The Shifted-Contour Remainder in the Smoothed High-Frequency Barrier

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Abstract

Let

$$H_0(s) = -\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}$$

be the logarithmic derivative of the Riemann zeta-function with the pole at $s = 1$ removed. In a smoothed Perron decomposition of a high-frequency remainder one obtains a shifted-contour term

$$\mathcal{R}_{X,K,A}(s) = -\frac{1}{2\pi i}\int_{(-A)} \left(-\frac{\zeta'}{\zeta}(s+w)\right) X^w K(w)\, dw.$$

This paper analyzes the structure of this term.

For

$$s = \frac{1}{2} + a + it, \qquad 0 < a < \frac{1}{2},$$

and $w = -A + i\tau$, the shifted line is

$$s + w = \frac{1}{2} + a - A + i(t+\tau).$$

Thus the shifted-contour remainder is a vertical convolution of $-\zeta'/\zeta$ on the line

$$\sigma_A = \frac{1}{2} + a - A.$$

More precisely, if

$$k_A(\tau) = K(-A + i\tau), \qquad f_\sigma(u) = -\frac{\zeta'}{\zeta}(\sigma + iu),$$

then

$$\mathcal{R}_{X,K,A}\left(\frac{1}{2} + a + it\right) = -\frac{X^{-A}}{2\pi}\int_{-\infty}^{\infty} f_{\sigma_A}(t+\tau) k_A(\tau)\, d\tau.$$

Consequently, Young–Minkowski inequalities reduce estimates for $\mathcal{R}_{X,K,A}$ to vertical mean-square estimates for $-\zeta'/\zeta$ on the shifted line.

The paper separates three regimes. If $0 < A < a$, then $\sigma_A > 1/2$, and the contour remainder is controlled by a right-side mean-square estimate for ζ'/ζ. If $A = a$, then $\sigma_A = 1/2$, and the shifted line passes through critical-line zeros; in this regime the direct L^2-contour formulation is singular. If $A > a$, then $\sigma_A < 1/2$. In the near-left regime $0 < A - a < 1/2$, the functional equation transfers $-\zeta'/\zeta(\sigma_A + iu)$ to a logarithmic derivative on the right-side line $1 - \sigma_A = 1/2 + A - a$, plus gamma and trigonometric terms. The latter are controlled by Stirling's formula and contribute only $O(T \log^2 T)$ in mean square on intervals of length T.

Thus the shifted-contour term is not an independent obstruction: it reduces to vertical mean-square estimates for ζ'/ζ on lines to the right of the critical line, together with harmless gamma-factor terms. The paper does not prove RH and does not prove the required right-side mean-square estimates. It gives a structural reduction of the shifted-contour regularity condition to the same type of vertical mean-square problem appearing elsewhere in the high-frequency barrier.

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1 Introduction

Let

$$H_0(s) = -\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}.$$

The poles of H_0 are precisely the zeros of $\zeta(s)$, since the pole at $s = 1$ has been removed.

In a smoothed Perron framework, one studies remainders of the form

$$E_{X,K}(s) = H_0(s) - P_{X,K}(s),$$

where $P_{X,K}$ is a smoothed prime sum. Shifting the Perron contour gives a decomposition

$$E_{X,K} = \mathcal{Z} + \mathcal{T} + \mathcal{M} + \mathcal{R}.$$

Here $\mathcal{Z}$ is the nontrivial-zero packet term, $\mathcal{T}$ is the trivial-zero contribution, $\mathcal{M}$ is the main-pole correction, and $\mathcal{R}$ is the shifted-contour remainder.

The trivial-zero and main-pole terms can be estimated directly for admissible kernels with rapid vertical decay. The nontrivial-zero term is the zero-packet Gram obstruction. The shifted-contour term is more subtle. It is

$$\mathcal{R}_{X,K,A}(s) = -\frac{1}{2\pi i}\int_{(-A)} \left(-\frac{\zeta'}{\zeta}(s+w)\right) X^w K(w)\, dw.$$

This paper studies this term.

Let

$$s = \frac{1}{2} + a + it, \qquad 0 < a < \frac{1}{2},$$

and let

$$w = -A + i\tau.$$

Then

$$s + w = \frac{1}{2} + a - A + i(t + \tau).$$

Thus the shifted-contour remainder is a weighted average of $-\zeta'/\zeta$ on the vertical line

$$\sigma_A = \frac{1}{2} + a - A.$$

The averaging kernel is $K(-A + i\tau)$, and the contour shift supplies a factor X^{-A}.

The central observation is:

$$\boxed{\mathcal{R}_{X,K,A} = X^{-A} \times \text{vertical convolution of } -\zeta'/\zeta \text{ on } \operatorname{Re} s = \sigma_A.}$$

This immediately reduces L^2-bounds for $\mathcal{R}$ to vertical mean-square bounds for $-\zeta'/\zeta$.

There are three regimes.

(i) If $0 < A < a$, then $\sigma_A > 1/2$. This is a right-side shifted contour.

(ii) If $A = a$, then $\sigma_A = 1/2$. This is the singular critical-line contour.

(iii) If $A > a$, then $\sigma_A < 1/2$. This is a left-side shifted contour. In the near-left range $0 < A - a < 1/2$, the functional equation transfers the problem to the right-side line $1 - \sigma_A = 1/2 + A - a$, plus explicit gamma and trigonometric terms.

The outcome is a conditional but useful conclusion: shifted-contour regularity is not a new independent mystery. It is reducible to vertical mean-square estimates for ζ'/ζ on lines to the right of the critical line, plus harmless gamma-factor contributions.

The paper does not prove RH. It also does not prove the needed right-side vertical mean-square estimates. It clarifies the analytic structure of the shifted-contour remainder.

2 The shifted-contour remainder

Let $K(w)$ be a meromorphic smoothing kernel with rapid decay on vertical lines. For $A > 0$, define

$$\mathcal{R}_{X,K,A}(s) = -\frac{1}{2\pi i} \int_{(-A)} \left(-\frac{\zeta'}{\zeta}(s + w) \right) X^w K(w)\, dw.$$

We write

$$w = -A + i\tau, \qquad dw = i\, d\tau.$$

Then

$$\mathcal{R}_{X,K,A}(s) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(-\frac{\zeta'}{\zeta}(s - A + i\tau) \right) X^{-A+i\tau} K(-A + i\tau)\, d\tau.$$

Since

$$X^{-A+i\tau} = X^{-A} e^{i\tau \log X},$$

we have

$$|\mathcal{R}_{X,K,A}(s)| \le \frac{X^{-A}}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\zeta'}{\zeta}(s - A + i\tau) \right| |K(-A + i\tau)|\, d\tau.$$

Now take

$$s = \frac{1}{2} + a + it.$$

Set

$$\sigma_A = \frac{1}{2} + a - A.$$

Then

$$s - A + i\tau = \sigma_A + i(t + \tau).$$

Define

$$f_\sigma(u) = -\frac{\zeta'}{\zeta}(\sigma + iu),$$

and

$$k_A(\tau) = e^{i\tau \log X} K(-A + i\tau).$$

Since the exponential factor has modulus one,

$$|k_A(\tau)| = |K(-A + i\tau)|.$$

Then

$$\boxed{\mathcal{R}_{X,K,A}\left(\frac{1}{2} + a + it\right) = -\frac{X^{-A}}{2\pi} \int_{-\infty}^{\infty} f_{\sigma_A}(t + \tau) k_A(\tau)\, d\tau.}$$

Remark 2.1. *The factor $e^{i\tau \log X}$ changes only the phase of the convolution kernel. It is irrelevant for L^2-upper bounds based on absolute values.*

3 A convolution inequality

Let $I = [T, 2T]$. For a locally square-integrable function f and an integrable kernel k, define

$$(C_k f)(t) = \int_{\mathbb{R}} f(t + \tau) k(\tau)\, d\tau.$$

Lemma 3.1 (Localized Young–Minkowski inequality)**.** *Let $k \in L^1(\mathbb{R})$. Then*

$$\boxed{\|C_k f\|_{L^2(I)} \le \int_{\mathbb{R}} |k(\tau)|\, \|f\|_{L^2(I+\tau)}\, d\tau,}$$

where

$$I + \tau = \{t + \tau : t \in I\}.$$

Proof. By Minkowski's integral inequality,

$$\begin{aligned} \|C_k f\|_{L^2(I)} &= \left(\int_I \left| \int_{\mathbb{R}} f(t + \tau) k(\tau)\, d\tau \right|^2 dt \right)^{1/2} \\ &\le \int_{\mathbb{R}} |k(\tau)| \left(\int_I |f(t + \tau)|^2 dt \right)^{1/2} d\tau \\ &= \int_{\mathbb{R}} |k(\tau)|\, \|f\|_{L^2(I+\tau)}\, d\tau. \end{aligned}$$

□

Applying this to the shifted-contour remainder gives the following.

Theorem 3.2 (Convolution bound for the shifted contour)**.** *Let*

$$I = [T, 2T], \qquad \sigma_A = \frac{1}{2} + a - A.$$

Assume $K(-A + i\tau) \in L^1(\mathbb{R})$. *Then*

$$\boxed{\left\| \mathcal{R}_{X,K,A}\left(\frac{1}{2} + a + i\cdot\right) \right\|_{L^2(I)} \leq \frac{X^{-A}}{2\pi} \int_{\mathbb{R}} |K(-A + i\tau)| \left\| \frac{\zeta'}{\zeta}(\sigma_A + i\cdot) \right\|_{L^2(I+\tau)} d\tau.}$$

Proof. Use the convolution representation and the localized Young–Minkowski inequality. □

Remark 3.3. *This theorem is the basic reduction. It does not assume RH. If the line* $\operatorname{Re} s = \sigma_A$ *contains a zero of* $\zeta(s)$*, then the right-hand side is infinite; the theorem correctly records this obstruction.*

4 A local vertical mean-square input

The convolution bound becomes useful once one has a local vertical mean-square estimate for ζ'/ζ.

Hypothesis 4.1 (Translated vertical mean-square bound)**.** *Let* $\sigma > 1/2$. *We say that a translated vertical mean-square bound holds with loss parameters* M, B *if, for every interval* J *of length* T,

$$\boxed{\int_J \left| \frac{\zeta'}{\zeta}(\sigma + iu) \right|^2 du \ll T \Lambda_\sigma(T)^2,}$$

where

$$\Lambda_\sigma(T)^2 = d(\sigma)^{-M} \log^B(2 + T + d(\sigma)^{-1}),$$

and

$$d(\sigma) = \sigma - \frac{1}{2}.$$

Remark 4.2. *This hypothesis is intentionally stated as an input. The present paper does not prove such estimates. In applications one may take* $d(\sigma)$ *comparable to* a*, in which case the loss becomes* a^{-M} *up to constants.*

Corollary 4.3 (Contour regularity from vertical mean square)**.** *Assume* $\sigma_A > 1/2$*, and suppose the translated vertical mean-square bound holds on the line* $\operatorname{Re} s = \sigma_A$. *If* $K(-A + i\tau) \in L^1(\mathbb{R})$*, then*

$$\boxed{\int_T^{2T} |\mathcal{R}_{X,K,A}(1/2 + a + it)|^2 dt \ll X^{-2A} \|K(-A + i\cdot)\|_1^2 T \Lambda_{\sigma_A}(T)^2.}$$

Proof. The translated mean-square bound gives

$$\left\| \frac{\zeta'}{\zeta}(\sigma_A + i\cdot) \right\|_{L^2(I+\tau)} \ll T^{1/2} \Lambda_{\sigma_A}(T)$$

uniformly in τ. Insert this into the convolution bound and square. □

5 The right-side regime $0 < A < a$

If

$$0 < A < a,$$

then

$$\sigma_A = \frac{1}{2} + a - A > \frac{1}{2}.$$

This is the right-side shifted-contour regime.

Let

$$d_A = a - A > 0.$$

Then

$$\sigma_A - \frac{1}{2} = d_A.$$

Theorem 5.1 (Right-side shifted-contour criterion)**.** *Assume $0 < A < a$. Suppose the translated vertical mean-square bound holds on the line*

$$\operatorname{Re} s = \frac{1}{2} + d_A.$$

Then

$$\boxed{\int_T^{2T} |\mathcal{R}_{X,K,A}(1/2 + a + it)|^2 dt \ll X^{-2A} T d_A^{-M} \log^B(2 + T + d_A^{-1}).}$$

Proof. Apply the previous corollary with

$$\sigma_A = \frac{1}{2} + d_A.$$

The L^1-norm of $K(-A + i\tau)$ is absorbed into the implied constant. □

Corollary 5.2 (Comparable shift)**.** *Let $A = \eta a$, where $0 < \eta < 1$ is fixed. Then*

$$d_A = (1 - \eta)a.$$

Under the same vertical mean-square input,

$$\boxed{\int_T^{2T} |\mathcal{R}_{X,K,A}(1/2 + a + it)|^2 dt \ll X^{-2A} T a^{-M} \log^B(2 + T + a^{-1}).}$$

Proof. Since $d_A \asymp a$, the loss

$$d_A^{-M} \log^B(2 + T + d_A^{-1})$$

is comparable to

$$a^{-M} \log^B(2 + T + a^{-1}).$$

□

Remark 5.3. *The factor $X^{-2A} \leq 1$ is favorable. The essential input is the vertical mean-square estimate for ζ'/ζ on the right-side line $\operatorname{Re} s = 1/2 + d_A$.*

6 The singular regime $A = a$

If

$$A = a,$$

then

$$\sigma_A = \frac{1}{2}.$$

The shifted line is the critical line.

This regime is singular. If $\zeta(s)$ has a zero on the critical line, then $-\zeta'/\zeta(s)$ has a pole on the line.

Lemma 6.1 (Critical-line poles produce L^2-divergence)**.** *Suppose $\zeta(s)$ has a zero*

$$\rho = \frac{1}{2} + i\gamma$$

of multiplicity m_ρ. Then

$$-\frac{\zeta'}{\zeta}\left(\frac{1}{2} + iu\right) = -\frac{m_\rho}{i(u-\gamma)} + O(1)$$

near $u = \gamma$. Consequently,

$$\int_{\gamma-\delta}^{\gamma+\delta} \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2} + iu\right)\right|^2 du = +\infty$$

for every $\delta > 0$.

Proof. Near a zero ρ of multiplicity m_ρ,

$$\frac{\zeta'}{\zeta}(s) = \frac{m_\rho}{s-\rho} + O(1).$$

Putting $s = 1/2 + iu$ gives

$$s - \rho = i(u - \gamma).$$

Thus

$$\left|\frac{\zeta'}{\zeta}\left(\frac{1}{2} + iu\right)\right|^2 \asymp \frac{m_\rho^2}{(u-\gamma)^2}$$

near $u = \gamma$, and the integral diverges. □

Proposition 6.2 (The critical shifted contour should be avoided)**.** *The choice $A = a$ cannot be used as an ordinary L^2-regular shifted contour unless the critical-line poles are detoured or otherwise regularized. In the direct vertical-line formulation, it is singular.*

Proof. Hardy proved that $\zeta(s)$ has infinitely many zeros on the critical line. At each such zero, the preceding lemma gives local L^2-divergence of ζ'/ζ on the line. □

Remark 6.3. *The conclusion is not that contour methods fail. Rather, the contour must avoid poles by indentation, smoothing, or by choosing $A \neq a$. The direct line $A = a$ is not a regular L^2-contour.*

7 The left-side regime $A > a$

Assume now that

$$A > a.$$

Set

$$\alpha = A - a > 0.$$

Then

$$\sigma_A = \frac{1}{2} - \alpha.$$

We focus on the near-left range

$$0 < \alpha < \frac{1}{2}.$$

Then

$$0 < \sigma_A < \frac{1}{2}.$$

This avoids the trivial zeros on the real axis and permits a clean use of the functional equation.

The functional equation may be written as

$$\zeta(s) = \chi(s)\zeta(1-s),$$

where

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s).$$

Taking logarithmic derivatives gives

$$\frac{\zeta'}{\zeta}(s) = \frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(1-s).$$

Hence

$$\boxed{-\frac{\zeta'}{\zeta}(s) = -\frac{\chi'}{\chi}(s) + \frac{\zeta'}{\zeta}(1-s).}$$

For

$$s = \frac{1}{2} - \alpha + iu,$$

we have

$$1 - s = \frac{1}{2} + \alpha - iu.$$

Thus the logarithmic derivative on the left-side line is expressed in terms of a logarithmic derivative on the right-side line

$$\operatorname{Re} s = \frac{1}{2} + \alpha$$

plus the explicit gamma-trigonometric factor

$$\frac{\chi'}{\chi}(s).$$

8 Gamma and trigonometric factor estimates

We now estimate χ'/χ.

Since

$$\chi(s) = 2^s \pi^{s-1} \sin\Big(\frac{\pi s}{2}\Big)\, \Gamma(1-s),$$

we have

$$\boxed{\frac{\chi'}{\chi}(s) = \log(2\pi) + \frac{\pi}{2}\cot\Big(\frac{\pi s}{2}\Big) - \frac{\Gamma'}{\Gamma}(1-s).}$$

Lemma 8.1 (Gamma-factor bound)**.** *Let* $0 < \alpha < 1/2$. *Uniformly for*

$$s = \frac{1}{2} - \alpha + iu,$$

one has

$$\boxed{\frac{\chi'}{\chi}(s) \ll_\alpha \log(2 + |u|).}$$

More generally, the bound is uniform when α *ranges in a compact subinterval of* $(0, 1/2)$.

Proof. The term $\log(2\pi)$ is constant. Since

$$\operatorname{Re} s = \frac{1}{2} - \alpha \in (0, 1/2),$$

the function

$$\cot\Big(\frac{\pi s}{2}\Big)$$

has no pole on this vertical line and is bounded for $|u| \le 1$. For $|u| \ge 1$, it is bounded uniformly on vertical strips away from its real-axis poles.

For the gamma term, Stirling's formula gives

$$\frac{\Gamma'}{\Gamma}(1-s) = \log(1-s) + O\Big(\frac{1}{|1-s|}\Big)$$

on vertical strips. Hence

$$\frac{\Gamma'}{\Gamma}(1-s) \ll \log(2 + |u|).$$

Combining the estimates proves the lemma. $\square$

Corollary 8.2 (Mean square of the gamma factor)**.** *Let* $I = [T, 2T]$ *and* $0 < \alpha < 1/2$. *Then*

$$\boxed{\int_I \left| \frac{\chi'}{\chi}\Big(\frac{1}{2} - \alpha + iu\Big) \right|^2 du \ll_\alpha T \log^2(2 + T).}$$

The bound is uniform for α *in compact subintervals of* $(0, 1/2)$.

Proof. Use the pointwise bound

$$\left| \frac{\chi'}{\chi}\Big(\frac{1}{2} - \alpha + iu\Big) \right| \ll_\alpha \log(2 + |u|)$$

and integrate over $T \le u \le 2T$. $\square$

9 Functional-equation transfer for $A > a$

We now transfer the left-side shifted contour to the right side.

Let

$$A > a, \qquad \alpha = A - a, \qquad 0 < \alpha < \frac{1}{2}.$$

Then

$$\sigma_A = \frac{1}{2} - \alpha.$$

Define

$$g_\alpha(u) = -\frac{\zeta'}{\zeta}\left(\frac{1}{2} - \alpha + iu\right).$$

By the functional equation,

$$g_\alpha(u) = -\frac{\chi'}{\chi}\left(\frac{1}{2} - \alpha + iu\right) + \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \alpha - iu\right).$$

Taking absolute values,

$$|g_\alpha(u)| \le \left|\frac{\chi'}{\chi}\left(\frac{1}{2} - \alpha + iu\right)\right| + \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2} + \alpha - iu\right)\right|.$$

Theorem 9.1 (Left-side contour transfer)**.** *Assume*

$$A > a, \qquad \alpha = A - a, \qquad 0 < \alpha < \frac{1}{2}.$$

Suppose a translated vertical mean-square bound holds for

$$\frac{\zeta'}{\zeta}\left(\frac{1}{2} + \alpha + iv\right).$$

Then

$$\boxed{\int_T^{2T} |\mathcal{R}_{X,K,A}(1/2 + a + it)|^2 dt \ll X^{-2A} T \left[\alpha^{-M} \log^B(2 + T + \alpha^{-1}) + \log^2(2 + T)\right].}$$

Proof. By the convolution bound, it is enough to bound local L^2-norms of

$$-\frac{\zeta'}{\zeta}\left(\frac{1}{2} - \alpha + iu\right).$$

Using the functional equation,

$$-\frac{\zeta'}{\zeta}\left(\frac{1}{2} - \alpha + iu\right) = -\frac{\chi'}{\chi}\left(\frac{1}{2} - \alpha + iu\right) + \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \alpha - iu\right).$$

The right-side logarithmic derivative is controlled by the translated vertical mean-square input on the line $\operatorname{Re} s = 1/2 + \alpha$. The gamma and trigonometric factor contributes

$$\ll T \log^2(2 + T)$$

by the preceding corollary. Applying the convolution inequality and absorbing $\|K(-A + i\cdot)\|_1$ into the implied constant gives the result. $\square$

Corollary 9.2 (Comparable left shift)**.** *If*

$$A = a + \eta a = (1+\eta)a$$

with fixed $0 < \eta < 1$*, then*

$$\alpha = A - a = \eta a.$$

Under the same translated vertical mean-square input,

$$\boxed{\int_T^{2T} |\mathcal{R}_{X,K,A}(1/2+a+it)|^2 dt \ll X^{-2A} T \left[a^{-M} \log^B(2+T+a^{-1}) + \log^2(2+T)\right].}$$

Proof. Since $\alpha \asymp a$, the loss involving α is comparable to the loss involving a. □

10 Subcritical contour criteria

The previous sections show that shifted-contour regularity follows from right-side vertical mean-square estimates.

Theorem 10.1 (Subcritical shifted-contour criterion)**.** *Let* $0 \le M < 1$*. Suppose either:*

(i) $0 < A < a$*, with* $a - A \asymp a$*, and a translated vertical mean-square bound holds on*

$$\operatorname{Re} s = \frac{1}{2} + a - A;$$

or

(ii) $A > a$*, with* $A - a \asymp a$ *and* $0 < A - a < 1/2$*, and a translated vertical mean-square bound holds on*

$$\operatorname{Re} s = \frac{1}{2} + A - a.$$

Then

$$\boxed{\int_T^{2T} |\mathcal{R}_{X,K,A}(1/2+a+it)|^2 dt \ll X^{-2A} T a^{-M} \log^B(2+T+a^{-1}) + X^{-2A} T \log^2(2+T).}$$

In particular, since $X^{-2A} \le 1$*, the shifted-contour remainder is subcritical whenever the corresponding right-side vertical mean-square input is subcritical.*

Proof. Case (i) follows from the right-side shifted-contour criterion. Case (ii) follows from the functional-equation transfer. The gamma-factor term is $O(T \log^2 T)$, which has no positive a^{-1}-power loss. □

Remark 10.2. *The shifted-contour remainder is therefore not an independent obstruction in this formulation. It reduces to the same type of vertical mean-square problem for* ζ'/ζ *on lines to the right of* $1/2$*.*

11 Consequences for the smoothed Perron remainder

Suppose the smoothed Perron remainder decomposes as

$$E_{X,K} = \mathcal{Z} + \mathcal{T} + \mathcal{M} + \mathcal{R}.$$

The terms $\mathcal{T}$ and $\mathcal{M}$ are directly harmless for admissible kernels. The term $\mathcal{R}$ is controlled by the criteria above. Hence the smoothed remainder is controlled once:

(i) the nontrivial-zero packet term $\mathcal{Z}$ is controlled;

(ii) the appropriate right-side vertical mean-square estimate for ζ'/ζ is available.

This gives the schematic implication

$$\boxed{\begin{array}{c}\text{zero-packet Gram control}\\ +\text{right-side vertical mean-square control for } \zeta'/\zeta\\ \Longrightarrow \text{smoothed high-frequency remainder control.}\end{array}}$$

Remark 11.1. *The statement remains conditional. The purpose is to identify the inputs. The shifted-contour term does not introduce a new type of obstruction beyond vertical mean-square control for ζ'/ζ.*

12 Limitations

The paper has several limitations.

First, it does not prove the translated vertical mean-square estimates for ζ'/ζ on lines to the right of $1/2$. These estimates remain analytic inputs.

Second, the singular regime $A = a$ is not regularized here. If one wants to use the critical-line contour, one must introduce indentation or another regularization around critical-line zeros.

Third, the left-side transfer is stated in the near-left range

$$0 < A - a < \frac{1}{2}.$$

This avoids trivial-zero complications on the real axis and keeps the gamma-factor estimates clean. Larger left shifts require additional bookkeeping for trivial zeros and possible poles of the trigonometric factor.

Fourth, all estimates depend on the admissibility and vertical decay of the smoothing kernel K.

13 Conclusion

We studied the shifted-contour remainder

$$\mathcal{R}_{X,K,A}(s) = -\frac{1}{2\pi i}\int_{(-A)}\left(-\frac{\zeta'}{\zeta}(s+w)\right)X^{w}K(w)\,dw$$

appearing in a smoothed Perron decomposition of a high-frequency remainder.

For

$$s = \frac{1}{2} + a + it, \qquad w = -A + i\tau,$$

the shifted line is

$$s + w = \frac{1}{2} + a - A + i(t+\tau).$$

Thus

$$\mathcal{R}_{X,K,A}$$

is a vertical convolution of $-\zeta'/\zeta$ on the line

$$\operatorname{Re} s = \frac{1}{2} + a - A,$$

with kernel $K(-A+i\tau)$ and prefactor X^{-A}.

If $0 < A < a$, the shifted line lies to the right of the critical line, and $\mathcal{R}$ is controlled by a right-side vertical mean-square estimate for ζ'/ζ.

If $A = a$, the shifted line is the critical line. Because $\zeta(s)$ has zeros on the critical line, the direct L^2-contour formulation is singular.

If $A > a$ and $0 < A - a < 1/2$, the shifted line lies to the left of the critical line. The functional equation transfers

$$-\frac{\zeta'}{\zeta}\left(\frac{1}{2} - (A-a) + iu\right)$$

to

$$\frac{\zeta'}{\zeta}\left(\frac{1}{2} + (A-a) - iu\right)$$

plus the explicit factor $-\chi'/\chi$. The gamma and trigonometric terms contribute only

$$O(T \log^2 T)$$

in mean square on intervals of length T.

Hence shifted-contour regularity reduces to vertical mean-square estimates for ζ'/ζ on lines to the right of the critical line, plus harmless gamma-factor terms.

The paper does not prove RH. It does not prove the required right-side mean-square estimates. Its contribution is to show that the shifted contour is not a new independent obstruction: it is another expression of the same vertical mean-square barrier.

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