The Shifted-Contour Remainder in the Smoothed High-Frequency Barrier

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Abstract

Let

$$H_0(s) = -\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}$$

be the logarithmic derivative of the Riemann zeta-function with the pole at $s = 1$ removed. In a smoothed Perron decomposition of a high-frequency remainder one obtains a shifted-contour term

$$\mathcal{R}_{X,K,A}(s) = -\frac{1}{2\pi i}\int_{(-A)} \left(-\frac{\zeta'}{\zeta}(s+w)\right) X^w K(w)\, dw.$$

This paper analyzes the structure of this term.

For

$$s = \frac{1}{2} + a + it, \qquad 0 < a < \frac{1}{2},$$

and $w = -A + i\tau$, the shifted line is

$$s + w = \frac{1}{2} + a - A + i(t+\tau).$$

Thus the shifted-contour remainder is a vertical convolution of $-\zeta'/\zeta$ on the line

$$\sigma_A = \frac{1}{2} + a - A.$$

More precisely, if

$$k_A(\tau) = K(-A + i\tau), \qquad f_\sigma(u) = -\frac{\zeta'}{\zeta}(\sigma + iu),$$

then

$$\mathcal{R}_{X,K,A}\left(\frac{1}{2} + a + it\right) = -\frac{X^{-A}}{2\pi}\int_{-\infty}^{\infty} f_{\sigma_A}(t+\tau) e^{i\tau \log X} k_A(\tau)\, d\tau.$$

Consequently, Young–Minkowski inequalities reduce estimates for $\mathcal{R}_{X,K,A}$ to translated vertical mean-square estimates for $-\zeta'/\zeta$ on the shifted line.

The paper separates three regimes. If $0 < A < a$, then $\sigma_A > 1/2$, and the contour remainder is controlled by a right-side mean-square estimate for ζ'/ζ. If $A = a$, then $\sigma_A = 1/2$, and the shifted line passes through critical-line zeros; in this regime the direct L^2-contour formulation is singular. If $A > a$, then $\sigma_A < 1/2$. In the near-left regime $0 < A - a < 1/2$, the functional equation transfers $-\zeta'/\zeta(\sigma_A + iu)$ to a logarithmic derivative on the right-side line $1 - \sigma_A = 1/2 + A - a$, plus gamma and trigonometric terms. The reflected argument $u \mapsto -u$ is handled explicitly using the symmetry

$$\overline{\zeta'/\zeta(\sigma + iu)} = \zeta'/\zeta(\sigma - iu).$$

The gamma and trigonometric terms are controlled by Stirling's formula and contribute only $O(T \log^2 T)$ in mean square on intervals of length T, after convolution against the rapidly decaying kernel.

Thus the shifted-contour term is not an independent obstruction: it reduces to translated vertical mean-square estimates for ζ'/ζ on lines to the right of the critical line, together with harmless gamma-factor terms. The paper does not prove RH and does not prove the required right-side mean-square estimates. It gives a structural reduction of the shifted-contour regularity condition to the same type of vertical mean-square problem appearing elsewhere in the high-frequency barrier.

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Contents

1 Introduction

Let

$$H_0(s) = -\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}.$$

The poles of H_0 are precisely the zeros of $\zeta(s)$, since the pole at $s = 1$ has been removed.

In a smoothed Perron framework, one studies remainders of the form

$$E_{X,K}(s) = H_0(s) - P_{X,K}(s),$$

where $P_{X,K}$ is a smoothed prime sum. Shifting the Perron contour gives a decomposition

$$E_{X,K} = \mathcal{Z} + \mathcal{T} + \mathcal{M} + \mathcal{R}.$$

Here $\mathcal{Z}$ is the nontrivial-zero packet term, $\mathcal{T}$ is the trivial-zero contribution, $\mathcal{M}$ is the main-pole correction, and $\mathcal{R}$ is the shifted-contour remainder.

The trivial-zero and main-pole terms can be estimated directly for admissible kernels with rapid vertical decay. The nontrivial-zero term is the zero-packet Gram obstruction. The shifted-contour term is more subtle. It is

$$\mathcal{R}_{X,K,A}(s) = -\frac{1}{2\pi i}\int_{(-A)} \left(-\frac{\zeta'}{\zeta}(s+w)\right) X^w K(w)\,dw.$$

This paper studies this term.

Let

$$s = \frac{1}{2} + a + it, \qquad 0 < a < \frac{1}{2},$$

and let

$$w = -A + i\tau.$$

Then

$$s + w = \frac{1}{2} + a - A + i(t+\tau).$$

Thus the shifted-contour remainder is a weighted average of $-\zeta'/\zeta$ on the vertical line

$$\sigma_A = \frac{1}{2} + a - A.$$

The averaging kernel is $K(-A+i\tau)$, and the contour shift supplies a factor X^{-A}.

The central observation is:

$$\boxed{\mathcal{R}_{X,K,A} = X^{-A} \times \text{vertical convolution of } -\zeta'/\zeta \text{ on } \operatorname{Re} s = \sigma_A.}$$

This reduces L^2-bounds for $\mathcal{R}$ to translated vertical mean-square bounds for $-\zeta'/\zeta$.

There are three regimes.

(i) If $0 < A < a$, then $\sigma_A > 1/2$. This is a right-side shifted contour.

(ii) If $A = a$, then $\sigma_A = 1/2$. This is the singular critical-line contour.

(iii) If $A > a$, then $\sigma_A < 1/2$. This is a left-side shifted contour. In the near-left range $0 < A - a < 1/2$, the functional equation transfers the problem to the right-side line $1 - \sigma_A = 1/2 + A - a$, plus explicit gamma and trigonometric terms.

The outcome is conditional but useful: shifted-contour regularity is reducible to translated vertical mean-square estimates for ζ'/ζ on lines to the right of the critical line, plus harmless gamma-factor contributions.

The paper does not prove RH. It also does not prove the needed right-side vertical mean-square estimates. It clarifies the analytic structure of the shifted-contour remainder.

2 The shifted-contour remainder

Let $K(w)$ be a meromorphic smoothing kernel with rapid decay on vertical lines. For $A > 0$, define

$$\mathcal{R}_{X,K,A}(s) = -\frac{1}{2\pi i}\int_{(-A)} \left(-\frac{\zeta'}{\zeta}(s+w)\right) X^w K(w)\,dw.$$

We write

$$w = -A + i\tau, \qquad dw = i\,d\tau.$$

Then

$$\mathcal{R}_{X,K,A}(s) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-\frac{\zeta'}{\zeta}(s-A+i\tau)\right)X^{-A+i\tau}K(-A+i\tau)\,d\tau.$$

Since

$$X^{-A+i\tau} = X^{-A}e^{i\tau\log X},$$

we have

$$|\mathcal{R}_{X,K,A}(s)| \le \frac{X^{-A}}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\zeta'}{\zeta}(s-A+i\tau)\right||K(-A+i\tau)|\,d\tau.$$

Now take

$$s = \frac{1}{2}+a+it.$$

Set

$$\sigma_A = \frac{1}{2}+a-A.$$

Then

$$s-A+i\tau = \sigma_A + i(t+\tau).$$

Define

$$f_\sigma(u) = -\frac{\zeta'}{\zeta}(\sigma+iu),$$

and

$$k_A(\tau) = e^{i\tau\log X}K(-A+i\tau).$$

Since the exponential factor has modulus one,

$$|k_A(\tau)| = |K(-A+i\tau)|.$$

Then

$$\boxed{\mathcal{R}_{X,K,A}\left(\frac{1}{2}+a+it\right) = -\frac{X^{-A}}{2\pi}\int_{-\infty}^{\infty}f_{\sigma_A}(t+\tau)k_A(\tau)\,d\tau.}$$

Remark 2.1. *The factor $e^{i\tau\log X}$ changes only the phase of the convolution kernel. It is irrelevant for L^2-upper bounds based on absolute values.*

3 Convolution inequalities

Let $I=[T,2T]$. For a locally square-integrable function f and an integrable kernel k, define

$$(C_kf)(t) = \int_{\mathbb{R}} f(t+\tau)k(\tau)\,d\tau.$$

Lemma 3.1 (Localized Young–Minkowski inequality)**.** *Let $k\in L^1(\mathbb{R})$. Then*

$$\boxed{\|C_kf\|_{L^2(I)} \le \int_{\mathbb{R}}|k(\tau)|\,\|f\|_{L^2(I+\tau)}\,d\tau,}$$

where

$$I+\tau = \{t+\tau : t\in I\}.$$

Proof. By Minkowski's integral inequality,

$$\begin{aligned}\|C_k f\|_{L^2(I)} &= \left(\int_I \left|\int_{\mathbb{R}} f(t+\tau)k(\tau)\,d\tau\right|^2 dt\right)^{1/2}\\ &\le \int_{\mathbb{R}} |k(\tau)| \left(\int_I |f(t+\tau)|^2 dt\right)^{1/2} d\tau\\ &= \int_{\mathbb{R}} |k(\tau)|\,\|f\|_{L^2(I+\tau)}\,d\tau.\end{aligned}$$

□

For later use we introduce logarithmic kernel moments.

Definition 3.2 (Logarithmic kernel moment)**.** *For $r \ge 0$, define*

$$\mathcal{K}_{A,r} = \int_{-\infty}^{\infty} |K(-A+i\tau)|\,(1+\log(2+|\tau|))^r\,d\tau.$$

If K has rapid vertical decay on the line $\operatorname{Re} w = -A$, then

$$\mathcal{K}_{A,r} < \infty$$

for every fixed $r \ge 0$.

Applying the convolution inequality to the shifted-contour remainder gives the following.

Theorem 3.3 (Convolution bound for the shifted contour)**.** *Let*

$$I = [T, 2T], \qquad \sigma_A = \frac{1}{2} + a - A.$$

Assume $K(-A+i\tau) \in L^1(\mathbb{R})$. Then

$$\boxed{\left\|\mathcal{R}_{X,K,A}\left(\frac{1}{2}+a+i\cdot\right)\right\|_{L^2(I)} \le \frac{X^{-A}}{2\pi}\int_{\mathbb{R}} |K(-A+i\tau)|\left\|\frac{\zeta'}{\zeta}(\sigma_A+i\cdot)\right\|_{L^2(I+\tau)} d\tau.}$$

Proof. Use the convolution representation and the localized Young–Minkowski inequality. □

Remark 3.4. *This theorem is the basic reduction. It does not assume RH. If the line $\operatorname{Re} s = \sigma_A$ contains a zero of $\zeta(s)$, then the right-hand side is infinite; the theorem correctly records this obstruction.*

4 Translated vertical mean-square inputs

The convolution bound becomes useful once one has vertical mean-square estimates for ζ'/ζ on intervals that may be translated by τ. We therefore state the input in a form that is uniform in the position of the interval.

For an interval $J \subset \mathbb{R}$, define

$$H(J) = 2 + \sup_{u \in J} |u|.$$

Hypothesis 4.1 (Uniform translated vertical mean-square bound)**.** *Let $\sigma > 1/2$, and put*

$$d(\sigma) = \sigma - \frac{1}{2}.$$

We say that a uniform translated vertical mean-square bound holds with loss parameters M, B if, for every interval $J \subset \mathbb{R}$ of length $T \geq 2$,

$$\boxed{\int_J \left|\frac{\zeta'}{\zeta}(\sigma + iu)\right|^2 du \ll T\, d(\sigma)^{-M} \log^B\big(H(J) + d(\sigma)^{-1}\big)\,.}$$

The implied constant is independent of the position of J.

Remark 4.2. *This is a translated estimate. It is stronger than a bound stated only for $J = [T, 2T]$, but it is exactly the form needed after convolution, because $I + \tau = [T + \tau, 2T + \tau]$ moves with τ.*

Lemma 4.3 (Logarithmic comparison under translation)**.** *Let $I = [T, 2T]$. For every $\tau \in \mathbb{R}$ and $d > 0$,*

$$\log\big(H(I + \tau) + d^{-1}\big) \ll \log(2 + T + d^{-1}) + \log(2 + |\tau|).$$

Consequently, for every $r \geq 0$,

$$\log^r\big(H(I + \tau) + d^{-1}\big) \ll_r \log^r(2 + T + d^{-1})\,(1 + \log(2 + |\tau|))^r\,.$$

Proof. Since $I + \tau = [T + \tau, 2T + \tau]$,

$$H(I + \tau) \leq 2 + 2T + |\tau|.$$

Therefore

$$H(I + \tau) + d^{-1} \leq (2 + 2T + d^{-1})(2 + |\tau|).$$

Taking logarithms gives the first estimate. The second follows from

$$(x + y)^r \ll_r x^r(1 + y)^r$$

for $x, y \geq 0$, after increasing constants. $\square$

Corollary 4.4 (Contour regularity from translated mean square)**.** *Assume $\sigma_A > 1/2$, and suppose the uniform translated vertical mean-square bound holds on the line* $\operatorname{Re} s = \sigma_A$*. Put*

$$d_A = \sigma_A - \frac{1}{2}.$$

If $K(-A + i\tau)$ has finite logarithmic moments, then

$$\boxed{\int_T^{2T} |\mathcal{R}_{X,K,A}(1/2 + a + it)|^2 dt \ll X^{-2A} T d_A^{-M} \log^B(2 + T + d_A^{-1}).}$$

Proof. The translated mean-square bound gives, for $I = [T, 2T]$,

$$\left\|\frac{\zeta'}{\zeta}(\sigma_A + i\cdot)\right\|_{L^2(I+\tau)} \ll T^{1/2} d_A^{-M/2} \log^{B/2}\big(H(I + \tau) + d_A^{-1}\big)\,.$$

Using the logarithmic comparison lemma, we obtain

$$\int_{\mathbb{R}} |K(-A + i\tau)| \left\|\frac{\zeta'}{\zeta}(\sigma_A + i\cdot)\right\|_{L^2(I+\tau)} d\tau$$
$$\ll T^{1/2} d_A^{-M/2} \log^{B/2}(2 + T + d_A^{-1}) \mathcal{K}_{A,B/2}.$$

The convolution bound and the finiteness of $\mathcal{K}_{A,B/2}$ give

$$\|\mathcal{R}_{X,K,A}\|_{L^2(I)} \ll X^{-A} T^{1/2} d_A^{-M/2} \log^{B/2}(2 + T + d_A^{-1}).$$

Squaring proves the estimate. $\square$

5 The right-side regime $0 < A < a$

If

$$0 < A < a,$$

then

$$\sigma_A = \frac{1}{2} + a - A > \frac{1}{2}.$$

This is the right-side shifted-contour regime.

Let

$$d_A = a - A > 0.$$

Then

$$\sigma_A - \frac{1}{2} = d_A.$$

Theorem 5.1 (Right-side shifted-contour criterion)**.** *Assume $0 < A < a$. Suppose the uniform translated vertical mean-square bound holds on the line*

$$\operatorname{Re} s = \frac{1}{2} + d_A.$$

Then

$$\boxed{\int_T^{2T} |\mathcal{R}_{X,K,A}(1/2 + a + it)|^2 dt \ll X^{-2A} T d_A^{-M} \log^B(2 + T + d_A^{-1}).}$$

Proof. Apply the previous corollary with

$$\sigma_A = \frac{1}{2} + d_A.$$

$\square$

Corollary 5.2 (Comparable right shift)**.** *Let $A = \eta a$, where $0 < \eta < 1$ is fixed. Then*

$$d_A = (1 - \eta)a.$$

Under the same translated vertical mean-square input,

$$\boxed{\int_T^{2T} |\mathcal{R}_{X,K,A}(1/2 + a + it)|^2 dt \ll X^{-2A} T a^{-M} \log^B(2 + T + a^{-1}).}$$

Proof. Since $d_A \asymp a$, the loss

$$d_A^{-M} \log^B(2 + T + d_A^{-1})$$

is comparable to

$$a^{-M} \log^B(2 + T + a^{-1}).$$

$\square$

Remark 5.3. *The factor $X^{-2A} \leq 1$ is favorable. The essential input is the translated vertical mean-square estimate for ζ'/ζ on the right-side line $\operatorname{Re} s = 1/2 + d_A$.*

6 The singular regime $A = a$

If

$$A = a,$$

then

$$\sigma_A = \frac{1}{2}.$$

The shifted line is the critical line.

This regime is singular. If $\zeta(s)$ has a zero on the critical line, then $-\zeta'/\zeta(s)$ has a pole on the line.

Lemma 6.1 (Critical-line poles produce L^2-divergence)**.** *Suppose $\zeta(s)$ has a zero*

$$\rho = \frac{1}{2} + i\gamma$$

of multiplicity m_ρ. Then

$$-\frac{\zeta'}{\zeta}\left(\frac{1}{2} + iu\right) = -\frac{m_\rho}{i(u-\gamma)} + O(1)$$

near $u = \gamma$. Consequently,

$$\int_{\gamma-\delta}^{\gamma+\delta} \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2} + iu\right)\right|^2 du = +\infty$$

for every $\delta > 0$.

Proof. Near a zero ρ of multiplicity m_ρ,

$$\frac{\zeta'}{\zeta}(s) = \frac{m_\rho}{s-\rho} + O(1).$$

Putting $s = 1/2 + iu$ gives

$$s - \rho = i(u-\gamma).$$

Thus

$$\left|\frac{\zeta'}{\zeta}\left(\frac{1}{2} + iu\right)\right|^2 \asymp \frac{m_\rho^2}{(u-\gamma)^2}$$

near $u = \gamma$, and the integral diverges. □

Proposition 6.2 (The critical shifted contour should be avoided)**.** *The choice $A = a$ cannot be used as an ordinary L^2-regular shifted contour unless the critical-line poles are detoured or otherwise regularized. In the direct vertical-line formulation, it is singular.*

Proof. Hardy proved that $\zeta(s)$ has infinitely many zeros on the critical line. At each such zero, the preceding lemma gives local L^2-divergence of ζ'/ζ on the line. □

Remark 6.3. *The conclusion is not that contour methods fail. Rather, the contour must avoid poles by indentation, smoothing, or by choosing $A \neq a$. The direct line $A = a$ is not a regular L^2-contour.*

7 Reflection symmetry

The left-side transfer uses both the functional equation and reflection symmetry. We record the latter explicitly.

Lemma 7.1 (Reflection symmetry)**.** *For real σ and real u, away from zeros and poles of ζ,*

$$\boxed{\overline{\frac{\zeta'}{\zeta}(\sigma+iu)}=\frac{\zeta'}{\zeta}(\sigma-iu).}$$

Consequently,

$$\boxed{\left|\frac{\zeta'}{\zeta}(\sigma-iu)\right|=\left|\frac{\zeta'}{\zeta}(\sigma+iu)\right|.}$$

Proof. The Dirichlet series for $\zeta(s)$ has real coefficients in its half-plane of absolute convergence, so

$$\overline{\zeta(s)}=\zeta(\overline{s})$$

there. By analytic continuation the identity holds throughout the domain where both sides are defined. Differentiating gives

$$\overline{\zeta'(s)}=\zeta'(\overline{s}).$$

Dividing by the corresponding identity for ζ proves

$$\overline{\frac{\zeta'}{\zeta}(s)}=\frac{\zeta'}{\zeta}(\overline{s}).$$

Taking $s=\sigma+iu$ gives the result. □

Corollary 7.2 (Reflected interval control)**.** *Assume the uniform translated vertical mean-square bound holds on the line* $\operatorname{Re}s=\sigma>1/2$*. Then it also controls*

$$\frac{\zeta'}{\zeta}(\sigma-iu)$$

on every interval J of length T:

$$\int_J\left|\frac{\zeta'}{\zeta}(\sigma-iu)\right|^2du\ll T\,d(\sigma)^{-M}\log^B\bigl(H(J)+d(\sigma)^{-1}\bigr)\,.$$

Proof. By the reflection symmetry,

$$\left|\frac{\zeta'}{\zeta}(\sigma-iu)\right|=\left|\frac{\zeta'}{\zeta}(\sigma+iu)\right|.$$

Alternatively, after the change of variables $v=-u$, the interval J is reflected to $-J$, which has the same length. Since the translated mean-square bound is uniform over all intervals of length T, the estimate follows. □

8 The left-side regime $A>a$

Assume now that

$$A>a.$$

Set

$$\alpha=A-a>0.$$

Then

$$\sigma_A = \frac{1}{2} - \alpha.$$

We focus on the near-left range

$$0 < \alpha < \frac{1}{2}.$$

Then

$$0 < \sigma_A < \frac{1}{2}.$$

This avoids the trivial zeros on the real axis and permits a clean use of the functional equation.

The functional equation may be written as

$$\zeta(s) = \chi(s)\zeta(1-s),$$

where

$$\chi(s) = 2^s \pi^{s-1} \sin\Big(\frac{\pi s}{2}\Big)\,\Gamma(1-s).$$

Taking logarithmic derivatives gives

$$\frac{\zeta'}{\zeta}(s) = \frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(1-s).$$

Hence

$$\boxed{-\frac{\zeta'}{\zeta}(s) = -\frac{\chi'}{\chi}(s) + \frac{\zeta'}{\zeta}(1-s).}$$

For

$$s = \frac{1}{2} - \alpha + iu,$$

we have

$$1 - s = \frac{1}{2} + \alpha - iu.$$

Thus the logarithmic derivative on the left-side line is expressed in terms of a logarithmic derivative on the right-side line

$$\operatorname{Re} s = \frac{1}{2} + \alpha$$

with the reflected argument $-u$, plus the explicit gamma-trigonometric factor

$$\frac{\chi'}{\chi}(s).$$

9 Gamma and trigonometric factor estimates

We now estimate χ'/χ.

Since

$$\chi(s) = 2^s \pi^{s-1} \sin\Big(\frac{\pi s}{2}\Big)\,\Gamma(1-s),$$

we have

$$\boxed{\frac{\chi'}{\chi}(s) = \log(2\pi) + \frac{\pi}{2}\cot\Big(\frac{\pi s}{2}\Big) - \frac{\Gamma'}{\Gamma}(1-s).}$$

Lemma 9.1 (Cotangent bound on the near-left line)**.** *Let $0 < \alpha < 1/2$. On the line*

$$s = \frac{1}{2} - \alpha + iu,$$

the cotangent term satisfies

$$\cot\Big(\frac{\pi s}{2}\Big) \ll_\alpha 1.$$

The bound is uniform when α ranges in a compact subinterval of $(0, 1/2)$. It is also uniform for $0 < \alpha \le \alpha_0 < 1/2$.

Proof. The poles of $\cot(\pi s/2)$ occur at even integers. The line

$$\operatorname{Re} s = \frac{1}{2} - \alpha$$

stays away from these poles when $0 < \alpha < 1/2$, except that the constant may deteriorate as $\alpha \to 1/2$, where the line approaches $\operatorname{Re} s = 0$. For fixed α, the function is continuous for $|u| \le 1$, hence bounded there. For $|u| \to \infty$, the standard formula for $\cot(x + iy)$ shows that it tends to $-i\ \mathrm{sgn}(y)$, and is therefore bounded. This proves the claim. □

Lemma 9.2 (Gamma-factor bound)**.** *Let $0 < \alpha < 1/2$. Uniformly for*

$$s = \frac{1}{2} - \alpha + iu,$$

one has

$$\boxed{\frac{\chi'}{\chi}(s) \ll_\alpha \log(2 + |u|).}$$

The bound is uniform when α ranges in a compact subinterval of $(0, 1/2)$, and also for $0 < \alpha \le \alpha_0 < 1/2$.

Proof. The term $\log(2\pi)$ is constant. The cotangent term is bounded by the preceding lemma.

For the gamma term, put

$$z = 1 - s = \frac{1}{2} + \alpha - iu.$$

This lies in a fixed vertical strip with positive real part. Stirling's formula for the logarithmic derivative of the gamma function gives

$$\frac{\Gamma'}{\Gamma}(z) = \log z + O(|z|^{-1})$$

uniformly in such vertical strips, with the principal branch of the logarithm. Since $|\arg z| \le \pi$, one has

$$|\log z| \le \log(2 + |z|) + O(1) \ll \log(2 + |u|).$$

Therefore

$$\frac{\Gamma'}{\Gamma}(1 - s) \ll_\alpha \log(2 + |u|).$$

Combining the estimates proves the lemma. □

Corollary 9.3 (Translated mean square of the gamma factor)**.** *Let $J \subset \mathbb{R}$ be an interval of length $T \ge 2$, and let $0 < \alpha < 1/2$. Then*

$$\boxed{\int_J \left|\frac{\chi'}{\chi}\Big(\frac{1}{2} - \alpha + iu\Big)\right|^2 du \ll_\alpha T \log^2 H(J).}$$

The bound is uniform in α in the same ranges as above.

Proof. Use the pointwise bound

$$\left|\frac{\chi'}{\chi}\Big(\frac{1}{2} - \alpha + iu\Big)\right| \ll_\alpha \log(2 + |u|) \le \log H(J)$$

for $u \in J$, and integrate over an interval of length T. □

10 Functional-equation transfer for $A > a$

We now transfer the left-side shifted contour to the right side.

Let

$$A > a, \qquad \alpha = A - a, \qquad 0 < \alpha < \frac{1}{2}.$$

Then

$$\sigma_A = \frac{1}{2} - \alpha.$$

Define

$$g_\alpha(u) = -\frac{\zeta'}{\zeta}\left(\frac{1}{2} - \alpha + iu\right).$$

By the functional equation,

$$g_\alpha(u) = -\frac{\chi'}{\chi}\left(\frac{1}{2} - \alpha + iu\right) + \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \alpha - iu\right).$$

Taking absolute values and using reflection symmetry,

$$|g_\alpha(u)| \le \left|\frac{\chi'}{\chi}\left(\frac{1}{2} - \alpha + iu\right)\right| + \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2} + \alpha + iu\right)\right|.$$

More precisely, for intervals, the second term is controlled by reflecting $u \mapsto -u$, as in the reflection lemma.

Theorem 10.1 (Left-side contour transfer)**.** *Assume*

$$A > a, \qquad \alpha = A - a, \qquad 0 < \alpha < \frac{1}{2}.$$

Suppose a uniform translated vertical mean-square bound holds for

$$\frac{\zeta'}{\zeta}\left(\frac{1}{2} + \alpha + iv\right).$$

Then

$$\boxed{\int_T^{2T} |\mathcal{R}_{X,K,A}(1/2 + a + it)|^2 dt \ll X^{-2A} T \left[\alpha^{-M} \log^B(2 + T + \alpha^{-1}) + \log^2(2 + T)\right].}$$

The logarithmic powers may be enlarged by constants depending on the logarithmic moments of K.

Proof. By the convolution bound, it is enough to estimate local L^2-norms on translated intervals $I + \tau$ of

$$-\frac{\zeta'}{\zeta}\left(\frac{1}{2} - \alpha + iu\right).$$

Using the functional equation,

$$-\frac{\zeta'}{\zeta}\left(\frac{1}{2} - \alpha + iu\right) = -\frac{\chi'}{\chi}\left(\frac{1}{2} - \alpha + iu\right) + \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \alpha - iu\right).$$

The reflected logarithmic derivative is controlled by the uniform translated mean-square input on the right-side line, because by reflection symmetry

$$\left|\frac{\zeta'}{\zeta}\left(\frac{1}{2} + \alpha - iu\right)\right| = \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2} + \alpha + iu\right)\right|,$$

and after $v = -u$ the interval $I + \tau$ is reflected to another interval of the same length.

The gamma-trigonometric factor contributes, on every translated interval J of length T,

$$\ll T \log^2 H(J).$$

Using the logarithmic comparison lemma and the finite logarithmic moments of K, the convolution integral gives

$$\|\mathcal{R}_{X,K,A}\|_{L^2(I)}^2 \ll X^{-2A} T \left[\alpha^{-M} \log^B(2 + T + \alpha^{-1}) + \log^2(2 + T)\right],$$

after enlarging the implied constants and logarithmic exponents if necessary. □

Corollary 10.2 (Comparable left shift)**.** *If*

$$A = a + \eta a = (1 + \eta)a$$

with fixed $0 < \eta < 1$*, then*

$$\alpha = A - a = \eta a.$$

Under the same translated vertical mean-square input,

$$\boxed{\int_T^{2T} |\mathcal{R}_{X,K,A}(1/2 + a + it)|^2 dt \ll X^{-2A} T \left[a^{-M} \log^B(2 + T + a^{-1}) + \log^2(2 + T)\right].}$$

Proof. Since $\alpha \asymp a$, the loss involving α is comparable to the loss involving a. □

11 Subcritical contour criteria

The previous sections show that shifted-contour regularity follows from right-side vertical mean-square estimates.

Theorem 11.1 (Subcritical shifted-contour criterion)**.** *Let* $0 \le M < 1$*. Suppose either:*

(i) $0 < A < a$*, with* $a - A \asymp a$*, and a uniform translated vertical mean-square bound holds on*

$$\operatorname{Re} s = \frac{1}{2} + a - A;$$

or

(ii) $A > a$*, with* $A - a \asymp a$ *and* $0 < A - a < 1/2$*, and a uniform translated vertical mean-square bound holds on*

$$\operatorname{Re} s = \frac{1}{2} + A - a.$$

Then

$$\boxed{\int_T^{2T} |\mathcal{R}_{X,K,A}(1/2 + a + it)|^2 dt \ll X^{-2A} T a^{-M} \log^{B'}(2 + T + a^{-1}),}$$

for some $B' \ge \max\{B, 2\}$*. In particular, since* $X^{-2A} \le 1$*, the shifted-contour remainder is subcritical whenever the corresponding right-side vertical mean-square input is subcritical.*

Proof. Case (i) follows from the right-side shifted-contour criterion. Case (ii) follows from the functional-equation transfer.

The gamma contribution has the form

$$X^{-2A} T \log^2(2 + T).$$

This is subcritical in the present loss accounting: it corresponds to $M = 0$ and $B = 2$. When it is combined with a bound of the form

$$T a^{-M} \log^B(2 + T + a^{-1}), \qquad M \ge 0,$$

it is absorbed after replacing B by $B' = \max\{B, 2\}$, since $a^{-M} \ge 1$ for $0 < a < 1$. □

Remark 11.2. *The shifted-contour remainder is therefore not an independent obstruction in this formulation. It reduces to the same type of vertical mean-square problem for* ζ'/ζ *on lines to the right of* $1/2$*, plus an explicit gamma-factor contribution.*

12 Consequences for the smoothed Perron remainder

Suppose the smoothed Perron remainder decomposes as

$$E_{X,K} = \mathcal{Z} + \mathcal{T} + \mathcal{M} + \mathcal{R}.$$

The terms $\mathcal{T}$ and $\mathcal{M}$ are directly harmless for admissible kernels. The term $\mathcal{R}$ is controlled by the criteria above. Hence the smoothed remainder is controlled once:

(i) the nontrivial-zero packet term $\mathcal{Z}$ is controlled;

(ii) the appropriate right-side translated vertical mean-square estimate for ζ'/ζ is available.

This gives the schematic implication

$$\boxed{\begin{array}{c}\text{zero-packet Gram control}\\ +\text{right-side translated vertical mean-square control for } \zeta'/\zeta\\ \Longrightarrow \text{smoothed high-frequency remainder control.}\end{array}}$$

Remark 12.1. *The statement remains conditional. The purpose is to identify the inputs. The shifted-contour term does not introduce a new type of obstruction beyond translated vertical mean-square control for* ζ'/ζ*.*

13 Limitations

The paper has several limitations.

First, it does not prove the translated vertical mean-square estimates for ζ'/ζ on lines to the right of $1/2$. These estimates remain analytic inputs.

Second, the singular regime $A = a$ is not regularized here. If one wants to use the critical-line contour, one must introduce indentation or another regularization around critical-line zeros.

Third, the left-side transfer is stated in the near-left range

$$0 < A - a < \frac{1}{2}.$$

This avoids trivial-zero complications on the real axis and keeps the gamma-factor estimates clean. Larger left shifts require additional bookkeeping for trivial zeros and possible poles of the trigonometric factor.

Fourth, the translated mean-square input must be uniform over intervals whose position varies. This uniformity is necessary because the convolution kernel shifts the interval $I = [T, 2T]$ to $I + \tau$.

Fifth, all estimates depend on the admissibility and vertical decay of the smoothing kernel K.

14 Conclusion

We studied the shifted-contour remainder

$$\mathcal{R}_{X,K,A}(s) = -\frac{1}{2\pi i}\int_{(-A)} \left(-\frac{\zeta'}{\zeta}(s+w)\right) X^w K(w)\, dw$$

appearing in a smoothed Perron decomposition of a high-frequency remainder.

For

$$s = \frac{1}{2} + a + it, \qquad w = -A + i\tau,$$

the shifted line is

$$s + w = \frac{1}{2} + a - A + i(t+\tau).$$

Thus

$$\mathcal{R}_{X,K,A}$$

is a vertical convolution of $-\zeta'/\zeta$ on the line

$$\operatorname{Re} s = \frac{1}{2} + a - A,$$

with kernel $K(-A+i\tau)$, phase $e^{i\tau\log X}$, and prefactor X^{-A}.

If $0 < A < a$, the shifted line lies to the right of the critical line, and $\mathcal{R}$ is controlled by a translated right-side vertical mean-square estimate for ζ'/ζ.

If $A = a$, the shifted line is the critical line. Because $\zeta(s)$ has zeros on the critical line, the direct L^2-contour formulation is singular.

If $A > a$ and $0 < A - a < 1/2$, the shifted line lies to the left of the critical line. The functional equation transfers

$$-\frac{\zeta'}{\zeta}\left(\frac{1}{2} - (A-a) + iu\right)$$

to

$$\frac{\zeta'}{\zeta}\left(\frac{1}{2} + (A-a) - iu\right)$$

plus the explicit factor $-\chi'/\chi$. The reflected argument is controlled by the symmetry

$$\left|\frac{\zeta'}{\zeta}(\sigma - iu)\right| = \left|\frac{\zeta'}{\zeta}(\sigma + iu)\right|.$$

The gamma and trigonometric terms contribute only

$$O(T\log^2 T)$$

in mean square on intervals of length T, after convolution against the rapidly decaying kernel.

Hence shifted-contour regularity reduces to translated vertical mean-square estimates for ζ'/ζ on lines to the right of the critical line, plus harmless gamma-factor terms.

The paper does not prove RH. It does not prove the required right-side mean-square estimates. Its contribution is to show that the shifted contour is not a new independent obstruction: it is another expression of the same translated vertical mean-square barrier.

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