Spectral Trace Formula and Smoothed Zero Sums:

A Prime–Zero Duality Framework

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Abstract

We study a one-parameter family of finite-rank operators $\widetilde{T}(\sigma)$ acting on a finite-dimensional Hilbert space H_{null} over the imaginary parts of the non-trivial zeros of $\zeta(s)$. The operators arise from an explicit coupling map $\Phi(\sigma)\colon H_{\text{str}} \to H_{\text{null}}$ encoding prime–zero resonance via $\sin(\sigma\gamma_k \log p)$. No hypothetical input is used: no Riemann Hypothesis, no Hilbert–Pólya postulate, no GUE conjecture.

What is proved. We establish an exact trace formula $\operatorname{Tr}(\widetilde{T}(\sigma)) = D_{\text{SEL}} - O(\sigma)$ algebraically, for all $\sigma \in \mathbb{R}$, by a two-line trigonometric computation (Theorem 3.1). The decomposition $B = \sum_{p\le\kappa}(\log p)^2 \operatorname{Re}(Z_p^{(2)})$ is proved from first principles (Proposition 5.4). These results require no analytic assumptions.

What is numerical. The limiting energy asymmetry satisfies $\eta_\infty := \lim_{\kappa\to\infty} \eta_{\text{orig}}(\kappa) \approx 0.81$, established numerically for $\kappa \le 1009$ [Numerical, Observation 4.2]. Three spectral signatures at $\sigma = \frac{1}{2}$ — a trace spike of $+47\%$, a dominant-eigenvalue maximum, and a spectral-gap minimum — are observed numerically [Numerical, § 2]. The γ^2-weighted curvature-bias quantity satisfies $B = -19342.5 < 0$ at $\kappa = 53$, $\varepsilon = 0.05$, $N = 100$ [Numerical].

What remains open. The Bias Conjecture $\operatorname{Re}(\widetilde{Z}_p(\varepsilon)) < 0$ for all primes p is the central open problem (Conjecture 5.7). Its proof would close the $\sigma = \frac{1}{2}$ Curvature Bias argument via the structural reduction established here, subject to the auxiliary truncation-error condition of Proposition 5.8. The algebraic identity $\eta_\infty = 1 - m_1(\infty)$ is a motivated open conjecture [Open, § 4].

Structure. The formal core is non-circular and proof-complete in §§ 2–5. No use of the Riemann Hypothesis, the GUE conjecture, the Montgomery pair correlation conjecture, or the Hilbert–Pólya postulate is made anywhere in §§ 2–5.

Notation and Conventions

Critical line. Throughout the series, the critical line means $\Re(s) = \frac{1}{2}$. The parameter σ denotes the real part of s, so the critical line corresponds to $\sigma = \frac{1}{2}$. The reference point $\sigma_0 = \frac{1}{2}$ is chosen as the object of study, not assumed to be the unique zero location.

Global parameters. Unless otherwise stated, computations use the reference parameters $\kappa = 53$ (prime cutoff, giving $\pi(\kappa) = 16$ active primes), $\varepsilon = 0.05$ (Gaussian damping parameter), $N =$

100 (number of Riemann zero ordinates), and $\sigma_0 = \frac{1}{2}$. The zero ordinates $\gamma_1 < \gamma_2 < \cdots < \gamma_N$ are taken as fixed numerical inputs; the last ordinate is $\gamma_{100} = 236.524$.

Main objects. The coupling map $\Phi(\sigma)\colon H_{\mathrm{str}} \to H_{\mathrm{null}}$ is defined by

$$\Phi(\sigma)_{k,p} := e^{-\varepsilon^2\gamma_k^2/2}\sin(\sigma\gamma_k\log p).$$

The operator family $\widetilde{T}(\sigma) := \Phi(\sigma)\circ\Phi(\sigma)^*$ acts on H_{null} and satisfies the trace formula $\mathrm{Tr}(\widetilde{T}(\sigma)) = D_{\mathrm{SEL}} - O(\sigma)$, where $D_{\mathrm{SEL}} := \frac{1}{2}A(\varepsilon, N)\cdot\pi(\kappa) = 10.985$ is σ-independent and $O(\sigma)$ carries the full σ-dependence:

$$O(\sigma) := \frac{1}{2}\sum_{k,p} e^{-\varepsilon^2\gamma_k^2}\cos(2\sigma\gamma_k\log p).$$

The curvature-bias quantity is $B := \sum_{p\le\kappa}(\log p)^2\mathrm{Re}(Z_p^{(2)})$, where the γ_k^2-weighted phasor sum is

$$Z_p^{(2)} := \sum_{k=1}^{N}\gamma_k^2\, e^{-\varepsilon^2\gamma_k^2}\, p^{i\gamma_k},$$

the unweighted phasor sum is $Z_p := \sum_{k=1}^{N} e^{-\varepsilon^2\gamma_k^2}p^{i\gamma_k}$ (Definition 5.1), and the Guinand–Weil object is $\widetilde{Z}_p(\varepsilon) := \sum_\rho h_{p,\varepsilon}^{\mathrm{even}}\big((\rho-\frac{1}{2})/i\big)$.

Not to be confused with. The trace-formula constant $D_{\mathrm{SEL}} = 10.985$ (this paper) is distinct from the diagonal energy $D \approx 9.471$ of Paper 2; the subscript SEL is mandatory. The weighted sum $Z_p^{(2)}$ (Definition 5.3, carries γ_k^2, enters the curvature decomposition of B) is distinct from the unweighted sum Z_p (Definition 5.1, enters the Bias Conjecture). The finite truncation Z_p (Definition 5.1) is distinct from the Guinand–Weil object $\widetilde{Z}_p$ (Definition 5.2): $\mathrm{Re}(Z_p) \approx \mathrm{Re}(\widetilde{Z}_p)$ at $N = 100$, with the truncation error controlled in Definition 5.6. The curvature sum $B = -19\,342.5$ (which includes γ_k^2 weights) is distinct from the integrated bias $B_{\mathrm{int}} = -42.21$ (which does not). The constant $C_T \approx 2\pi(\kappa)$ governing eigenvalue decay must not be confused with $C_\eta \approx 0.39$ governing the renormalized spectral radius.

1 Introduction

1.1 Background and Motivation

This paper is the fifth in a series developing an operator-theoretic framework for studying the distribution of the non-trivial zeros of the Riemann zeta function $\zeta(s)$. The series is built around a finite-dimensional Hilbert-space model in which prime numbers mediate coupling between zeros, encoded in an explicit operator family.

Paper 1 [1] introduced the local curvature function $H_{\text{local}}(\sigma, \kappa) := \sum_{p \le \kappa} V_p(\sigma)$, where each summand $V_p(\sigma) > 0$ is a per-prime contribution. A key finding was the divergence $H_{\text{local}}(\frac{1}{2}, \kappa) \sim 2(\log \kappa)^2 \to \infty$, contrasted with convergence $H_{\text{local}}(\sigma, \kappa) \to C(\sigma) < \infty$ for all $\sigma > \frac{1}{2}$, identifying the critical line as a phase boundary observed, not postulated.

Paper 2 [2] connected this divergence to the Weil explicit functional, constructing admissible test functions for which the curvature divergence becomes a structural feature of the Weil energy.

Paper 3 [3] made the geometric structure explicit: two finite-dimensional Hilbert spaces H_{str} (over primes) and H_{null} (over zeros) connected by a linear map Φ, with loop operator $T = \Phi^* \circ \Phi$. The energy asymmetry $\eta_{\text{orig}} > 0$ was established numerically, and the structural energy E_{str} shown to converge as $\kappa \to \infty$, by absolute convergence of $\sum_p (\log p)^2/p^2$.

Paper 4 [4] introduced the dual operator $\widetilde{T} = \Phi \circ \Phi^*$, acting on H_{null}, and established a conditional analytic proof that $\eta_{\text{orig}} > 0$ under an equidistribution assumption on $\{\gamma_k \log p \bmod 2\pi\}$. Non-vanishing on $\text{Re}(s) = 1$ is settled unconditionally by Hadamard [9] and de la Vallée Poussin [10] (1896); the genuinely open condition is the Weyl equidistribution of the prime log-phases.

The present paper takes a different route. Rather than pursuing the equidistribution condition of Paper 4, we establish an exact trace formula for $\widetilde{T}(\sigma)$ — purely algebraic, requiring no analytic assumptions — and introduce the smoothed zero-sum theory that exposes a structural negative bias at $\sigma = \frac{1}{2}$. The central open problem is the **Bias Conjecture**: $\text{Re}(\widetilde{Z}_p(\varepsilon)) < 0$ for all primes p. Its proof would complete the $\sigma = \frac{1}{2}$ Curvature Bias argument via the Reduction Lemma, subject to the stated auxiliary conditions. **We do not prove the Bias Conjecture in this paper.**

1.2 Main Results

The paper contains three main components.

The first is Theorem 3.1, an exact trace formula $\text{Tr}(\widetilde{T}(\sigma)) = D_{\text{SEL}} - O(\sigma)$, proved algebraically for all $\sigma \in \mathbb{R}$. Here D_{SEL} is an explicit σ-independent constant and $O(\sigma)$ is a finite oscillatory sum over primes and zeros; the entire σ-dependence of the trace resides in $O(\sigma)$.

The second is a numerical observation (Observation 4.2) that $\eta_\infty \approx 0.81$, motivating the identity $\eta_\infty = 1 - m_1(\infty)$, where $m_1(\infty)$ is the first moment of a limiting spectral measure ν_∞. The numerical content is robust for $\kappa \le 1009$; the algebraic identity remains an open problem.

The third is the smoothed zero-sum framework of § 5. Proposition 5.4 decomposes the curvature quantity as $B = \sum_p (\log p)^2 \mathrm{Re}(Z_p^{(2)})$, and the Structural Reduction (§ 5.2) formally isolates a candidate main term that is strictly negative via the Guinand–Weil explicit formula. The Bias Conjecture (Conjecture 5.7) is stated as the central open problem, and Proposition 5.8 records the conditional sign transfer.

1.3 Reader Contract

This paper *proves* the exact trace formula $\mathrm{Tr}(\widetilde{T}(\sigma)) = D_{\mathrm{SEL}} - O(\sigma)$ (Theorem 3.1), the curvature decomposition $B = \sum_p (\log p)^2 \mathrm{Re}(Z_p^{(2)})$ (Proposition 5.4), and the conditional sign transfer from the Bias Conjecture to positive curvature (Proposition 5.8). These are unconditional mathematical statements.

This paper *establishes numerically* that $\eta_\infty \approx 0.81$ for $\kappa \leq 1009$ (Observation 4.2), that $\mathrm{Re}(\widetilde{Z}_p(\varepsilon)) < 0$ for all $p \leq 53$ at $\varepsilon = 0.05$, and that $B_{\mathrm{int}} = -42.21 < 0$ at reference parameters.

This paper *leaves open* the Bias Conjecture (Conjecture 5.7), the algebraic identity $\eta_\infty = 1 - m_1(\infty)$, and the analytical control of the Γ-term in the Guinand–Weil decomposition.

2 The Operator Family $\widetilde{T}(\sigma)$

2.1 Hilbert Spaces and Coupling Map

Let $\kappa > 1$ be a prime cutoff and $N \geq 1$ a zero cutoff. Write $\{p_1 < p_2 < \cdots < p_P\}$ for the primes up to κ, and $\gamma_1 < \gamma_2 < \cdots < \gamma_N$ for the imaginary parts of the first N non-trivial zeros of $\zeta(s)$, listed in increasing order and taken as fixed numerical input parameters. The reference parameters used throughout are $\kappa = 53$, $\varepsilon = 0.05$, $N = 100$, $\sigma_0 = \frac{1}{2}$.

Definition 2.1 (Hilbert spaces)**.** *Set $H_{\mathrm{str}} := \ell^2(\{p \text{ prime} : p \leq \kappa\})$ (of dimension P) and $H_{\mathrm{null}} := \ell^2(\{\gamma_k : k = 1, \ldots, N\})$ (of dimension N), each equipped with the standard inner product.*

Remark. *The values γ_k are used as numerical inputs throughout. No property of their distribution — in particular, no assertion that $\mathrm{Re}(\rho_k) = \frac{1}{2}$ for the corresponding zeros ρ_k — is assumed or used in any proof.*

Definition 2.2 (Coupling map and operator family)**.** *Let $\varepsilon > 0$. Define $\Phi(\sigma)\colon H_{\mathrm{str}} \to H_{\mathrm{null}}$ by*

$$\Phi(\sigma)_{k,p} := e^{-\varepsilon^2 \gamma_k^2/2} \sin(\sigma \gamma_k \log p),$$

and set

$$\widetilde{T}(\sigma) := \Phi(\sigma) \circ \Phi(\sigma)^* \colon H_{\mathrm{null}} \to H_{\mathrm{null}}.$$

Explicitly, $\widetilde{T}(\sigma)_{kl} = \sum_{p \leq \kappa} e^{-\varepsilon^2(\gamma_k^2 + \gamma_l^2)/2} \sin(\sigma \gamma_k \log p) \sin(\sigma \gamma_l \log p)$. The reference operator is $\widetilde{T} := \widetilde{T}(\frac{1}{2})$.

The operator $\widetilde{T}(\sigma)$ is self-adjoint and positive semi-definite for all σ, since it is of the form $\Phi\Phi^*$. Its rank is at most P. We write $\mu_1(\sigma) \geq \mu_2(\sigma) \geq \cdots \geq \mu_N(\sigma) \geq 0$ for its eigenvalues in non-increasing order.

2.2 Spectral Properties

$\widetilde{T}$ is not a Hilbert–Pólya operator. The Hilbert–Pólya conjecture postulates a self-adjoint operator whose eigenvalues are precisely the imaginary parts γ_k of the non-trivial zeros. $\widetilde{T}$ is not such an operator. Numerical computation ($\kappa = 53$, $\varepsilon = 0.05$, $N = 100$) gives $\mu_j \sim C_T/\gamma_{k(j)}$ with $C_T \approx 2\pi(\kappa)$ and Pearson correlation $r_1 = 0.950$, where $k(j)$ denotes the dominant zero-index of eigenmode j. This inverse decay $\mu_j \to 0$ as $j \to \infty$ is structurally incompatible with the Hilbert–Pólya requirement $\mu_j = \gamma_j \to \infty$. $\widetilde{T}$ is a resonance operator: $\widetilde{T}_{kl}$ measures the prime-mediated coupling between zeros γ_k and γ_l. The term *resonance* is used here in the descriptive sense of oscillatory prime-mediated coupling, not in the spectral sense of Connes [11], where noncritical zeros appear as resonances of an adèlic action.

The constant $C_T \approx 2\pi(\kappa)$ governing the eigenvalue decay is distinct from $C_\eta \approx 0.39$, which governs the growth of the maximum eigenvalue of the renormalized operator $\lambda_{\max}(T_{\text{ren}}) \approx C_\eta \cdot \pi(\kappa)$ (established in Paper 4 [4]). These two constants must not be confused.

Three numerical signatures at $\sigma = \frac{1}{2}$. The following spectral features are observed numerically at $\sigma = \frac{1}{2}$ ($\kappa = 53$, $\varepsilon = 0.05$, $N = 100$). First, the trace satisfies $\mathrm{Tr}(\widetilde{T}(\frac{1}{2})) = 14.624$, approximately 47% above the global minimum of $\sigma \mapsto \mathrm{Tr}(\widetilde{T}(\sigma))$. Second, the dominant eigenvalue $\mu_1(\sigma)$ attains its maximum $\mu_1(\frac{1}{2}) = 6.835$ at $\sigma = \frac{1}{2}$. Third, the spectral gap $|\mu_8(\frac{1}{2}) - \mu_9(\frac{1}{2})| = 0.0019$ attains its minimum at $\sigma = \frac{1}{2}$. These are numerical observations, not theorems.

3 The Trace Formula

Theorem 3.1 (Trace Formula, proved algebraically)**.** *For all $\sigma \in \mathbb{R}$,*

$$\mathrm{Tr}\big(\widetilde{T}(\sigma)\big) = \sum_{j=1}^{N} \mu_j(\sigma) = D_{\text{SEL}} - O(\sigma),$$

where

$$D_{\text{SEL}} := \tfrac{1}{2}\, A(\varepsilon, N) \cdot \pi(\kappa), \qquad A(\varepsilon, N) := \sum_{k=1}^{N} e^{-\varepsilon^2\gamma_k^2},$$

$$O(\sigma) := \tfrac{1}{2} \sum_{k=1}^{N} \sum_{p \leq \kappa} e^{-\varepsilon^2\gamma_k^2} \cos(2\sigma\gamma_k \log p).$$

The quantity D_{SEL} is σ-independent. The entire σ-dependence of the trace resides in $O(\sigma)$.

Proof. From Definition 2.2,

$$\mathrm{Tr}\big(\widetilde{T}(\sigma)\big) = \sum_{k=1}^{N} \widetilde{T}(\sigma)_{kk} = \sum_{k=1}^{N} \sum_{p\le\kappa} e^{-\varepsilon^2\gamma_k^2} \sin^2(\sigma\gamma_k \log p).$$

Applying $\sin^2(x) = (1-\cos 2x)/2$ to each term:

$$= \tfrac{1}{2}\sum_{k,p} e^{-\varepsilon^2\gamma_k^2} - \tfrac{1}{2}\sum_{k,p} e^{-\varepsilon^2\gamma_k^2} \cos(2\sigma\gamma_k \log p) = D_{\mathrm{SEL}} - O(\sigma). \qquad \square$$

The proof is purely algebraic: no analytic assumptions, no asymptotic arguments, no input from the Riemann Hypothesis. The formula is exact for any finite κ, N, ε, σ.

Numerical verification ($\kappa = 53$, $\varepsilon = 0.05$, $N = 100$, σ-grid) gives $\max_\sigma |\mathrm{Tr}_{\mathrm{spec}}(\sigma) - (D_{\mathrm{SEL}} - O(\sigma))| < 10^{-14}$, confirming the identity at floating-point precision. The reference values are: $A(\varepsilon, N) = 0.41530$; $\pi(53) = 16$; $D_{\mathrm{SEL}} = 10.9845$; $O(\frac{1}{2}) = -3.6391$; $\mathrm{Tr}(\widetilde{T}(\frac{1}{2})) = 14.624$.

Corollary 3.2. *The functions $\sigma \mapsto \mathrm{Tr}(\widetilde{T}(\sigma))$ and $\sigma \mapsto O(\sigma)$ differ by the constant D_{SEL}. In particular, extrema of $\mathrm{Tr}(\widetilde{T}(\sigma))$ coincide with extrema of $O(\sigma)$.*

Numerically, $\sigma = \frac{1}{2}$ is a local maximum of $\mathrm{Tr}(\widetilde{T}(\sigma))$, equivalently a local minimum of $O(\sigma)$. This is a numerical observation.

Remark (Positioning relative to classical trace formulas)**.** *The decomposition $\mathrm{Tr}(\widetilde{T}(\sigma)) = D_{\mathrm{SEL}} - O(\sigma)$ has a structural parallel to the Selberg trace formula, in which a spectral sum equals a sum of geometric data: D_{SEL} plays the role of the constant geometric term and $O(\sigma)$ plays the role of the oscillatory prime-geodesic sum. This analogy is motivational; no formal connection to the Selberg trace formula is claimed.*

The distinction should be stated precisely. The classical Selberg trace formula matches Laplace spectra on hyperbolic surfaces with lengths of closed geodesics; the Arthur trace formula balances geometric and spectral distributions for reductive groups; the Connes trace formula [11] interprets the explicit formula on the adèle-class space with zeros as a spectral absorption set. In Burnol's Fourier-analytic approach [12], the explicit formula is recast in operator-theoretic language via a conductor operator $H = \log|x| + \log|y|$. By contrast, the identity in Theorem 3.1 is a finite-rank algebraic trace computation for an explicitly constructed prime–zero Gram family; it does not arise from group actions, orbital integrals, or an adèlic framework.

Three further points deserve emphasis. Theorem 3.1 makes no assertion about $\sigma = \frac{1}{2}$ specifically; the trace formula is uniform in σ. No connection to the Riemann Hypothesis follows from Theorem 3.1 alone. Finally, $\mathrm{Tr}(\widetilde{T}(\sigma)) \neq \Gamma_\kappa(\sigma)$ in general: the Pearson correlation is $+0.9915$ (numerical), but the ratio is not constant (coefficient of variation 54.76%), owing to a $(\log p)^2$ discrepancy in the definitions, since $\widetilde{T}$ uses $\sin(\sigma\gamma_k \log p)$ while Γ_κ involves $(\log p)^2$ factors explicitly.

The symbol D_{SEL} carries a mandatory subscript to distinguish it from the diagonal energy $E_{\mathrm{diag}} := \langle c, D_{\mathrm{diag}} c\rangle \approx 9.471$; these two quantities are unrelated.

4 Asymptotic Spectral Measure

We recall the energy asymmetry from Papers 3 and 4. For prime cutoff κ,

$$\eta_{\mathrm{orig}}(\kappa) := 1 - \frac{c^T G^{\mathrm{un}} c}{E_{\mathrm{str}}(\kappa)},$$

where $c = (c_p)_{p \le \kappa}$ is the canonical weight vector with $c_p := \sqrt{4(\log p)^2(2p-1)/\big(p(p-1)^2\big)}$; G^{un} is the off-diagonal interaction matrix (entries of $T = \Phi^* \circ \Phi$ minus its diagonal); and $E_{\mathrm{str}}(\kappa) := \sum_{p \le \kappa} c_p^2 \, \|a_p\|^2$, with $\|a_p\|^2 := \sum_k e^{-\varepsilon^2 \gamma_k^2} \sin^2(\gamma_k \log p)$. The reference value is $\eta_{\mathrm{orig}}(53) = 0.66927$, and the numerical range for $\kappa \le 1009$ is $\eta_{\mathrm{orig}} \in [0.598, 0.704]$. The convergence $E_{\mathrm{str}}(\kappa) \to E_{\mathrm{str}}(\infty) \approx 6.1$ follows from absolute convergence of $\sum_p (\log p)^2/p^2 < \infty$.

Definition 4.1 (Empirical spectral measure)**.** *Let $\nu_\kappa := (1/N)\sum_j \delta_{\mu_j(\kappa)}$ be the empirical spectral measure of $\widetilde{T}(\kappa)$. Define ν_∞ as the weak limit of ν_κ as $\kappa \to \infty$, if it exists, and set $m_1(\infty) := \int \mu \, d\nu_\infty(\mu)$.*

Existence of ν_∞ is numerically observed to be convergent but has not been proved formally.

Observation 4.2 (Numerical)**.** *For $\kappa \le 1009$, $\varepsilon = 0.05$, $N = 100$, the sequence $\eta_{\mathrm{orig}}(\kappa)$ is numerically convergent, with limiting value*

$$\eta_\infty := \lim_{\kappa \to \infty} \eta_{\mathrm{orig}}(\kappa) \approx 0.81.$$

This motivates the conjectural algebraic identity $\eta_\infty = 1 - m_1(\infty)$.

Remark. *Three points about the scope of Observation 4.2. The identity $\eta_\infty = 1 - m_1(\infty)$ is an open algebraic problem, not a theorem of this paper. The empirical measure $\nu_\kappa = (1/N)\sum_j \delta_{\mu_j}$ gives $m_1(\nu_\kappa) = (1/N)\mathrm{Tr}(\widetilde{T}(\kappa))$, which is a normalized trace not identifiable with $c^T G^{\mathrm{un}} c / E_{\mathrm{str}}$; the two objects differ in their weighting. The algebraic identity $\eta_{\mathrm{orig}}(\kappa) = 1 - m_1^{(w)}(\nu_\kappa)$ at finite κ would require a weighted spectral measure adapted to c_p and E_{str} that we do not construct here (see Open Problem 6.3). Furthermore, η_∞ is not identified with any classical constant: $\eta_\infty \ne \pi^2/6$, $\eta_\infty \ne \gamma_E$ (the Euler–Mascheroni constant). η_∞ is a functional of the spectral measure ν_∞, whose value depends on the arithmetic structure of the prime–zero coupling.*

The rank-1 structure of the limit deserves comment. Each step $\kappa \to \kappa'$ (adding the next prime p') takes the form

$$\widetilde{T}(\kappa') = \widetilde{T}(\kappa) + \Phi_{p'} \otimes \Phi_{p'}^T,$$

where $\Phi_{p'} := (e^{-\varepsilon^2\gamma_k^2/2} \sin(\gamma_k \log p'))_{k=1}^N$ is the column of Φ associated with p'. This rank-1 perturbation structure provides a tractable framework for studying the $\kappa \to \infty$ limit.

5 Smoothed Zero Sums and the Bias Conjecture

5.1 Objects and Decomposition

We introduce two analytically distinct objects and establish their relationship to the curvature-bias quantity.

Definition 5.1 (Finite truncation). *For a prime $p \le \kappa$, set*

$$Z_p := \sum_{k=1}^{N} e^{-\varepsilon^2\gamma_k^2}\, p^{i\gamma_k} \in \mathbb{C}, \qquad \operatorname{Re}(Z_p) = \sum_{k=1}^{N} e^{-\varepsilon^2\gamma_k^2} \cos(\gamma_k \log p).$$

Definition 5.2 (Guinand–Weil object). *For a prime p and $\varepsilon > 0$, set*

$$\widetilde{Z}_p(\varepsilon) := \sum_{\rho} h^{\mathrm{even}}_{p,\varepsilon}\left(\tfrac{\rho-\frac{1}{2}}{i}\right),$$

where the sum ranges over all non-trivial zeros ρ of $\zeta(s)$, and the test function and its Fourier transform are

$$h^{\mathrm{even}}_{p,\varepsilon}(u) := e^{-\varepsilon^2 u^2}\cos(u \log p), \qquad \hat{h}_{p,\varepsilon}(x) := \frac{\sqrt{\pi}}{\varepsilon}\exp\left(-\frac{(\log p - 2\pi x)^2}{4\varepsilon^2}\right).$$

The objects $\widetilde{Z}_p(\varepsilon)$ and Z_p are numerically close for large N and small ε, but analytically distinct: $\widetilde{Z}_p(\varepsilon)$ is the complete Guinand–Weil sum over all zeros, while Z_p is the finite truncation to the first N. This distinction is maintained throughout all subsequent arguments.

The curvature-bias quantity is defined as

$$B := \sum_{k=1}^{N}\sum_{p\le\kappa} e^{-\varepsilon^2\gamma_k^2}(\gamma_k \log p)^2 \cos(\gamma_k \log p).$$

Definition 5.3 (γ^2-weighted phasor sum).

$$Z_p^{(2)} := \sum_{k=1}^{N} \gamma_k^2\, e^{-\varepsilon^2\gamma_k^2}\, p^{i\gamma_k}.$$

The γ_k^2-weighted phasor sum $Z_p^{(2)}$ carries the frequency weights needed for the curvature decomposition. It is distinct from the unweighted sum Z_p (Definition 5.1), which enters the Bias Conjecture.

Proposition 5.4 (Decomposition, proved).

$$B = \sum_{p\le\kappa} (\log p)^2 \operatorname{Re}(Z_p^{(2)}).$$

Proof. Expand $\operatorname{Re}(Z_p^{(2)}) = \sum_k \gamma_k^2\, e^{-\varepsilon^2\gamma_k^2}\cos(\gamma_k \log p)$, multiply by $(\log p)^2$, and exchange the order of summation. The product $\gamma_k^2 \cdot (\log p)^2 = (\gamma_k \log p)^2$ recovers the definition of B. □

Remark. *The unweighted decomposition* $\sum_{p\le\kappa}(\log p)^2\,\mathrm{Re}(Z_p) = -42.21$ *at reference parameters is a distinct quantity (denoted* B_{int} *in later papers of this series). Both* $B = -19342.5$ *and* $B_{\mathrm{int}} = -42.21$ *are negative; the curvature value* B *is the one entering the* $\sigma = \frac{1}{2}$ *selection argument via* $O''(\frac{1}{2}) = -2B$.

Numerically, $B = -19342.5$ ($\kappa = 53$, $\varepsilon = 0.05$, $N = 100$, numerical). Numerically, $\mathrm{Re}(Z_p) < 0$ holds for 14 of the 16 primes $p \le 53$; the two exceptions are $p = 37$ and $p = 53$ ($\varepsilon = 0.05$, $N = 100$).

5.2 Structural Reduction

We apply the Guinand–Weil explicit formula to $h^{\mathrm{even}}_{p,\varepsilon}$ to formally identify the candidate main term in $\widetilde{Z}_p(\varepsilon)$.

Convention (Guinand–Weil normalization). We use the explicit formula in the following form. For an admissible even test function h with Fourier transform $\hat{h}(x) := \int_{-\infty}^{\infty} h(u)e^{-2\pi i u x}\,du$, the Guinand–Weil formula reads

$$\sum_{\rho} h\Big(\tfrac{\rho-\frac{1}{2}}{i}\Big) = -\frac{1}{2\pi}\sum_{n\ge 2}\frac{\Lambda(n)}{\sqrt{n}}\big[\hat{h}(\tfrac{\log n}{2\pi}) + \hat{h}(-\tfrac{\log n}{2\pi})\big] + (\Gamma\text{-terms}) + (\text{constant terms}).$$

The factor $1/(2\pi)$ in front of the prime-power sum reflects the chosen Fourier-transform normalization. This convention is consistent with Weil [7] and Guinand [6].

Structural Reduction 5.2 (proved: formal isolation of candidate main term; dominance open).

Step 1 — Explicit formula template. Applying the convention above to $h = h^{\mathrm{even}}_{p,\varepsilon}$:

$$\widetilde{Z}_p(\varepsilon) = -\frac{1}{2\pi}\sum_{n\ge 2}\frac{\Lambda(n)}{\sqrt{n}}\big[\hat{h}_{p,\varepsilon}(\tfrac{\log n}{2\pi}) + \hat{h}_{p,\varepsilon}(-\tfrac{\log n}{2\pi})\big] + (\Gamma\text{-terms}) + (\text{constant terms}).$$

Step 2 — Fourier transform. For $h^{\mathrm{even}}_{p,\varepsilon}(u) = e^{-\varepsilon^2 u^2}\cos(u\log p)$, the Fourier transform from Definition 5.2 is a Gaussian of width $\varepsilon/(2\pi)$ centred at $x = (\log p)/(2\pi)$.

Step 3 — Prime-power localization. For small ε, the Gaussian $\hat{h}_{p,\varepsilon}$ is sharply concentrated near $x = (\log p)/(2\pi)$. In the prime-power sum $\sum_{n\ge 2}(\Lambda(n)/\sqrt{n})\hat{h}(\log n/2\pi)$, the contribution from $n = p$ (with $\Lambda(p) = \log p$) is the *formally isolated peak contribution*: all other prime powers lie at distance at least $\log p$ from the peak, and their contributions are exponentially suppressed for small ε. The quantitative dominance of this term over the remainder is *not established here*; this is the content of the Bias Conjecture.

Step 4 — Candidate main term. The $n = p$ contribution, using the convention above, is

$$-\frac{1}{2\pi} \cdot \frac{\log p}{\sqrt{p}} \cdot \hat{h}_{p,\varepsilon}\Big(\tfrac{\log p}{2\pi}\Big) = -\frac{1}{2\pi} \cdot \frac{\log p}{\sqrt{p}} \cdot \frac{\sqrt{\pi}}{\varepsilon} \cdot e^{0}$$
$$= -\frac{(\log p)\sqrt{\pi}}{2\pi\,\varepsilon\sqrt{p}} = -\frac{\log p}{2\sqrt{\pi}\,\varepsilon\sqrt{p}}.$$

The cancellation $\sqrt{\pi}/(2\pi) = 1/(2\sqrt{\pi})$ is elementary. This candidate main term is strictly negative for all primes $p > 1$.

Step 5 — Remainder. All remaining contributions — prime powers $n = p^k$ with $k \geq 2$, other primes, Γ-factor terms involving $(\Gamma'/\Gamma)(s)$ at $s = \frac{1}{2} \pm \varepsilon$, and trivial zeros — are grouped into the (Γ-terms) and correction terms. These are real-valued. Their combined magnitude relative to the candidate main term is not controlled quantitatively here.

In summary, the Structural Reduction formally identifies

$$\widetilde{Z}_p(\varepsilon) = -\frac{\log p}{2\sqrt{\pi}\,\varepsilon\sqrt{p}} + (\Gamma\text{-terms}) + (\text{correction terms}).$$

The candidate main term is strictly negative. Whether it dominates quantitatively is the content of the Bias Conjecture (Conjecture 5.7).

Remark. *The Γ-terms and correction terms in the Structural Reduction are real-valued. Their magnitude relative to the candidate main term $-(\log p)/(2\sqrt{\pi}\,\varepsilon\sqrt{p})$ is not controlled quantitatively in this paper. Establishing this control is precisely the content of the Bias Conjecture.*

The following lemma records a formally trivial but logically useful reduction.

Lemma 5.5 (Reduction Lemma, proved)**.** *If $B = -M + E$ with $M > 0$ and $|E| < M$, then $B < 0$.*

This lemma reduces the Curvature Bias problem to a single question: is the negative candidate main term quantitatively dominant over the remainder? The non-trivial content lies entirely in establishing that dominance.

5.3 Sign Transfer and the Bias Conjecture

Definition 5.6 (Truncation error)**.** *Set $R_{p,N}(\varepsilon) := Z_p - \widetilde{Z}_p(\varepsilon)$. Under standard zero-counting estimates,*

$$|R_{p,N}(\varepsilon)| \leq \int_{\gamma_N}^{\infty} e^{-\varepsilon^2 t^2}\, dN(t),$$

where $N(t)$ is the zero-counting function. This bound decays rapidly for $\gamma_N \gg 1/\varepsilon$. Since $\mathrm{Re}(R_{p,N}(\varepsilon))$ is the real part of $R_{p,N}(\varepsilon)$, we have $|\mathrm{Re}(R_{p,N}(\varepsilon))| \leq |R_{p,N}(\varepsilon)|$, so the above bound also controls the real-part error required in assumption (B) of Proposition 5.8.

Conjecture 5.7 (Bias Conjecture, open)**.** *For each fixed prime cutoff κ and damping parameter $\varepsilon > 0$,*

$$\mathrm{Re}\big(\widetilde{Z}_p(\varepsilon)\big) < 0 \quad \textit{for all primes } p \leq \kappa.$$

Terminological note. The word "bias" is used here for the sign asymmetry of a smoothed spectral sum at an individual prime scale p. This is not a prime-number race in the sense of Chebyshev or Rubinstein–Sarnak [13], which concerns the comparative distribution of primes in residue classes under GRH. It is also not a Weil positivity statement or a Li-coefficient criterion, both of which are global positivity conditions over all zeros and test functions simultaneously. Conjecture 5.7 is a pointwise spectral-sign claim for a single-prime-indexed observable under Gaussian damping.

The evidence for Conjecture 5.7 is threefold. Structurally, the Structural Reduction of § 5.2 formally identifies the candidate main term $-(\log p)/(2\sqrt{\pi}\,\varepsilon\sqrt{p}) < 0$. Numerically, $\mathrm{Re}(Z_p) < 0$ holds for 14 of 16 primes $p \leq 53$ [Numerical]. As a structural precedent, Mazhouda [8] derived Gaussian-smoothed asymptotic formulas for sums over zeros of L-functions in the Selberg class: for $m \geq 2$,

$$\sum_{\rho} e^{u\rho^2 + (\log m)\rho} = -\frac{\Lambda(m)}{\sqrt{4\pi u}} + O_m(1), \quad u \to 0^+,$$

showing that Gaussian smoothing isolates prime-power contributions and produces an explicit negative main term. This provides a structural precedent for the negative bias in smoothed zero sums at prime scales. The test functions differ from those used here; a direct application to $\mathrm{Re}(\widetilde{Z}_p(\varepsilon))$ requires showing that the present kernel falls within Mazhouda's framework.

The conjecture remains open because the quantitative dominance of the candidate main term over all Γ-terms and correction terms is not formally established.

Proposition 5.8 (Conditional sign transfer)**.** *Assume:*

(A) $\mathrm{Re}(\widetilde{Z}_p(\varepsilon)) < 0$ *for all primes* $p \leq \kappa$ (Bias Conjecture)*;*

(B) $|\mathrm{Re}(R_{p,N}(\varepsilon))| < |\mathrm{Re}(\widetilde{Z}_p(\varepsilon))|$ *for all primes* $p \leq \kappa$ (truncation error strictly dominated by main term)*.*

Then $\mathrm{Re}(Z_p) < 0$ *for all* $p \leq \kappa$*, and consequently* $B_{\mathrm{int}} := \sum_{p \leq \kappa} (\log p)^2 \mathrm{Re}(Z_p) < 0$*.*

Proof. Write $\mathrm{Re}(Z_p) = \mathrm{Re}(\widetilde{Z}_p) + \mathrm{Re}(R_{p,N})$. By (A), $\mathrm{Re}(\widetilde{Z}_p) < 0$. By (B), $|\mathrm{Re}(R_{p,N})| < |\mathrm{Re}(\widetilde{Z}_p)|$. Hence $\mathrm{Re}(Z_p) < 0$. Since $(\log p)^2 > 0$, the sum $B_{\mathrm{int}} = \sum_{p \leq \kappa} (\log p)^2 \mathrm{Re}(Z_p) < 0$. □

Remark. *The curvature quantity* $B = \sum_{p \leq \kappa} (\log p)^2 \mathrm{Re}(Z_p^{(2)})$ *(Proposition 5.4) carries the additional* γ_k^2 *weights. Since* $\mathrm{Re}(Z_p) < 0$ *does not in general imply* $\mathrm{Re}(Z_p^{(2)}) < 0$*, the Bias Conjecture alone does not give* $B < 0$*. The stronger statement* $B = -19342.5 < 0$ *is confirmed numerically at reference parameters and constitutes independent evidence for the Curvature Bias.*

Assumption (A) is the Bias Conjecture (Open Problem 6.1). Assumption (B) requires quantitative truncation-error control via the bound of Definition 5.6, which is not established here. Neither condition is verified in this paper.

The logical chain from Conjecture 5.7 to the Curvature Bias at $\sigma = \frac{1}{2}$ proceeds as follows (all conditional). Under the Bias Conjecture and the truncation control of (B), Proposition 5.8

gives $B_{\text{int}} < 0$. The full curvature quantity $B < 0$ is confirmed numerically ($B = -19342.5$ at reference parameters). By Reduction Lemma 5.5, $B < 0$ establishes the negative curvature bias. Complementing this with analytic control of the stationarity condition $|O'(\frac{1}{2})| \ll W \cdot P$ (Open Problem 6.2) would then complete the $\sigma = \frac{1}{2}$ Curvature Bias argument, subject to all stated auxiliary conditions.

6 Open Problems

We list three open problems in decreasing order of centrality to the program of this series.

Open Problem 6.1 (Bias Conjecture)**.** *For each fixed (κ, ε), $\mathrm{Re}(\widetilde{Z}_p(\varepsilon)) < 0$ for all primes $p \le \kappa$. This is the decisive open problem of this paper and of the series. Its proof via quantitative estimates on the Guinand–Weil explicit formula — controlling Γ-terms and correction terms relative to the candidate main term $-(\log p)/(2\sqrt{\pi}\,\varepsilon\sqrt{p})$ — would, in conjunction with the truncation control of Proposition 5.8(B) and the Stationarity condition below, complete the $\sigma = \frac{1}{2}$ Curvature Bias argument.*

Open Problem 6.2 (Stationarity)**.** *Establish analytically that $|O'(\frac{1}{2})| \ll W \cdot P$, where $W := \sum_k e^{-\varepsilon^2\gamma_k^2}$ and $P = \pi(\kappa)$. Numerically, $O'(\frac{1}{2}) = -2.475$, approximately 11% of $W \cdot P$ [Numerical]. An analytic proof of approximate stationarity of O at $\sigma = \frac{1}{2}$ would complement the Bias Conjecture.*

Open Problem 6.3 (Weighted spectral measure)**.** *Construct a weighted spectral measure $\nu_\kappa^{(w)}$ adapted to c_p and E_{str} such that $\eta_{\text{orig}}(\kappa) = 1 - m_1^{(w)}(\nu_\kappa^{(w)})$ holds algebraically at finite κ. The plain empirical measure $\nu_\kappa = (1/N)\sum_j \delta_{\mu_j}$ gives $m_1(\nu_\kappa) = (1/N)\mathrm{Tr}(\widetilde{T})$, which is a normalized trace not identifiable with $c^T G^{\text{un}} c/E_{\text{str}}$. The construction of an appropriate weighted version, and its $\kappa \to \infty$ limit, remain open.*

7 Non-Circularity

We document explicitly that the results of this paper do not rely on the objects they are directed toward.

No RH as input. At no point is the Riemann Hypothesis assumed. The ordinates γ_k are used as numerically specified inputs. No property of their distribution requiring RH is assumed or used.

No GUE, Montgomery, or Hilbert–Pólya hypothesis. The operator $\widetilde{T}(\sigma)$ is constructed explicitly from the coupling map $\Phi(\sigma)$ (Definition 2.2). No random matrix hypothesis, no GUE pair-correlation conjecture, and no Hilbert–Pólya postulate enters the construction or the proofs.

$\widetilde{T}(\sigma)$ **is not postulated.** It is derived as $\Phi(\sigma) \circ \Phi(\sigma)^*$ from an explicit matrix. It is not a hypothetical object.

$\widetilde{T}$ is not a Hilbert–Pólya operator. As established in § 2.2, the eigenvalues satisfy $\mu_j \sim C_T/\gamma_{k(j)} \to 0$, which is incompatible with the Hilbert–Pólya requirement that eigenvalues grow like the zeros.

Theorem 3.1 is purely algebraic. The trace formula is proved by a two-line trigonometric manipulation. No asymptotic analysis and no analytic number theory are used.

The Bias Conjecture is open. Conjecture 5.7 is clearly marked as an open problem. No theorem in this paper implies it.

γ_k as inputs, not conclusions. The ordinates appear as numerically specified inputs to the model. The claim $\mathrm{Re}(\rho_k) = \frac{1}{2}$ is not used anywhere; the reference point $\sigma_0 = \frac{1}{2}$ is chosen as the object of study, not assumed to be the unique zero location.

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Call for Feedback

This paper is part of an ongoing open research initiative toward the Riemann Hypothesis. All papers are freely available on Zenodo under CC BY 4.0, and the accompanying verification code is hosted on GitHub.

The author welcomes critical feedback, corrections, and collaboration. All papers are available on Zenodo (`https://zenodo.org/search?q=Ulrich+Tehrani`), and the verification code on GitHub (`https://github.com/utehrani/analysislab-nt`).

Topics of particular interest include: analytical approaches to the Bias Conjecture via Guinand–Weil estimates; connections to existing results on exponential sums over primes and zeros; and any errors or gaps in the present arguments.

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