On certain General Limitations affecting Hyper-magic Squares. *By* Samuel Roberts.

Received and read November 10th, 1892.

1. This paper does not aim at making any addition to the known ways of constructing magic squares.*

Hyper-magic squares, as I regard them, include those called by the late M. E. Lucas† "carrés diaboliques," and also treated of by Rev. A. H. Frost, under the designation "nasik squares."‡ The special form is of ancient origin. The second method given in the fragment by Moschopulus (probably of the fourteenth century) is a general one for forming such squares, and they have been discussed by various modern authors. My object is to show some limitations to which they are subject when the elements are positive or negative integers. Incidentally it will appear that hyper-magic squares of oddly even orders cannot be formed of series of consecutive natural numbers.§ There is some reason to believe that much ingenuity has

---

* Notwithstanding this remark, it has been imagined that I contemplated the actual construction of hyper-magic squares having consecutive natural numbers as elements. So far is this from being the case, I have not, unless inadvertently, shown that such squares exist. It was not necessary, since my conclusions are of a negative kind.

† The subject has been brought into connexion with the "Geometry of Tissues," by M. Lucas, and others (*Principii fondamentali della Geometria dei Tessuti*, per Edoardo Lucas, Torino, 1880; see also *Récréations Mathématiques*, par M. E. Lucas, Introduction, t. I., p. XVIII.).

‡ I do not say that hyper-magic squares include nasik squares, but that they include "carrés diaboliques," which, I take it, are hyper-magic squares made up of natural numbers from 1 to n^2 (v. § 2).

The first definition of nasik squares (*Quarterly Journal of Mathematics*, VII., pp. 93, 94) apparently makes "carrés diaboliques" coextensive with them.

A later definition (*Quar. Jour.*, XV., p. 34) is in the following terms: "A square containing n cells on each side, in which are placed the natural numbers from 1 to n^2, in such an order that the constant sum $\frac{1}{2}n(n^2+1)$ is obtained by adding the numbers on n of the cells, those n cells lying in a variety of different directions, and their relative position in each direction being defined by simple laws."

I should not presume to limit the comprehensiveness of this definition.

§ With regard to this, I have been referred to the following passage in Rev. A. H. Frost's paper (*Quarterly Journal of Mathematics*, XV., p. 49):—

"Nasik Squares of the form $2(2n+1)$ cannot be filled with consecutive natural numbers from 1 to $4(2n+1)^2$ either by this or the process adopted in the previous paper; for it will be found that, as in the squares of the form $4n$, we have to give (*e.g.*, the case of 6^2) $p_1, p_2, \ldots p_6$ such values that the sum of 3 equals the sum of the been fruitlessly employed in trying to form such squares. It may be well to mention here that a very interesting historical essay on the subject of magic squares has been published by Dr. Siegmund Günther, in his work entitled *Vermischte Untersuchungen zur Geschichte der Mathematischen Wissenschaften*, Leipsic, 1876. This work contains the fragment of Moschopulus. The short historical notices found in ordinary books of reference are necessarily very inadequate.

2. Consider the square array

$$\left.\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \ldots\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \ldots\ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \ldots\ldots & a_{nn} \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

There are n rows, n columns, and two principal diagonals. We may, however, reckon $2(n-1)$ secondary or broken diagonals, each being made up of a series of literal elements parallel to a principal diagonal together with a complementary series parallel to the same diagonal, but on the other side of it, the two series being composed of n elements. The claim of these pairs of series to be regarded as diagonals is apparent if we suppose the arrangement to be applied by deformation to the surface of an anchor ring. The broken diagonals then become complete.

Further, let the numerical values of the literal elements be such that the sum of each row, of each column, and of each diagonal, is the same. The square is then hyper-magic. The number n is the order, and the common sum-value of the rows, columns, and diagonals is the weight of the square (A).*

3. It is necessary to take the cases of odd and even orders separately.

---

other 3; but, as the sum of $6 = \frac{1}{3}\, 6\cdot 7$, an odd number, they cannot be thus divided; but if we pass over one of the p's and r's, making the r's, 1, 2, 3, 4, 5, 7, and the p's = 0, 1, 2, 3, 4, 6, and multiply the p's by 6, we get a Nasik Square, but not in consecutive numbers."

If any one is satisfied that the foregoing is a proof or even an enunciation of the absolute negation in the text, I must leave the matter so, and can only say that I am unable to read the passage in that way.

* See additional note at the end of this paper.

When $n = 3$, the hyper-magic form is impossible except for equal elements.

The simply magic form is

$$\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ \dfrac{4a_{13}-2a_{11}+a_{12}}{3} & \dfrac{a_{11}+a_{12}+a_{13}}{3} & \dfrac{4a_{11}-2a_{13}+a_{12}}{3} \\ \dfrac{2a_{11}+2a_{12}-a_{13}}{3} & \dfrac{2a_{11}+2a_{13}-a_{12}}{3} & \dfrac{2a_{12}+2a_{13}-a_{11}}{3}. \end{array}$$

Here the sum-values of the two principal diagonals, the rows and columns are the same. For the elements 1, 2, 3 ... 9, the form is one of the aspects of

$$\begin{array}{ccc} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{array}$$

We can make a complete set of parallel diagonals fulfil the weight condition by making

$$a_{11}+a_{12} = 2a_{13} \quad \text{or} \quad a_{12}+a_{13} = 2a_{11}.$$

When $n = 5$, the direct hyper-magic conditions are 20 in number, but not all independent. It is convenient to write $(p, q, r \ldots)$ for $a_{1p}+a_{2q}+a_{3r}+\ldots$, or simply $(p)_m$ when $p = q = r = \ldots$, and m is the number of elements involved. The order of the left-hand suffixes is supposed to remain unchanged.

The conditions are expressed directly by

$$\sum_{\mu=1}^{\mu=5} a_{1_\mu} = \sum_{\mu=1}^{\mu=5} a_{2_\mu} = \sum_{\mu=1}^{\mu=5} a_{3_\mu} = \sum_{\mu=1}^{\mu=5} a_{4_\mu} = \sum_{\mu=1}^{\mu=5} a_{5_\mu} = W,$$

$$\begin{aligned} a_{31}+a_{41}+a_{51} &= W-(1)_2, & a_{31}+a_{42}+a_{53} &= W-(45), \\ a_{32}+a_{42}+a_{52} &= W-(2)_2, & a_{32}+a_{43}+a_{54} &= W-(51), \\ a_{33}+a_{43}+a_{53} &= W-(3)_2, & a_{33}+a_{44}+a_{55} &= W-(12), \\ a_{34}+a_{44}+a_{54} &= W-(4)_2, & a_{34}+a_{45}+a_{51} &= W-(23), \\ a_{35}+a_{45}+a_{55} &= W-(5)_2, & a_{35}+a_{41}+a_{52} &= W-(34), \end{aligned}$$

$$\begin{aligned} a_{31}+a_{45}+a_{54} &= W-(32), \\ a_{32}+a_{41}+a_{55} &= W-(43), \\ a_{33}+a_{42}+a_{51} &= W-(54), \\ a_{34}+a_{43}+a_{52} &= W-(15), \\ a_{35}+a_{44}+a_{53} &= W-(21). \end{aligned}$$

Writing C_{pq} for $a_{3p}-a_{3q}$, we get

$$(54)+(34)-(1)_2-(2)_2 = C_{13}-C_{52} = K_1,$$
$$(15)+(45)-(2)_2-(3)_2 = C_{24}-C_{13} = K_2,$$
$$(21)+(51)-(3)_2-(4)_2 = C_{35}-C_{24} = K_3,$$
$$(32)+(12)-(4)_2-(5)_2 = C_{41}-C_{35} = K_4,$$
$$(43)+(23)-(5)_2-(1)_2 = C_{52}-C_{41} = K_5.$$

But $-5a_{31}+W = -4a_{31}+a_{32}+a_{33}+a_{34}+a_{35} = 4C_{31}+3C_{53}+2C_{25}+C_{42}$
$$= 2K_2+K_3+2K_4$$

(because $C_{31}+C_{53}+C_{25}+C_{42}+C_{14} = 0$), and finally

$$a_{31} = a_{23}+a_{24}-a_{11}.$$

A hyper-magic square can always be varied, without losing its characteristic properties, by putting the last row first, or the last column first, or *vice versâ*.* This is self-evident when the square is represented on an anchor ring, or is rolled round a cylinder. We can therefore obtain $a_{32}, a_{33}, a_{34}, a_{35}$ by increasing the right-hand suffixes by unity successively, and rejecting multiples of 5 when the rule gives a suffix exceeding 5.

The elements of the last two rows are found by using the two next preceding rows in each case. Thus

$$a_{41} = (a_{25}+a_{21}-a_{13})+(a_{21}+a_{22}-a_{14})-a_{31} = W-a_{24}-a_{23}-a_{13}-a_{14}.$$

The general solution is sufficiently indicated by setting down the first and last columns, as follows :—

$$\begin{array}{ccccc}
a_{11} & . & . & . & a_{15} \\
a_{21} & . & . & . & a_{25} \\
a_{23}+a_{24}-a_{11} & . & . & . & a_{22}+a_{23}-a_{15} \\
W-a_{23}-a_{24}-a_{13}-a_{14} & . & . & . & W-a_{12}-a_{13}-a_{22}-a_{23} \\
a_{13}+a_{14}-a_{31} & . & . & . & a_{12}+a_{13}-a_{25},
\end{array}$$

with $$\sum_{\mu=1}^{\mu=5} a_{1\mu} = \sum_{\mu=1}^{\mu=5} a_{2\mu} = W.$$

---

* The definition of "carrés diaboliques" given by M. Lucas (*Récréations Math.*, Introduction, t. I., p. XVII.) is founded on this property.

4. If $n = 7$, these are the conditions :—

$$\sum_{\mu=1}^{\mu=7} a_{1\mu} = \sum_{\mu=1}^{\mu=7} a_{2\mu} = \sum_{\mu=1}^{\mu=7} a_{3\mu} = \sum_{\mu=1}^{\mu=7} a_{4\mu} = \sum_{\mu=1}^{\mu=7} a_{5\mu} = \sum_{\mu=1}^{\mu=7} a_{6\mu} = \sum_{\mu=1}^{\mu=7} a_{7\mu} = W,$$

and three other sets of conditions, the leading equations of which, expressed in the present notation, are

$$a_{51}+a_{61}+a_{71} = W-(1)_4, \quad a_{51}+a_{64}+a_{73} = W-(4567),$$

$$a_{51}+a_{67}+a_{76} = W-(5432).$$

The remaining conditions are formed by adding successively unity to each right-hand suffix, and rejecting multiples of 7 when a suffix exceeds 7.

Writing E_{pq} for $a_{6p}-a_{5q}$, we get, from the conditions,

$$(7654)+(3456)-(1)_4-(2)_4 = E_{13}-E_{72} = K_1,$$

$$(1765)+(4567)-(2)_4-(3)_4 = E_{24}-E_{13} = K_2,$$

$$(2176)+(5671)-(3)_4-(4)_4 = E_{35}-E_{24} = K_3,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$(6543)+(2345)-(7)_4-(1)_4 = E_{72}-E_{61} = K_7.$$

Also
$$-7a_{51}+W = -6a_{51}+a_{52}+a_{53}+a_{54}+a_{55}+a_{56}+a_{57}$$
$$= 6E_{31}+5E_{53}+4E_{75}+3E_{27}+2E_{42}+E_{64}$$
$$= 3K_2+2K_3+4K_4+2K_5+3K_6,$$

by means of the identity

$$E_{13}+E_{35}+E_{57}+\dots+E_{61} = 0.$$

Substituting the values of $K_2 \dots K_6$ in terms of elements, we get

$$-7a_{51}+W = 7a_{11}-W+7\,(a_{21}+a_{22}+a_{27})-3W+7\,(a_{31}-a_{34}-a_{35})+W$$
$$+7\,(a_{41}+a_{42}+a_{47})-3W,$$

or
$$a_{51} = W-a_{11}-a_{21}-a_{22}-a_{27}-a_{31}+a_{34}+a_{35}-a_{41}-a_{42}-a_{47}.$$

The values of $a_{52} \dots a_{57}$ follow by symmetry.

Making use now of the second, third, fourth, and fifth rows of elements, we get

$$a_{61} = a_{11}+a_{12}+a_{17}+a_{21}+a_{23}+a_{27}-a_{24}-a_{25}+a_{31}+a_{32}+a_{37}-a_{34}-a_{35}$$
$$+a_{41}+a_{42}+a_{47}-W.$$

The full solution is sufficiently shown by setting down the first and last columns, thus :—

$$\begin{array}{ccccccc} a_{11} & . & . & . & . & . & a_{17} \\ a_{21} & . & . & . & . & . & a_{27} \\ a_{31} & . & . & . & . & . & a_{37} \\ a_{41} & . & . & . & . & . & a_{47} \end{array}$$

$$W-a_{11}-a_{21}-a_{22}-a_{27}-a_{31}+a_{34}+a_{35}-a_{41}-a_{42}-a_{47} \dots W-a_{17}-a_{27}-a_{21}-a_{26}-a_{37}+a_{33}+a_{34}-a_{47}-a_{41}-a_{46}$$

$$\left.\begin{array}{l} a_{11}+a_{12}+a_{17}+a_{21}+a_{22}+a_{27}-a_{24}-a_{25} \\ +a_{31}+a_{32}+a_{37}-a_{34}-a_{35}+a_{41}+a_{42}+a_{47}-W \end{array}\right\} \dots \left\{\begin{array}{l} a_{17}+a_{11}+a_{16}+a_{27}+a_{21}+a_{26}-a_{23}-a_{24} \\ +a_{37}+a_{31}+a_{36}-a_{33}-a_{34}+a_{47}+a_{41}+a_{46}-W \end{array}\right.$$

$$W-a_{41}-a_{31}-a_{32}-a_{37}-a_{21}+a_{24}+a_{25}-a_{11}-a_{12}-a_{17} \dots W-a_{47}-a_{37}-a_{31}-a_{36}-a_{27}+a_{23}+a_{24}-a_{17}-a_{11}-a_{16},$$

with $$\sum_{\mu=1}^{\mu=7} a_{1\mu} = \sum_{\mu=1}^{\mu=7} a_{2\mu} = \sum_{\mu=1}^{\mu=7} a_{3\mu} = \sum_{\mu=1}^{\mu=7} a_{4\mu} = W.$$

5. We can now see the form of the general solution for $n = 2m+1$.

Using the same notation, and writing Pab for $a_{2m-1,a}-a_{2m-1,b}$, we get, from the hyper-magic conditions,

$$(2m+1,\ 2m,\ 2m-1 \dots 4)+(345 \dots 2m)-(1)_{2m-2}-(2)_{2m-2}$$
$$= P_{13}-P_{2m+1,2} = K_1,$$
$$(1,\ 2m+1,\ 2m\ \dots\ 5)+(456 \dots 2m+1)-(2)_{2m-2}-(3)_{2m-2}$$
$$= P_{24}-P_{13} = K_2,$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$(2m,\ 2m-1,\ 2m-2 \dots 3)+(234 \dots 2n-1)-(2m+1)_{2m-2}-(1)_{2m-2}$$
$$= P_{2m+1,2}-P_{2m,1} = K_{2m+1},$$

and also

$$\begin{aligned} -(2m+1)\,a_{2m-1,1}+W &= 2mP_{31}+(2m-1)\,P_{53}+(2m-2)\,P_{75}+\dots \\ &\quad \dots+(m+1)\,P_{2m+1,2m-1}+mP_{2,2m+1}+(m-1)\,P_{42} \\ &\quad +(m-2)\,P_{64}+\dots+P_{2m,2m-2} \\ &= mK_2+(m-1)\,K_3+2\,(m-1)\,K_4+2\,(m-2)\,K_5 \\ &\quad +3\,(m-2)\,K_6+3\,(m-3)\,K_7+\dots \\ &\quad \dots+2\,(m-1)\,K_{2m-2}+(m-1)\,K_{2m-1}+mK_{2m}. \end{aligned}$$

The last right-hand equivalent is verified by adding the zero value $m\,(P_{31}+P_{42}+\dots+P_{2,2m+1})$, and replacing $K_2 \dots K_{2m}$ by their P-values.

The values of $a_{2m-1,2} \dots a_{2m-1,2m+1}$ are obtained by symmetry, and the remaining rows also by repetition of same process applied to the $2m-2$ next preceding rows in each case.

6. In order, however, to reach definite conclusions, we must go into a little further detail.

It will be shown that the numerical coefficients of the literal elements of any one of the first $2m-2$ rows in the value of $a_{2m-1,1}$ have the same residues relative to $2m+1$ as modulus. To this end, the result of the last article is written more fully, as follows:—

$$\left.\begin{aligned}
&-(2m+1)\,a_{2m-1,1}+W\\
=\;& \quad m\,\{(1,2m+1,2m\ldots5)+(4,5,6\ldots2m+1)-(2)_{2m-2}-(3)_{2m-2}\}\\
&+(m-1)\{(2,1,2m+1\ldots6)+(5,6,7\ldots2m+1,1)-(3)_{2m-2}-(4)_{2m-2}\}\\
&+2(m-1)\{3,2,1\ldots7)+(6,7,8\ldots1,2)-(4)_{2m-2}-(5)_{2m-2}\}\\
&+\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\
&+(m-1)\{(2m-2,2m-3,2m-4\ldots1)+(2m+1,1,2\ldots2m-3)\\
&\qquad\qquad\qquad\qquad\qquad -(2m-1)_{2m-2}-(2m)_{2m-2}\}\\
&+m\,\{(2m-1,2m-2,2m-3\ldots2)+(1,2,3\ldots2m-2)\\
&\qquad\qquad\qquad\qquad\qquad -(2m)_{2m-2}-(2m+1)_{2m-2}\}
\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots(\mathrm{B}).$$

For uniformity of law, we may consider unity in the first row as equivalent to $2m+2$.

Let p be a suffix of the series $(2m+2, 2m+1 \ldots 5)$, and let r be the $r+1^{\text{th}}$ suffix in the column of which the leading term is p. Then

$$p+r = \tau+\rho\,(2m+1) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\alpha),$$

where ρ is zero or unity. Similarly, let q be a suffix of the series $(4, 5, 6 \ldots 2m+1)$ of the same rank (say σ) as p, and let τ be the $s+1^{\text{th}}$ suffix of the column whose leading term is q. Then

$$q+s = \tau+\rho'\,(2m+1) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta),$$

when ρ' is zero or unity. Also we have

$$p+q = 2m+6 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\gamma).$$

If r is odd, the numerical coefficient, outside the brackets of the $r+1^{\text{th}}$ row, is

$$\frac{r+1}{2}\left(m-\frac{r+1}{2}\right) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\delta);$$

if r is even, the coefficient is

$$\frac{r+2}{2}\left(m-\frac{r}{2}\right) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\epsilon).$$

The same rules apply to s. For $r = 2m-1$ or $2m$, the value of (δ) or (ϵ) is zero.

There are several cases to consider.

I., r and s both odd: the coefficient of a_{rr} in the right-hand member of (B), is, by (α), (β), (γ), and (δ),

$$\frac{r-p+1+\rho\,(2m+1)}{2}\cdot\frac{(1-\rho)(2m+1)+p-r-2}{2}$$

$$+\frac{r+p-4-(1-\rho')(2m+1)}{2}\cdot\frac{(2-\rho')(2m+1)-p-r+3}{2}$$

$$-\frac{(r-1)(2m-r+2)}{2},$$

the last term being due to terms $(r)_{2m-2}$.

Since ρ, ρ' are in this case both zero or both unity, we write ρ for ρ', and the expanded coefficient is

$$\tfrac{1}{4}\left\{-2\,(1-\rho)^2\,(2m+1)^2+\rho\,(4-4r)(2m+1)+(2r+2p-8)(2m+1)\right.$$
$$\left.-2\,(p-2)(p-3)\right\}.$$

The residue, relative to $2m+1$ as modulus, is independent of r, since the explicit multiplier of $2m+1$ is integer.

II., r and s are both even: the coefficient of a_{rr} is by (α), (β), (γ) and (ϵ),

$$\frac{r-p+2+\rho\,(2m+1)}{2}\cdot\frac{(1-\rho)(2m+1)+p-r-1}{2}$$

$$+\frac{r+p-3-(1-\rho')(2m+1)}{2}\cdot\frac{(2-\rho')(2m+1)-r-p+4}{2}$$

$$-\frac{(r-1)(2m-r+2)}{2}.$$

We may write ρ for ρ', since they have the same values, and the expanded expression is

$$\tfrac{1}{4}\left\{-2\,(1-\rho)^2\,(2m+1)^2+\rho\,(4-4r)(2m+1)+(2r+2p-6)(2m+1)\right.$$
$$\left.-2\,(p-2)(p-3)\right\}.$$

The residue, relative to $2m+1$ as modulus, is independent of r, and the same as before.

In the cases III., r odd and s even, IV., r even, s odd, we may put $1-\rho$ for ρ'. The coefficient of a_{or} is, in both cases,

$$\tfrac{1}{4}\left\{-2\rho^2(2m+1)^2+\rho(4p-10)(2m+1)-2(p-2)(p-3)\right\}.$$

The residue is the same, relative to $2m+1$ as modulus.

7. It appears then that

$$-(2m+1)\,a_{2m-1,1}=(2m+1)\,L+\mu W,$$

where L is a linear and integral function of the elements of the first $2m-2$ rows, and

$$\mu=-1-\sum_{p=5}^{p=2m+2}\frac{p-2\,.\,p-3}{2}=-\frac{2m-1\,.\,2m\,.\,2m+1}{2\,.\,3}.$$

Hence, if all the elements are to be integers, and $2m+1$ is a multiple of 3, W must also be a multiple of 3. This condition is satisfied when the elements are in arithmetical progression, and in fact hyper-magic squares can be formed in this case. Yet there is a well-known fundamental distinction between the cases of orders prime and not prime to 3, as to manner of construction.

This difference is due to the circumstance that if we have $2m+1$ distinct literal elements repeated $2m+1$ times, we cannot form a hyper-magic square with them when the order is a multiple of 3. The conditions

$$\sum_{\mu=1}^{\mu=2m+1}a_{1\mu}=\sum_{\mu=1}^{\mu=2m+1}a_{2\mu}=\ldots=\sum_{\mu=1}^{\mu=2m+1}a_{2m-2,\mu}=W$$

are to be satisfied.

8. I pass now to the case of an even order. For the order 4 the general form is

$$\begin{array}{cccc}
a_1, & a_2, & a_3, & a_4, \\
\frac{W}{2}-a_1+s, & \frac{W}{2}-a_2-s, & \frac{W}{2}-a_3+s, & \frac{W}{2}-a_4-s, \\
\frac{W}{2}-a_3, & \frac{W}{2}-a_4, & \frac{W}{2}-a_1, & \frac{W}{2}-a_2, \\
a_3-s, & a_4+s, & a_1-s, & a_2+s,
\end{array}$$

of course, with the condition

$$a_1+a_2+a_3+a_4 = W.$$

For integral elements, therefore, W must be even. This condition is fulfilled by numbers in arithmetical progression. A number s is introduced, for which there is no equivalent when the order is odd. The next even order, 6, shows the same peculiarity. We have

$$\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ \dots & & \dots & & \dots & \dots \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}. \end{array}$$

The sums of these rows must satisfy the weight condition, and then

$$\begin{array}{ll} a_{41}+a_{51}+a_{61} = W-(1)_3, & a_{41}+a_{52}+a_{63} = W-(456), \\ a_{42}+a_{52}+a_{62} = W-(2)_3, & a_{42}+a_{53}+a_{64} = W-(561), \\ \dots\quad\dots\quad\dots\quad\dots & \dots\quad\dots\quad\dots\quad\dots \\ a_{46}+a_{56}+a_{66} = W-(6)_3, & a_{46}+a_{51}+a_{62} = W-(345), \end{array}$$

$$a_{41}+a_{56}+a_{65} = W-(432),$$
$$a_{42}+a_{51}+a_{66} = W-(543),$$
$$\dots\quad\dots\quad\dots\quad\dots$$
$$a_{46}+a_{55}+a_{64} = W-(321),$$

whence we get

$$\begin{array}{ll} a_{41}-a_{43}+a_{51}-a_{53} = (654)-(1)_3, & a_{41}-a_{45}+a_{51}-a_{56} = (234)-(1)_3, \\ a_{42}-a_{44}+a_{52}-a_{53} = (165)-(2)_3, & a_{42}-a_{46}+a_{52}-a_{51} = (345)-(2)_3, \\ \dots\quad\dots\quad\dots\quad\dots\quad\dots & \dots\quad\dots\quad\dots\quad\dots\quad\dots \\ a_{46}-a_{42}+a_{56}-a_{51} = (543)-(6)_3, & a_{46}-a_{44}+a_{56}-a_{55} = (123)-(6)_3; \end{array}$$

and then, writing P_{ab} for $a_{4a}-a_{4b}$,

$$a_{41}-a_{43}+a_{42}-a_{46} = (654)+(345)-(1)_3-(2)_3 = P_{13}+P_{26} = K_1,$$
$$a_{42}-a_{44}+a_{43}-a_{41} = (165)+(456)-(2)_3-(3)_3 = P_{24}+P_{31} = K_2,$$
$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$
$$a_{46}-a_{42}+a_{41}-a_{45} = (543)+(234)-(6)_3-(1)_3 = P_{62}+P_{15} = K_6,$$

and $$\sum_{\mu=1}^{\mu=6} a_{1\mu} = \sum_{\mu=1}^{\mu=6} a_{2\mu} = \dots = \sum_{\mu=1}^{\mu=6} a_{6\mu} = W.$$

Also we have

$$P_{31} = P_{31},$$
$$P_{51} = P_{31}+P_{53},$$
$$P_{21} = P_{31}+P_{53}+P_{26}+P_{65},$$
$$P_{41} = P_{31}+P_{53}+P_{26}+P_{42}+P_{65},$$
$$P_{61} = P_{31}+P_{53}+P_{26}+P_{42}+P_{64}+P_{65},$$

whence $\quad -6a_{41}+W = 5P_{31}+4P_{53}+3P_{26}+2P_{42}+P_{64}+3P_{65};$

similarly $\quad -6a_{42}+W = 5P_{42}+4P_{64}+3P_{31}+2P_{53}+P_{15}+3P_{16},$

and $\quad -6(a_{41}+a_{42})+2W = 8P_{31}+6P_{53}+3P_{26}+7P_{42}+5P_{64}+4P_{15},$

which can be expressed in terms of the K's.

For, taking

$$pK_1+qK_2+rK_3+sK_4+tK_5+uK_6+k(P_{31}+P_{53}+P_{15})+l(P_{42}+P_{26}+P_{64}),$$

and identifying the expression with the value of $-6(a_{41}+a_{42})+2W$, we may make $p = u = 0$ (because $K_1+K_3+K_5 = K_2+K_4+K_6 = 0$), so that

$$k+q = 8, \quad k+s-r = 6, \quad k-t = 4, \quad l+r-q = 7, \quad l = 3,$$
$$l+t-s = 5,$$

the solution of which is

$$q = 0, \quad k = 8, \quad t = 4, \quad s = 2, \quad r = 4;$$

and therefore $\quad -3(a_{41}+a_{42})+W = 2K_3+K_4+2K_5;$

similarly $\quad -3(a_{42}+a_{43})+W = 2K_4+K_5+2K_6,$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

These equations show that one of the elements of the fourth row remains undetermined.

Substituting the values of the K's, we get

$$-(a_{41}+a_{42}) = a_{11}+a_{12}+a_{21}+a_{22}-a_{24}-a_{25}+a_{31}+a_{32}-a_{34}-a_{35}-\frac{2W}{3}.$$

Put $\quad -(\delta+a_{41}) = a_{11}+a_{21}-a_{24}+a_{31}-a_{34}-\frac{W}{3},$

and then we have

$$a_{41} = -a_{11} - a_{21} + a_{24} - a_{31} + a_{34} + \frac{W}{3} - \delta,$$

$$a_{42} = -a_{12} - a_{22} + a_{25} - a_{32} + a_{35} + \frac{W}{3} + \delta,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$a_{46} = -a_{16} - a_{26} + a_{23} - a_{36} + a_{33} + \frac{W}{3} + \delta.$$

This shows that for integral elements the weight must be a multiple of 3.

But W must also be even; for, if we take as given the elements of the first four rows, we get

$$\begin{aligned} -6a_{51} + W &= 5(a_{52} - a_{51}) + 4(a_{53} - a_{51}) + 3(a_{54} - a_{53}) \\ &\qquad + 2(a_{55} - a_{54}) + a_{56} - a_{55} \\ &= 5\left[(3456) - (2)_4\right] + 4\left[(4561) - (3)_4\right] \\ &\quad + 3\left[(5612) - (4)_4\right] + \left[(6123) - (5)_4\right] + (1234) - (6)_4 \\ &= W - 6a_{12} + 2W - 6a_{22} - 6a_{23} + 3W - 6a_{32} - 6a_{33} - 6a_{34} \\ &\qquad + 4W - 6a_{42} - 6a_{43} - 6a_{44} - 6a_{45}, \end{aligned}$$

or
$$a_{51} = a_{12} + a_{22} + a_{23} + a_{32} + a_{32} + a_{33} + a_{34} - a_{41} - a_{46} - \frac{W}{2}.$$

Substituting for $a_{41} + a_{46}$,

$$a_{51} = a_{11} + a_{12} + a_{16} + a_{21} + a_{22} + a_{26} - a_{24} + a_{31} + a_{32} + a_{36} - \tfrac{7}{6}W.$$

The first column of the general form of solution is therefore

$$a_{11},$$
$$a_{21},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{31},$$
$$-a_{11} - a_{21} + a_{24} - a_{31} + a_{34} + \frac{W}{3} - \delta,$$
$$a_{11} + a_{12} + a_{16} + a_{21} + a_{22} + a_{26} - a_{24} + a_{31} + a_{32} + a_{36} - \tfrac{7}{6}W,$$
$$-a_{11} - a_{12} - a_{16} - a_{21} - a_{22} - a_{26} - a_{31} - a_{32} - a_{36} - a_{34} + \tfrac{11}{6}W + \delta.$$

The other columns are obtained by successively adding unity to the right-hand suffixes and changing the signs of δ.

9. Let $n = 2m$, and denote $a_{2m-2,\mu} - a_{2m-2,\nu}$ by $P_{\mu\nu}$.

Then, analogously to previous cases,

$$P_{13}+P_{2,2m} = (2m, 2m-1 \dots 4) - (1)_{2m-3} + (3, 4 \dots 2m-1) - (2)_{2m-3} = K_1,$$
$$P_{24}+P_{31} = (1, 2m \dots 5) - (2)_{2m-3} + (4, 5 \dots 2m) - (3)_{2m-3} = K_2,$$
$$P_{35}+P_{42} = (2, 1 \dots 6) - (3)_{2m-3} + (5, 6 \dots 2m, 1) - (4)_{2m-3} = K_3,$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$P_{2m,2}+P_{1,2m-1} = (2m-1, 2m-2 \dots 3) - (2m)_{2m-3} + (2, 3 \dots 2m-2) - (1)_{2m-3} = K_{2m};$$

also, since

$$\begin{aligned} P_{31} &= P_{31}, \\ P_{51} &= P_{31}+P_{53}, \\ &\dots \quad \dots \quad \dots \\ P_{21} &= P_{31}+P_{53}+\dots+P_{2,2m}+P_{2m,2m-1}, \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \\ P_{2m,1} &= P_{31}+P_{53}+\dots P_{2m,2m-2}+P_{2m,2m-1}, \end{aligned}$$

we have

$$-2ma_{2m-2,1}+W = (2m-1)\,P_{31}+(2m-2)\,P_{53}+\dots+(m+1)\,P_{2m-1,2m-3}$$
$$+mP_{2,2m}+(m-1)\,P_{42}+\dots+P_{2m,2m-2}+mP_{2m,2m-1},$$

and the values of $a_{2m-2,2} \dots a_{2m-2,2m}$ follow by symmetry; so that we have

$$-2m\,(a_{2m-2,1}+a_{2m-2,2})+2W$$
$$= (3m-1)\,P_{31}+(3m-3)\,P_{53}+(3m-5)\,P_{75}+\dots$$
$$+(m+3)\,P_{2m-1,2m-3}+(m+1)\,P_{1,2m-1}+mP_{2,2m}$$
$$+(3m-2)\,P_{42}+(3m-4)\,P_{64}+\dots+(m+2)\,P_{2m,2m-2}.$$

Identifying the right-hand member with

$$p_1K_1+p_2K_2+p_3K_3+\dots+p_{2m}K_{2m}$$
$$+k\,(P_{31}+P_{53}+\dots+P_{2m-1,2m-3}+P_{1,2m-1})+l\,(P_{2,2m}+P_{42}+\dots+P_{2m,2m-2}),$$

we get

$$k+p_2-p_1 = 3m-1, \quad k+p_4-p_3 = 3m-3, \quad \dots \quad k+p_{2m}-p_{2m-1} = m+1,$$
$$l+p_3-p_4 = 3m-2, \quad l+p_5-p_4 = 3m-4, \quad \dots \quad l+p_{2m-1}-p_{2m-2} = m+2,$$
$$l+p_1-p_{2m} = m.$$

The solution is (when, by means of the two zero periods of the K's, we make $p_1 = p_{2m} = 0$), $k = 3m-1$, $l = m$,

$$p_2 = 0,\quad p_3 = 2m-2,\quad p_5 = 4m-8,\quad p_7 = 6m-18,\ \ldots\ p_{2m-1} = 2m-2.$$
$$p_4 = 2m-4,\quad p_6 = 4m-12,\quad p_8 = 6m-24,\ \ldots\ p_{2m-2} = 2m-4;$$

whence

$$\begin{aligned} -m\,(a_{2m-2,1}+a_{2m-2,2})+W \\ = \ & (m-1)\left\{(2,1,2m\ldots 6)-(3)_{2m-3}+(5,6\ldots 2m,1)-(4)_{2m-3}\right\} \\ & +(m-2)\left\{(3,2,1\ldots 7)-(4)_{2m-3}+(6,7\ldots 1,2)-(5)_{2m-3}\right\} \\ & +2(m-2)\left\{(4,3,2\ldots 8)-(5)_{2m-3}+(7,8\ldots 2,3)-(6)_{2m-3}\right\} \\ & +\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \\ & +(m-1)\left\{(2m-2,2m-3\ldots 1,2)-(2m-1)_{2m-3}\right. \\ & \qquad\qquad \left.+(1,2,3\ldots 2m-3)-(2m)_{2m-3}\right\}. \end{aligned}$$

For uniformity of law we may write $2m+2$, $2m+1$ for 2, 1 in the first row of the right-hand member.

Let p be the σ^{th} suffix in the series $(2m+2\ldots 6)$, and let q be the σ^{th} suffix in the series $(5, 6\ldots 2m-1)$. Then, for the suffix r, in the $r+1^{\text{th}}$ place of the column to which p belongs, and in the $s+1^{\text{th}}$ place in the column to which q belongs, we have

$$p+r = \tau+\rho\,(2m),\quad q+s = \tau+\rho'\,(2m),\quad p+q = 2m+7,$$

when ρ, ρ' are zero or unity. Observe also that the coefficient outside the brackets of the $r+1^{\text{th}}$ row is $\frac{r+1}{2}\left(m-\frac{r+3}{2}\right)$ if r is odd, and $\frac{r+2}{2}\left(m-\frac{r+2}{2}\right)$ if r is even. The coefficient of $(\tau)_{2m-3}$ is

$$-\frac{(\tau-2)(2m-\tau+1)}{2}.$$

Now r and s are not of the same parity, because p and q are not. If r is even and s odd, the coefficient of $a_{\sigma\tau}$ on right-hand side of the equation is

$$\begin{aligned} & \frac{\tau-p+2+\rho\,(2m)}{2}\cdot\frac{(1-\rho)(2m)-\tau+p-2}{2} \\ & +\frac{\tau+p-6+(\rho'-1)(2m)}{2}\cdot\frac{(2-\rho')(2m)-\tau-p+4}{2} \\ & -\frac{(\tau-2)(2m-\tau+1)}{2}. \end{aligned}$$

Expanding this, we find, after taking away the multiple of m, as it appears explicitly, a residue $-\frac{p-3 \,.\, p-4}{2}$, which is independent of r; the result is the same if r is odd and s even. Hence

$$-m\left(a_{2m-2,1}+a_{2m-2,2}\right) = mL+\mu W,$$

where L is a linear and integral function of the elements, and

$$\mu = -1-\sum_{p=6}^{p=2m+2}\frac{p-3 \,.\, p-4}{2} = -\frac{2m-2 \,.\, 2m-1 \,.\, 2m}{2 \,.\, 3}.$$

Hence, if the elements are all integral and m is a multiple of 3, W must also be a multiple of 3. The conditions

$$\sum_{\mu=1}^{\mu=2m} a_{1\mu} = \ldots = \sum_{\mu=1}^{\mu=2m} a_{2m-3,\mu} = W$$

must be satisfied.

10. To show that W must also be even, recourse must be had to the value of $a_{2m-1,1}$, when the elements of the first $2m-2$ rows are assumed to be given. We may take the form for n generally, viz.,

$$\left.\begin{matrix} a_{1,1}, & a_{1,2} & \ldots & a_{1,n} \\ \ldots & \ldots & & \ldots \\ a_{n,1}, & a_{n,2} & \ldots & a_{n,n} \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (C).$$

Then
$$\begin{aligned} a_{n-1,1}-a_{n-1,n} &= (2, 3, 4 \ldots n-1)-(1)_{n-2} = K_1, \\ a_{n-1,2}-a_{n-1,1} &= (3, 4, 5 \ldots n) \quad -(2)_{n-2} = K_2, \\ \ldots \quad & \ldots \quad \ldots \quad \ldots \\ a_{n-1,n}-a_{n-1,n-1} &= (1, 2, 3 \ldots n-2)-(n)_{n-2} = K_n. \end{aligned}$$

Because $a_{n-1,2} = K_2+a_{n-1,1}$, $a_{n-1,3} = K_3+K_2-a_{n-1,1}$,

$$a_{n-1,4} = K_4+K_3+K_2-a_{n-1,1}, \text{ \&c.},$$

and
$$a_{n-1,1}+a_{n-1,2}+\ldots+a_{n-1,n} = w,$$

we get $-na_{n-1,1}+W = (n-1)K_2+(n-2)K_3+(n-3)K_4+\ldots+K_n$.

This may be written more conveniently

$$\begin{aligned} -na_{n-1,1}+W = \; & (n-1)\left\{(3, 4, 5 \ldots n)-(2)_{n-2}\right\} \\ & +(n-2)\left\{(4, 5, 6 \ldots 1)-(3)_{n-2}\right\} \\ & +(n-3)\left\{(5, 6, 7 \ldots 2)-(4)_{n-2}\right\} \\ & + \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ & + \quad \left\{(1, 2, 3 \ldots n-2)-(n)_{n-2}\right\}. \end{aligned}$$

Let p be the σ^{th} suffix of the set $(3, 4, 5 \dots n)$, and let r be the suffix in the $r+1^{\text{th}}$ row on the right-hand side of the equation, and in the column under p. Then

$$p+r = \tau+\rho n,$$

where ρ is zero or unity. The corresponding multiplier outside the brackets is $n-r-1$, and the whole coefficient of $a_{\sigma\tau}$ on the right-hand side of the equation is $(n-\tau-\rho n+p-1)-(n-\tau+1)$, and this is congruent with $p-2$ as to the mod n.

Hence $$-na_{n-1,1} = nL+\mu W,$$

where L is an integral linear function of the elements of the first $n-2$ rows of (C), and

$$\mu = \sum_{p=3}^{p=n}(p-2)-1 = \frac{n-2 \cdot n-1}{2}-1 = \frac{n(n-3)}{2}.$$

It follows that for integer elements W must be even, if n is even.

Hence a hyper-magic square of even order cannot be formed with integer elements unless the weight is even, nor if the order is a multiple of 3, unless the weight is also a multiple of 3.

The elements $1, 2, 3 \dots 4(2m+1)^2$ give the weight

$$= (2m+1)\left[4(2m+1)^2+1\right],$$

which is odd. Consequently, a hyper-magic square with these elements is impossible in every case.

Moreover, it is not possible to form an oddly even hyper-magic square with integer elements in arithmetical progression, as $a+d$, $a+2d$, &c. For such a square is the sum of two squares, one of them having equal elements, and the other having the elements $1, 2, 3 \dots 4(2m+1)^2$, each multiplied by the common difference, and this is the case though we make the weight even by appropriate values of a and d.

Additional Note.

[I did not think it necessary to justify my use of the names "magic square" and "hyper-magic square" for squares filled up with any numbers fulfilling the usual magical conditions as to summation. But some mathematicians still insist on a narrow meaning, and, therefore, I now add several extracts which bear me out in a more liberal application of the terms.

It is quite true that the earlier (and, as I think, unfortunately, some later) definitions require a magic square to have for elements the natural numbers from 1 to n^2, where n is the root or order. But it was found unnecessary and mathematically undesirable so to restrict the meaning. Thus, Schottus (1664, *Curiosa Technica*, Lib. XI., Cap. XIV., quoted by Günther) formulates the problem as follows: "Numeros quoscunque quadratos ita in quadrata disponere, ut quævis series additæ, sive transversim sumantur sive a summo deorsum sive decussatim seu diagonaliter semper eandem summam conficiant." The historical notice connected with M. Sauveur's paper (1710, *Mém. de l'Académie Royale des Sciences*) contains the following passage: "De tout cela il suit qu'au lieu qu'on prenoit pour la construction des Quarrés Magiques que des nombres en progression arithmétique et même naturelle, la choix est beaucoup plus libre qu'on ne pensoit. C'est telle liberté, reconnue par M. Sauveur dans toute son étendue et avec les seules restrictions absolument necessaires, qui lui a fait naître la pensée de construire les Quarrés Magiques par lettres, c'est-à-dire d'une manière beaucoup plus générale que l'on n'a jamais fait et aussi générale qu'il soit possible. Car dès que les nombres ont quelque chose en général et d'indéterminé, les lettres sont propres à exprimer toute leur généralité et leur indétermination."

Then, in Hutton's edition of *Ozanam's Recreations* (1803) we find that "the name 'magic square' is given to a square divided into several other small equal squares or cells, filled up with the terms of any progression of numbers, but generally an arithmetical one, in such a manner that those in each band, whether horizontal or vertical or diagonal, shall always form the same sum." In the *Penny Cyclopædia*, it is, I suppose, Professor de Morgan who writes: "Magic square. — This term is applied to a set of numbers arranged in a square, in such a manner that the vertical, horizontal, and diagonal columns shall give the same sums." There are intermediate definitions. In fact, two things strike anyone who looks into the history of the subject: (1) the vacillation and ambiguity of definition, (2) the frequent reproduction and development of old methods of formation without due recognition of previous results.]

Thursday, December 8th, 1892.

A. B. KEMPE, Esq., F.R.S., President, in the Chair.

Messrs. H. G. Dawson, M.A., Fellow of Christ's College, Cambridge, W. J. Greenstreet, M.A., formerly of St. John's College, Cambridge, and W. Welsh, M.A., Fellow and Mathematical Tutor of Jesus College, Cambridge, were elected members.

The Auditor, Mr. Heppel, having read his report, upon the motion of Professor Greenhill, seconded by Lieut.-Col. Cunningham, the Treasurer's report was adopted, and Mr. Heppel thanked for the trouble he had taken.

The following communications were made:—

On a Theorem in Differentiation, and its Application to Spherical Harmonics: Dr. Hobson.

On Cauchy's Condensation Test for the Convergency of Series: Dr. M. J. M. Hill.

Additional Note on Secondary Tucker Circles: Mr. J. Griffiths.

Notes on Determinants: Mr. J. E. Campbell.

A Geometrical Note: Mr. R. Tucker.

The President (Major MacMahon in the chair) made an impromptu communication upon a problem which he thought to be subsidiary to that of the "Stamp-folding" Problem.

The following presents were received:—

"Vector Algebra and Trigonometry," by R. Baldwin Hayward; 8vo, 1892. From the Author.

"Zeittafeln zur Geschichte der Mathematik, Physik, und Astronomie, bis zum Jahre 1500," von Dr. Felix Müller; 8vo, Leipzig, 1892.

"Beiblätter zu den Annalen der Physik und Chemie," Band XVI., Stück 10.

"Proceedings of the Royal Society," Vol. LII., No. 316.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig," 1892, III.

"Bulletin of the New York Mathematical Society," Vol. II., No. 2.

"Bulletin de la Société Mathématique de France," Tome XX., No. 6.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXVI., 3me Livraison; Harlem.

"Entwurf einer neuen Integralrechnung auf Grund der Potenzial-, Logarithmal-, und Numeralrechnung," von Dr. Julius Bergbohm; Pamphlet, 8vo, Leipzig, 1892.

"Kansas University Quarterly," Vol. I., No. 2; October, 1892.

"Bulletin des Sciences Mathématiques," Tome XVI.; October, 1892.

"Rendiconti del Circolo Matematico di Palermo," Tome VI., Fasc. 5.

"Bestimmung der Trägheitsmomente des menschlichen Körpers und seiner Glieder," von W. Braune und O. Fischer, No. VIII. des XVIII. Bandes der Abhandlungen der mathematisch-physischen Classe der K.-S. Gesells. der Wissenchaften zu Leipzig.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. I., Fasc. 8–9, 2° Semestre; Roma, 1892.

"Educational Times," December, 1892.

"Annales de la Faculté des Sciences de Toulouse," Tome VI., Fasc. 3; 1892.

"Indian Engineering," Vol. XII., Nos. 18, 19, 20.

"Invention," Vol. XIV., No. 706, N.S.

On a Theorem in Differentiation, and its application to Spherical Harmonics. By E. W. HOBSON. Received and read December 8th, 1892.

It has been shown by Clebsch,* in a paper entitled "Ueber eine Eigenschaft der Kugelfunctionen," that, if $f_n(x, y, z)$ denote any rational homogeneous function of x, y, z of degree n, the expression

$$f_n - \frac{r^2 V^2 f_n}{2 \cdot 2n-1} + \frac{r^4 V^4 f_n}{2 \cdot 4 \cdot 2n-1 \cdot 2n-3} - \dots$$

is a spherical harmonic, where $r^2 = x^2+y^2+z^2$, and V^2 is Laplace's operator. The consideration of this theorem has led me to a theorem in differentiation which it is the object of the present communication to investigate and to apply to the theory of spherical harmonics.

1. Let $f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ denote a rational homogeneous function of degree n of the three operators, and suppose it required to find an expression for $f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r}$, r denoting $(x^2+y^2+z^2)^{\frac{1}{2}}$. It is clear that the required expression is of the form

$$(-1)^n . 1.3.5 \dots (2n-1)\left[\frac{f_n(x, y, z)}{r^{2n+1}} + \frac{f_{n-2}}{r^{2n-1}} + \frac{f_{n-4}}{r^{2n-3}} + \dots\right],$$

---

* *Crelle's Journal*, Vol. LX., 1862.