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VIBRATION EQUATIONS OF A RECTANGULAR ORTHOTROPIC VISCOELASTIC PLATE WITH ATTACHED MASSES BASED ON THE TIMOSHENKO HYPOTHESIS

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Abstract: This paper investigates the derivation of governing equations for both free and forced vibrations of a rectangular orthotropic viscoelastic plate carrying attached masses. The formulation is based on the Timoshenko plate theory. The equations are obtained using an energy-based approach, specifically the Hamilton–Ostrogradsky principle. As a result, a system of integro-differential equations describing the coupled dynamics of the plate and the attached masses is derived.

Keywords: orthotropic plate, attached mass, viscoelasticity, natural frequency

Introduction. The study of vibrations in linear dissipative mechanical systems remains a challenging problem. Modern research often incorporates the Boltzmann–Volterra hereditary theory to describe the rheological behavior of materials. This approach leads to integro-differential equations, which are widely applied in vibration protection problems. Reducing system mass or dimensions typically increases the complexity of vibration analysis. In engineering applications such as electronic equipment, resonance effects depend strongly on structural parameters. To reduce vibration amplitudes, multilayer coatings are frequently used, although they increase the overall mass of the system.

Methods. We consider a rectangular orthotropic plate with several attached point masses connected via elastic elements. The plate edges are assumed to be fixed. The motion of each attached mass is described using Newton's second law, accounting for the stiffness of the connecting springs. The plate thickness is assumed to be small compared to its other dimensions, allowing the use of the Timoshenko hypothesis.

The deformation field is described in terms of displacements and rotations of the mid-surface. Stress-strain relations are defined using generalized Hooke's law, while viscoelastic effects are incorporated through relaxation kernels. The governing equations are obtained using the Hamilton–Ostrogradsky variational principle.[11-12]:

$$
\begin{aligned}
Э = {} & \frac{Eh}{2(1-\nu^2)}\int_0^{\zeta_1}\int_0^{2\pi}\left[\left(\frac{\partial u}{\partial \xi}+\frac{\partial v}{\partial \theta}-w\right)^2+2(1-\nu)\left(\frac{\partial u}{\partial \xi}\left(\frac{\partial v}{\partial \theta}-w\right)-\right.\right.\\
& \left.\left.-\frac{1}{4}\left(\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial \xi}\right)^2\right)\right]d\xi d\theta+\frac{Eh}{24(1-\nu^2)R^2}\int_0^{\xi_1}\int_0^{2\pi}\left[\left(\frac{\partial^2 w}{\partial \xi^2}+\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial v}{\partial \theta}\right)^2-\right.\\
& \left.-2(1-\nu)\left(\frac{\partial^2 w}{\partial \xi^2}\left(\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial v}{\partial \theta}\right)-\frac{1}{4}\left(\frac{\partial^2 w}{\partial \xi \partial \theta}+\frac{\partial v}{\partial \xi}\right)^2\right)\right]d\xi d\theta+\\
& +\frac{E_c}{2R}\sum_{l=1}^{k}\int_0^{\xi_1}\left[F_c\left(\frac{\partial u}{\partial \xi}-\frac{h_c}{R}\frac{\partial^2 w}{\partial \xi^2}\right)^2+\frac{I_{yc}}{R^2}\left(\frac{\partial^2 w}{\partial \xi^2}\right)^2+\frac{G_c}{E_c}I_{kp.c}\frac{G_c}{E_c}I_{kp.c}\left(\frac{\partial^2 w}{\partial \xi \partial \theta}+\frac{\partial v}{\partial \xi}\right)^2\right]_{\theta=\theta_1}d\xi-\\
& -\frac{\sigma_x h}{2}\int_0^{\xi_1}\int_0^{2\pi}\left(\frac{\partial w}{\partial \xi}\right)^2 d\xi d\theta-\frac{\sigma_x F_c}{2R}\sum_{i=1}^{k}\int_0^{\xi_1}\left(\frac{\partial w}{\partial \xi}\right)^2\Bigg|_{\theta=\theta_1}d\xi
\end{aligned}
\tag{1}
$$

Here $\xi_1 = \frac{L}{R}$, $\xi = \frac{x}{R}$, $\theta = \frac{y}{R}$; *x , y,z*- coordinates, E_c, G_c - elasticity and shear moduli of the longitudinal ribs material, k – number of longitudinal ribs, σ_x - axial compressive stresses, u, ν, w - components of the shell displacement vector,*h* and *R* – the thickness and radius of the shell, respectively,*E*, ν - Young's modulus and Poisson's ratio of the shell material, F_c , I_{yc} , $I_{kp.c}$ - respectively, the areas and moments of inertia of the cross-section of the longitudinal rod relative to the axis *OX* and *OZ,* and also the moment of inertia during torsion.

The kinetic energy of the shell is:

$$K = \frac{Eh}{2(1-\nu^2)} \int_0^{\xi_1}\int_0^{2\pi} \left[\left(\frac{\partial u}{\partial t_1}\right)^2 + \left(\frac{\partial \nu}{\partial t_1}\right)^2 + \left(\frac{\partial w}{\partial t_1}\right)^2 \right] d\xi d\theta$$

$$+ \frac{\overline{\rho_c} E_c F_c}{2R(1-\nu^2)} \sum_{i=1}^{k_1} \int_0^{\xi_1} \left[\left(\frac{\partial u}{\partial t_1}\right)^2 + \left(\frac{\partial w}{\partial t_1}\right) \right]_{\theta=\theta_1} d\xi \tag{2}$$

Here $\overline{\rho_c} = \frac{\rho_c}{\rho_0}$, were ρ_0, ρ_c - the densities of the shell and longitudinal rod materials, respectively, $\theta_i = \frac{2\pi}{k_1} i$.

The interaction of the filler with the shell is represented as a surface load applied to the shell, which performs work on the displacements of the contact surface when transferring the system from a deformed state to the initial undeformed state.

$$A_0 = -\int_0^{\xi_1}\int_0^{2\pi} (q_x u + q_\theta \nu + q_z w) d\xi d\theta + \int_0^{\xi_1}\int_0^{2\pi} f q_z (u+v) d\xi d\theta \tag{3}$$

were q_x, q_θ, q_z - pressure from the filler on the shell, f – coefficient of friction.

The total energy of the system is:

$$\Pi = \mathrm{Э} + K + A_0 \tag{4}$$

The equation of motion of the medium in vector form has the form [2,3]: $a_e^2 grad\ div\ \vec{S} - a_t^2 rot\ rot\ \vec{S} + \omega^2 \vec{S} = 0$, $0 \le x \le L$, $0 \le r \le R$ (5)

Were $a^2{}_t = (\lambda + 2\mu)/\rho$, $a^2{}_e = \mu/\rho$, a a_t, a_e - the propagation speeds of longitudinal and transverse waves in the filler, respectively; $S = S(S_x, S_\theta, S_z)$ - displacement vector; λ, μ - Lame coefficients. Contact conditions are added to the systems of equations of motion of the medium (5). It is assumed that the contact between the shell and the filler is rigid, i.e. when $r = R$:

$$u = S_x;\ \ \nu = S_\theta;\ \ \ w = S_z \tag{6}$$

$$q_x = -\sigma_{rx}, q_y = -\sigma_{r\theta},\ \ \ q_z = -\sigma_{rr}, w = S_r \tag{7}$$

Components $\sigma_{rx}, \sigma_{r\theta}$, σ_{rr} - stress tensors are defined as follows [13-15]:

$$\sigma_{rx} = \mu_s \left(\frac{\partial S_x}{\partial r} + \frac{\partial S_r}{\partial x}\right);\ \ \sigma_{r\theta} = \mu_s \left(r\frac{\partial}{\partial r}\left(\frac{S_r}{r}\right) + \frac{1}{r}\frac{\partial S_r}{\partial \theta} \right),$$

$$\sigma_{rr} = \lambda_s \left(\frac{\partial S_r}{\partial x} + r\frac{\partial}{\partial r}\left(\frac{S_r}{r}\right) + \frac{1}{r}\frac{\partial S_\theta}{\partial \theta} \right) + 2\mu_s \frac{\partial S_r}{r} \tag{8}$$

λ_s, μ_s - Lame coefficients for the environment.

Supplementing the equations of motion of the filler (5) with contact conditions (6) and (7), we arrive at a contact problem of vibrations of a cylindrical shell reinforced with cross-rib systems filled with a medium. In other words, the problem of vibrations of a cylindrical shell with a filler reinforced with cross-rib systems under axial compression is reduced to the joint integration of the equations of shell theory and the equations of motion of the filler when the specified conditions are met on the surface of their contact.

Further, we will consider shells whose edges are hinged. We seek the components of the displacement vector of such shells in the form:

$$u = A\cos kx\cos n\varphi\exp(i\omega_1 t_1),$$
$$\vartheta = B\sin kx\sin n\varphi\exp(i\omega_1 t_1), \qquad (9)$$
$$w = C\sin kx\cos n\varphi\exp(i\omega_1 t_1)$$

Where,A, B, C – unknown constants; $k = \frac{m\pi}{L}$ $(m = 1,2,....)$, m, n - wave numbers in the longitudinal and circumferential directions, respectively, L - length of the shell,

$$\omega_1 = \frac{\omega}{\omega_0}, \quad t_1 = \omega_0 t, \ \omega_0 = \sqrt{\frac{E}{(1-\nu^2)\rho_0 R^2}}, \quad \omega_1 = \sqrt{\frac{(1-\nu^2)\rho_0 R^2\omega^2}{E}}$$

For equal weights of the reinforced shell and the shell without reinforcement, their natural frequencies are denoted by ω and ω_0.

The solutions of system (5) have the form [4, 15]:

a) with small inertial effects from the filler on the process of system oscillations:

$$S_x = \left[-kr\frac{\partial I_n(kr)}{\partial r} - 4(1-\nu_s)kI_n(kr)\right]A_s + kI_n(kr)B_s \ \cos n\varphi\cos kx\exp(i\omega_1 t_1)$$

$$S_\varphi = \left[-\frac{n}{r}I_n(kr)B_s - \frac{\partial I_n(kr)}{\partial r}\gamma_1 r)C_s\right]\sin\varphi\cos kx\exp(i\omega_1 t_1)$$

$$S_r = \left[-k^3 rI_n(kr)A_s + \frac{\partial I_n(kr)}{\partial r}B_s + \frac{n}{r}I_n(kr)C_s\right]\cos n\varphi\sin kx\exp(i\omega_1 t_1) \qquad (10)$$

b) the inertial effects of the filler on the process of system oscillations are significant:

$$S_x = \left[A_s kI_n(\gamma_e r) - \frac{C_s\gamma_t^2}{\partial r}I_n(\gamma_1 r)\right]\cos n\varphi\cos kx\exp(i\omega_1 t_1)$$

$$S_\varphi = \left[-\frac{A_s n}{r}I_n(\gamma_e r) - \frac{C_s nk}{r\mu}I_n(\gamma_1 r) - \frac{B_s}{n}\frac{\partial I_n(\gamma_1 r)}{\partial r}\right]\sin n\varphi\sin kx\exp(i\omega_1 t_1)$$

$$S_r = \left[A_s\frac{\partial I_n(\gamma_e r)}{\partial r} - \frac{C_s k}{\mu_1}\frac{\partial I_n(\gamma_1 r)}{\partial r} + \frac{B_s}{r}I_n(\gamma_1 r)\right]\cos n\varphi\sin kx\exp(i\omega_1 t_1)$$

(11)

Here I_n - modified Bessel function of the nth order of the first kind, A_s, B_s, C_s - permanent.

Using contact conditions (6), displacements of shells (9), solution of the equation of motion of the medium (10) and (11), we express the constants A_s, B_s, C_s through A, B, C. As a result, for q_x, q_θ, q_r we find:

$$q_x = \left(\widetilde{C}_{x1}A + \widetilde{C}_{x2}B + \widetilde{C}_{x3}C\right)\cos n\varphi\ \cos kx\ \exp(i\omega_1 t_1)$$
$$q_\theta = \left(\widetilde{C}_{\theta 1}A + \widetilde{C}_{\theta 2}B + \widetilde{C}_{\theta 3}C\right)\sin n\varphi\ \sin kx\ \exp(i\omega_1 t_1)$$
$$q_r = \left(\widetilde{C}_{x1}A + \widetilde{C}_{x2}B + \widetilde{C}_{x3}C\right)\cos n\varphi\ \sin kx\ \exp(i\omega_1 t_1) \qquad (12)$$

После подстановки (12) в (3) и интегрирования по ξ и θ получаем для работы распределенных нагрузок со стороны заполнителя, приложенных к оболочке:

$$A = -R^2\pi\left[S_2\widetilde{C}_{x1}A^2 + (S_2\widetilde{C}_{x2} + S_1\widetilde{C}_{\theta 1})AB + (S_2\widetilde{C}_{x3} + S_1\widetilde{C}_{r1})AC + \right.$$
$$+ S_1(\widetilde{C}_{\theta 3} + \widetilde{C}_{r2})BC + S_1\widetilde{C}_{\theta 2}B^2 + S_1\widetilde{C}_{r3}C^2 \qquad (13)$$

Here $\widetilde{C}_{ra}$ - constant, $S_1 = \dfrac{1}{2} - \dfrac{\sin 2k\xi_1}{4k}$.

Using (1), (2), (13) for the total energy of the system we obtain a second-order polynomial with respect to the constant parameters A,B,C:

$$\Pi = (\breve{\varphi}_{11} - S_2\breve{C}_{x1} - \psi_{11}\omega_1^2)A^2 + (\breve{\varphi}_{22} - S_1\breve{C}_{\theta 2} - \psi_{22}\omega_1^2)B^2 + +(\breve{\varphi}_{23} - S_1\breve{C}_{r3} - \psi_{33}\omega_1^2 + I_1\sigma_x)C^2 +$$
$$(\breve{\varphi}_{44} - S_2\breve{C}_{x2} + S_1\breve{C}_{\theta 1})AB + +(\breve{\varphi}_{55} - S_2\breve{C}_{x3} + S_1\breve{C}_{r1})AC + S_1(\breve{\varphi}_{66} + \breve{C}_{\theta 3} + \breve{C}_{r2})BC$$

Note that the quantities $\breve{\varphi}_{ii}(i = 1,2,........,6)$, $\psi_{ii}(i = 1,2,........,6)$, $I_i(i = 1,2)$ have a bulky appearance, so we do not include them here.

The conditions of the extremum P for the parameters A, B, C reduce the solution of the problem of vibrations of a shell reinforced by longitudinal systems of ribs filled with a medium and subjected to longitudinal compression, taking into account friction in contact, to homogeneous systems of linear algebraic equations of the third order, non-trivial solutions of which are possible only if the determinant of this system is equal to zero. Equating the determinants of the indicated systems to zero, we obtain the following frequency equation:

$$2(\breve{\varphi}_{11} - S_2\breve{C}_{x1} - \psi_{11}\omega_1^2)A + (\breve{\varphi}_{44} + S_2\breve{C}_{x2} + S_1\breve{C}_{\theta 1})B + (\breve{\varphi}_{55} - S_2\breve{C}_{x3} + S_1\breve{C}_{r1})C = 0$$
$$(\breve{\varphi}_{44} + S_2\breve{C}_{x2} + S_1\breve{C}_{\theta 1})A + 2(\breve{\varphi}_{22} - S_1\breve{C}_{\theta 2} - \psi_{22}\omega_1^2)B + (\breve{\varphi}_{66} + \breve{C}_{\theta 3} + \breve{C}_{r2})C = 0$$
$$(\breve{\varphi}_{55} + S_2\breve{C}_{x3} + S_1\breve{C}_{r1})A + (\breve{\varphi}_{66} + \breve{C}_{\theta 3} + \breve{C}_{r2})B + 2(\breve{\varphi}_{33} - S_1\breve{C}_{r3} - \psi_{33}\omega_1^2 + I_1\sigma_x)C = 0 \qquad (14)$$

It is easy to see that in case a) the system of equations (14) is reduced to a cubic equation with respect to ω ω_1^2, otherwise it is transcendental. Since in what follows we will be interested only in low frequencies of bending vibrations, this equation in case a) can be simplified by discarding the terms with ω_1^4 and ω_1^6. B As a result we get ($\omega_1^2 = \lambda_a$):

$$\lambda_a = \frac{f_3^2 f_4 + f_1 f_5^2 + f_2^2 f_6}{2f_5^2\psi_{11} + f_2^2\psi_{33} - 4f_1 f_4\psi_{33} - 0{,}5f_6(f_1\psi_{22} + f_4\psi_{11})}$$
$$f_1 = \breve{\varphi}_{11} - S_2\breve{C}_{x1};\ f_2 = \breve{\varphi}_{44} + S_2\breve{C}_{x2} + S_1\breve{C}_{\theta 1};\quad f_3 = \breve{\varphi}_{55} + S_2\breve{C}_{x1} + S_1\breve{C}_{r1};$$
$$f_5 = \breve{\varphi}_{66} + \breve{C}_{\theta 3} + \breve{C}_{r2};\ f_6 = \breve{\varphi}_{33} - S_1\breve{C}_{r3} + I_1\sigma_x \qquad (15)$$

It is defined in a similar way λ_b for the occasion b).

Results and analysis. Let us present the results of the study of the influence of the number of ribs and the rigidity of the fillers on the critical stress of axial compression. The calculations were performed for the shell, medium and ribs with the following parameters:

$$E = E_c = E_h = 6{,}67\ 10^9 H/m^2;\ \nu = 0{,}3;\ \ x = 1;\ \ n = 8;\ \ h_h = 1{,}39mm;\ \ R = 160mm;$$

$$L_1 = 800mm;\quad \frac{F_c}{2\pi Rh} = 0{,}1591\ 10^{-1};\quad \frac{I_{yc}}{2\pi R^3 h}0{,}8289;\qquad h = 0{,}45mm;$$

$$F_x = 5{,}75mm^2;\quad I_{sh} = 19{,}9mm^4;\quad |h_c| = 0{,}1375\ 10^{-1}R;\quad \frac{I_{kpc}}{2\pi R^3 h} = 0{,}5305\ 10^{-6};$$

$$I_{kph} = 0{,}48mm^4;\ f = 0{,}25$$

The calculation results are presented in Fig. 1. The dependence of the axial compression stress is shown here. From Fig. 1 it is evident that with increasing stress the frequency of the system decreases. In addition, taking into account friction leads to a decrease in the value of the natural frequency of the structure under study. As noted, the method for determining the optimal reinforcement parameters is based on a comparison of the minimum vibration frequencies of a ribbed and smooth cylindrical shell, reinforced by longitudinal rib systems filled with a medium.

The following parameters are considered as variable: relative thickness of the shell $h^* = h/R$, distances between longitudinal and transverse ribs, related to the thickness of the shell ratio of the weight of all ribs to the weight of the shell φ_1 and the ratio of the weight of the longitudinal ribs to the weight of the transverse ribs φ_2. It is assumed that the radius and length of the shell, as well as the characteristics of the shape of the sections of the longitudinal and transverse ribs are predetermined. Note that for rectangular sections it is necessary to specify the relations ψ_1 and ψ_2 heights of longitudinal and annular ribs to their thicknesses, respectively. The dimensionless characteristics of the ribs included in (1), (2) are expressed through the specified parameters:

$$\overline{\gamma^{(1)}}_c = \frac{\varphi_1\varphi_2}{1+\varphi_2},\qquad \bar{\gamma}_s^{(2)} = \frac{\varphi_1}{1+\varphi_2},\qquad \frac{h_c}{R} = -\frac{h^*}{2}(1+\sqrt{a_1\varphi_1\bar{\gamma}_c^{(1)}}),$$

$$\mu_{s2} = \frac{1-\nu}{6}\frac{a_2}{\psi_2}(h^*)^2\left(\bar{\gamma}_s^{(2)}\right)^2;\quad \frac{h_c}{R} = -\frac{h^*}{2}\ 1+\ 1+\frac{1}{k_1}\ \sqrt{a_1\varphi_1\bar{\gamma}_c^{(1)}}\ ,$$

$$\eta_{s1}^{(2)} = \bar{\gamma}_{s1}^{(2)}\bar{\gamma}_s^{(2)}\frac{a_2\psi_2(h^*)^2}{12},\qquad \eta_{s1}^{(2)} = \bar{\gamma}_{s1}^{(2)}\bar{\gamma}_s^{(2)}\frac{a_2\psi_2(h^*)^2}{12},$$

$$\eta_c^{(1)} = \bar{\gamma}_c^{(1)}\ \frac{a_1}{12}\psi_1\bar{\gamma}_c^{(1)}(h^*)^2 + \ \frac{h_c}{R}\ ^2,\qquad \mu_{s1} = \frac{1-\nu}{6}(h^*)^2\left(\bar{\gamma}_c^{(1)}\right)^2\frac{a_1}{\psi_1}$$

With this formulation, the result of the study is practically independent of the characteristics of the shell material, since $(\omega^2_{\min})$, as is known, weakly depend on Poisson's ratio ν, and their attitude μ do not depend on the modulus of elasticity E. It should be noted that in order to improve the bearing capacity of the shell, it is necessary to find such a combination of parameters h^*, a_1, a_2, φ_1 and φ_2, under which μ takes on the greatest value.

As an example to illustrate the changes μ Depending on the relative weights of the ribs, the results of calculations of cylindrical shells filled with a medium reinforced by longitudinally supported rib systems are presented.

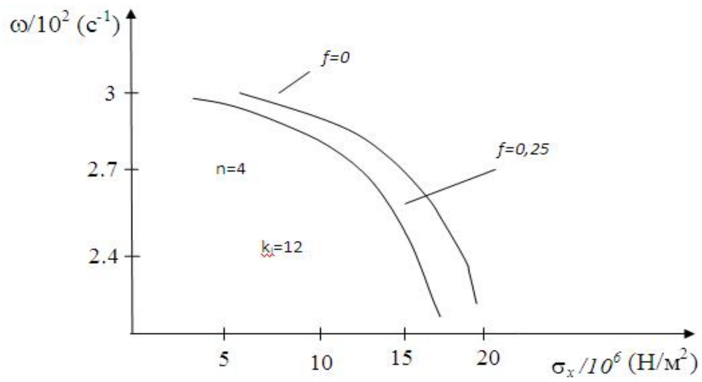

Fig. 1. System frequency dependencies $\omega = \omega_1 \omega_0$ from compressive stresses

4. Conclusions

In conclusion, a mathematical model for the vibration of a rectangular orthotropic viscoelastic plate with attached masses has been developed. The governing equations were derived using an energy-based approach and are expressed in the form of integro-differential equations. This model can be used for further analytical and numerical studies of complex mechanical systems.

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