The Prime Resonance Ecosystem: A Spectral Framework for the Riemann Hypothesis

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April 28, 2026

Abstract

We present a spectral framework for the Riemann zeta function based on prime shift operators. The Hilbert space $L^2(\mathbb{R}^+, dx/x)$, unitary operators $\mathcal{U}_p f(x) = f(px)$, and the Euler product operator $\mathcal{E} = \prod_p (I - \mathcal{U}_p)^{-1}$ define a lossless resonance network. A rigged Hilbert space $\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}'$ with $\mathcal{S} = S^{1/2}_{1/2}$ (a nuclear Gelfand–Shilov space) is introduced. The functional equation forces zeros to appear in symmetric pairs. The corresponding log-time resonant state would be $\phi(u) = 2\cosh((\sigma - 1/2)u)e^{i\gamma u}$. We prove that $\cosh(\alpha u) \notin \mathcal{S}'$ unless $\alpha = 0$. This reduces the Riemann Hypothesis to constructing a two-dimensional extension of $\mathcal{E}$ whose kernel projects continuously onto $\mathcal{S}'$. Two precise gaps are identified. The Hilbert–Pólya conjecture is reformulated but not proved. No claim of proof is made. A research program is offered.
Keywords: Riemann Hypothesis, Hilbert–Pólya conjecture, prime shift operators, rigged Hilbert spaces, Gelfand–Shilov spaces, spectral theory.

Contents

1 Prologue

The Riemann Hypothesis has stood for 167 years. Many have claimed proofs. All have failed. This paper makes no claim of proof.

Instead, we build a new language — a physical language of frequencies, resonances, and losslessness. We assemble a coherent mathematical framework. We identify two precise gaps. We show what remains to be done.

This is not a victory lap. It is a roadmap.

2 The Physical Intuition

The Riemann zeta function $\zeta(s)$ encodes the prime numbers. The Euler product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \qquad \mathrm{Re}(s) > 1$$

is the first hint of a deeper structure.

Physical reframing: Consider $\mathrm{Re}(s) = 1/2$. Write $s = 1/2 + it$. Then

$$p^{-s} = p^{-1/2} e^{-it \log p}.$$

The factor $e^{-it \log p}$ is a phase shift. The factor $p^{-1/2}$ is a constant attenuation.

Now think of each prime as a **resonator**. The operator $(1 - p^{-s})^{-1}$ is its transfer function. The product over all primes is the transfer function of an infinite network of resonators in series.

The zeros of $\zeta(s)$ are the frequencies where the network has **zero transmission** — complete destructive interference across all primes.

The question: Can such perfect cancellation happen at a frequency with $\sigma \neq 1/2$? That is the Riemann Hypothesis.

3 The Mathematical Framework

3.1 State Space

Let

$$\mathcal{H} = L^2(\mathbb{R}^+, dx/x), \qquad \langle f, g \rangle = \int_0^\infty f(x)\overline{g(x)}\frac{dx}{x}.$$

The measure dx/x is the Haar measure of $(\mathbb{R}^+, \times)$.

3.2 Prime Shift Operators

For each prime p, define

$$(\mathcal{U}_p f)(x) = f(px).$$

Proposition 3.1. *Each $\mathcal{U}_p$ is unitary on $\mathcal{H}$.*

Proof. $\int_0^\infty |f(px)|^2 dx/x = \int_0^\infty |f(y)|^2 dy/y$ by the substitution $y = px$. ■

This is the **losslessness** property. Energy is conserved.

3.3 Resonance Transform (Mellin)

Define

$$(\mathcal{M}f)(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x) x^{-1/2-it} \frac{dx}{x}, \qquad t \in \mathbb{R}.$$

Proposition 3.2. *$\mathcal{M} : \mathcal{H} \to L^2(\mathbb{R})$ is unitary.*

In resonance space,

$$\mathcal{M}\mathcal{U}_p\mathcal{M}^{-1}\hat{f}(t) = p^{-1/2-it}\hat{f}(t).$$

3.4 Euler Product Operator

Define

$$\mathcal{E} = \prod_p (I - \mathcal{U}_p)^{-1} = \prod_p \sum_{k=0}^{\infty} \mathcal{U}_p^k,$$

with domain

$$\mathrm{Dom}(\mathcal{E}) = \left\{ f \in \mathcal{H} : \zeta(1/2 + it)(\mathcal{M}f)(t) \in L^2(\mathbb{R}) \right\}.$$

Proposition 3.3. *In resonance space,*

$$\mathcal{M}\mathcal{E}\mathcal{M}^{-1}\hat{f}(t) = \zeta(1/2 + it)\hat{f}(t).$$

Proof. Each $(I - \mathcal{U}_p)$ acts as multiplication by $1 - p^{-1/2-it}$. The product over all primes gives $\prod_p (1 - p^{-1/2-it})^{-1} = \zeta(1/2 + it)$ by the Euler product formula. ■

3.5 Rigged Resonance Space

Let $\mathcal{S} = S_{1/2}^{1/2}$ be the Gelfand–Shilov space of smooth functions $\eta(u)$ such that there exists $a > 0$ with

$$|\eta(u)| \leq Ce^{-a|u|^{1/2}}, \qquad |\hat{\eta}(t)| \leq Ce^{-a|t|^{1/2}}.$$

Proposition 3.4. *$\mathcal{S}$ is nuclear, dense in $\mathcal{H}$, and*

$$\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}'$$

is a rigged Hilbert space. Dirac deltas $\delta(t-\gamma) \in \mathcal{S}'$ for all real γ.

3.6 Renormalized Inner Product

For $\epsilon > 0$ and $\gamma, \gamma' \in \mathbb{R}$, define

$$\langle\langle \delta_\gamma, \delta_{\gamma'} \rangle\rangle_\epsilon = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\epsilon|u|} e^{-i\gamma u} \overline{e^{-i\gamma' u}} du.$$

Proposition 3.5.

$$\lim_{\epsilon \to 0^+} \pi\epsilon \langle\langle \delta_\gamma, \delta_{\gamma'} \rangle\rangle_\epsilon = \delta_{\gamma,\gamma'}.$$

Define the renormalized inner product

$$\langle \delta_\gamma, \delta_{\gamma'} \rangle_{\text{ren}} = \delta_{\gamma,\gamma'}.$$

The completion

$$\mathcal{D} = \overline{\text{span}\{\delta(t-\gamma_n)\}}^{\|\cdot\|_{\text{ren}}}$$

is a Hilbert space $\mathcal{D} \cong \ell^2(\{\gamma_n\})$.

4 The Kernel of $\mathcal{E}$

Define

$$\mathcal{K} = \ker(\mathcal{E}) \cap \mathcal{S}'.$$

Theorem 4.1 (Resonant Frequency Theorem)**.**

$$\mathcal{K} = \overline{\text{span}}\{\delta(t-\gamma_n)\}$$

in the weak- topology of $\mathcal{S}'$, where $\zeta(1/2 + i\gamma_n) = 0$.*

Proof. From Proposition 3, $\hat{\psi}(t) = (\mathcal{M}\psi)(t)$ satisfies $\zeta(1/2+it)\hat{\psi}(t) = 0$ in the distribution sense. Since $\zeta(1/2+it)$ is analytic except at $t = 0$ and has isolated zeros, the general solution is $\hat{\psi}(t) = \sum_n c_n \delta(t-\gamma_n)$. ■

Note: This theorem describes the kernel of $\mathcal{E}$ on the critical line. It does not assume RH. It simply states that the kernel consists of deltas at the zeros of $\zeta(1/2 + it)$ — whatever those zeros are.

5 The Hilbert–Pólya Conjecture (Status)

Define $\mathcal{R}$ on $\mathcal{D}$ by

$$(\mathcal{R}\psi)(t) = t \cdot \psi(t).$$

Theorem 5.1. *$\mathcal{R}$ is essentially self-adjoint on $\mathcal{D}$ with pure point spectrum $\sigma(\mathcal{R}) = \{\gamma_n\}$.*

Status: This constructs an operator whose eigenvalues are the imaginary parts of zeros **on the critical line**. It does not prove that all nontrivial zeros are on the critical line. It assumes they are (by using $\zeta(1/2 + it)$ as the symbol). Therefore, HPC is not solved — it is reformulated.

The gap: To solve HPC, one must define $\mathcal{D}$ independently of the zeros (e.g., as $\ker(\mathcal{E})$) and prove its elements are deltas at real numbers that coincide with the zero set. This is not yet done.

6 The Riemann Hypothesis (Status)

6.1 The Functional Equation

The zeta function satisfies

$$\zeta(s) = \chi(s)\zeta(1 - s),$$

with $\chi(s) \neq 0$ for $0 < \mathrm{Re}(s) < 1$. Hence zeros come in symmetric pairs: if $\zeta(s_0) = 0$, then $\zeta(1 - s_0) = 0$.

6.2 Log-Time Coordinates

Let $u = \log x$, $\phi(u) = e^{-u/2} f(e^u)$. Then

$$(\mathcal{M}f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(u) e^{-itu} du.$$

6.3 The Symmetric Resonant State

For a symmetric pair $s_0 = \sigma + i\gamma$ and $1 - s_0$, the corresponding log-time state (from a hypothetical two-dimensional extension) would be

$$\phi(u) = e^{(\sigma-1/2)u} e^{i\gamma u} + e^{(1/2-\sigma)u} e^{-i\gamma u} = 2\cosh((\sigma - 1/2)u) e^{i\gamma u}.$$

6.4 The Growth Lemma

Lemma 6.1 (Exponential Forbidden)**.** *For $\phi(u) = 2\cosh(\alpha u) e^{i\gamma u}$,*

$$\phi \in \mathcal{S}' \iff \alpha = 0.$$

Proof. If $\alpha = 0$, then $\phi(u) = 2e^{i\gamma u}$ is bounded, hence $\phi \in \mathcal{S}'$. If $\alpha \neq 0$, then $|\phi(u)| \sim e^{|\alpha||u|}$ as $|u| \to \infty$, which grows faster than any $e^{b|u|^{1/2}}$ because

$$\lim_{|u|\to\infty} \frac{|\alpha||u|}{b|u|^{1/2}} = \infty.$$

This contradicts the definition of $\mathcal{S}'$, which requires subexponential growth. ∎

6.5 The Missing Construction

To prove RH, one would need:

1. A **two-dimensional extension** $\mathcal{E}_{\mathbb{C}}$ acting on a rigged space $\mathcal{T} \subset \mathcal{H}_2 \subset \mathcal{T}'$ where $\mathcal{T} = S^{1/2,1/2}_{1/2,1/2}(\mathbb{R}^2)$, such that $\mathcal{E}_{\mathbb{C}}$ multiplies by $\zeta(\sigma + it)$ in the appropriate transform.
2. **Proof** that the kernel of $\mathcal{E}_{\mathbb{C}}$ contains $\delta(s - s_0) + \delta(s - (1 - s_0))$ for any zero s_0.
3. A **continuous projection** $P : \mathcal{T}' \to \mathcal{S}'$ (restriction to the diagonal $u = v$) such that $P\Psi$ is exactly $2\cosh((\sigma - 1/2)u)e^{i\gamma u}$ for the symmetric pair.
4. **Conclusion:** By Lemma 8, $\alpha = 0$, hence $\sigma = 1/2$.

None of these steps are completed in this paper. They form a research program.

7 Two Precise Gaps

Gap	Description
Gap 1 (HPC)	Construct $\mathcal{D}$ independently of the zeros and prove its elements are deltas at real numbers that coincide with the zero set of $\zeta(1/2 + it)$.
Gap 2 (RH)	Construct the two-dimensional extension $\mathcal{E}_{\mathbb{C}}$ and prove the projection from its kernel to $\mathcal{S}'$ is continuous and yields $2\cosh((\sigma - 1/2)u)e^{i\gamma u}$.

8 What Has Been Achieved

Achievement	Status
Physical language of frequencies and resonances	✓Built
Hilbert space $L^2(\mathbb{R}^+, dx/x)$	✓Defined
Unitary prime shift operators	✓Defined
Euler product operator on critical line	✓Defined
Rigged Hilbert space $\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}'$	✓Built
Renormalized inner product	✓Built
Operator $\mathcal{R}$	✓Defined
Lemma (cosh forbidden)	✓Proved
Functional equation pairing	✓Used
Reduction of RH to a two-dimensional construction	✓Formulated
Two precise gaps	✓Identified

9 The Physical Heart

The prime resonance ecosystem is lossless. Its natural law is nuclearity — subexponential growth only. A zero at $\sigma \neq 1/2$ would require the ecosystem to sustain $\cosh((\sigma - 1/2)u)$. It cannot. Therefore no such zero exists.

This is the intuition. The mathematics to make it rigorous is within reach. The gaps are precise. The path is visible.

10 Epilogue: A New Language

This paper does not claim victory. It claims a new language.

- Zeros are frequencies.
- Primes are shift operators.
- The Euler product is a transfer function.
- The critical line is the real frequency axis.
- The Riemann Hypothesis is a conservation law.

This language will outlive any single proof. It will guide future work. It may, in time, lead to the proof.

We document the journey. We invite others to walk it with us.

Acknowledgments

The author thanks the primes for their resonance.

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