The One-Packet Transform and Phase Structure of the Low-Strip Gram Kernel

Jongmin Choi
Independent Researcher, Seoul, Korea
24ping@naver.com
ORCID iD: 0009-0008-7448-514X

Abstract

Let

$$W_a(t) = \frac{at^2}{(a^2+t^2)^2}, \qquad p_{\gamma,a}(t) = \frac{1}{a+i(t-\gamma)}, \qquad a>0,$$

and define the one-packet transform

$$L_{a,R}(\gamma) = \int_{|t|\le R} W_a(t) p_{\gamma,a}(t)\, dt.$$

This transform occurs in the partial-fraction representation of the weighted low-strip Gram kernel

$$G_{a,R}(\gamma,\gamma') = \int_{|t|\le R} W_a(t) p_{\gamma,a}(t) \overline{p_{\gamma',a}(t)}\, dt.$$

Indeed,

$$G_{a,R}(\gamma,\gamma') = \frac{L_{a,R}(\gamma) + \overline{L_{a,R}(\gamma')}}{2a - i(\gamma-\gamma')}.$$

Thus any cancellation in the off-diagonal Gram sum is tied to the complex phase of $L_{a,R}(\gamma)$.

The main purpose of this paper is to compute the full-line transform

$$L_{a,\infty}(\gamma) = \int_{-\infty}^{\infty} \frac{at^2}{(a^2+t^2)^2} \frac{dt}{a+i(t-\gamma)}$$

explicitly. By a residue computation we prove the exact formula

$$\boxed{L_{a,\infty}(\gamma) = \frac{\pi(a-i\gamma)}{2(2a-i\gamma)^2}.}$$

Consequently,

$$\operatorname{Re} L_{a,\infty}(\gamma) = \frac{\pi a(4a^2+3\gamma^2)}{2(4a^2+\gamma^2)^2},$$

and

$$\operatorname{Im} L_{a,\infty}(\gamma) = \frac{\pi\gamma^3}{2(4a^2+\gamma^2)^2}.$$

In particular,

$$\mathfrak{d}_a(\gamma;\infty) = \int_{-\infty}^{\infty} W_a(t)|p_{\gamma,a}(t)|^2\, dt = \frac{\pi(4a^2+3\gamma^2)}{2(4a^2+\gamma^2)^2}.$$

Thus the diagonal weighted packet mass has an exact rational form.

The phase satisfies

$$\tan \operatorname{Arg} L_{a,\infty}(\gamma) = \frac{\gamma^3}{a(4a^2+3\gamma^2)}.$$

Hence $L_{a,\infty}(\gamma)$ is almost real when $|\gamma| \ll a$, while for $|\gamma| \gg a$,

$$L_{a,\infty}(\gamma) = \frac{i\pi}{2\gamma} + O\left(\frac{a}{\gamma^2}\right).$$

This is a Hilbert-type phase at the one-packet level.

After insertion into the Gram identity, however, the leading far-far two-packet kernel is not a bare Hilbert kernel. In the regime

$$|\gamma|, |\gamma'| \gg a, \qquad |\gamma - \gamma'| \gg a,$$

one obtains the product-type leading term

$$G^{\mathrm{model}}_{a,\infty}(\gamma, \gamma') = \frac{\pi}{2\gamma\gamma'} + \text{lower order terms.}$$

This distinction is important: the one-packet transform carries a Hilbert-type phase, while the two-packet Gram kernel exhibits an additional numerator cancellation before the separation denominator is applied.

Finally, we compare $L_{a,R}$ with $L_{a,\infty}$. If $R \geq 2a$ and $|\gamma| \leq R$, then

$$L_{a,R}(\gamma) = L_{a,\infty}(\gamma) + O\left(\frac{a}{R^2} \log\left(2 + \frac{R}{a}\right)\right).$$

No proof of the Riemann Hypothesis is claimed. The paper computes the one-packet transform governing the weighted Gram kernel and identifies the phase structure that must be exploited in future off-diagonal operator estimates.

2020 Mathematics Subject Classification. 11M06, 11M26, 11N05, 30E20, 42A50, 46C05.

Keywords. Riemann zeta-function, logarithmic derivative, Riemann Hypothesis, Cauchy packets, Gram kernels, spectral weights, residue calculus, Hilbert-type phase, product kernel, phase cancellation.

Contents

1 Introduction

Let

$$W_a(t) = \frac{at^2}{(a^2+t^2)^2}, \qquad a > 0.$$

For a real ordinate γ, define the Cauchy packet

$$p_{\gamma,a}(t) = \frac{1}{a+i(t-\gamma)}.$$

The weighted low-strip Gram kernel is

$$G_{a,R}(\gamma,\gamma') = \int_{|t|\le R} W_a(t)p_{\gamma,a}(t)\overline{p_{\gamma',a}(t)}\,dt.$$

This kernel appears when zero packets attached to critical-line zeros of the Riemann zeta-function are studied on the vertical line

$$\operatorname{Re} s = \frac{1}{2} + a.$$

The elementary identity

$$p_{\gamma,a}(t)\overline{p_{\gamma',a}(t)} = \frac{p_{\gamma,a}(t)+\overline{p_{\gamma',a}(t)}}{2a-i(\gamma-\gamma')}$$

reduces the two-packet Gram kernel to a one-packet transform. Define

$$L_{a,R}(\gamma) = \int_{|t|\le R} W_a(t)p_{\gamma,a}(t)\,dt.$$

Then

$$G_{a,R}(\gamma,\gamma') = \frac{L_{a,R}(\gamma)+\overline{L_{a,R}(\gamma')}}{2a-i(\gamma-\gamma')}.$$

Thus the structure of $G_{a,R}$ depends strongly on the complex quantity $L_{a,R}(\gamma)$.

Absolute-value estimates give

$$|L_{a,R}(\gamma)| \ll (a^2+\gamma^2)^{-1/2}.$$

Such estimates are useful but discard phase information. If the off-diagonal Gram sum has cancellation, that cancellation must come from the complex phases of the terms

$$L_{a,R}(\gamma).$$

Therefore the next step is to compute $L_{a,R}$ itself, or at least an accurate approximation.

This paper carries out that computation for the full-line transform

$$L_{a,\infty}(\gamma) = \int_{-\infty}^{\infty} \frac{at^2}{(a^2+t^2)^2}\frac{dt}{a+i(t-\gamma)}.$$

The main formula is

$$L_{a,\infty}(\gamma) = \frac{\pi(a-i\gamma)}{2(2a-i\gamma)^2}.$$

This formula has several consequences.

First, the real and imaginary parts are explicit:

$$\operatorname{Re} L_{a,\infty}(\gamma) = \frac{\pi a(4a^2 + 3\gamma^2)}{2(4a^2 + \gamma^2)^2},$$

and

$$\operatorname{Im} L_{a,\infty}(\gamma) = \frac{\pi \gamma^3}{2(4a^2 + \gamma^2)^2}.$$

Second, the diagonal mass

$$\mathfrak{d}_a(\gamma; \infty) = \int_{-\infty}^{\infty} W_a(t) |p_{\gamma,a}(t)|^2 \, dt$$

is obtained exactly from the identity

$$\operatorname{Re} L_{a,\infty}(\gamma) = a \, \mathfrak{d}_a(\gamma; \infty).$$

Hence

$$\mathfrak{d}_a(\gamma; \infty) = \frac{\pi(4a^2 + 3\gamma^2)}{2(4a^2 + \gamma^2)^2}.$$

Third, the phase is explicit:

$$\tan \operatorname{Arg} L_{a,\infty}(\gamma) = \frac{\gamma^3}{a(4a^2 + 3\gamma^2)}.$$

In the far-ordinate regime

$$|\gamma| \gg a,$$

one has

$$L_{a,\infty}(\gamma) = \frac{i\pi}{2\gamma} + O\left(\frac{a}{\gamma^2}\right).$$

This is a Hilbert-type phase at the level of the one-packet transform.

However, it is important not to overstate this observation. The Gram kernel contains the combination

$$L_{a,\infty}(\gamma) + \overline{L_{a,\infty}(\gamma')}.$$

Since

$$\overline{L_{a,\infty}(\gamma')} = -\frac{i\pi}{2\gamma'} + O\left(\frac{a}{\gamma'^2}\right)$$

in the far-ordinate regime, the numerator has the leading form

$$\frac{i\pi}{2}\left(\frac{1}{\gamma} - \frac{1}{\gamma'}\right).$$

After division by

$$2a - i(\gamma - \gamma'),$$

this leading numerator cancels the difference

$$\gamma - \gamma'$$

in the far-far separated regime. The resulting leading kernel is product-type:

$$\frac{\pi}{2\gamma\gamma'}.$$

Thus the one-packet transform has Hilbert-type phase, but the full two-packet Gram kernel has a more subtle rational structure.

The finite low strip is handled by writing

$$L_{a,R}(\gamma) = L_{a,\infty}(\gamma) - T_{a,R}(\gamma),$$

where

$$T_{a,R}(\gamma) = \int_{|t|>R} W_a(t) p_{\gamma,a}(t)\, dt.$$

We prove that if $R \geq 2a$ and $|\gamma| \leq R$, then

$$|T_{a,R}(\gamma)| \ll \frac{a}{R^2} \log\left(2 + \frac{R}{a}\right).$$

Therefore

$$L_{a,R}(\gamma) = \frac{\pi(a - i\gamma)}{2(2a - i\gamma)^2} + O\left(\frac{a}{R^2} \log\left(2 + \frac{R}{a}\right)\right).$$

This paper is deliberately limited. It does not prove any zero-spacing estimate, any Bessel bound, or any mean-square estimate for the logarithmic derivative of $\zeta(s)$. Its contribution is the exact calculation of the one-packet transform and the extraction of its phase structure.

2 The one-packet transform

We begin with the basic definitions.

Definition 2.1 (Spectral weight)**.** *For $a > 0$, define*

$$W_a(t) = \frac{at^2}{(a^2 + t^2)^2}.$$

Definition 2.2 (Cauchy packet)**.** *For $\gamma \in \mathbb{R}$, define*

$$p_{\gamma,a}(t) = \frac{1}{a + i(t - \gamma)}.$$

For real t, the conjugate packet is

$$\overline{p_{\gamma,a}(t)} = \frac{1}{a - i(t - \gamma)}.$$

We shall consistently distinguish $p_{\gamma,a}$ from its conjugate.

Definition 2.3 (One-packet transform)**.** *For $R \in (0, \infty]$, define*

$$L_{a,R}(\gamma) = \int_{|t|\leq R} W_a(t) p_{\gamma,a}(t)\, dt.$$

When $R = \infty$, this means

$$L_{a,\infty}(\gamma) = \int_{-\infty}^{\infty} W_a(t) p_{\gamma,a}(t)\, dt.$$

Explicitly,

$$L_{a,R}(\gamma) = \int_{|t|\leq R} \frac{at^2}{(a^2 + t^2)^2} \frac{dt}{a + i(t - \gamma)}.$$

The integrand is absolutely integrable over $\mathbb{R}$. Indeed, as $|t| \to \infty$,

$$W_a(t) = O_a(t^{-2}), \qquad p_{\gamma,a}(t) = O(t^{-1}),$$

so the product is $O_a(t^{-3})$. Near $t = \gamma$, the packet is bounded by $1/a$, so no singularity occurs on the real line.

3 Connection with the Gram kernel

The weighted Gram kernel is

$$G_{a,R}(\gamma,\gamma') = \int_{|t|\le R} W_a(t) p_{\gamma,a}(t) \overline{p_{\gamma',a}(t)}\, dt.$$

The following identity explains why $L_{a,R}$ is the correct object.

Lemma 3.1 (Partial fractions for Cauchy packets)**.** *For all real t, γ, γ' and all $a > 0$,*

$$p_{\gamma,a}(t)\overline{p_{\gamma',a}(t)} = \frac{p_{\gamma,a}(t) + \overline{p_{\gamma',a}(t)}}{2a - i(\gamma - \gamma')}.$$

Proof. We have

$$p_{\gamma,a}(t) = \frac{1}{a + i(t-\gamma)}$$

and

$$\overline{p_{\gamma',a}(t)} = \frac{1}{a - i(t-\gamma')}.$$

Thus

$$\begin{aligned} p_{\gamma,a}(t) + \overline{p_{\gamma',a}(t)} &= \frac{1}{a+i(t-\gamma)} + \frac{1}{a-i(t-\gamma')} \\ &= \frac{a - i(t-\gamma') + a + i(t-\gamma)}{(a+i(t-\gamma))(a-i(t-\gamma'))} \\ &= \frac{2a - i(\gamma-\gamma')}{(a+i(t-\gamma))(a-i(t-\gamma'))}. \end{aligned}$$

Dividing by $2a - i(\gamma - \gamma')$ gives the claim. □

Corollary 3.2 (Gram kernel reduction)**.** *For all $a > 0$, $R > 0$, and real γ, γ',*

$$G_{a,R}(\gamma,\gamma') = \frac{L_{a,R}(\gamma) + \overline{L_{a,R}(\gamma')}}{2a - i(\gamma-\gamma')}.$$

Proof. Multiply the partial-fraction identity by $W_a(t)$, integrate over $|t| \le R$, and use that $W_a(t)$ is real-valued:

$$\int_{|t|\le R} W_a(t)\overline{p_{\gamma',a}(t)}\, dt = \overline{\int_{|t|\le R} W_a(t) p_{\gamma',a}(t)\, dt} = \overline{L_{a,R}(\gamma')}.$$

□

Thus the two-packet interaction is governed by the one-packet transform and by the separation denominator

$$2a - i(\gamma - \gamma').$$

4 Residue computation of $L_{a,\infty}$

We now compute the full-line transform exactly.

Theorem 4.1 (Exact full-line one-packet transform)**.** *For every $a > 0$ and every real γ,*

$$\boxed{L_{a,\infty}(\gamma) = \frac{\pi(a - i\gamma)}{2(2a - i\gamma)^2}.}$$

Proof. Consider the meromorphic function

$$F_{a,\gamma}(z) = \frac{az^2}{(a^2+z^2)^2} \frac{1}{a+i(z-\gamma)}.$$

For real t,

$$F_{a,\gamma}(t) = \frac{at^2}{(a^2+t^2)^2} \frac{1}{a+i(t-\gamma)}.$$

Moreover,

$$F_{a,\gamma}(z) = O(|z|^{-3})$$

as $|z| \to \infty$ away from the poles.

Assume first that $\gamma \neq 0$. Close the contour in the upper half-plane. The poles of $F_{a,\gamma}$ in the upper half-plane are at

$$z = ia$$

and

$$z = \gamma + ia.$$

The pole at $z = ia$ is a double pole, while $z = \gamma + ia$ is a simple pole. Since $F_{a,\gamma}(z) = O(|z|^{-3})$, the contribution of the upper semicircle tends to zero. Hence

$$L_{a,\infty}(\gamma) = 2\pi i \left(\operatorname*{Res}_{z=ia} F_{a,\gamma}(z) + \operatorname*{Res}_{z=\gamma+ia} F_{a,\gamma}(z) \right).$$

A direct residue computation gives

$$\operatorname*{Res}_{z=ia} F_{a,\gamma}(z) = \frac{i(a-i\gamma)}{4\gamma^2},$$

and

$$\operatorname*{Res}_{z=\gamma+ia} F_{a,\gamma}(z) = -\frac{ia(a-i\gamma)^2}{\gamma^2(2a-i\gamma)^2}.$$

Therefore

$$\begin{aligned} \operatorname*{Res}_{z=ia} F_{a,\gamma} + \operatorname*{Res}_{z=\gamma+ia} F_{a,\gamma} &= \frac{i(a-i\gamma)}{4\gamma^2} - \frac{ia(a-i\gamma)^2}{\gamma^2(2a-i\gamma)^2} \\ &= -\frac{i(a-i\gamma)}{4(2a-i\gamma)^2}. \end{aligned}$$

Thus

$$L_{a,\infty}(\gamma) = 2\pi i \left(-\frac{i(a-i\gamma)}{4(2a-i\gamma)^2} \right) = \frac{\pi(a-i\gamma)}{2(2a-i\gamma)^2}.$$

The formula extends to $\gamma = 0$ by continuity. Alternatively, direct substitution gives

$$L_{a,\infty}(0) = \frac{\pi}{8a},$$

which agrees with

$$\frac{\pi a}{2(2a)^2} = \frac{\pi}{8a}.$$

This proves the theorem. □

Remark 4.2. *The apparent singularities in the intermediate residue expressions at $\gamma = 0$ cancel in the sum of residues. The final formula is regular at $\gamma = 0$.*

5 Real and imaginary parts

We now extract the real and imaginary parts.

Theorem 5.1 (Real and imaginary parts)**.** *For every $a > 0$ and every real γ,*

$$\boxed{\operatorname{Re} L_{a,\infty}(\gamma) = \frac{\pi a(4a^2 + 3\gamma^2)}{2(4a^2 + \gamma^2)^2},}$$

and

$$\boxed{\operatorname{Im} L_{a,\infty}(\gamma) = \frac{\pi \gamma^3}{2(4a^2 + \gamma^2)^2}.}$$

Proof. From the exact formula,

$$L_{a,\infty}(\gamma) = \frac{\pi(a - i\gamma)}{2(2a - i\gamma)^2}.$$

Multiplying numerator and denominator by

$$(2a + i\gamma)^2$$

gives

$$L_{a,\infty}(\gamma) = \frac{\pi}{2} \frac{(a - i\gamma)(2a + i\gamma)^2}{(4a^2 + \gamma^2)^2}.$$

Since

$$(2a + i\gamma)^2 = 4a^2 - \gamma^2 + 4ia\gamma,$$

we compute

$$\begin{aligned}
(a - i\gamma)(2a + i\gamma)^2 &= (a - i\gamma)(4a^2 - \gamma^2 + 4ia\gamma) \\
&= a(4a^2 - \gamma^2) + 4ia^2\gamma - i\gamma(4a^2 - \gamma^2) + 4a\gamma^2 \\
&= a(4a^2 + 3\gamma^2) + i\gamma^3.
\end{aligned}$$

Therefore

$$L_{a,\infty}(\gamma) = \frac{\pi}{2} \frac{a(4a^2 + 3\gamma^2) + i\gamma^3}{(4a^2 + \gamma^2)^2}.$$

The real and imaginary parts follow. □

Corollary 5.2 (Parity)**.** *The function*

$$\operatorname{Re} L_{a,\infty}(\gamma)$$

is even in γ, while

$$\operatorname{Im} L_{a,\infty}(\gamma)$$

is odd in γ.

Proof. The real part depends on γ^2, while the imaginary part contains γ^3 divided by a function of γ^2. □

6 The exact diagonal mass

The real part of $L_{a,R}$ is related to the diagonal packet mass.

Definition 6.1 (Weighted diagonal packet mass)**.** *Define*

$$\mathfrak{d}_a(\gamma; R) = \int_{|t|\le R} W_a(t)|p_{\gamma,a}(t)|^2 \, dt.$$

Lemma 6.2 (Diagonal consistency)**.** *For every $R \in (0, \infty]$,*

$$\operatorname{Re} L_{a,R}(\gamma) = a \, \mathfrak{d}_a(\gamma; R).$$

Proof. Since

$$p_{\gamma,a}(t) = \frac{1}{a + i(t-\gamma)},$$

we have

$$\operatorname{Re} p_{\gamma,a}(t) = \frac{a}{a^2 + (t-\gamma)^2}.$$

Therefore

$$\begin{aligned}
\operatorname{Re} L_{a,R}(\gamma) &= \int_{|t|\le R} W_a(t) \operatorname{Re} p_{\gamma,a}(t) \, dt \\
&= \int_{|t|\le R} W_a(t) \frac{a}{a^2 + (t-\gamma)^2} \, dt \\
&= a \, \mathfrak{d}_a(\gamma; R).
\end{aligned}$$

□

Corollary 6.3 (Exact full-line diagonal mass)**.** *For every $a > 0$ and every real γ,*

$$\boxed{\mathfrak{d}_a(\gamma; \infty) = \frac{\pi(4a^2 + 3\gamma^2)}{2(4a^2 + \gamma^2)^2}.}$$

Proof. By diagonal consistency,

$$\mathfrak{d}_a(\gamma; \infty) = \frac{1}{a} \operatorname{Re} L_{a,\infty}(\gamma).$$

Substituting the formula for the real part gives the claim. □

Remark 6.4. *This exact identity refines the elementary estimate*

$$\mathfrak{d}_a(\gamma; \infty) \ll \frac{1}{a^2 + \gamma^2}.$$

7 Magnitude and phase

The exact formula also gives the magnitude and phase.

Proposition 7.1 (Magnitude)**.** *For every $a > 0$ and every real γ,*

$$\boxed{|L_{a,\infty}(\gamma)| = \frac{\pi\sqrt{a^2 + \gamma^2}}{2(4a^2 + \gamma^2)}.}$$

Proof. From

$$L_{a,\infty}(\gamma) = \frac{\pi(a - i\gamma)}{2(2a - i\gamma)^2},$$

we have

$$|a - i\gamma| = \sqrt{a^2 + \gamma^2}$$

and

$$|(2a - i\gamma)^2| = |2a - i\gamma|^2 = 4a^2 + \gamma^2.$$

The formula follows. ∎

Corollary 7.2 (Sharp absolute bound)**.** *For every $a > 0$ and every real γ,*

$$|L_{a,\infty}(\gamma)| \ll (a^2 + \gamma^2)^{-1/2}.$$

Proof. Since

$$4a^2 + \gamma^2 \geq a^2 + \gamma^2,$$

we have

$$\frac{\sqrt{a^2 + \gamma^2}}{4a^2 + \gamma^2} \leq \frac{1}{\sqrt{a^2 + \gamma^2}}.$$

∎

Proposition 7.3 (Phase)**.** *For $\gamma \neq 0$,*

$$\boxed{\tan \operatorname{Arg} L_{a,\infty}(\gamma) = \frac{\gamma^3}{a(4a^2 + 3\gamma^2)}.}$$

Here the argument is taken in the interval $(-\pi/2, \pi/2)$, since

$$\operatorname{Re} L_{a,\infty}(\gamma) > 0.$$

Proof. The real part is positive:

$$\operatorname{Re} L_{a,\infty}(\gamma) = \frac{\pi a(4a^2 + 3\gamma^2)}{2(4a^2 + \gamma^2)^2} > 0.$$

Thus

$$\tan \operatorname{Arg} L_{a,\infty}(\gamma) = \frac{\operatorname{Im} L_{a,\infty}(\gamma)}{\operatorname{Re} L_{a,\infty}(\gamma)}.$$

Using the real and imaginary part formulas gives

$$\frac{\pi\gamma^3/[2(4a^2 + \gamma^2)^2]}{\pi a(4a^2 + 3\gamma^2)/[2(4a^2 + \gamma^2)^2]} = \frac{\gamma^3}{a(4a^2 + 3\gamma^2)}.$$

∎

Remark 7.4. *For $\gamma > 0$, the imaginary part is positive. For $\gamma < 0$, the imaginary part is negative. Thus $L_{a,\infty}(\gamma)$ lies in the right half-plane, above or below the real axis according to the sign of γ.*

8 Asymptotic regimes

We record the two basic regimes.

Theorem 8.1 (Near-origin regime)**.** *If $|\gamma| \le a$, then*

$$L_{a,\infty}(\gamma) = \frac{\pi}{8a} + O\left(\frac{|\gamma|}{a^2}\right).$$

In particular, $L_{a,\infty}(\gamma)$ is predominantly real in this range.

Proof. At $\gamma = 0$,

$$L_{a,\infty}(0) = \frac{\pi}{8a}.$$

From the exact formula

$$L_{a,\infty}(\gamma) = \frac{\pi(a - i\gamma)}{2(2a - i\gamma)^2},$$

the derivative with respect to γ is $O(a^{-2})$ for $|\gamma| \le a$. Hence

$$L_{a,\infty}(\gamma) = L_{a,\infty}(0) + O\left(\frac{|\gamma|}{a^2}\right).$$

□

Theorem 8.2 (Far-ordinate regime)**.** *If $|\gamma| \ge 2a$, then*

$$\boxed{L_{a,\infty}(\gamma) = \frac{i\pi}{2\gamma} + O\left(\frac{a}{\gamma^2}\right).}$$

More precisely,

$$\operatorname{Re} L_{a,\infty}(\gamma) = \frac{3\pi a}{2\gamma^2} + O\left(\frac{a^3}{\gamma^4}\right),$$

and

$$\operatorname{Im} L_{a,\infty}(\gamma) = \frac{\pi}{2\gamma} + O\left(\frac{a^2}{\gamma^3}\right).$$

Proof. Using the exact real part,

$$\operatorname{Re} L_{a,\infty}(\gamma) = \frac{\pi a(4a^2 + 3\gamma^2)}{2(4a^2 + \gamma^2)^2}.$$

For $|\gamma| \ge 2a$,

$$4a^2 + \gamma^2 = \gamma^2 \left(1 + O\left(\frac{a^2}{\gamma^2}\right)\right),$$

and hence

$$\operatorname{Re} L_{a,\infty}(\gamma) = \frac{3\pi a}{2\gamma^2} + O\left(\frac{a^3}{\gamma^4}\right).$$

Similarly,

$$\operatorname{Im} L_{a,\infty}(\gamma) = \frac{\pi\gamma^3}{2(4a^2 + \gamma^2)^2} = \frac{\pi}{2\gamma} + O\left(\frac{a^2}{\gamma^3}\right).$$

Combining the estimates gives

$$L_{a,\infty}(\gamma) = \frac{i\pi}{2\gamma} + O\left(\frac{a}{\gamma^2}\right).$$

□

Remark 8.3 (One-packet Hilbert-type phase)**.** *The leading term*

$$\frac{i\pi}{2\gamma}$$

is purely imaginary and has a Hilbert-type phase at the level of the one-packet transform. This should not be confused with a direct Hilbert-transform bound for the full Gram matrix. The Gram kernel contains the combination

$$L_{a,\infty}(\gamma) + \overline{L_{a,\infty}(\gamma')},$$

and that combination changes the leading two-packet structure.

9 Finite-strip tail estimates

The actual low-strip transform is finite:

$$L_{a,R}(\gamma) = \int_{|t|\le R} W_a(t) p_{\gamma,a}(t)\, dt.$$

Write

$$L_{a,R}(\gamma) = L_{a,\infty}(\gamma) - T_{a,R}(\gamma),$$

where

$$T_{a,R}(\gamma) = \int_{|t|>R} W_a(t) p_{\gamma,a}(t)\, dt.$$

Theorem 9.1 (Finite-strip tail estimate)**.** *Assume*

$$R \ge 2a \qquad \text{and} \qquad |\gamma| \le R.$$

Then

$$\boxed{|T_{a,R}(\gamma)| \ll \frac{a}{R^2} \log\left(2 + \frac{R}{a}\right).}$$

The implied constant is absolute.

Proof. We estimate

$$|T_{a,R}(\gamma)| \le \int_{|t|>R} W_a(t) \frac{dt}{|a + i(t-\gamma)|}.$$

Since

$$|a + i(t-\gamma)| = \sqrt{a^2 + (t-\gamma)^2},$$

and since $|t| > R \ge 2a$, we have

$$W_a(t) \ll \frac{a}{t^2}.$$

Thus

$$|T_{a,R}(\gamma)| \ll a \int_{|t|>R} \frac{dt}{t^2 \sqrt{a^2 + (t-\gamma)^2}}.$$

We split into positive and negative tails.

For the positive tail, first take $R < t < 2R$. Since $t^{-2} \le R^{-2}$,

$$a \int_R^{2R} \frac{dt}{t^2 \sqrt{a^2 + (t-\gamma)^2}} \le \frac{a}{R^2} \int_R^{2R} \frac{dt}{\sqrt{a^2 + (t-\gamma)^2}}.$$

Because $\gamma \le R$, the change of variables $u = t - \gamma$ gives an interval contained in $[0, 3R]$. Therefore

$$\int_R^{2R} \frac{dt}{\sqrt{a^2 + (t-\gamma)^2}} \le \int_0^{3R} \frac{du}{\sqrt{a^2 + u^2}} \ll \log\left(2 + \frac{R}{a}\right).$$

Thus this part is

$$\ll \frac{a}{R^2}\log\left(2+\frac{R}{a}\right).$$

For $t \geq 2R$, since $|\gamma| \leq R$,

$$|t-\gamma| \geq t-R \geq \frac{t}{2}.$$

Hence

$$\sqrt{a^2+(t-\gamma)^2} \gg t,$$

and

$$a\int_{2R}^{\infty}\frac{dt}{t^2\sqrt{a^2+(t-\gamma)^2}} \ll a\int_{2R}^{\infty}\frac{dt}{t^3} \ll \frac{a}{R^2}.$$

The negative tail is analogous. On $-2R < t < -R$, use $t^{-2} \leq R^{-2}$ and the change of variables $u = t-\gamma$. Since $\gamma \geq -R$, the relevant interval is again within distance $O(R)$ from the possible minimum of $|u|$, and the same logarithmic bound follows. On $t \leq -2R$,

$$|t-\gamma| \geq |t|-|\gamma| \geq |t|/2,$$

so the contribution is $O(a/R^2)$.

Combining the pieces gives

$$|T_{a,R}(\gamma)| \ll \frac{a}{R^2}\log\left(2+\frac{R}{a}\right).$$

□

Corollary 9.2 (Finite-strip approximation)**.** *If $R \geq 2a$ and $|\gamma| \leq R$, then*

$$\boxed{L_{a,R}(\gamma) = \frac{\pi(a-i\gamma)}{2(2a-i\gamma)^2} + O\left(\frac{a}{R^2}\log\left(2+\frac{R}{a}\right)\right).}$$

Proof. Use

$$L_{a,R}(\gamma) = L_{a,\infty}(\gamma) - T_{a,R}(\gamma),$$

the exact formula for $L_{a,\infty}$, and the tail estimate. □

10 Application to logarithmic low strips

Let

$$R_D(a) = \log^D(e+a^{-1}), \qquad D > 0.$$

For $0 < a < 1/2$, this is the typical enlarged logarithmic low-strip scale.

Corollary 10.1 (Approximation in an enlarged logarithmic strip)**.** *Let*

$$R = R_D(a) = \log^D(e+a^{-1}).$$

For $0 < a < 1/2$ and $|\gamma| \leq R_D(a)$,

$$L_{a,R_D(a)}(\gamma) = \frac{\pi(a-i\gamma)}{2(2a-i\gamma)^2} + O_D\left(\frac{a}{\log^{2D}(e+a^{-1})}\log\left(2+\frac{\log^D(e+a^{-1})}{a}\right)\right).$$

In particular, the error tends to 0 as $a \to 0^+$.

Proof. For $0 < a < 1/2$, one has $R_D(a) \geq 2a$. Substitute

$$R = \log^D(e+a^{-1})$$

into the finite-strip approximation. The error becomes

$$\frac{a}{R^2}\log\left(2+\frac{R}{a}\right) = \frac{a}{\log^{2D}(e+a^{-1})}\log\left(2+\frac{\log^D(e+a^{-1})}{a}\right),$$

which tends to 0 as $a \to 0^+$. □

11 Consequences for the Gram kernel

The exact reduction formula gives

$$G_{a,R}(\gamma,\gamma') = \frac{L_{a,R}(\gamma) + \overline{L_{a,R}(\gamma')}}{2a - i(\gamma-\gamma')}.$$

Using the finite-strip approximation, one obtains a corresponding approximation for $G_{a,R}$.

Proposition 11.1 (Approximate finite-strip Gram kernel)**.** *Assume*

$$R \geq 2a, \qquad |\gamma| \leq R, \qquad |\gamma'| \leq R.$$

Then

$$\begin{aligned} G_{a,R}(\gamma,\gamma') = {} & \frac{L_{a,\infty}(\gamma) + \overline{L_{a,\infty}(\gamma')}}{2a - i(\gamma-\gamma')} \\ & + O\left(\frac{aR^{-2}\log(2+R/a)}{a+|\gamma-\gamma'|}\right). \end{aligned}$$

Proof. By the finite-strip approximation,

$$L_{a,R}(\gamma) = L_{a,\infty}(\gamma) + O\left(\frac{a}{R^2}\log\left(2+\frac{R}{a}\right)\right),$$

and similarly for γ'. Taking conjugates preserves the same error size. Substituting into

$$G_{a,R}(\gamma,\gamma') = \frac{L_{a,R}(\gamma) + \overline{L_{a,R}(\gamma')}}{2a - i(\gamma-\gamma')}$$

and using

$$|2a - i(\gamma-\gamma')| \asymp a + |\gamma-\gamma'|$$

gives the result. □

The main term is the explicit rational kernel

$$\boxed{G^{\mathrm{model}}_{a,\infty}(\gamma,\gamma') = \frac{\dfrac{\pi(a-i\gamma)}{2(2a-i\gamma)^2} + \dfrac{\pi(a+i\gamma')}{2(2a+i\gamma')^2}}{2a-i(\gamma-\gamma')}.}$$

Here the second term is the conjugate

$$\overline{L_{a,\infty}(\gamma')} = \frac{\pi(a+i\gamma')}{2(2a+i\gamma')^2}.$$

Remark 11.2. *The formula above is the correct full-line two-packet rational kernel. Its sign, real part, operator norm, and summation behavior are not settled in this paper. They form the next natural problem.*

12 One-packet phase and the far-far product kernel

We now clarify the far-ordinate structure.

From the far-ordinate asymptotic,

$$L_{a,\infty}(\gamma) = \frac{i\pi}{2\gamma} + O\left(\frac{a}{\gamma^2}\right)$$

for $|\gamma| \geq 2a$. Therefore

$$\overline{L_{a,\infty}(\gamma')} = -\frac{i\pi}{2\gamma'} + O\left(\frac{a}{\gamma'^2}\right)$$

for $|\gamma'| \geq 2a$. Hence

$$\boxed{L_{a,\infty}(\gamma) + \overline{L_{a,\infty}(\gamma')} = \frac{i\pi}{2}\left(\frac{1}{\gamma} - \frac{1}{\gamma'}\right) + O\left(\frac{a}{\gamma^2} + \frac{a}{\gamma'^2}\right).}$$

Substituting into the Gram identity gives

$$G^{\text{model}}_{a,\infty}(\gamma, \gamma') = \frac{\frac{i\pi}{2}\left(\gamma^{-1} - \gamma'^{-1}\right)}{2a - i(\gamma - \gamma')} + \text{error terms}.$$

If also

$$|\gamma - \gamma'| \gg a,$$

then

$$2a - i(\gamma - \gamma') = -i(\gamma - \gamma')\left(1 + O\left(\frac{a}{|\gamma - \gamma'|}\right)\right).$$

Thus the leading term becomes

$$-\frac{\pi}{2}\frac{\gamma^{-1} - \gamma'^{-1}}{\gamma - \gamma'}.$$

Since

$$\gamma^{-1} - \gamma'^{-1} = \frac{\gamma' - \gamma}{\gamma\gamma'} = -\frac{\gamma - \gamma'}{\gamma\gamma'},$$

we obtain

$$\boxed{-\frac{\pi}{2}\frac{\gamma^{-1} - \gamma'^{-1}}{\gamma - \gamma'} = \frac{\pi}{2\gamma\gamma'}.}$$

Therefore, in the far-far separated regime,

$$\boxed{G^{\text{model}}_{a,\infty}(\gamma, \gamma') = \frac{\pi}{2\gamma\gamma'} + \text{lower order terms.}}$$

Remark 12.1 (Hilbert-type phase versus Hilbert kernel)**.** *The one-packet transform has the Hilbert-type leading phase*

$$L_{a,\infty}(\gamma) \sim \frac{i\pi}{2\gamma}.$$

However, after insertion into the two-packet Gram identity, the combination

$$L_{a,\infty}(\gamma) + \overline{L_{a,\infty}(\gamma')}$$

cancels the factor

$$\gamma - \gamma'$$

in the far-far numerator. The resulting leading two-packet kernel is therefore product-type,

$$\frac{\pi}{2\gamma\gamma'},$$

rather than a bare Hilbert kernel. A separate operator analysis of the exact rational kernel is still needed.

13 Limitations

The paper has several limitations.

First, it does not prove RH.

Second, it does not prove any high-frequency mean-square estimate for the logarithmic derivative of $\zeta(s)$.

Third, it does not prove a Schur or Bessel bound for the low-strip Gram kernel.

Fourth, it does not solve the close-pair problem. When

$$|\gamma - \gamma'| \lesssim a,$$

the denominator

$$2a - i(\gamma - \gamma')$$

does not produce a large separation gain.

Fifth, the finite-strip approximation is proved here only under the condition

$$|\gamma| \le R.$$

This is the natural range for ordinates inside the low strip, but other ranges would require separate estimates.

Sixth, although the full-line transform is computed exactly, the full rational two-variable kernel

$$G^{\mathrm{model}}_{a,\infty}(\gamma, \gamma')$$

is not fully analyzed here. Its sign, operator behavior, and summation properties remain for later work.

Seventh, the far-ordinate asymptotic of $L_{a,\infty}$ should not be interpreted as an immediate Hilbert-transform bound for the Gram matrix. The Gram kernel contains

$$L_{a,\infty}(\gamma) + \overline{L_{a,\infty}(\gamma')},$$

and this combination produces a product-type far-far leading kernel in the separated regime.

Eighth, the passage from finite packet systems to infinite zero-packet expansions for ζ'/ζ requires additional convergence and regularization arguments.

14 Conclusion

The weighted low-strip Gram kernel satisfies the exact identity

$$G_{a,R}(\gamma, \gamma') = \frac{L_{a,R}(\gamma) + \overline{L_{a,R}(\gamma')}}{2a - i(\gamma - \gamma')}.$$

Thus the one-packet transform

$$L_{a,R}(\gamma) = \int_{|t| \le R} \frac{at^2}{(a^2 + t^2)^2} \frac{dt}{a + i(t - \gamma)}$$

is the fundamental object controlling the off-diagonal phase structure.

This paper computes the full-line transform exactly:

$$L_{a,\infty}(\gamma) = \frac{\pi(a - i\gamma)}{2(2a - i\gamma)^2}.$$

The real and imaginary parts are

$$\operatorname{Re} L_{a,\infty}(\gamma) = \frac{\pi a(4a^2 + 3\gamma^2)}{2(4a^2 + \gamma^2)^2},$$

and

$$\operatorname{Im} L_{a,\infty}(\gamma) = \frac{\pi\gamma^3}{2(4a^2+\gamma^2)^2}.$$

In particular,

$$\mathfrak{d}_a(\gamma;\infty) = \frac{\pi(4a^2+3\gamma^2)}{2(4a^2+\gamma^2)^2}.$$

The far-ordinate one-packet transform has the Hilbert-type phase

$$L_{a,\infty}(\gamma) = \frac{i\pi}{2\gamma} + O\left(\frac{a}{\gamma^2}\right).$$

However, after taking the conjugate in the second packet and inserting the result into the Gram identity, the far-far separated leading kernel becomes

$$\frac{\pi}{2\gamma\gamma'}.$$

Thus the full two-packet problem is not merely a Hilbert-transform problem. It is an exact rational-kernel problem.

For finite strips, if $R \geq 2a$ and $|\gamma| \leq R$, then

$$L_{a,R}(\gamma) = L_{a,\infty}(\gamma) + O\left(\frac{a}{R^2}\log\left(2+\frac{R}{a}\right)\right).$$

Thus, in logarithmic low strips, the finite one-packet transform is approximated by an explicit rational function.

The next natural step is to analyze the induced full-line rational Gram kernel

$$\frac{L_{a,\infty}(\gamma) + \overline{L_{a,\infty}(\gamma')}}{2a - i(\gamma-\gamma')},$$

especially its real part, sign structure, and operator behavior. That analysis is required before one can turn the phase information obtained here into a genuine off-diagonal cancellation estimate.

References

[1] L. V. Ahlfors, *Complex Analysis*, 3rd ed., McGraw–Hill, New York, 1979.

[2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

[3] J. B. Conrey, The Riemann Hypothesis, *Notices Amer. Math. Soc.* **50** (2003), no. 3, 341–353.

[4] H. Davenport, *Multiplicative Number Theory*, 3rd ed., revised by H. L. Montgomery, Graduate Texts in Mathematics, Vol. 74, Springer, New York, 2000.

[5] H. M. Edwards, *Riemann's Zeta Function*, Academic Press, New York, 1974.

[6] S. M. Gonek, Mean values of the Riemann zeta-function and its derivatives, *Invent. Math.* **75** (1984), 123–141.

[7] D. A. Goldston, On the function $S(T)$ in the theory of the Riemann zeta-function, *J. Number Theory* **27** (1987), 149–177.

[8] G. H. Hardy, Sur les zéros de la fonction $\zeta(s)$ de Riemann, *C. R. Acad. Sci. Paris* **158** (1914), 1012–1014.

[9] A. E. Ingham, *The Distribution of Prime Numbers*, Cambridge University Press, Cambridge, 1932.

[10] A. Ivić, *The Riemann Zeta-Function: Theory and Applications*, Dover Publications, Mineola, NY, 2003.

[11] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, Vol. 53, American Mathematical Society, Providence, RI, 2004.

[12] H. L. Montgomery, The pair correlation of zeros of the zeta function, *Analytic Number Theory, Proc. Sympos. Pure Math.* **24** (1973), 181–193.

[13] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge Studies in Advanced Mathematics, Vol. 97, Cambridge University Press, Cambridge, 2007.

[14] B. Riemann, Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monatsberichte der Berliner Akademie* (1859), 671–680.

[15] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw–Hill, New York, 1987.

[16] A. Selberg, Contributions to the theory of the Riemann zeta-function, *Arch. Math. Naturvid.* **48** (1946), 89–155.

[17] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Clarendon Press, Oxford, 1986.