An $M = 1$ Admissible Framework for the Dyadic High-Frequency Barrier

Jongmin Choi
Independent Researcher, Seoul, Korea
24ping@naver.com
ORCID iD: 0009-0008-7448-514X

Abstract

Let

$$H_0(s) = -\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1},$$

and put

$$H_a(t) = H_0\left(\frac{1}{2} + a + it\right), \qquad 0 < a < \frac{1}{2}.$$

In a spectral formulation of the high-frequency barrier one is led to the weighted integral

$$\mathcal{G}_{\mathrm{spec}}(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{at^2}{(a^2+t^2)^2} |H_a(t)|^2 \, dt,$$

with the convention that the integral is $+\infty$ if a pole occurs on the line of integration.

Dyadic reductions compare the shell

$$T < |t| \le 2T$$

against the spectral threshold

$$\frac{T^2}{a}.$$

A translated vertical mean-square input with loss exponent $M < 1$ is strong enough to push the remaining obstruction into a shrinking low-frequency cone. However, the natural diagonal packet scale for the unsmoothed vertical logarithmic derivative near the critical line is $M = 1$. This paper analyzes what happens if one accepts the natural $M = 1$ loss

$$\int_{T<|t|\le 2T} |H_a(t)|^2 \, dt \ll Ta^{-1} \log^B(2 + T + a^{-1}).$$

The main result is that the $M = 1$ loss is still sufficient to clear all dyadic shells with

$$T \ge \log^D(e + a^{-1})$$

for every fixed $D > B$. Indeed, on such shells the spectral weight contributes a factor a/T^2, so the shell contribution is bounded by

$$\frac{\log^B(2 + T + a^{-1})}{T},$$

and the dyadic tail is uniformly summable.

Thus accepting $M = 1$ does not destroy the dyadic high-frequency framework. It does, however, enlarge the remaining low-frequency region from a shrinking cone to the strip

$$|t| \le \log^D(e + a^{-1}).$$

This region grows as $a \to 0^+$, so local regularity near $s = 1/2$ does not automatically control it. In the notation of a shrinking cone $|t| \le aR(a)$, this enlarged region would correspond to $aR(a) = \log^D(e + a^{-1}) \to \infty$, not to $aR(a) \to 0$.

The paper therefore identifies the tradeoff:

$$M = 1 \quad \text{is analytically natural but geometrically expensive.}$$

No proof of the Riemann Hypothesis is claimed. The purpose is to rebalance the dyadic framework at the natural $M = 1$ scale and to isolate the enlarged low-frequency obstruction.

2020 Mathematics Subject Classification. 11M06, 11M26, 11N05, 42A38.

Keywords. Riemann Hypothesis, Riemann zeta-function, logarithmic derivative, mean-square estimates, dyadic decomposition, spectral weights, low-frequency obstruction.

Contents

1 Introduction

Let

$$H_0(s) = -\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}.$$

The pole of $-\zeta'/\zeta$ at $s = 1$ has been removed. The remaining poles of H_0 are precisely the zeros of $\zeta(s)$, with residues given by their multiplicities up to sign.

For $0 < a < 1/2$, define

$$H_a(t) = H_0\left(\frac{1}{2} + a + it\right).$$

The spectral high-frequency quantity studied in this paper is

$$\mathcal{G}_{\mathrm{spec}}(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_a(t) |H_a(t)|^2 \, dt,$$

where

$$W_a(t) = \frac{at^2}{(a^2 + t^2)^2}.$$

We use the convention that this integral is $+\infty$ if H_a has a pole on the line of integration.

The dyadic decomposition of this integral leads to shells

$$T < |t| \leq 2T, \qquad T \asymp 2^k a.$$

On a high-frequency shell $T \gg a$, one has

$$W_a(t) \asymp \frac{a}{T^2}.$$

Therefore the contribution of a shell is approximately

$$\frac{a}{T^2} \int_{T<|t|\leq 2T} |H_a(t)|^2 \, dt.$$

The natural diagonal packet scale for the unsmoothed vertical logarithmic derivative near the critical line is

$$Ta^{-1} \log T.$$

Indeed, the model packet

$$p_{\gamma,a}(t) = \frac{1}{a + i(t - \gamma)}$$

has

$$\int_{-\infty}^{\infty} |p_{\gamma,a}(t)|^2 \, dt = \frac{\pi}{a}.$$

Together with the Riemann–von Mangoldt density $T \log T$, this gives a formal diagonal scale of order

$$\frac{T \log T}{a}.$$

Thus it is natural to consider an $M = 1$-type mean-square input:

$$\int_{T<|t|\leq 2T} |H_a(t)|^2 \, dt \ll Ta^{-1} \log^B(2 + T + a^{-1}).$$

Substituting this into the shell contribution gives

$$\frac{a}{T^2} \cdot Ta^{-1} \log^B(2 + T + a^{-1}) = \frac{\log^B(2 + T + a^{-1})}{T}.$$

Thus high-frequency shells are harmless once

$$T \gg \log^B(2 + T + a^{-1}).$$

This observation is the main point of the paper.

Accepting the $M = 1$ loss does not destroy the dyadic high-frequency framework. It clears the tail of the spectral integral. The cost is geometric: the low-frequency region that remains is no longer a shrinking cone. Instead it has size

$$|t| \lesssim \log^D(e + a^{-1})$$

for a fixed $D > B$. This region grows as $a \to 0^+$.

This paper treats $\mathcal{G}_{\rm spec}(a)$ as a spectral target quantity. It does not, by itself, prove that uniform boundedness of $\mathcal{G}_{\rm spec}(a)$ is equivalent to the Riemann Hypothesis. Any implication from

$$\sup_{0<a<1/2} \mathcal{G}_{\rm spec}(a) < \infty$$

to RH must be supplied by an independent spectral criterion. The present paper only proves the conditional reduction of the high-frequency part of $\mathcal{G}_{\rm spec}(a)$ to the enlarged low-frequency energy under the $M = 1$ mean-square input.

The paper is organized as follows.

(i) We define the spectral weight and the dyadic shells.

(ii) We recall the dyadic threshold T^2/a.

(iii) We formulate the natural $M = 1$ mean-square input.

(iv) We prove that $M = 1$ clears all shells $T \geq \log^D(e + a^{-1})$, $D > B$.

(v) We isolate the enlarged low-frequency region.

(vi) We compare the $M = 1$ framework with a general $M < 1$ framework.

(vii) We explain why the enlarged low-frequency region is not a shrinking cone.

(viii) We record the zero-exclusion condition forced by finite enlarged low-frequency energy.

No proof of RH is claimed. The paper is a structural rebalancing of the dyadic spectral framework at the natural $M = 1$ scale.

Remark 1.1 (On the phrase "natural $M = 1$ scale")**.** *The phrase "natural $M = 1$ scale" refers in this paper to the unsmoothed vertical logarithmic-derivative packet*

$$(a + i(t - \gamma))^{-1},$$

whose $L^2(\mathbb{R})$-mass is π/a. It is not a statement about every X-dependent smoothed Perron packet. Smoothed remainder packets may contain additional X-dependent factors. The present $M = 1$ input is a vertical mean-square hypothesis for $H_a(t)$ itself.

Remark 1.2 (On the role of external spectral criteria)**.** *The present paper is intentionally formulated as a conditional spectral reduction. It does not invoke any external criterion connecting*

$$\sup_a \mathcal{G}_{\rm spec}(a)$$

with RH. If such a criterion is combined with the present result, then the conclusions here may be used as a high-frequency reduction step. In this paper, however, the object of study is only the decomposition of $\mathcal{G}_{\rm spec}$ under the $M = 1$ mean-square input.

2 The spectral weight

Define

$$W_a(t) = \frac{at^2}{(a^2 + t^2)^2}, \qquad 0 < a < \frac{1}{2}.$$

The associated spectral quantity is

$$\mathcal{G}_{\rm spec}(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_a(t) |H_a(t)|^2 \, dt.$$

If H_a has a pole on the line $\operatorname{Re} s = 1/2 + a$, then the integral is understood as $+\infty$.

Lemma 2.1 (Basic bounds for the spectral weight)**.** *For all $t \in \mathbb{R}$,*

$$0 \le W_a(t) \le \frac{1}{4a}.$$

Moreover, if $|t| \ge 2a$, then

$$\frac{16}{25}\frac{a}{t^2} \le W_a(t) \le \frac{a}{t^2}.$$

If $|t| \le 2a$, then

$$W_a(t) \asymp \frac{t^2}{a^3},$$

with absolute implied constants.

Proof. The function

$$W_a(t) = \frac{at^2}{(a^2+t^2)^2}$$

is nonnegative. Put $u = t^2$. Then

$$W_a(t) = f(u), \qquad f(u) = \frac{au}{(a^2+u)^2}.$$

A direct differentiation gives

$$f'(u) = a\,\frac{a^2-u}{(a^2+u)^3}.$$

Thus the maximum occurs at $u = a^2$, i.e. at $t = \pm a$. Therefore

$$W_a(\pm a) = \frac{a \cdot a^2}{(2a^2)^2} = \frac{1}{4a}.$$

Hence

$$W_a(t) \le \frac{1}{4a}.$$

If $|t| \ge 2a$, then

$$a^2 \le \frac{t^2}{4},$$

and therefore

$$t^2 \le a^2 + t^2 \le \frac{5}{4}t^2.$$

Hence

$$(a^2+t^2)^2 \ge t^4$$

and

$$(a^2+t^2)^2 \le \frac{25}{16}t^4.$$

Consequently,

$$W_a(t) = \frac{at^2}{(a^2+t^2)^2} \le \frac{at^2}{t^4} = \frac{a}{t^2},$$

while

$$W_a(t) = \frac{at^2}{(a^2+t^2)^2} \ge \frac{at^2}{(25/16)t^4} = \frac{16}{25}\frac{a}{t^2}.$$

If $|t| \le 2a$, then

$$a^2 \le a^2 + t^2 \le 5a^2.$$

Thus

$$W_a(t) = \frac{at^2}{(a^2+t^2)^2} \asymp \frac{at^2}{a^4} = \frac{t^2}{a^3}.$$

□

3 Dyadic shells and the threshold T^2/a

For $k \geq 0$, define dyadic shells

$$I_0(a) = \{t : |t| \leq 2a\},$$

and for $k \geq 1$,

$$I_k(a) = \{t : 2^k a < |t| \leq 2^{k+1} a\}.$$

Let

$$T_k = 2^k a.$$

Then

$$I_k(a) = \{t : T_k < |t| \leq 2T_k\}, \qquad k \geq 1.$$

For high-frequency shells $k \geq 1$, the spectral weight satisfies

$$W_a(t) \asymp \frac{a}{T_k^2} \qquad (t \in I_k(a)).$$

Thus the shell contribution is comparable to

$$\frac{a}{T_k^2} \int_{I_k(a)} |H_a(t)|^2 \, dt.$$

This shows why the dyadic threshold for the unweighted mean square is

$$\boxed{\int_{I_k(a)} |H_a(t)|^2 \, dt \lesssim \frac{T_k^2}{a}.}$$

Indeed, if the shell integral is $O(T_k^2/a)$, then its weighted contribution is $O(1)$.

Definition 3.1 (Dyadic shell contribution)**.** *For $k \geq 0$, define*

$$\mathcal{S}_k(a) = \int_{I_k(a)} W_a(t) |H_a(t)|^2 \, dt.$$

Thus

$$2\pi \mathcal{G}_{\text{spec}}(a) = \sum_{k \geq 0} \mathcal{S}_k(a),$$

with the convention that either side may be $+\infty$.

Lemma 3.2 (High-shell comparison)**.** *For $k \geq 1$,*

$$\boxed{\mathcal{S}_k(a) \ll \frac{a}{T_k^2} \int_{I_k(a)} |H_a(t)|^2 \, dt.}$$

Proof. For $t \in I_k(a)$ with $k \geq 1$, one has $|t| > 2a$, hence

$$W_a(t) \ll \frac{a}{t^2} \ll \frac{a}{T_k^2}.$$

Therefore

$$\mathcal{S}_k(a) = \int_{I_k(a)} W_a(t) |H_a(t)|^2 \, dt \ll \frac{a}{T_k^2} \int_{I_k(a)} |H_a(t)|^2 \, dt.$$

$\square$

4 The natural $M = 1$ mean-square input

Motivated by the diagonal packet scale for the unsmoothed vertical logarithmic derivative, we consider the following $M = 1$ mean-square input.

Hypothesis 4.1 ($M = 1$ dyadic mean-square input). *There exist constants $C > 0$ and $B \geq 0$ such that, for every $0 < a < 1/2$ and every $T \geq 1$,*

$$\boxed{\int_{T<|t|\leq 2T} |H_a(t)|^2\, dt \leq C\, T a^{-1} \log^B(2 + T + a^{-1}).}$$

Remark 4.2. *This is not proved in this paper. It is an analytic input. The point of the present work is to determine what this natural $M = 1$ input would imply for the dyadic spectral barrier.*

Remark 4.3. *The power a^{-1} is the natural diagonal packet loss near the critical line for the vertical packet $(a + i(t - \gamma))^{-1}$. It is weaker than an $M < 1$ input but more realistic from that zero-packet viewpoint.*

Under this hypothesis, each high shell satisfies a simple bound.

Proposition 4.4 (Single-shell bound under $M = 1$). *Assume the $M = 1$ dyadic mean-square input. Then for every $k \geq 1$,*

$$\boxed{\mathcal{S}_k(a) \ll \frac{\log^B(2 + T_k + a^{-1})}{T_k}.}$$

Proof. By the high-shell comparison,

$$\mathcal{S}_k(a) \ll \frac{a}{T_k^2} \int_{I_k(a)} |H_a(t)|^2\, dt.$$

The $M = 1$ input gives

$$\int_{I_k(a)} |H_a(t)|^2\, dt \ll T_k a^{-1} \log^B(2 + T_k + a^{-1}).$$

Therefore

$$\mathcal{S}_k(a) \ll \frac{a}{T_k^2} \cdot T_k a^{-1} \log^B(2 + T_k + a^{-1}) = \frac{\log^B(2 + T_k + a^{-1})}{T_k}.$$

□

5 A dyadic logarithmic summation lemma

We need a dyadic summation estimate.

Lemma 5.1 (Dyadic comparison). *Let $F : [1, \infty) \to [0, \infty)$ be nondecreasing, and let $T_k = 2^k a$. Then for every $A \geq 1$,*

$$\sum_{T_k > A} \frac{F(T_k)}{T_k} \ll \int_{A/2}^{\infty} \frac{F(2x)}{x^2}\, dx,$$

with an absolute implied constant.

Proof. For $x \in [T_k/2, T_k]$, one has $T_k \le 2x$. Since F is nondecreasing,

$$F(T_k) \le F(2x).$$

Also $x \asymp T_k$ on this interval. Hence

$$\frac{F(T_k)}{T_k} \ll \int_{T_k/2}^{T_k} \frac{F(2x)}{x^2}\, dx.$$

Summing over all k with $T_k > A$ gives

$$\sum_{T_k > A} \frac{F(T_k)}{T_k} \ll \int_{A/2}^{\infty} \frac{F(2x)}{x^2}\, dx.$$

□

Lemma 5.2 (Dyadic logarithmic tail)**.** *Let $B \ge 0$, $D > B$, and*

$$R_D(a) = \log^D(e + a^{-1}).$$

Then

$$\boxed{\sum_{\substack{k \ge 1 \\ T_k \ge R_D(a)}} \frac{\log^B(2 + T_k + a^{-1})}{T_k} \ll_{B,D} 1}$$

uniformly for $0 < a < 1/2$. More generally, the same conclusion holds with $R_D(a)$ replaced by $cR_D(a)$ for any fixed $c > 0$. Moreover, the part of the sum with

$$R_D(a) \le T_k \le e + a^{-1}$$

is

$$O_{B,D}\left(\log^{B-D}(e + a^{-1})\right).$$

Proof. Put

$$A = e + a^{-1}, \qquad R = \log^D A.$$

We first prove the estimate with lower cutoff R. The case cR with fixed $c > 0$ is identical, since c only changes absolute constants in the dyadic summation.

Split the dyadic indices into two ranges.

First suppose

$$R \le T_k \le A.$$

Then

$$\log(2 + T_k + a^{-1}) \ll \log A,$$

and hence

$$\sum_{\substack{k \ge 1 \\ R \le T_k \le A}} \frac{\log^B(2 + T_k + a^{-1})}{T_k} \ll \log^B A \sum_{\substack{k \ge 1 \\ T_k \ge R}} \frac{1}{T_k}.$$

Since $T_k = 2^k a$ is dyadic,

$$\sum_{T_k \ge R} \frac{1}{T_k} \ll \frac{1}{R}.$$

Thus this part is

$$\ll \frac{\log^B A}{R} = \log^{B-D} A.$$

Next suppose

$$T_k > A.$$

Then

$$\log(2 + T_k + a^{-1}) \ll \log(2 + T_k).$$

Apply the dyadic comparison lemma with

$$F(x) = \log^B(2 + x).$$

This gives

$$\sum_{T_k > A} \frac{\log^B(2 + T_k)}{T_k} \ll \int_{A/2}^{\infty} \frac{\log^B(2 + 2x)}{x^2}\, dx.$$

Since $A = e + a^{-1} \geq e$, the portion $A/2 \leq x \leq A$ contributes

$$\ll_B \frac{\log^B(2 + A)}{A}.$$

The portion $x \geq A$ satisfies the same bound:

$$\int_A^{\infty} \frac{\log^B(2 + 2x)}{x^2}\, dx \ll_B \frac{\log^B(2 + A)}{A}.$$

Hence

$$\int_{A/2}^{\infty} \frac{\log^B(2 + 2x)}{x^2}\, dx \ll_B \frac{\log^B(2 + A)}{A} \ll_B 1.$$

Combining the two ranges proves the lemma. □

6 High-frequency clearance under $M = 1$

We now prove the main high-frequency clearance theorem.

Theorem 6.1 ($M = 1$ high-frequency clearance)**.** *Assume the $M = 1$ dyadic mean-square input with logarithmic exponent B. Let $D > B$, and define*

$$R_D(a) = \log^D(e + a^{-1}).$$

Then

$$\boxed{\sum_{\substack{k \geq 1 \\ T_k \geq R_D(a)}} \mathcal{S}_k(a) \ll_{B,D} 1}$$

uniformly for $0 < a < 1/2$. More generally, the same conclusion holds with $R_D(a)$ replaced by $cR_D(a)$ for any fixed $c > 0$.

Proof. By the single-shell bound under $M = 1$,

$$\mathcal{S}_k(a) \ll \frac{\log^B(2 + T_k + a^{-1})}{T_k}.$$

Therefore

$$\sum_{\substack{k \geq 1 \\ T_k \geq R_D(a)}} \mathcal{S}_k(a) \ll \sum_{\substack{k \geq 1 \\ T_k \geq R_D(a)}} \frac{\log^B(2 + T_k + a^{-1})}{T_k}.$$

The dyadic logarithmic tail lemma gives the desired uniform bound. The case $cR_D(a)$ follows from the corresponding general form of the same lemma. □

Corollary 6.2 (Tail decay with extra logarithmic room). *Assume the $M = 1$ dyadic mean-square input with exponent B. If $D > B$, then*

$$\sum_{\substack{k \geq 1 \\ T_k \geq R_D(a)}} \mathcal{S}_k(a) = O_{B,D}(1).$$

Moreover, the part of the tail with

$$R_D(a) \leq T_k \leq e + a^{-1}$$

is

$$O_{B,D}\big(\log^{B-D}(e + a^{-1})\big) = o(1)$$

as $a \to 0^+$.

Proof. This is the quantitative statement already obtained in the proof of the dyadic logarithmic tail lemma. ☐

7 The enlarged low-frequency region

The high-frequency clearance theorem leaves the complementary region

$$|t| \leq R_D(a), \qquad R_D(a) = \log^D(e + a^{-1}).$$

We define the corresponding low-frequency energy.

Definition 7.1 (Enlarged low-frequency energy). *For $D > 0$, define*

$$\boxed{\mathcal{C}_D(a) = \frac{1}{2\pi} \int_{|t| \leq R_D(a)} W_a(t)|H_a(t)|^2 \, dt, \qquad R_D(a) = \log^D(e + a^{-1}).}$$

Theorem 7.2 ($M = 1$-conditional reduction to the enlarged low energy). *Assume the $M = 1$ dyadic mean-square input with logarithmic exponent B. Let $D > B$. Then*

$$\boxed{\sup_{0<a<1/2} \mathcal{G}_{\mathrm{spec}}(a) < \infty \quad \Longleftrightarrow \quad \sup_{0<a<1/2} \mathcal{C}_D(a) < \infty.}$$

More precisely,

$$\mathcal{C}_D(a) \leq \mathcal{G}_{\mathrm{spec}}(a)$$

for every a, and

$$\mathcal{G}_{\mathrm{spec}}(a) \leq \mathcal{C}_D(a) + O_{B,D}(1)$$

uniformly in $0 < a < 1/2$, where the second estimate uses the $M = 1$ dyadic mean-square input.

Proof. By definition,

$$2\pi\mathcal{G}_{\mathrm{spec}}(a) = \int_{|t| \leq R_D(a)} W_a(t)|H_a(t)|^2 \, dt + \int_{|t| > R_D(a)} W_a(t)|H_a(t)|^2 \, dt.$$

The first term is $2\pi\mathcal{C}_D(a)$.

It remains to bound the tail. Since $0 < a < 1/2$, the low shell

$$I_0(a) = \{|t| \leq 2a\}$$

is contained in $\{|t| \leq 1\}$. Also

$$R_D(a) = \log^D(e + a^{-1}) \geq 1.$$

Hence $I_0(a)$ is contained in the low-frequency region.

For the remaining shells,

$$\{|t| > R_D(a)\} \subset \bigcup_{\substack{k\geq 1\\ T_k > R_D(a)/2}} I_k(a).$$

Indeed, if $t \in I_k(a)$ and $I_k(a)$ meets the set $\{|t| > R_D(a)\}$, then $2T_k > R_D(a)$, hence $T_k > R_D(a)/2$. Therefore

$$\int_{|t|>R_D(a)} W_a(t)|H_a(t)|^2\,dt \leq \sum_{\substack{k\geq 1\\ T_k > R_D(a)/2}} \mathcal{S}_k(a).$$

By the $M = 1$ high-frequency clearance theorem, in its fixed-multiple form with $c = 1/2$, this sum is $O_{B,D}(1)$. Thus

$$\mathcal{G}_{\text{spec}}(a) \leq \mathcal{C}_D(a) + O_{B,D}(1).$$

The inequality

$$\mathcal{C}_D(a) \leq \mathcal{G}_{\text{spec}}(a)$$

is immediate from nonnegativity of the integrand. The equivalence of uniform boundedness follows. □

Remark 7.3. *The equivalence in this theorem is conditional on the $M = 1$ dyadic mean-square input. Without that input, boundedness of the enlarged low-frequency energy alone does not control the high-frequency tail.*

Remark 7.4. *The $M = 1$ input clears the high-frequency tail, but it leaves a low-frequency region whose height is logarithmic in a^{-1}, not of order a or $a\log^D(a^{-1})$.*

8 Comparison with an $M < 1$ framework

It is useful to compare the $M = 1$ situation with a general mean-square input of the form

$$\int_{T<|t|\leq 2T} |H_a(t)|^2\,dt \ll Ta^{-M}\log^B(2 + T + a^{-1}).$$

For a high shell $T = T_k$, this gives

$$\mathcal{S}_k(a) \ll \frac{a}{T^2}\cdot Ta^{-M}\log^B(2 + T + a^{-1}) = \frac{a^{1-M}}{T}\log^B(2 + T + a^{-1}).$$

Thus the transition scale is approximately

$$T \gtrsim a^{1-M}\log^D(e + a^{-1}), \qquad D > B.$$

Proposition 8.1 (Transition scale for general M)**.** *Assume a dyadic mean-square input with exponent M:*

$$\int_{T<|t|\leq 2T} |H_a(t)|^2\,dt \ll Ta^{-M}\log^B(2 + T + a^{-1}).$$

Let $D > B$. Then all shells with

$$T \geq a^{1-M}\log^D(e + a^{-1})$$

give a uniformly bounded dyadic tail.

Proof. The shell contribution is

$$\ll \frac{a^{1-M}}{T} \log^B(2 + T + a^{-1}).$$

Summing over dyadic $T \geq a^{1-M} \log^D(e + a^{-1})$ gives the same dyadic logarithmic summation as before, with an additional prefactor a^{1-M} and lower limit $a^{1-M} \log^D(e + a^{-1})$. The ratio of the prefactor to the lower limit is

$$\frac{a^{1-M}}{a^{1-M} \log^D(e + a^{-1})} = \log^{-D}(e + a^{-1}),$$

which absorbs the logarithmic loss when $D > B$. □

Corollary 8.2 (The difference between $M < 1$ and $M = 1$)**.** *If $M < 1$, then*

$$a^{1-M} \log^D(e + a^{-1}) \to 0 \qquad (a \to 0^+).$$

Thus the remaining low-frequency region shrinks toward $t = 0$.

If $M = 1$, then the transition scale becomes

$$\log^D(e + a^{-1}),$$

which grows as $a \to 0^+$.

Proof. If $M < 1$, then $1 - M > 0$, so the power a^{1-M} dominates the logarithm and tends to 0. If $M = 1$, the power factor disappears. □

Remark 8.3. *This is the main geometric distinction. The $M < 1$ input pushes the remaining obstruction into a shrinking region near $t = 0$. The $M = 1$ input clears the high-frequency tail but leaves a growing logarithmic strip.*

9 The zero-exclusion condition forced by finite enlarged low energy

Accepting the natural $M = 1$ loss is analytically reasonable, but it has a geometric cost.

The remaining low-frequency region is

$$|t| \leq R_D(a), \qquad R_D(a) = \log^D(e + a^{-1}).$$

As $a \to 0^+$, this region expands:

$$R_D(a) \to \infty.$$

Therefore one cannot control $\mathcal{C}_D(a)$ merely from local regularity of H_0 near $s = 1/2$.

Definition 9.1 (Enlarged low strip)**.** *For $D > 0$, define*

$$\mathcal{K}_D = \left\{ \frac{1}{2} + a + it : 0 < a < \frac{1}{2},\ |t| \leq \log^D(e + a^{-1}) \right\}.$$

Proposition 9.2 (Zero-exclusion forced by finite enlarged low energy)**.** *Let $0 < a < 1/2$. If*

$$\mathcal{C}_D(a) < \infty,$$

then

$$\zeta\left(\frac{1}{2} + a + it\right) \neq 0 \qquad (|t| \leq R_D(a)).$$

Equivalently, if there exists a zero

$$\rho = \frac{1}{2} + a + i\gamma$$

with

$$|\gamma| \le R_D(a),$$

then

$$\mathcal{C}_D(a) = +\infty.$$

Proof. Suppose that

$$\rho = \frac{1}{2} + a + i\gamma$$

is a zero of $\zeta(s)$ with

$$|\gamma| \le R_D(a).$$

Then

$$H_a(t) = H_0\left(\frac{1}{2} + a + it\right)$$

has a simple pole at $t = \gamma$, with nonzero principal part. Hence, near $t = \gamma$,

$$|H_a(t)|^2 \asymp \frac{1}{(t-\gamma)^2}.$$

If $\gamma \neq 0$, then

$$W_a(\gamma) = \frac{a\gamma^2}{(a^2+\gamma^2)^2} > 0.$$

Thus, for sufficiently small $\delta > 0$,

$$W_a(t) \gg W_a(\gamma) > 0 \qquad (|t-\gamma| < \delta),$$

and so

$$\int_{|t-\gamma|<\delta} W_a(t)|H_a(t)|^2 \, dt \gg \int_{|t-\gamma|<\delta} \frac{dt}{(t-\gamma)^2} = +\infty.$$

Therefore $\mathcal{C}_D(a) = +\infty$.

If $\gamma = 0$, then the zero would lie at the real point

$$s = \frac{1}{2} + a, \qquad 0 < s < 1.$$

It is classical that $\zeta(s) \neq 0$ for $0 < s < 1$. Hence this case cannot occur.

Thus any zero of $\zeta(s)$ on the segment

$$\left\{\frac{1}{2} + a + it : \ |t| \le R_D(a)\right\}$$

forces $\mathcal{C}_D(a) = +\infty$. The contrapositive gives the zero-exclusion statement. □

Remark 9.3. *This proposition should be read as a necessary zero-exclusion condition for finite enlarged low energy:*

$$\mathcal{C}_D(a) < \infty \quad \Rightarrow \quad \textit{no zeros on the corresponding vertical segment.}$$

It is not a proof of any zero-free region. It identifies the obstruction.

10 Why the enlarged region is not a shrinking cone

A shrinking low-frequency cone has the form

$$|t| \le aR(a), \qquad aR(a) \to 0.$$

In that case the region

$$\frac{1}{2} + a + it$$

approaches 1/2, and the known fact

$$\zeta(1/2) \neq 0$$

can imply local boundedness of H_0 for sufficiently small a.

The enlarged $M = 1$ low region is different:

$$|t| \le R_D(a), \qquad R_D(a) = \log^D(e + a^{-1}).$$

This does not approach $t = 0$. Therefore local holomorphy of H_0 near $s = 1/2$ gives no uniform control of the entire region.

Proposition 10.1 (Failure of shrinking-cone localization)**.** *Let*

$$R_D(a) = \log^D(e + a^{-1}).$$

Then

$$R_D(a) \to \infty \qquad (a \to 0^+).$$

In particular, the region

$$\left\{ \frac{1}{2} + a + it : \ |t| \le R_D(a) \right\}$$

does not shrink to $s = 1/2$.

Proof. This follows immediately from

$$\log(e + a^{-1}) \to \infty$$

as $a \to 0^+$. □

Remark 10.2 (Comparison with shrinking-cone notation)**.** *A shrinking low-cone condition is usually written in the form*

$$|t| \le aR(a)$$

with

$$aR(a) \to 0.$$

The enlarged low region in the present $M = 1$ framework is instead

$$|t| \le R_D(a), \qquad R_D(a) = \log^D(e + a^{-1}).$$

In shrinking-cone notation this would correspond to

$$aR(a) = R_D(a), \qquad R(a) = \frac{R_D(a)}{a}.$$

Hence

$$aR(a) = R_D(a) \to \infty,$$

not 0*. Therefore shrinking-cone local regularity near $s = 1/2$ does not apply to the enlarged $M = 1$ low region.*

Remark 10.3. *This is the geometric cost of accepting $M = 1$. The high-frequency tail is cleared, but the remaining low-frequency region is no longer local.*

11 Rebalanced criterion at the $M = 1$ scale

Combining the preceding results gives a clean conditional criterion.

Theorem 11.1 ($M = 1$-admissible spectral reduction)**.** *Assume the $M = 1$ dyadic mean-square input:*

$$\int_{T<|t|\leq 2T} |H_a(t)|^2\, dt \ll Ta^{-1} \log^B(2 + T + a^{-1})$$

uniformly for $0 < a < 1/2$ and $T \geq 1$. Let $D > B$. Then

$$\sup_{0<a<1/2} \mathcal{G}_{\mathrm{spec}}(a) < \infty$$

if and only if

$$\sup_{0<a<1/2} \mathcal{C}_D(a) < \infty.$$

Moreover, finite enlarged low energy forces the zero-exclusion condition

$$\zeta\left(\frac{1}{2} + a + it\right) \neq 0 \qquad (|t| \leq R_D(a)).$$

Proof. The conditional equivalence of $\mathcal{G}_{\mathrm{spec}}$ and $\mathcal{C}_D$ follows from the $M = 1$-conditional reduction to the enlarged low energy. The zero-exclusion condition follows from the preceding proposition. □

Remark 11.2. *The theorem should not be read as a proof of RH. It says that at the natural $M = 1$ scale, the high-frequency tail is not the main obstacle. The obstacle is the enlarged low-frequency energy, which includes a zero-exclusion requirement on a growing strip.*

12 Interpretation

The $M = 1$ framework gives a balanced picture.

$$\text{Mean-square input at } M = 1 \quad \Longrightarrow \quad \text{high-frequency tail cleared.}$$

But the remaining low-frequency region is

$$|t| \leq \log^D(e + a^{-1}),$$

which grows as $a \to 0^+$. Thus

$$M = 1 \quad \text{is analytically natural}$$

because it matches the unsmoothed vertical packet scale, but

$$M = 1 \quad \text{is geometrically expensive}$$

because it leaves a large low-frequency region.

By contrast, an $M < 1$ input is analytically stronger but geometrically more favorable: the remaining low-frequency region shrinks.

Input scale	High-frequency effect	Remaining region
$M < 1$	clears tail	shrinking low region
$M = 1$	clears tail	enlarged logarithmic strip

13 Limitations

The paper has several limitations.

First, it does not prove the $M = 1$ dyadic mean-square input. That input remains a serious zero-sensitive analytic estimate for ζ'/ζ.

Second, it does not prove boundedness of the enlarged low-frequency energy

$$\mathcal{C}_D(a).$$

This low energy may contain poles from hypothetical zeros off the critical line.

Third, it does not prove RH. The paper only shows that the natural $M = 1$ scale is compatible with high-frequency clearance and identifies the enlarged low-frequency obstruction.

Fourth, this paper treats $\mathcal{G}_{\text{spec}}(a)$ as a spectral target quantity. It does not, by itself, establish that

$$\sup_{0<a<1/2} \mathcal{G}_{\text{spec}}(a) < \infty$$

is equivalent to RH, or that it implies RH. Such a conclusion would require an independent spectral criterion.

Fifth, the logarithmic exponent $D > B$ is not optimized. The purpose is structural rather than sharp.

Sixth, the analysis uses dyadic shell estimates. Passing from global translated mean-square bounds to such dyadic estimates is standard but still an input.

14 Conclusion

The natural diagonal packet scale for the unsmoothed vertical logarithmic derivative

$$\frac{\zeta'}{\zeta}\left(\frac{1}{2} + a + it\right)$$

near the critical line is

$$Ta^{-1} \log T,$$

corresponding to $M = 1$. This paper examined the dyadic spectral framework at that natural scale.

The key calculation is that on a high shell $T < |t| \le 2T$,

$$W_a(t) \asymp \frac{a}{T^2}.$$

Thus an $M = 1$ mean-square input gives shell contribution

$$\frac{a}{T^2} \cdot Ta^{-1} \log^B(2 + T + a^{-1}) = \frac{\log^B(2 + T + a^{-1})}{T}.$$

The dyadic sum of these contributions is uniformly bounded once

$$T \ge \log^D(e + a^{-1})$$

with $D > B$.

Therefore accepting $M = 1$ does not destroy the high-frequency reduction. It clears the dyadic tail. The price is that the remaining low-frequency region becomes

$$|t| \le \log^D(e + a^{-1}),$$

which grows as $a \to 0^+$. In shrinking-cone notation this region would correspond to $aR(a) \to \infty$, not $aR(a) \to 0$. Local regularity near $s = 1/2$ therefore no longer suffices to control it.

Thus the result is a conditional spectral reduction, not an RH criterion by itself.

The final message is:

$$\boxed{M = 1 \text{ is analytically natural but geometrically expensive.}}$$

The high-frequency barrier moves into an enlarged low-frequency obstruction.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

[2] J. B. Conrey, The Riemann Hypothesis, *Notices Amer. Math. Soc.* **50** (2003), no. 3, 341–353.

[3] H. Davenport, *Multiplicative Number Theory*, 3rd ed., revised by H. L. Montgomery, Graduate Texts in Mathematics, Vol. 74, Springer, New York, 2000.

[4] H. M. Edwards, *Riemann's Zeta Function*, Academic Press, New York, 1974.

[5] S. M. Gonek, Mean values of the Riemann zeta-function and its derivatives, *Invent. Math.* **75** (1984), 123–141.

[6] D. A. Goldston, On the function $S(T)$ in the theory of the Riemann zeta-function, *J. Number Theory* **27** (1987), 149–177.

[7] G. H. Hardy, Sur les zéros de la fonction $\zeta(s)$ de Riemann, *C. R. Acad. Sci. Paris* **158** (1914), 1012–1014.

[8] A. E. Ingham, *The Distribution of Prime Numbers*, Cambridge University Press, Cambridge, 1932.

[9] A. Ivić, *The Riemann Zeta-Function: Theory and Applications*, Dover Publications, Mineola, NY, 2003.

[10] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, Vol. 53, American Mathematical Society, Providence, RI, 2004.

[11] H. L. Montgomery, The pair correlation of zeros of the zeta function, *Analytic Number Theory, Proc. Sympos. Pure Math.* **24** (1973), 181–193.

[12] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge Studies in Advanced Mathematics, Vol. 97, Cambridge University Press, Cambridge, 2007.

[13] B. Riemann, Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monatsberichte der Berliner Akademie* (1859), 671–680.

[14] A. Selberg, Contributions to the theory of the Riemann zeta-function, *Arch. Math. Naturvid.* **48** (1946), 89–155.

[15] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Clarendon Press, Oxford, 1986.