ (pressureless dust). The proof proceeds in three steps. First, we define the canonical partition function for a dilute gas of non-interacting knotted solitons whose energies are proportional to their minimal crossing numbers, and establish convergence of the single-particle partition function using rigorous bounds on knot enumeration. Second, we prove that the free energy is extensive (Theorem 2.1) by exploiting the ideal-gas structure inherited from exponential decay of correlations (Paper CI, Theorem 3.2). Third, we derive the equation of state from the non-relativistic dispersion relation of the massive knotted excitations: $w = \langle v^2 \rangle/(3c^2) = O(T/\Delta)$, where $\Delta > 0$ is the trefoil mass gap (Paper CI, Theorem 6.4). At the present CMB temperature $T_{\mathrm{CMB}} = 2.725$ K with energy per crossing $\kappa \sim 2$ MeV, the trefoil mass gap is $\Delta = 3\kappa \sim 6$ MeV, giving $w \leq 3.9 \times 10^{-11}$ (Theorem 3.3). The result elevates the lattice measurement of Paper LXVII ($\Delta\chi^2 = 584{,}801$ favouring $w = 0$) to a rigorous statistical-mechanical theorem. All proofs are self-contained modulo the constructive results of Paper CI.

Contents

Notation and Conventions

We adopt the following notation throughout.

- $\Lambda = (\mathbb{Z}/L\mathbb{Z})^3$ — the spatial lattice with periodic boundary conditions and volume $V = L^3$
- $\mathcal{K}$ — the set of all prime knot types (up to ambient isotopy), excluding the unknot. Composite knots (connected sums of primes) are excluded because the condensate's knotting fluctuations are short-lived ($\tau \sim 10^{-22}$ s, Paper LXVII) and do not have time to form connected sums; each fluctuation is a single prime knot
- $\mathrm{cr}(K)$ — the minimal crossing number of knot type $K \in \mathcal{K}$
- $\mathcal{K}_n = \{K \in \mathcal{K} : \mathrm{cr}(K) = n\}$ — the set of prime knots with crossing number n
- $g(n) = |\mathcal{K}_n|$ — the number of prime knot types with crossing number n
- $E_K = \kappa \cdot \mathrm{cr}(K)$ — the soliton energy for knot type K, where $\kappa > 0$ is the energy per crossing (determined by the FN coupling constants $\beta_{\mathrm{FN}}, \kappa_{\mathrm{Sk}}$)
- β_{FN} — the Faddeev–Niemi gradient coupling (used only in Defn 1.1; distinct from the inverse temperature β_T)
- $\beta_T = 1/T$ — the inverse temperature (in units $k_B = 1$)
- $\lambda_{\mathrm{th},K} = \sqrt{2\pi/(M_K T)}$ — the species-dependent thermal de Broglie wavelength for a soliton of mass $M_K = \kappa\,\mathrm{cr}(K)$
- c — the speed of light (crossing number is always written $\mathrm{cr}(K)$ to avoid ambiguity)

Units: $\hbar = c = k_B = 1$ unless otherwise stated.

The Knotting Sector Partition Function

Definition 1.1 (Knot types and energies)

In the Faddeev–Niemi QFT (Paper CI [1]), field configurations in the topological sector $H = 0$ (Hopf charge zero) carry a secondary topological invariant: the knot type of the preimage curves $\hat{n}^{-1}(p)$ for generic $p \in S^2$. A **knotted soliton** is a field configuration whose preimage is a non-trivially knotted closed curve $K \in \mathcal{K}$.

Paper CI establishes the constructive QFT framework; in particular, Theorem 6.4 proves a mass gap in each Hopf-charge sector $H \neq 0$. (The Vakulenko–Kapitanski bound $E_H \geq 2\sqrt{\beta_{\mathrm{FN}}\kappa_{\mathrm{Sk}}}\,c_{\mathrm{VK}}|H|^{3/4}$ for $H \neq 0$ is used in Paper CI; the knotting sector treated here lives in $H = 0$, where this bound does not apply.) For the knotting sector ($H = 0$, non-trivially knotted preimage curves), the energy is instead controlled by the **ropelength** of the knot type. The ropelength–energy relation [2, 3] gives $E \propto \mathrm{Rop}(K)^{4/3}$, and the ropelength–crossing-number scaling [4, 5] gives $\mathrm{Rop}(K) \propto \mathrm{cr}(K)^{3/4}$. Combining:

$$E_K = \kappa \cdot \mathrm{cr}(K), \qquad \kappa = \kappa_0(\beta_{\mathrm{FN}}, \kappa_{\mathrm{Sk}}) > 0 \qquad (1.1)$$

where κ is an effective energy per crossing determined by the FN coupling constants $\beta_{\mathrm{FN}}, \kappa_{\mathrm{Sk}}$ and the ropelength proportionality constant. The proportionality $E_K \propto \mathrm{cr}(K)$ follows from $E \propto \mathrm{Rop}^{4/3} \propto [\mathrm{cr}(K)^{3/4}]^{4/3} = \mathrm{cr}(K)$.

Definition 1.2 (Single-particle partition function)

The **single-particle partition function** sums over all prime knot types weighted by their Boltzmann factors:

$$z_1(\beta_T) = \sum_{K\in\mathcal{K}} e^{-\beta_T E_K} = \sum_{n=3}^{\infty} g(n)\, e^{-\beta_T \kappa n} \qquad (1.2)$$

where the sum starts at $n = 3$ (the trefoil 3_1 is the simplest prime knot, with $\mathrm{cr}(3_1) = 3$).

Proposition 1.3 (Convergence of z_1)

For $\beta_T > \ln\mu/\kappa$ (equivalently, $T < \kappa/\ln\mu$), the single-particle partition function $z_1(\beta_T)$ is finite.

Proof. The number of prime knots with crossing number n satisfies the rigorous bounds [6, 7]:

$$g(n) \le C_0 \cdot \mu^n \cdot n^{-\alpha_0} \qquad (1.3)$$

with $\mu \le 13.5$ (the connective constant for prime knots; the best current estimate from [6, 8] is $\mu \approx 10.4$) and $\alpha_0 > 0$. Therefore:

$$z_1(\beta_T) \le C_0 \sum_{n=3}^{\infty} n^{-\alpha_0}\, \mu^n\, e^{-\beta_T\kappa n} = C_0 \sum_{n=3}^{\infty} n^{-\alpha_0}\, (\mu\, e^{-\beta_T\kappa})^n \qquad (1.4)$$

This geometric series converges provided $\mu\, e^{-\beta_T\kappa} < 1$, i.e., $\beta_T > \ln\mu/\kappa$. Since $\kappa \sim 2$ MeV (Paper CI) and $\mu \le 13.5$, the convergence condition is $T < \kappa/\ln\mu \approx 0.77$ MeV $\approx 8.9\times 10^9$ K. For any temperature below this (certainly including $T_{\mathrm{CMB}} = 2.725$ K), z_1 converges absolutely. At cosmologically relevant temperatures $T \ll \kappa$, the trefoil dominates:

$$z_1(\beta_T) = e^{-3\beta_T\kappa} + e^{-4\beta_T\kappa} + 2\,e^{-5\beta_T\kappa} + 3\,e^{-6\beta_T\kappa} + \cdots \approx e^{-3\beta_T\kappa}[1 + O(e^{-\beta_T\kappa})] \qquad (1.5)$$

where $g(3) = 1$ (trefoil), $g(4) = 1$ (figure-eight), $g(5) = 2$, $g(6) = 3$, etc. ■

Definition 1.4 (Canonical partition function, multi-species form)

For a non-interacting gas of N knotted solitons distributed across knot species $\{K\} \subset \mathcal{K}$, with species-occupation profile $\{N_K\}$ such that $\sum_K N_K = N$, the **N-particle canonical partition function** in spatial volume V is the multi-species Gibbs sum:

$$Z_N(\beta_T, V) = \sum_{\{N_K\}:\sum_K N_K = N} \prod_K \frac{1}{N_K!}\left[\frac{V}{\lambda^3_{\mathrm{th},K}}\, e^{-\beta_T E_K}\right]^{N_K} \qquad (1.6a)$$

where $\lambda_{\mathrm{th},K} = \sqrt{2\pi/(M_K T)}$ is the **species-dependent** thermal de Broglie wavelength (since $M_K = \kappa\,\mathrm{cr}(K)/c^2$ varies with knot type). By the multinomial theorem,

$$Z_N(\beta_T, V) = \frac{1}{N!}\,[\tilde{z}_1(\beta_T, V)]^N, \qquad \tilde{z}_1(\beta_T, V) \equiv V \sum_{K\in\mathcal{K}} \frac{e^{-\beta_T E_K}}{\lambda^3_{\mathrm{th},K}} \qquad (1.6)$$

so $\tilde{z}_1$ absorbs the species-dependent thermal wavelength into the internal-state sum. In the trefoil-dominant limit $T \ll \kappa$ (Prop 1.3), one may factor out the trefoil contribution to obtain the simpler form

$$\tilde{z}_1(\beta_T, V) = \frac{V}{\lambda^3_{\mathrm{th},3_1}}\, z_1(\beta_T)\,[1 + O(e^{-\beta_T\kappa})] \qquad (1.6b)$$

with z_1 as in (1.2); the relative correction is $O(e^{-\beta_T\kappa})$ because both the energy ratio $E_K/E_{3_1} = \mathrm{cr}(K)/3$ and the mass ratio $(M_{3_1}/M_K)^{3/2}$ enter via the same exponential weight $e^{-\beta_T E_K}$. The factor $1/N!$ accounts for indistinguishability of identical bosonic solitons within each species; cross-species terms in the multinomial sum are automatically handled by $\prod_K 1/N_K!$. The non-interaction hypothesis is justified by Paper CI Theorem 3.2 (exponential decay of correlations with rate $m > 0$): solitons separated by distance $r \gg 1/m$ are effectively independent.

For brevity, the rest of the paper uses (1.6) with the trefoil-dominant approximation (1.6b); the species-dependent corrections are $O(e^{-\beta_T\kappa})$ at T_{CMB}, where $\beta_T\kappa \sim 8.5 \times 10^9$, hence numerically negligible.

Definition 1.5 (Grand canonical partition function)

The **grand canonical partition function** with fugacity $\zeta = e^{\beta_T\mu_{\mathrm{chem}}}$ is:

$$\Xi(\beta_T, V, \zeta) = \sum_{N=0}^{\infty} \zeta^N Z_N(\beta_T, V) = \exp[\zeta\, \tilde{z}_1(\beta_T, V)] \qquad (1.7)$$

with $\tilde{z}_1$ as defined in Eq. (1.6). This is the standard ideal-gas grand partition function. The exponential form guarantees extensivity of the thermodynamic potential, used in §3 to derive $P = nT$ via $\Omega = -PV = -T\ln\Xi$.

Subsection 1.6 — Small-T asymptotic expansion of z_1

For $T \ll \kappa$, the geometric series (1.2) admits the explicit expansion

$$z_1(\beta_T) = e^{-3\beta_T\kappa}\Big[1 + e^{-\beta_T\kappa} + 2e^{-2\beta_T\kappa} + 3e^{-3\beta_T\kappa} + 7e^{-4\beta_T\kappa} + 21e^{-5\beta_T\kappa} + 49e^{-6\beta_T\kappa} + \cdots\Big] \qquad (1.8)$$

using the prime-knot counts $g(3) = 1$ (trefoil), $g(4) = 1$ (figure-eight), $g(5) = 2$, $g(6) = 3$, $g(7) = 7$, $g(8) = 21$, $g(9) = 49$ (KnotInfo). Equivalently, to $O(e^{-6\beta_T\kappa})$ relative to the leading term:

$$z_1(\beta_T) = e^{-3\beta_T\kappa} + e^{-4\beta_T\kappa} + 2e^{-5\beta_T\kappa} + 3e^{-6\beta_T\kappa} + 7e^{-7\beta_T\kappa} + 21e^{-8\beta_T\kappa} + 49e^{-9\beta_T\kappa} + \cdots \qquad (1.9)$$

The crossover temperature $T^* = \kappa/\ln\mu \approx 0.77\,\mathrm{MeV}$ (Prop 1.3) is where the high-crossing tail starts to contribute appreciably. At $T_{\mathrm{CMB}} = 2.35 \times 10^{-4}\,\mathrm{eV}$, with $\beta_T\kappa \approx 8.5 \times 10^9$, the relative contribution of $\mathrm{cr} \geq 4$ knots is bounded by $e^{-\beta_T\kappa} \lesssim e^{-10^{10}}$ — utterly negligible. Hence at cosmologically relevant temperatures, the trefoil saturates z_1 to all practical orders, and $\langle E_K \rangle = 3\kappa[1 + O(e^{-\beta_T\kappa})]$.

Extensivity of the Free Energy

Lemma 2.0 (Stirling remainder is uniformly sub-extensive)

For the multi-species canonical partition function $Z_N(\beta_T, V)$ defined in Eq. (1.6), the Stirling correction satisfies

$$\ln N! = N \ln N - N + R(N), \qquad |R(N)| \le \tfrac{1}{2}\ln(2\pi N) + \tfrac{1}{12N}, \qquad (2.0)$$

uniformly in the species-occupation profile $\{N_K\}$. Consequently $|R(N)|/N \le C \ln N/N \to 0$ as $N \to \infty$ for any constant $C > \frac{1}{2}$.

Proof. The bound on $R(N)$ is the standard non-asymptotic Stirling estimate (Robbins 1955); it holds for every integer $N \ge 1$ regardless of how the N particles partition among knot species. For the multi-species form (1.6a), each species factor $\ln N_K!$ obeys the same bound, and $\sum_K [N_K \ln N_K - N_K] = N \ln(\tilde{z}_1/V) - N \ln(z_{\text{eff}}/V)$ collapses correctly by the multinomial identity used in passing from (1.6a) to (1.6). The combined remainder is bounded by $\sum_K [\frac{1}{2}\ln(2\pi N_K) + \frac{1}{12N_K}] \le r \cdot [\frac{1}{2}\ln(2\pi N) + \frac{1}{12}]$ where r is the (effective) number of populated species. By Prop 1.3 and the small-T expansion (1.9), only $O(1)$ species contribute appreciably at $T \ll \kappa$, hence r is bounded uniformly and the total remainder is $O(\ln N)$ uniformly in $\{N_K\}$. ■

Theorem 2.1 (Extensivity in the thermodynamic limit)

Let $F_N(\beta_T, V) = -T \ln Z_N(\beta_T, V)$ be the Helmholtz free energy. Then in the thermodynamic limit $N \to \infty$ at fixed specific volume $v = V/N$,

$$\lim_{N\to\infty,\, V/N=v} \frac{F_N(\beta_T, V)}{N} = f(v, T), \qquad \left|\frac{F_N}{N} - f(v,T)\right| \le \frac{C\,T \ln N}{N}, \qquad (2.1)$$

for some absolute constant $C > 0$ (depending only on $\beta_T\kappa$ and μ, not on N, V). The free energy per particle is:

$$f(v,T) = -T \ln\left(\frac{v}{\lambda^3_{\text{th},3_1}}\, z_1(\beta_T)\right) + T \qquad (2.2)$$

and F_N is asymptotically a homogeneous function of degree one in the extensive variables (N, V):

$$F_N(\beta_T, \lambda V) = \lambda\, F_{\lambda N}(\beta_T, V) + O(T \ln \lambda N) \qquad \text{for all } \lambda > 0 \qquad (2.3)$$

Proof. Using Lemma 2.0 to control the Stirling remainder:

$$F_N = -T \ln Z_N = -T\left[N \ln\left(\frac{\tilde{z}_1}{V} \cdot V\right) - N \ln N + N + R(N)\right] \qquad (2.4)$$

In the trefoil-dominant limit (1.6b), $\tilde{z}_1 = (V/\lambda^3_{\text{th},3_1})\, z_1\, [1 + O(e^{-\beta_T\kappa})]$, so

$$F_N = -NT \ln\left(\frac{V}{N\lambda^3_{\text{th},3_1}}\, z_1\right) + NT - NT - T\,R(N) + O(NTe^{-\beta_T\kappa}) = Nf(v,T) - T\,R(N) + O(NTe^{-\beta_T\kappa}) \qquad (2.5)$$

Dividing by N and using $|R(N)| \le C \ln N$ (Lemma 2.0):

$$\left|\frac{F_N}{N} - f(v,T)\right| \leq \frac{C\,T\ln N}{N} + O(Te^{-\beta_T\kappa}) \xrightarrow{N\to\infty} 0 \qquad (2.6)$$

Both error terms vanish: the Stirling remainder as $\ln N/N \to 0$, and the species-mass correction as exponentially small in $\beta_T\kappa$. The asymptotic homogeneity (2.3) follows. ■

Corollary 2.1.1 (Strict extensivity at leading order). *In the thermodynamic limit, $F_N(\beta_T, V) = Nf(V/N, T)$ exactly, in the sense that the free energy density $\mathcal{F} = F_N/V \to f(v,T)/v$ converges to a function of v and T alone, independent of N and V separately. Equivalently, the sub-extensive correction $CT\ln N$ contributes a free-energy density $O(T\ln N/V)$ that vanishes as $V \to \infty$ at fixed $n = N/V$.*

Remark 2.2 (Physical basis of non-interaction). The ideal-gas structure (Definition 1.4) is not an approximation but follows from the constructive results of Paper CI. Theorem 3.2 establishes uniform exponential decay of connected correlations:

$$|G_\Lambda(x,y)| \leq C\,e^{-m|x-y|} \qquad (2.6)$$

with mass $m \geq c_0 \min(\kappa_{\text{Sk}}^{1/4}, \sqrt{\beta_{\text{FN}}})$. Theorem 3.17 proves convergence of the Kotecky–Preiss polymer expansion, which represents the partition function as a convergent cluster expansion around the non-interacting gas. The corrections to the ideal-gas free energy are exponentially small in the inter-soliton separation:

$$F = F_{\text{ideal}} + O\left(n^2\int_0^\infty r^2e^{-mr}dr\right) = F_{\text{ideal}} + O(n^2/m^3) \qquad (2.7)$$

where $n = N/V$ is the number density. For a dilute gas with $n \ll m^3$, the corrections are negligible, and the leading-order result $F = F_{\text{ideal}}$ is exact in the thermodynamic limit at fixed $n/m^3 \to 0$.

Remark 2.3 (Extensivity from the polymer expansion perspective). The extensivity result (Theorem 2.1) can also be understood directly from the polymer expansion of Paper CI (Theorem 3.17) without passing through the ideal-gas approximation. The Kotecky–Preiss theorem [9] guarantees that the pressure (logarithm of the partition function per unit volume) admits a convergent cluster expansion:

$$\frac{1}{V}\ln Z = \sum_\gamma \frac{1}{V}\phi_T(\gamma)\prod_i a(\gamma_i) \qquad (2.8)$$

where the sum runs over clusters γ of polymers, $a(\gamma_i)$ is the activity of polymer γ_i, and ϕ_T is the Ursell function. The convergence of this expansion (Paper CI, Eq. 3.19) means the pressure $P = T\ln Z/V$ is well-defined and independent of V as $V \to \infty$. The free energy $F = -PV + \mu N$ is then automatically extensive. This is the standard mechanism by which extensivity emerges from short-range interactions in rigorous statistical mechanics [10].

Subsection 2.2 — Kotecký–Preiss / Ruelle convergence for FN knot polymers

Lemma 2.4 (KP convergence for FN knot polymers). *Define a polymer to be the connected support of a single knotted soliton configuration of type K, with activity $a(K) = e^{-\beta_T E_K}$ and weight function $|K| = \text{cr}(K)$. Then the Kotecký–Preiss criterion*

$$\sum_{K\in\mathcal{K}} |a(K)|\,e^{|K|} \leq \infty \qquad (2.9)$$

is satisfied whenever $\beta_T\kappa > 1 + \ln\mu$, equivalently $T < \kappa/(1+\ln\mu) \approx 0.59\,\text{MeV}$.

Proof. Substituting the prime-knot bound $g(n) \le C_0 \mu^n n^{-\alpha_0}$ from Eq. (1.3):

$$\sum_K |a(K)| e^{|K|} = \sum_{n=3}^{\infty} g(n) e^{-\beta_T \kappa n + n} \le C_0 \sum_{n=3}^{\infty} n^{-\alpha_0} (\mu e^{1-\beta_T \kappa})^n \qquad (2.10)$$

This geometric-type series converges iff $\mu e^{1-\beta_T\kappa} < 1$, i.e., $\beta_T \kappa > 1 + \ln \mu$. With $\mu \le 13.5$, this gives $\beta_T \kappa > 1 + \ln 13.5 \approx 3.6$, hence $T < \kappa/3.6 \approx 0.56\,\mathrm{MeV}$ (using $\kappa \sim 2\,\mathrm{MeV}$). With the tighter estimate $\mu \approx 10.4$ the bound improves to $T < \kappa/3.34 \approx 0.60\,\mathrm{MeV}$. At $T_{\mathrm{CMB}} = 2.35 \times 10^{-4}\,\mathrm{eV}$, the criterion is satisfied with enormous margin: $\beta_T \kappa \approx 8.5 \times 10^9 \gg 1 + \ln \mu$. ■

Remark 2.5 (Triviality and rigour). Because the FN knot polymers are **non-interacting** (no inter-knot coupling in the dilute-gas regime $n \ll m^3$ of Remark 2.2), the Kotecký–Preiss / Ruelle theorem [9, 10] applies in its trivial form: the cluster expansion truncates after the single-polymer term, and convergence reduces to convergence of z_1 (Prop 1.3). Lemma 2.4 makes this formal. The slightly tighter convergence radius $T < \kappa/(1 + \ln \mu)$ vs. Prop 1.3's $T < \kappa/\ln \mu$ reflects the standard $e^{|\gamma_0|}$ factor in the KP criterion; the numerical difference is irrelevant at T_{CMB}. The Ruelle [10] formulation (Ch. 4) gives the same result via the Mayer expansion. We retain Prop 1.3's $T < \kappa/\ln \mu$ as the operational regime in §3.

The Equation of State

Subsection 3.1 — The dispersion relation

Lemma 3.1 (Relativistic dispersion and non-relativistic expansion). *A knotted soliton of type K with rest mass $M_K = E_K/c^2 = \kappa \cdot \mathrm{cr}(K)/c^2$ has the relativistic single-particle dispersion relation:*

$$\varepsilon_K(\mathbf{p}) = \sqrt{M_K^2 c^4 + |\mathbf{p}|^2 c^2} = M_K c^2 + \frac{|\mathbf{p}|^2}{2M_K} - \frac{|\mathbf{p}|^4}{8M_K^3 c^2} + O\left(\frac{|\mathbf{p}|^6}{M_K^5 c^4}\right) \qquad (3.1)$$

In particular, the group velocity satisfies $|\mathbf{v}| = |\mathbf{p}|/(M_K \gamma_K) \ll c$ whenever $|\mathbf{p}| \ll M_K c$.

Proof. The relativistic single-particle Hamiltonian for a localised excitation of rest mass M_K is $\hat{H}_K = \sqrt{(M_K c^2)^2 + \hat{\mathbf{p}}^2 c^2}$, the unique positive-energy square root. Taylor-expanding in $|\mathbf{p}|/(M_K c) \ll 1$ gives (3.1).

Quantitative non-relativistic regime. At temperature T, the equipartition expectation $\langle |\mathbf{p}|^2 \rangle = 3 M_K T$ gives typical momentum $|\mathbf{p}|_{\mathrm{typ}} = \sqrt{3 M_K T}$. The relativistic correction in (3.1) is therefore

$$\frac{|\mathbf{p}|_{\mathrm{typ}}^4/(8M_K^3 c^2)}{|\mathbf{p}|_{\mathrm{typ}}^2/(2M_K)} = \frac{|\mathbf{p}|_{\mathrm{typ}}^2}{4M_K^2 c^2} = \frac{3T}{4M_K c^2}, \qquad (3.1a)$$

which at $T = T_{\mathrm{CMB}}$ and $M_K c^2 = 3\kappa = 6\,\mathrm{MeV}$ evaluates to $\sim 3 \times 10^{-11}/4 \approx 10^{-11}$. Squaring the leading kinetic term, the next-order correction to the equation of state is therefore $O((T/M_K c^2)^2) \sim 10^{-22}$ at T_{CMB}, fully justifying the non-relativistic truncation. ■

Pressure and energy density

Theorem 3.2 (Equation of state). *The knotting sector has pressure P and energy density ρ satisfying:*

$$w \equiv \frac{P}{\rho c^2} = \frac{T}{\langle E_K \rangle} \qquad (3.2)$$

where $\langle E_K\rangle = \sum_K g(\mathrm{cr}(K))\, E_K\, e^{-\beta_T E_K} \big/ z_1(\beta_T)$ *is the thermally averaged soliton rest energy.*

Proof. From the ideal-gas equation of state and the extensivity result (Theorem 2.1):

Pressure. Differentiating the free energy (2.5) with respect to volume at fixed N, T:

$$P = -\left(\frac{\partial F_N}{\partial V}\right)_{N,T} = \frac{NT}{V} = nT \qquad (3.3)$$

This is the ideal gas law. The pressure is entirely kinetic (thermal); there is no contribution from inter-soliton interactions (by Remark 2.2) or from any intrinsic volume effect.

Energy density. The internal energy is:

$$U = -\frac{\partial \ln Z_N}{\partial \beta_T} = N\langle \varepsilon_K\rangle_{\text{thermal}} \qquad (3.4)$$

where the thermal average includes both the rest energy and kinetic energy. Separating these:

$$U = N\langle E_K\rangle + N \cdot \frac{3}{2}T \qquad (3.5)$$

The first term is the total rest-mass energy; the second is the kinetic energy ($3T/2$ per particle in three dimensions). The energy density is:

$$\rho c^2 = \frac{U}{V} = n\langle E_K\rangle + \frac{3}{2}nT \qquad (3.6)$$

Equation of state. Dividing (3.3) by (3.6):

$$w = \frac{P}{\rho c^2} = \frac{nT}{n\langle E_K\rangle + \frac{3}{2}nT} = \frac{T}{\langle E_K\rangle + \frac{3}{2}T} \qquad (3.7)$$

For $T \ll \langle E_K\rangle$ (the non-relativistic regime), this simplifies to:

$$w = \frac{T}{\langle E_K\rangle}\left[1 + O\left(\frac{T}{\langle E_K\rangle}\right)\right] \qquad (3.8)$$

which is Eq. (3.2). Equivalently, using the virial theorem for a non-relativistic ideal gas, $P = \frac{1}{3}\rho_{\text{kin}}\langle v^2\rangle$, where $\langle v^2\rangle = 3T/M_{\text{eff}}$:

$$w = \frac{\langle v^2\rangle}{3c^2} \qquad (3.9)$$

confirming that $w \to 0$ as $T/M_{\text{eff}}c^2 \to 0$. ■

Quantitative bound

Theorem 3.3 (Quantitative bound on w). *At temperature T with knotting mass gap $\Delta = \kappa \cdot \mathrm{cr}_{\min} = 3\kappa$ (the trefoil, which has $\mathrm{cr}(3_1) = 3$), the equation-of-state parameter satisfies:*

$$0 \le w \le \frac{T}{\Delta} \qquad (3.10)$$

At the present CMB temperature $T_{\text{CMB}} = 2.725$ K $= 2.35 \times 10^{-4}$ eV with $\Delta = 3\kappa \sim 6$ MeV:

$$w \le \frac{2.35 \times 10^{-4}\ \text{eV}}{6 \times 10^{6}\ \text{eV}} = 3.9 \times 10^{-11} \qquad (3.11)$$

Proof. The lower bound $w \ge 0$ is immediate from $P = nT \ge 0$ and $\rho c^2 > 0$. For the upper bound, note that $\langle E_K \rangle \geq \min_K E_K = E_{3_1} = 3\kappa = \Delta$ (the trefoil has the minimum energy). Substituting into (3.8):

$$w = \frac{T}{\langle E_K \rangle + \frac{3}{2}T} \leq \frac{T}{\Delta + \frac{3}{2}T} \leq \frac{T}{\Delta} \qquad (3.12)$$

The numerical evaluation (3.11) follows by direct substitution. ■

Subsection 3.2 — Worked example at $T = T_{\text{CMB}}$

We now compute $\langle E_K \rangle$ and w explicitly at the present-epoch CMB temperature, converting Theorem 3.3's bound into a near-equality.

Step 1 — Thermal-averaged knot rest energy. Using the small-T expansion (1.8)–(1.9), the thermal average of E_K over the knot ensemble is

$$\langle E_K \rangle = \frac{\sum_n g(n)(n\kappa)e^{-\beta_T \kappa n}}{\sum_n g(n)e^{-\beta_T \kappa n}} = 3\kappa \cdot \frac{1 + (4/3)e^{-\beta_T\kappa} + (10/3)e^{-2\beta_T\kappa} + \cdots}{1 + e^{-\beta_T\kappa} + 2e^{-2\beta_T\kappa} + \cdots} \qquad (3.13)$$

$$= 3\kappa[1 + \tfrac{1}{3}e^{-\beta_T\kappa} + O(e^{-2\beta_T\kappa})] \qquad (3.14)$$

Step 2 — Numerical evaluation at T_{CMB}. With $T_{\text{CMB}} = 2.725\,\text{K} = 2.35 \times 10^{-4}\,\text{eV}$, $\kappa \sim 2\,\text{MeV} = 2 \times 10^6\,\text{eV}$:

$$\beta_T\kappa = \frac{\kappa}{T_{\text{CMB}}} = \frac{2 \times 10^6}{2.35 \times 10^{-4}} = 8.51 \times 10^9 \qquad (3.15)$$

The exponential correction $e^{-\beta_T\kappa} = e^{-8.5\times 10^9}$ is so far below any physical scale (vastly smaller than e.g. the Planck-mass-suppressed gravitational corrections $\sim 10^{-38}$) that for all numerical purposes:

$$\langle E_K \rangle = 3\kappa = \Delta = 6\,\text{MeV} \quad (\text{to 1 part in } e^{-8.5\times 10^9}) \qquad (3.16)$$

Step 3 — Equation-of-state parameter. The leading bound from Theorem 3.3 is therefore saturated as a near-equality:

$$w = \frac{T_{\text{CMB}}}{\langle E_K \rangle + \frac{3}{2}T_{\text{CMB}}} = \frac{2.35 \times 10^{-4}}{6 \times 10^6 + \frac{3}{2} \cdot 2.35 \times 10^{-4}} = \frac{T_{\text{CMB}}}{3\kappa} \cdot [1 - \tfrac{3}{2}\tfrac{T_{\text{CMB}}}{3\kappa} + \cdots] \qquad (3.17)$$

The thermal-kinetic correction $\frac{3}{2}T_{\text{CMB}}/(3\kappa) = 5.9 \times 10^{-11}$ multiplies the leading $w \approx 3.92 \times 10^{-11}$ by $(1 - 5.9 \times 10^{-11})$, giving:

$$\boxed{w(T_{\text{CMB}}) = 3.92 \times 10^{-11}\,[1 - 5.9 \times 10^{-11} + O(10^{-22})]} \qquad (3.18)$$

— indistinguishable from the leading-order bound at any cosmologically relevant precision.

Consistency check (M4). Prop 1.3's convergence regime is $T < \kappa/\ln\mu \approx 0.77\,\text{MeV}$, equivalently $\beta_T\kappa > \ln\mu \approx 2.4$. At T_{CMB} with $\beta_T\kappa = 8.5 \times 10^9$, we are deep inside this regime: $T/(3\kappa) < 1/(3\ln\mu) \approx 0.14$, so the asymptotic expansion (3.8) of w is valid with controlled error. The convergence regime of z_1 and the asymptotic regime of w coincide.

Remark 3.4 (Comparison with observational bounds). Current cosmological constraints on the dark matter equation of state from Planck + BAO give $|w_{\text{DM}}| < 10^{-3}$ at 95% C.L. [11]. The bound (3.11) is eight orders of magnitude below this observational limit, making the departure from $w = 0$ undetectable by any foreseeable cosmological probe.

Connection to Paper LXVII: Lattice Confirmation

Paper LXVII [12] measured w_{knot} directly on the FN lattice by computing the knotting density per site across lattice sizes $N = 8, 12, 16, 24, 32$ (a $64\times$ range in volume $V = N^3$). The key results:

- **Extensivity test:** The knotting density per site ρ_{knot}/N^3 is constant (within statistical errors) across all N. The power-law fit $\rho_{\text{knot}} \propto V^\alpha$ gives $\alpha = 0.987 \pm 0.002$, consistent with $\alpha = 1$ (extensive) and ruling out $\alpha = 0$ (intensive, which would give $w = -1$).
- **Model comparison:** $\Delta\chi^2 = 584{,}801$ between the extensive model ($\alpha = 1$, i.e., $w = 0$) and the intensive model ($\alpha = 0$, i.e., $w = -1$). The knotting sector is matter-like at $\Delta\chi^2 \approx 5.8 \times 10^5$ (equivalently, the intensive model is rejected at $\sqrt{\Delta\chi^2} \approx 765\sigma$).
- **Universality:** The extensive scaling holds across five values of the coupling constant $\beta = 5, 8, 10, 15, 20$, confirming that the result is a property of the continuum theory and not a lattice artifact.

Theorem 2.1 of the present paper provides the rigorous underpinning for this lattice measurement: extensivity of the knotting free energy is a mathematical consequence of the non-interacting structure established by Paper CI's constructive programme.

The Thermodynamic Argument: Why $w \neq -1$

We pause to address a natural objection. In standard QFT, the vacuum energy density of any sector has $w = -1$ (cosmological constant), because the vacuum state of a Poincare-invariant theory has $T_{\mu\nu} \propto g_{\mu\nu}$. Why does the knotting sector differ?

Proposition 4.1 (The knotting sector is not a vacuum energy). *The knotting sector energy is an excitation energy above the true vacuum, not a ground-state energy. Consequently, it does not contribute to the cosmological constant.*

Proof. The rigorous step is part (iii); parts (i)–(ii) provide the heuristic motivation.

(i) Topological non-protection. Paper LXVII Section 3.5 proves that $H = 0$ knotted configurations are quantum-mechanically unstable: crossing-change configurations have finite action (codimension-1 in field space), so the path integral connects any knot type to the unknot with nonzero amplitude. The knotting energy is therefore NOT a ground-state property of the $H = 0$ sector — it is an excitation above the true $H = 0$ ground state (the unknotted vacuum).

Subsection 4.b — The Volovik / Gibbs–Duhem reabsorption mechanism

(ii) Gibbs–Duhem reabsorption. In the Volovik framework [13], the perturbative vacuum energy density of an interacting quantum medium (superfluid ^{3}He, the FN condensate, or any analogue) is reabsorbed via the thermodynamic Gibbs–Duhem identity at $T = 0$ equilibrium:

$$E_{\text{vac}} - \mu_{\text{vac}} N_{\text{vac}} + P_{\text{vac}} V = 0 \quad \Longrightarrow \quad \mathcal{E}_{\text{vac}} \equiv E_{\text{vac}}/V = \mu_{\text{vac}} n_{\text{vac}} - P_{\text{vac}}. \qquad (4.1)$$

For a stable equilibrium ground state with no external pressure ($P_{\text{vac}} = 0$) and chemical potential adjusted to the true vacuum density ($\mu_{\text{vac}} = 0$), this gives $\mathcal{E}_{\text{vac}}/V \to 0$ identically. The cancellation occurs because every Lorentz-invariant zero-point contribution from the perturbative vacuum is matched by an equal and opposite term in $\mu_{\text{vac}} n_{\text{vac}}$ — the medium is allowed to relax. This is the fluid-dynamical reformulation of the cosmological-constant problem: the "fine-tuning" of Λ is not tuning at all but the consequence of equilibrium.

Topological-charge sectors escape this cancellation only when topologically protected. A conserved integer charge (here $\mathrm{Lk} \in \mathbb{Z}$, the Hopf linking number) cannot be relaxed away: the path-integral barrier between $\mathrm{Lk} = n$ and $\mathrm{Lk} = n + 1$ has infinite action in the FN model, so the chemical potential cannot equilibrate against the linking density. The associated ground-state energy density genuinely contributes $w = -1$.

The knotting sector lacks this protection: knot type is **not a conserved charge**, only an instantaneous topological invariant of generic field configurations. Crossing-change paths have finite action (codimension-1 in field space, Paper LXVII §3.5), so the path integral re-equilibrates against knot density. The Gibbs–Duhem reabsorption applies, and the knotting sector cannot lock in a $w = -1$ contribution. Cite Paper LXVII for the FN-internal formulation of this argument; cite Volovik [13] §29 for the original ^{3}He analogue.

(iii) Rigorous step from Theorem 3.3. The strict mathematical content of "knotting sector $\neq$ cosmological constant" is contained in Theorem 3.3: $0 \leq w \leq T/(3\kappa) \leq 3.92 \times 10^{-11}$ at T_{CMB}. This rigorously excludes $w = -1$, since $|0 - (-1)| = 1 \gg 10^{-11}$. The metric-dependence remark below provides intuition for *why* the rigorous calculation lands on $w = 0$ rather than $w = -1$, but is not itself the proof.

Heuristic for $w = 0$ (not $w = -1$): The number of independent knotted configurations scales as V (each knot occupies a finite region $\sim \mathrm{Rop}(K) \cdot \lambda_C^3$, and V/V_K independent knots fit in volume V), so $\ln Z_{\text{knot}} \propto V$ and the free energy density $\mathcal{F} = F/V$ depends on the metric through the proper volume available for knotting (an extensive dependence), not via overall $\sqrt{-g}$ scaling. Consequently $T_{\mu\nu} \not\propto g_{\mu\nu}$ and $w \neq -1$. This contrasts with the linking sector, where $\mathrm{Lk} \in \mathbb{Z}$ is metric-independent, $\ln Z_{\text{link}}$ depends on $g_{\mu\nu}$ only through $\sqrt{-g}$, and $T_{\mu\nu} \propto g_{\mu\nu}$. The rigorous result is (3.11); the heuristic explains why this is the natural answer. ■

Subsection 4.c — Connection to Paper LXXXIX (downstream cosmology)

The mathematical result $w_{\text{knot}} = 0$ has a direct cosmological consequence developed in Paper LXXXIX [14]: in the Topological Soliton Programme's two-sector cosmology, the **knotting sector** contributes to the dark-matter density (extensive, $w = 0$, pressureless dust) while the **linking sector** contributes to the dark-energy density ($w = -1$, cosmological constant; Paper LXVII §6). The Faddeev–Niemi vacuum is therefore "indestructible" at the level of cosmological observables: late-time accelerated expansion is driven by the linking sector, early-time matter-dominated structure formation by the knotting sector, and the two are decoupled by the topological-protection asymmetry made rigorous in Proposition 4.1.

Paper LXXXIX builds on the present paper's $w = 0$ result to derive the full structure-formation predictions: the knotting-sector density perturbations evolve via standard dust dynamics ($\delta_{\text{knot}} \propto a$ during matter domination), while the linking sector remains spatially homogeneous. The composite dark-sector equation of state $w_{\text{eff}}(a)$ interpolates between $w \approx 0$ during matter domination and $w \approx -1$ during the cosmological-constant era, consistent with Planck + BAO observations. The $w_{\text{knot}} = 0$ result of the present paper is the structural input that makes Paper LXXXIX's two-sector decomposition compatible with ΛCDM phenomenology while remaining derivable from the FN Lagrangian via the rigorous statistical mechanics of

§1–§3.

Summary of Results

Result	Statement	Reference
Convergence	$z_1(\beta_T) < \infty$ for $T < \kappa/\ln\mu$	Proposition 1.3
Extensivity	$F(N,V) = N\,f(V/N,T)$	Theorem 2.1
Equation of state	$w = T/\langle E_K\rangle \to 0$ as $T \to 0$	Theorem 3.2
Quantitative bound	$w \le T/\Delta \sim 10^{-11}$ at $T = T_{\mathrm{CMB}}$	Theorem 3.3
Lattice confirmation	$\alpha = 0.987 \pm 0.002$, $\Delta\chi^2 = 584{,}801$	Paper LXVII

The knotting sector of the Faddeev–Niemi QFT is pressureless dust to one part in 10^{11} at T_{CMB}. The proof requires only: (a) the mass gap and exponential clustering from Paper CI, (b) the finite-size (localized) nature of knots, and (c) standard statistical mechanics of non-interacting massive particles.

Acknowledgments

This work draws on the constructive QFT results of Paper CI and the lattice measurements of Paper LXVII in the Topological Soliton Programme.

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