Foundations of an E_8 group field theory: action uniqueness, vacuum selection, and a four-dimensional algebraic substrate*

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Abstract

We formulate a local quantum field theory for an adjoint-valued scalar field $\Phi : E_8 \to \mathfrak{e}_8$ on the compact real form of the exceptional Lie group, from four standard meta-principles of local Lagrangian QFT and four $\mathfrak{e}_8$-specific postulates, and trace its consequences in four steps. (i) Separate $E_8^L \times E_8^R \times E_8^{\mathrm{Ad}}$-invariance, the Casimir-degree spectrum of $\mathfrak{e}_8$, Ostrogradski exclusion, and stability collapse the IR action to two leading and nine sub-leading Wilson coefficients with a bounded remainder. (ii) On the open half-line $c_2 < 0$ the action admits a Bose–Einstein-type symmetry-breaking vacuum on the round 247-sphere of $\mathfrak{e}_8$, stratified by Levi sub-root-systems. (iii) Two algebraic filters (cubic-anomaly safety; existence of an SU(2) substrate for a Skyrme-type soliton) together with one structural-geometric input (compact quaternion-Kähler structure) select the unique adjoint orbit $\mathrm{EIX} = E_8/(E_7 \times \mathrm{SU}(2))$. (iv) Suter's rank–antichain identity yields a four-dimensional abelian sector $\mathfrak{a} \subset \mathfrak{m}_{\mathrm{EIX}}$ with $\dim_{\mathbb{R}} \mathfrak{a} = 4$ and $[P_\mu, P_\nu] = 0$, and the lower homotopy of EIX vanishes, ruling out Kibble-type defects from the phase transition. The promotion to a smooth Lorentzian four-manifold is recorded as a hypothesis: dimension and abelian closure follow from (iv); Lorentzian signature and the global Poincaré subgroup are closed at leading bosonic Gaussian (and at a $\mathcal{D}_{\mathrm{stab}}$-interior point) by Osterwalder–Schrader reconstruction; metric reconstruction is closed at leading + BV-BRST sub-leading Sakharov order via a Camporesi–Higuchi spectral-zeta computation on EIX, with structural coefficient $\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})} = 432/3 = 144$ to within a $\leq 3.7\%$ finite-part bound. We record explicitly the structural selections (compact real form; single-copy adjoint-valued field; truncation $(n, k) \leq (4, 4)$; BEC branch; Cas_2 vacuum truncation; Wolf-space input F3) and the residual open problems. The construction is parallel to the non-compact E_8 programs of Lisi [1], Manogue et al. [2], Wilson [3] and Furey [4], and is not addressed by the Distler–Garibaldi no-go theorem [5]. Standard-Model embedding, particle content, and the Wilsonian calibration of Newton's constant are deferred. Verification scripts are available at https://github.com/lukasbednarik/E8-GFT.

Keywords: exceptional Lie group E_8; group field theory; adjoint orbits and Wolf spaces; spontaneous symmetry breaking; emergent Lorentzian spacetime; Sakharov induced gravity.

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Contents

1 Introduction

The exceptional Lie algebra $\mathfrak{e}_8$ is the largest finite-dimensional simple complex Lie algebra and combines several distinguishing structural features. The simply connected compact group E_8 has trivial centre, trivial outer automorphism group, and is its own universal cover. The adjoint is the smallest non-trivial irreducible representation, of dimension 248. The algebra of Ad-invariant polynomials on $\mathfrak{e}_8$ is generated in degrees $\{2, 8, 12, 14, 18, 20, 24, 30\}$ (Bourbaki [6]), with no primitive Casimir of degree 3 and no primitive Casimir of degree 4 beyond the squared quadratic. These features have prompted recurrent attempts to treat $\mathfrak{e}_8$ as a fundamental algebraic substrate of physics, ranging from heterotic-string compactifications to direct E_8 unification proposals [1–4], and one of those routes was closed by the Distler–Garibaldi no-go theorem [5] on E_8 Yang–Mills embeddings of three Standard Model generations. The present paper takes a different route, in which $\mathfrak{e}_8$ enters not as a Yang–Mills gauge algebra on a fixed spacetime but as the value space of a single scalar field on the group manifold itself.

1.1 Strategic context

The framework analysed here is a *group field theory* in the sense of Boulatov, Ooguri, and Oriti [7–9], specialised to a single copy of the compact group manifold and to an adjoint-valued field. The fundamental dynamical degree of freedom is a real scalar $\Phi : E_8 \to \mathfrak{e}_8$ on the compact real form E_8^c (P2, P3). The dynamics is governed by four standard meta-principles of local Lagrangian QFT (action principle, locality, Wigner symmetry principle, stability with no higher derivatives; M1–M4) together with four $\mathfrak{e}_8$-specific postulates (the algebra; its compact real form; the field-domain variant; a Wilsonian effective-operator expansion; P1–P4). No additional inputs from spacetime physics are introduced at the level of the postulates: there is no fundamental gauge field, no fundamental fermion, and no *a priori* four-dimensional manifold.

Within this setting we trace the consequences of M1–P4 in four steps. (i) The separate $E_8^L \times E_8^R \times E_8^{\text{Ad}}$ symmetry of the field, the Casimir-degree spectrum of $\mathfrak{e}_8$, an Ostrogradski exclusion of higher derivatives, and stability collapse the IR action to two leading and nine sub-leading Wilson coefficients with a quantitatively bounded remainder (Theorem 3.5). (ii) On the open half-line $c_2 < 0$ the action admits a Bose–Einstein-type symmetry-breaking vacuum on the round 247-sphere of $\mathfrak{e}_8$ (Theorem 4.3), stratified by adjoint-orbit type through the Levi sub-root-system catalogue of E_8 (Proposition 4.4). (iii) Two algebraic filters (cubic-anomaly safety and a non-abelian SU(2) substrate for a Skyrme-type soliton) together with one structural-geometric input (compact quaternion-Kähler structure) select the unique adjoint orbit EIX $= E_8/(E_7 \times \mathrm{SU}(2))$, the Wolf space of E_8 [10, 11] (Theorem 5.9). (iv) The rank of EIX as a symmetric space is 4, and Suter's rank–antichain identity [12] produces an abelian sector $\mathfrak{a} \subset \mathfrak{m}_{\mathrm{EIX}}$ with $\dim_{\mathbb{R}} \mathfrak{a} = 4$ and $[P_\mu, P_\nu] = 0$ (Proposition 6.7); the lower homotopy of EIX vanishes (Theorem 6.21), eliminating Kibble-type defects from the corresponding phase transition.

The geometric promotion of the abelian substrate $(\mathfrak{a}, [P_\mu, P_\nu] = 0)$ to a smooth Lorentzian four-manifold is recorded as Hypothesis 6.11, with five sub-claims of differing epistemic status. The dimension and abelian-closure sub-claims follow unconditionally from (iv); the Lorentzian-signature and global-Poincaré sub-claims are closed at the leading bosonic Gaussian level, and at a representative interior point of the convex stability domain $\mathcal{D}_{\mathrm{stab}}$ of Theorem 3.5, by Osterwalder–Schrader reconstruction [13, 14] (Propositions 6.12, 6.15, 6.17); the metric-reconstruction sub-claim is closed at leading + BV-BRST sub-leading Sakharov order through a Camporesi–Higuchi spectral-zeta computation [15] on EIX, with structural coefficient $\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})} = 432/3 = 144$ to within a $\leq 3.7\%$ residual finite-part bound (Proposition 6.19). The full local $\mathrm{Diff}(\mathcal{M}^{1,3})$-content of the diffeomorphism sub-claim is then inherited by the standard induced-gravity argument at the same leading + sub-leading level.

Several steps in the chain involve structural *selections* where the postulates admit more than one branch and the present paper investigates one of them; representative examples are the compact real form E_8^{c}, the single-copy adjoint-valued field domain, the Wilsonian truncation $(n, k) \leq (4, 4)$, the BEC branch $c_2 < 0$, and the Wolf-space input F3. We adopt throughout §§2–6 the convention that every such selection is flagged explicitly at the point at which it is taken, so that no selection is silently absorbed into a postulate. The complete catalogue, with the alternative branch and the downstream statements that depend on each selection, is collected in §7.2; the residual analytic open problems are collected in §7.3.

1.2 Position relative to existing E_8-based programs

The construction here is structurally parallel to several E_8-based programs in the literature; we record the relevant distinctions briefly.

The model of Lisi [1] attempts to embed the Standard Model fermion content directly in an E_8 Yang–Mills theory on the non-compact real form $E_{8(8)}$. The Distler–Garibaldi theorem [5] establishes that no E_8 Yang–Mills theory accommodates three full Standard Model generations with the correct chirality content, in the sense made precise there. The present framework is not an E_8 Yang–Mills theory in that sense: there is no fundamental gauge field A_μ valued in $\mathfrak{e}_8$, and the field is a scalar on the group manifold rather than a one-form on spacetime. The Distler–Garibaldi obstruction therefore does not address the present construction, and we make no claim beyond it.

The programs of Wilson [3] and Manogue et al. [2] work with the intermediate non-compact real form $E_{8(-24)}$ together with octonionic or $\mathrm{Spin}(7,3)$-type structures, with spacetime taken as fundamental. We adopt the compact real form E_8^{c} (Remark 2.2); spacetime is taken as emergent. The two structural choices are recorded as parallel branches that would each require their own derivation, not as a critique of either alternative.

The construction of Furey [4] builds Standard-Model degrees of freedom on the tensor product $\mathbb{O} \otimes \mathbb{H}$ of octonions and quaternions ($8 \times 4 = 32$ real dimensions). The present framework is structurally compatible: the Wolf space EIX that emerges as the selected vacuum orbit has reduced holonomy Sp(28)·Sp(1) [10, 11] and a quaternionic-fibre structure on its tangent bundle; the quaternionic content appears here as a property of the vacuum geometry rather than as an algebraic input (Remark 6.9).

The classical chain $G_{\mathrm{SM}} \subset \mathrm{SU}(5) \subset \mathrm{Spin}(10) \subset E_6 \subset E_7 \subset H_1$ (Slansky [16]) is used in Remark 5.8 only as an *a posteriori* algebraic consistency check on the selected stabilizer; it is not used as an orbit-selection filter and no Standard-Model embedding is claimed. The Standard-Model-embedding consequences of H_1, the matter representation content, and the Wilsonian calibration of the induced Newton constant are deferred to subsequent work.

1.3 Plan of the paper

§2 states the four meta-principles M1–M4 and the four postulates P1–P4, and records the conditional content of P4. §3 derives the action of Theorem 3.5 from M1–M4 and P1–P4. §4 analyses the symmetry-breaking vacuum on $c_2 < 0$ and its tree-level fluctuation spectrum in the Cas_2 truncation. §5 applies the three operational filters F1–F3 to the Borel–de Siebenthal-type catalogue of stabilizers and selects the EIX orbit. §6 extracts the four-dimensional algebraic substrate from EIX, establishes the lower-homotopy vanishing of EIX, and closes the leading and leading + sub-leading geometric sub-claims of Hypothesis 6.11 via Osterwalder–Schrader reconstruction and Camporesi–Higuchi spectral-zeta on EIX. §7 collects the established results, the structural selections, the residual open analytic completions, and the statements that this paper does *not* claim. Conventions are collected in Appendix A; the technical content of Theorem 3.5 is supplied in Appendix B.

2 Postulates and meta-principles

The assumptions of the present framework consist of four standard meta-principles of local Lagrangian quantum field theory, denoted M1–M4, and four theory-specific postulates, denoted P1–P4. The meta-principles are stated first because the postulates refer to them by name.

2.1 Meta-principles

The following four assumptions are standard for any local Lagrangian quantum field theory and contain no $\mathfrak{e}_8$-specific content [17–19]. We do *not* regard them as postulates of the present theory.

Principle M1 (Action principle)**.** Dynamics is generated by a local functional $S[\Phi]$ of the fundamental field Φ. The classical equations of motion are $\delta S/\delta\Phi = 0$, and quantization proceeds via the standard path integral

$$Z = \int \mathcal{D}\Phi \; e^{iS[\Phi]/\hbar}. \tag{1}$$

1

Principle M2 (Locality)**.** The action is the integral of a local Lagrangian density,

$$S[\Phi] = \int_{\mathcal{M}} \mathrm{d}\mu(x)\; \mathcal{L}(\Phi(x), \nabla\Phi(x), \nabla^2\Phi(x), \dots), \tag{2}$$

depending on the field and finitely many of its derivatives at a single point.

[1] The construction of the formal measure $\mathcal{D}\Phi$ for an $\mathfrak{e}_8$-valued field on a high-dimensional compact Lie group is non-trivial; we adopt the standard heat-kernel (Gaussian) regularisation around the leading quadratic kinetic part, in the convention of Oriti [9], Glimm and Jaffe [20].

Under P2 and P3 below, (2) specialises to $\mathcal{M} = E_8$, $\nabla = L_A$, and $\mathrm{d}\mu = \mathrm{d}\mu_{\mathrm{Haar}}$, the form used throughout the rest of the paper.

Principle M3 (Symmetry, Wigner principle)**.** The action is invariant under the full group of canonical symmetries of its structural data, acting independently on each independent structural factor:

$$S[\Phi] = S[g \cdot \Phi] \qquad \forall\, g \in \mathrm{Sym}(\text{structural data}). \tag{3}$$

When the structural data decompose into a product of independent canonical actions $G_1 \times \cdots \times G_r$, the canonical symmetry group of (3) is the full product; identifying any two factors along an isomorphism counts as additional structural input not contained in the data themselves. This non-identification clause is the standard rule that two formally distinct symmetry factors are kept separate without explicit dynamical input, as for the chiral $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ symmetry of massless QCD, which is reduced to the diagonal $\mathrm{SU}(N)_V$ only by an explicit mass term or a condensate.

Principle M4 (Stability and unitarity)**.** The theory is physically consistent in the strong, pre-emergent sense: there exist a Euclidean action $S_E[\Phi]$ and a vacuum configuration Φ_0 such that

(a) *Action bounded below:* $S_E[\Phi] \geq S_0 > -\infty$ for all sufficiently regular configurations;

(b) *Positive Hessian at the vacuum:* $\delta^2 S_E/\delta\Phi^2|_{\Phi_0} > 0$;

(c) *No higher derivatives:* the Lagrangian contains at most first derivatives of the field.

Condition M4(c) is a conservative form of the Ostrogradski exclusion [21, 22]: it is strictly stronger than the Ostrogradski theorem itself requires, and forbids degenerate higher-derivative theories (Galileons, DHOST, $f(R)$) that are admissible under that theorem.

2.2 Postulates

Postulate P1 (Fundamental algebra)**.** The fundamental algebraic object of the theory is the exceptional Lie algebra $\mathfrak{e}_8$, with $\dim \mathfrak{e}_8 = 248$ and $\operatorname{rank} \mathfrak{e}_8 = 8$.

Remark 2.1 (Structural features of $\mathfrak{e}_8$). Beyond its dimension and rank, $\mathfrak{e}_8$ has four further properties that are used repeatedly in the sequel and that distinguish it among the simple Lie algebras [6, 23]: (i) $\mathrm{Out}(\mathfrak{e}_8) = 1$, so $\mathrm{Aut}(\mathfrak{e}_8) = \mathrm{Inn}(\mathfrak{e}_8)$ and there is no discrete outer-automorphism ambiguity (in contrast to $\mathfrak{so}(8)$, $\mathfrak{su}(n \geq 3)$, or $\mathfrak{e}_6$); (ii) the centre of the simply connected group is trivial, $Z(E_8) = 1$, so the adjoint and simply connected forms coincide and the global form is unique; (iii) $\pi_1(E_8) = 1$, so E_8 is its own universal cover; (iv) the fundamental Casimir invariants of $\mathfrak{e}_8$ have degrees $\{2, 8, 12, 14, 18, 20, 24, 30\}$, so in particular there is no symmetric invariant of degree 3 (no d-symbol). Property (iv) is the structural origin of the cubic anomaly cancellation of Corollary 3.6 (§3.6); properties (i)–(iii) ensure that the choice of global E_8 in P3 is unambiguous.

Postulate P2 (Real form)**.** We adopt the *compact* real form E_8^{c} of $\mathfrak{e}_8$, characterized by negative-definite Killing form

$$\mathrm{Kill}(X, Y) \;:=\; \mathrm{Tr}(\mathrm{ad}_X\ \mathrm{ad}_Y), \tag{4}$$

i.e. $\mathrm{Kill}(X, X) < 0$ for all $X \neq 0$ in $\mathfrak{e}_8$. Throughout the paper we work with the rescaled, positive-definite form $\kappa(X, Y) := -\mathrm{Kill}(X, Y)/h^\vee_{E_8}$ (with $h^\vee_{E_8} = 30$); the full Lie-algebra normalisation is collected in Appendix A.2.

Remark 2.2 (Alternative real forms and the role of compactness). The complex Lie algebra of type E_8 admits three inequivalent real forms [24, 25]: the compact form E_8^{c} adopted in P2, the split form $E_{8(8)}$ (signature $(128, 120)$ on the Killing form, in the (#positive, #negative) convention with positive eigenvalues on the non-compact directions $\mathfrak{p}$), and the intermediate form $E_{8(-24)}$ (signature $(112, 136)$). P1 fixes the complex algebra but does not single out any one of these; the choice in P2 is therefore a genuine sub-postulate, not a derived statement, and it carries downstream consequences that are recorded explicitly here.

Two such consequences fix the choice. First, the quadratic kinetic invariant $\mathcal{H}_2 := \kappa^{AB}\,(L_A\Phi)\cdot(L_B\Phi)$ of the field (5) is non-negative on the configuration space precisely when κ is positive definite, which holds only on the compact form; on the non-compact forms κ has indefinite signature and $\mathcal{H}_2$ flips sign on the non-compact directions, in conflict with the positivity of the Hessian required by M4. Second, the integration measure on E_8 in (2) is the bi-invariant Haar measure, which is finite on the compact form and infinite on the non-compact forms; the Wilsonian expansion of P4 is posed directly with $\mathrm{d}\mu_{\mathrm{Haar}}$ and assumes finite total volume. The compact-form choice is hence the minimal way of making M4 and P4 simultaneously well-posed; relaxing it would require either a separate volume regularisation or a modification of M4.

Several competing E_8-based programs adopt non-compact forms instead—most notably $E_{8(-24)}$ in Lisi [1], Manogue et al. [2], Wilson [3] (and the no-go critique of Distler and Garibaldi [5], which surveys both $E_{8(8)}$ and $E_{8(-24)}$ and argues that no real form of E_8 accommodates three full Standard Model generations with the correct chirality content)—where spacetime is taken as fundamental rather than emergent. The present framework is structurally parallel to those programs but not equivalent: the compact-form choice is tied to the algebraic interpretation of P1 and to the emergent-spacetime track pursued in the later sections. We do not exclude the non-compact alternatives on physical grounds; we record them as parallel branches that would require a separate derivation.

Postulate P3 (Fundamental field)**.** The fundamental dynamical degree of freedom is a real scalar field

$$\Phi : E_8 \longrightarrow \mathfrak{e}_8 \tag{5}$$

on the group manifold $E_8 = \exp(\mathfrak{e}_8)$, valued in the adjoint representation of $\mathfrak{e}_8$.

Remark 2.3 (Field-domain alternatives). The framework of P1 admits four canonical choices of field domain. P3 adopts variant (A): a single copy of the group manifold E_8 with values in the adjoint representation. The three alternatives are (B) the Cartesian power E_8^d, which is the domain of the Boulatov–Ooguri group field theories [7–9] (with field values in $\mathbb{C}$ or $\mathfrak{e}_8$ depending on the formulation); (C) the algebra itself, $\mathfrak{e}_8 \cong \mathbb{R}^{248}$, viewed as the space of infinitesimal generators and carrying $\mathbb{R}$-, $\mathbb{C}$-, or $\mathfrak{e}_8$-valued functions; (D) higher gauge structures with sections of 2- or 3-bundles over E_8. The choice of variant (A) rests on three structural points. First, the adjoint representation is the smallest non-trivial irreducible representation of E_8 and the unique irreducible representation distinguished by P1 alone (the next irreducibles have dimensions $3875, 27000, 30380, \ldots$, see Slansky [16]); no other representation enters the postulate at this stage. Second, the compact group of P2 carries a canonical bi-invariant Haar measure of finite total volume, while $\mathfrak{e}_8 \cong \mathbb{R}^{248}$ in variant (C) admits no normalisable canonical measure and would require a volume regulator beyond P2. Third, variant (B) introduces the discrete parameter d (the Cartesian power), which is not contained in P1–P2; under structural economy this counts as additional input. Variants (B) and (D) are not excluded but lie outside the scope of the present framework.

Postulate P4 (Wilsonian effective-operator expansion)**.** There exists a UV cutoff scale Λ such that, in any neighbourhood of the trivial configuration $\Phi = 0$, the Lagrangian density of P3 admits a Wilsonian effective-operator expansion in a basis of Ad-invariant local monomials $\mathcal{P}_{n,k}$ of bi-degree n in Φ and k in left-invariant derivatives L_A,

$$\mathcal{L}(g) \;=\; \sum_{n,k\geq 0} c_{n,k}\,\mathcal{P}_{n,k}(\Phi(g),\, L_A\Phi(g), \ldots), \tag{6}$$

with bare Wilson coefficients $c_{n,k} \in \mathbb{R}$ bounded by the *naturalness condition*

$$|c_{n,k}| \;\le\; M\,\Lambda^{D-[\mathcal{P}_{n,k}]} \;=\; M\,\Lambda^{248-123n-k}, \qquad M = \mathcal{O}(1), \tag{7}$$

with $D = \dim E_8 = 248$ and the canonical engineering dimension $[\Phi] = (D-2)/2 = 123$ (see Remark 2.4(i) for the physical interpretation of Λ on a compact group of finite Haar volume).

Postulate P4 is the Wilsonian formulation of an effective field theory [26]; it is the standard organizing axiom of modern EFT and is not derivable from M1–M4 alone. The trailing $\ldots$ in (6) indicates that, at the level of the bare expansion, monomials in higher derivatives $L_A L_B \Phi$, $\ldots$ are admissible; M4(c) projects the surviving basis onto monomials in Φ and $L_A\Phi$ only, so that the catalogue of admissible $\mathcal{P}_{n,k}$ in each (n,k) sector—the content of subsequent sections—involves at most first derivatives.

Remark 2.4 (Conditional content of P4)*.* Two aspects of P4 are conditional rather than self-contained and are recorded explicitly here so that subsequent chapters can refer to them.

(i) *Physical content of* Λ *and* ϕ_0*.* The bound (7) is the standard Wilsonian recipe of [26] written in flat space with Lebesgue measure. On the compact group E_8^{c} of P2 with $\int \mathrm{d}\mu_{\mathrm{Haar}} = 1$, the engineering dimension $[\Phi] = (D-2)/2 = 123$ and the cutoff Λ raised to the 123rd power are dimensionally consistent but acquire a quantitative physical interpretation only after a length scale is identified; on a compact group of finite Haar volume this requires a downstream Sakharov-type emergent-spacetime construction external to P4 itself. Within the present chapter the bound is used as a formal book-keeping device, and the resulting relative-size inequality (21) of Section 3.4 is correct as a formal inequality in the dimensionless parameters ϕ_0/Λ^{123} and Λ_0/Λ. Its interpretation as a quantitative Wilsonian hierarchy depends on those parameters being matched to physical scales by the emergent-spacetime structure developed in later sections.

(ii) *Choice of truncation layer.* The naturalness bound (7) controls the *relative* size of operators in different (n,k) sectors but does not by itself prescribe which layer to retain. The truncation $(n,k) \le (4,4)$ used in Section 3 is a modelling choice — the leading plus first-sub-leading layer in the Wilson–Polchinski tag $\theta_{n,k} := D - 123n - k$ — and not a derived statement of P4. Truncating at a different layer would change the catalogue of surviving operators but not the underlying expansion or the master inequality (7).

Remark 2.5 (Polynomial form is a property of the basis, not of $\mathcal{L}$)*.* M2 controls the pointwise dependence of the Lagrangian density on the field and its derivatives but not its analytic structure: local Lagrangians of physical interest are not necessarily polynomial in their arguments (Born–Infeld $\mathcal{L} \sim \sqrt{1 - F^2/2}$, DBI sigma models, group sigma models $\mathrm{Tr}(g^{-1}\mathrm{d}g)^2$ with $g = \exp(\ldots)$ transcendental in the algebra coordinates). The polynomial expansion in P4 is therefore a representation of $\mathcal{L}$ in the operator basis $\{\mathcal{P}_{n,k}\}$, valid in a neighbourhood of $\Phi = 0$, rather than a restriction on the global analytic form of $\mathcal{L}$. UV-completed Lagrangians that are non-polynomial in Φ admit the same effective-operator expansion in the IR, with their UV-specific structure encoded in correlated patterns of the $c_{n,k}$ at all orders. The polynomial structure derived in subsequent sections is consequently a statement about the IR EFT representation, not about the underlying Lagrangian.

3 Action uniqueness on E_8

This section determines the form of the local action functional $S[\Phi]$ for the field of P3 from the meta-principles M1–M4 and the postulates P1–P4. The principal output is Theorem 3.5 (§3.5): in the IR regime the action is fixed up to two leading and nine sub-leading real coefficients, with a quantitatively controlled remainder. Two structural corollaries follow (§3.6): the adjoint anomaly coefficient D_{ABC} vanishes identically, and no primitive degree-four Casimir is available to build a Yang–Mills type quartic.

3.1 Setup: derivatives and the symmetry trio

Field and derivatives. With the orthonormal basis $\{T_A\}_{A=1}^{248}$ of $\mathfrak{e}_8$ fixed in Appendix A.2, the field of P3 decomposes as $\Phi(g) = \Phi^A(g)\, T_A$ with $\Phi^A \in C^\infty(E_8, \mathbb{R})$. The natural first-order differential operators on E_8 are the left-invariant vector fields L_A associated with T_A,

$$(L_A f)(g) \;:=\; \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(g\,\exp(t\,T_A)), \qquad f \in C^\infty(E_8), \tag{8}$$

which span the algebra $[L_A, L_B] = f_{AB}{}^C\, L_C$ with the totally antisymmetric structure constants $f_{ABC} = f_{AB}{}^D \kappa_{DC}$ of Helgason [24], Knapp [25].

The symmetry trio. A field of the form $\Phi : E_8 \to \mathfrak{e}_8$ admits three independent canonical E_8-actions:

- E_8^L (left translation): $\Phi(g) \mapsto \Phi(h^{-1}g)$;
- E_8^R (right translation): $\Phi(g) \mapsto \Phi(gh)$, which transforms L_A in the adjoint of E_8^R via $L_A \mapsto (\mathrm{Ad}_{h^{-1}})^B{}_A\, L_B$;
- E_8^{Ad} (internal gauge): $\Phi(g) \mapsto \mathrm{Ad}_h\, \Phi(g)$, which transforms the value index Φ^B in the adjoint of E_8^{Ad}.

The three factors $E_8^L, E_8^R, E_8^{\mathrm{Ad}}$ are the independent canonical actions of M3 on the structural data of $\Phi : E_8 \to \mathfrak{e}_8$: E_8^L and E_8^R act on the domain E_8, and E_8^{Ad} acts on the value $\mathfrak{e}_8$. By M3 the action is invariant under the full product $E_8^L \times E_8^R \times E_8^{\mathrm{Ad}}$. Bi-invariance of the Haar measure (Appendix A.2) makes E_8^L-invariance *automatic* for any local Lagrangian; the non-trivial restrictions come from the remaining two factors, which act independently on the manifold index A (from L_A) and on the internal value index B (from Φ^B). Indices of the two types must therefore be contracted *separately* through Ad-invariant tensors of $\mathfrak{e}_8$.

Remark 3.1 (Diagonal vs. separate $E_8^R \times E_8^{\mathrm{Ad}}$). Identifying E_8^R with E_8^{Ad} along the canonical isomorphism $T_A \leftrightarrow T_A$ would reduce the symmetry group to a single diagonal copy of E_8 and admit additional "divergence-type" invariants such as $L^A\Phi_A$ and $(L^A\Phi_A)^2$. The diagonal identification is not canonical in the sense of M3: E_8^R acts on the manifold whereas E_8^{Ad} acts on the value, so they belong to two distinct structural factors of the data and the final clause of M3 forbids their identification without additional structural input not contained in P1–P3. The separate trio is hence the direct content of M3 on the field of P3, in line with the standard group-field-theory literature [7–9].

3.2 Catalogue of Ad-invariant monomials

By P4 the Lagrangian density admits the expansion (6) in homogeneous local monomials $\mathcal{P}_{n,k}$ of bi-degree (n,k) in Φ and the $L_A\Phi$. Together with the separability of §3.1, the catalogue is controlled, sector by sector, by the Ad-invariant tensor theory of $\mathfrak{e}_8$.

Lemma 3.2 (Primitive Ad-invariant tensors of degree ≤ 3)**.** *On the adjoint representation of* $\mathfrak{e}_8$*, the primitive* Ad*-invariant tensors in degree at most three are exactly two: the symmetric Killing form* κ_{AB} *and the totally antisymmetric structure constants* f_{ABC}*. There is no symmetric* Ad*-invariant tensor of degree* 3*.*

Proof. Schur's lemma applied to the irreducible adjoint representation makes the Killing form the unique (up to scale) symmetric Ad-invariant bilinear form; the structure constants are Ad-invariant by the Jacobi identity. The absence of a symmetric Ad-invariant 3-tensor follows from the absence of a primitive Casimir of degree 3 in Lemma 3.3. □

Lemma 3.3 (Casimir degrees of $\mathfrak{e}_8$)**.** *The algebra of* Ad*-invariant polynomials on* $\mathfrak{e}_8$ *is a polynomial algebra in eight generators of degrees*

$$\deg(\textit{primitive Casimirs of } \mathfrak{e}_8) \;=\; \{\,2,\,8,\,12,\,14,\,18,\,20,\,24,\,30\,\}. \tag{9}$$

Proof. Chevalley restriction theorem (Bourbaki [6], Plate VII): Ad-invariant polynomials on a simple Lie algebra are generated in degrees $e_i + 1$, with the e_i the Weyl exponents. The exponents of $\mathfrak{e}_8$ are $\{1, 7, 11, 13, 17, 19, 23, 29\}$ [6], giving the eight degrees (9). □

Identically vanishing sectors. Two consequences of Lemma 3.2 eliminate entire sectors of the expansion (6):

- The cubic potential $f_{ABC}\,\Phi^A\Phi^B\Phi^C$ vanishes identically by antisymmetry of f paired against the symmetric product $\Phi^A\Phi^B\Phi^C$.
- Every Ad-invariant contraction with odd n at $k \in \{0, 2\}$ contains at least one factor of f saturated against a symmetric combination of fields, and vanishes by the same argument. The sectors $(3, 0)$ and $(3, 2)$ therefore carry zero independent invariants.

Notation for the small-degree generators. We write $M_{AB} := (L_A\Phi)^B$ for the matrix of first derivatives, and define the leading bilinears

$$C_2 := \kappa_{AB}\,\Phi^A\Phi^B, \tag{10}$$

$$\mathcal{H}_2 := \kappa^{AA'}\kappa_{BB'}\,(L_A\Phi^B)(L_{A'}\Phi^{B'}) = \mathrm{Tr}(L_A\Phi\cdot L^A\Phi), \tag{11}$$

$$\mathcal{H}_2^{\mathrm{grad}} := \tfrac{1}{4}\,\kappa^{AA'}(L_AC_2)(L_{A'}C_2), \tag{12}$$

$$\mathcal{H}_2^{\mathrm{mix}} := \kappa^{AA'}\kappa^{EE'}\big(f_{BCE}\,M_{AB}\,\Phi^C\big)\big(f_{B'C'E'}\,M_{A'B'}\,\Phi^{C'}\big). \tag{13}$$

The five quartic-derivative generators are

$$\mathcal{S}_a := \mathcal{H}_2^2 = \big(\mathrm{Tr}(L_A\Phi\cdot L^A\Phi)\big)^2, \tag{14}$$

$$\mathcal{S}_b := \kappa^{AA'}\kappa^{CC'}\kappa_{BD}\kappa_{B'D'}\,(L_A\Phi^B)(L_{A'}\Phi^{B'})(L_C\Phi^D)(L_{C'}\Phi^{D'})\Big|_{\perp\,\mathcal{S}_a}, \tag{15}$$

$$\mathcal{S}_c := f_{AA'E}\,f^E{}_{CC'}\,\kappa^{BD}\kappa^{B'D'}\,(L^A\Phi_B)(L^{A'}\Phi_{B'})(L^C\Phi_D)(L^{C'}\Phi_{D'}), \tag{16}$$

$$\mathcal{S}_{c'} := \kappa^{AA'}\kappa^{CC'}\,f^{BB'E}\,f^{DD'}{}_E\,(L_A\Phi^B)(L_{A'}\Phi^{B'})(L_C\Phi^D)(L_{C'}\Phi^{D'}), \tag{17}$$

$$\mathcal{S}_e := f_{AA'E}\,f^E{}_{CC'}\,f^{BB'F}\,f^{DD'}{}_F\,(L^A\Phi_B)(L^{A'}\Phi_{B'})(L^C\Phi_D)(L^{C'}\Phi_{D'}), \tag{18}$$

where $\big|_{\perp\,\mathcal{S}_a}$ denotes the structural Cauchy–Howe projection onto the $\lambda = (2, 2)$ isotypic component of the κ-only sector (Appendix B.3, Eq. (83)), $\mathcal{S}_c$ has f on the manifold indices and $\mathcal{S}_{c'}$ on the internal indices, and $\mathcal{S}_e$ uses f on both. The Ostrogradski-risky sector is generated by $\mathcal{S}_d := \mathrm{Tr}(L_AL_B\Phi\cdot L^AL^B\Phi)$, which contains second derivatives of Φ and is excluded by M4(c) below.

Lemma 3.4 (Catalogue of (n, k) sectors with $n, k \le 4$)**.** *Under the separate symmetry of §3.1 the admissible* Ad*-invariant sectors with $n, k \le 4$ have the dimensions of Table 1.*

Proof. The entries with $n + k$ odd or $n \in \{1, 3\}$, $k \in \{0, 2\}$ are zero by Lemma 3.2 and the vanishing argument above. The dimension of the $(4, 0)$ sector is one because the only admissible degree-4 scalar is C_2^2 (Lemma 3.3). The dimensions of $(2, 0)$ and $(2, 2)$ are immediate. For $(4, 4)$, Cauchy–Howe duality gives

$$\dim^{(4,4)} \;=\; \sum_{\lambda\vdash 4}\Big[\dim\,\mathrm{Inv}_{\mathfrak{e}_8}(S^\lambda\,\mathrm{adj}\,\mathfrak{e}_8)\Big]^2. \tag{19}$$

(n,k)	dim	Generators
$(2,0)$	1	C_2
$(3,0)$	0	— (cubic potential vanishes)
$(4,0)$	1	C_2^2 (no primitive degree-4 Casimir, eq. (9))
$(2,2)$	1	$\mathcal{H}_2$
$(3,2)$	0	—
$(4,2)$	3	$C_2\,\mathcal{H}_2$, $\mathcal{H}_2^{\mathrm{grad}}$, $\mathcal{H}_2^{\mathrm{mix}}$
$(2,4)$	—	all generators contain $L_A L_B \Phi$, eliminated by M4(c)
$(4,4)^*$	5	$\mathcal{S}_a$, $\mathcal{S}_b$, $\mathcal{S}_c$, $\mathcal{S}_{c'}$, $\mathcal{S}_e$

Table 1: Dimensions of the admissible Ad-invariant (n,k)-sectors with $n,k \le 4$ under separate $E_8^R \times E_8^{\mathrm{Ad}}$-invariance. The starred entry $(4,4)^*$ is the count of generators in which all four field factors are first-order derivatives $L_A\Phi$ (i.e. $\mathrm{Sym}^4(V_R \otimes V_{\mathrm{Ad}})$); mixed sub-distributions of $(n,k) = (4,4)$ such as $\Phi^2(L_A L_B\Phi)^2$ or $\Phi(L_A\Phi)(L_B L_C\Phi)(L_D\Phi)$ contain a factor $L_A L_B \Phi$ and are eliminated by M4(c) on the same footing as the $(2,4)$ row.

Combining the decompositions $\mathrm{Sym}^2\,\mathrm{adj}\,\mathfrak{e}_8 = \mathbf{1} \oplus \mathbf{3875} \oplus \mathbf{27000}$ of Slansky [16], McKay and Patera [27] with Lemma 3.3 (no primitives in degrees 3 and 4 in Sym, no primitive in degree 4 in Λ, by Mimura and Toda [28]) and the real-orthogonal Frobenius–Schur indicator of every $\mathfrak{e}_8$-irrep, the partitions (4), $(3,1)$, $(2,2)$, $(2,1,1)$, (1^4) contribute respectively $1,0,4,0,0$ to (19), giving $\dim^{(4,4)} = 5$. The five generators $\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_c, \mathcal{S}_{c'}, \mathcal{S}_e$ are explicitly constructed and linearly independent (Appendix B). The $(4,2)$ sector is analysed by the same plethysm argument applied to $\mathrm{Sym}^2 \otimes \mathrm{Sym}^2$ on the bare/derivative split, yielding three generators $C_2\mathcal{H}_2, \mathcal{H}_2^{\mathrm{grad}}, \mathcal{H}_2^{\mathrm{mix}}$. □

3.3 Stability constraints

Ostrogradski exclusion. By M4(c) the Lagrangian contains at most first derivatives of the field. The candidate $\mathcal{S}_d = \mathrm{Tr}(L_A L_B \Phi \cdot L^A L^B \Phi)$ has Hessian $\partial^2 \mathcal{S}_d / \partial(L_A L_B \Phi)^2 = 2\,\kappa^{AC}\kappa^{BD}$, manifestly non-degenerate; the Ostrogradski theorem [21, 22] then forbids it. The same argument eliminates the entire $(2,4)$ sector and any higher-n sector that necessarily contains $L_A L_B \Phi$. The four-field, four-derivative generators $\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_c, \mathcal{S}_{c'}, \mathcal{S}_e$ contain only first derivatives raised to the fourth power and are of Skyrme type [29, 30]; they are admissible.

Sign restrictions. Boundedness of the Hessian and of the potential imposed by M4(a)–(b) forces

$$\kappa_2 \;>\; 0, \qquad c_4 \;>\; 0, \tag{20}$$

where κ_2 is the coefficient of $\mathcal{H}_2$ and c_4 the coefficient of C_2^2. The remaining coefficients are real and range over a convex stability domain $\mathcal{D}_{\mathrm{stab}} \subset \mathbb{R}^9$ specified by positivity of the kinetic spectrum around the vacuum; the explicit description of $\mathcal{D}_{\mathrm{stab}}$ depends on the choice of vacuum configuration. The sign of c_2 is unconstrained by M4 alone.

3.4 Wilsonian hierarchy

By P4 the field has engineering dimension $[\Phi] = (D-2)/2 = 123$ on E_8 ($D = 248$); each derivative L_A contributes one mass dimension. The operator $\mathcal{P}_{n,k}$ then carries dimension $123n + k$, and the Wilson–Polchinski tag $\theta_{n,k} := D - 123n - k$ takes the values listed in Table 2.

For a configuration with characteristic field amplitude ϕ_0 and characteristic derivative scale Λ_0, the naturalness bound (7) translates into the relative size, normalised by the relevant term $|c_2\, C_2|$ which dominates the leading action in the IR regime $\Lambda_0 \ll \Lambda$ (by a factor $(\Lambda/\Lambda_0)^2$ over

Operator	(n,k)	$[\mathcal{P}_{n,k}]$	$\theta = 248 - [\mathcal{P}]$
C_2	$(2,0)$	246	$+2$ (relevant)
$\mathcal{H}_2$	$(2,2)$	248	0 (marginal)
C_2^2	$(4,0)$	492	-244 (irrelevant)
$C_2\mathcal{H}_2, \mathcal{H}_2^{\mathrm{grad}}, \mathcal{H}_2^{\mathrm{mix}}$	$(4,2)$	494	-246 (irrelevant)
$\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_c, \mathcal{S}_{c'}, \mathcal{S}_e$	$(4,4)$	496	-248 (irrelevant)
C_2^3 (†)	$(6,0)$	738	-490 (irrelevant)

Table 2: Power-counting tags of the surviving operators on the 248-dimensional manifold E_8. The entry marked (†) is illustrative: $n = 6$ lies outside the truncation $(n,k) \le (4,4)$, so C_2^3 belongs to $\mathcal{R}_{\mathrm{higher}}$ and not to S.

the marginal $|\kappa_2\, \mathcal{H}_2|$, generic for $c_2 \neq 0$):

$$\frac{|c_{n,k}\, \mathcal{P}_{n,k}|}{|c_2\, C_2|} \;\le\; M\left(\frac{\phi_0}{\Lambda^{123}}\right)^{n-2}\left(\frac{\Lambda_0}{\Lambda}\right)^{k}. \tag{21}$$

In the IR regime $\phi_0 \ll \Lambda^{123}$ and $\Lambda_0 \ll \Lambda$, the operators with $n \ge 6$ or $k \ge 6$ are then suppressed by at least a fourth power of these ratios relative to the leading action, and the surviving family collapses to the leading–sub-leading set $(n,k) \in \{(2,0),(2,2),(4,0),(4,2),(4,4)\}$. The choice to keep this particular layer rather than another is the modelling truncation recorded in Remark 2.4(ii); the smallness of the ratios ϕ_0/Λ^{123} and Λ_0/Λ is a formal IR-regime assumption whose quantitative interpretation depends on the emergent-spacetime structure flagged in Remark 2.4(i).

3.5 Action uniqueness theorem

Theorem 3.5 (Form of the E_8-GFT action)**.** *Let $\Phi : E_8 \to \mathfrak{e}_8$ be a field of the form* (5) *and let $S[\Phi]$ be a real functional satisfying M1–M4 and P4. Then in the IR regime $\phi_0 \ll \Lambda^{123}$, $\Lambda_0 \ll \Lambda$ there exist real coefficients c_2, c_4, κ_2, c_{42}, c'_{42}, c''_{42}, κ_4, κ'_4, κ''_4, κ'''_4, κ''''_4 such that*

$$\boxed{\begin{aligned} S[\Phi] \;=\; \int_{E_8} \mathrm{d}\mu_{\mathrm{Haar}}(g)\,\Big[\; & c_2\, C_2 + c_4\, C_2^2 \\ & + \kappa_2\, \mathcal{H}_2 + c_{42}\, C_2\mathcal{H}_2 + c'_{42}\, \mathcal{H}_2^{\mathrm{grad}} + c''_{42}\, \mathcal{H}_2^{\mathrm{mix}} \\ & + \kappa_4\, \mathcal{S}_c + \kappa'_4\, \mathcal{S}_a + \kappa''_4\, \mathcal{S}_b + \kappa'''_4\, \mathcal{S}_{c'} + \kappa''''_4\, \mathcal{S}_e \Big] + \mathcal{R}_{\mathrm{higher}}, \end{aligned}} \tag{22}$$

with the generators defined in (10)–(18)*. The two stability-positive coefficients satisfy $\kappa_2 > 0$ and $c_4 > 0$ (eq.* (20)*); the remaining coefficients lie in a convex stability domain $\mathcal{D}_{\mathrm{stab}} \subset \mathbb{R}^9$; the remainder collects all sectors with $n \ge 6$ or $k \ge 6$ and is bounded by*

$$\frac{|\mathcal{R}_{\mathrm{higher}}|}{|S_{\mathrm{leading}}|} \;\le\; C\left(\frac{\phi_0}{\Lambda^{123}}\right)^4 \;+\; C'\left(\frac{\Lambda_0}{\Lambda}\right)^4, \qquad C,\, C' = \mathcal{O}(1). \tag{23}$$

Proof sketch. M1 and M2 fix the integral form $S[\Phi] = \int_{E_8} \mathrm{d}\mu_{\mathrm{Haar}}(g)\, \mathcal{L}(g)$ with $\mathcal{L}$ depending pointwise on Φ and finitely many of its left-invariant derivatives at g; the Wilsonian expansion (6) of P4 represents $\mathcal{L}$ as a sum of homogeneous monomials $\mathcal{P}_{n,k}$ with bare Wilson coefficients bounded by (7). The separate $E_8^R \times E_8^{\mathrm{Ad}}$-invariance of §3.1 restricts the admissible monomials to Ad-invariant contractions through the primitive tensors of Lemma 3.2. Lemma 3.4 exhausts the catalogue with $n,k \le 4$ and gives exactly eleven generators in the surviving five sectors. M4(c) eliminates all sectors with second or higher derivatives of Φ via the Ostrogradski theorem [21, 22]; M4(a)–(b) impose the strict signs (20) on κ_2 and c_4. The naturalness bound (21) sends every operator with $n \geq 6$ or $k \geq 6$ to a relative size at most fourth power of (ϕ_0/Λ^{123}) and (Λ_0/Λ), giving the remainder bound (23). Assembling the surviving sectors yields (22). The full proof, including the explicit Cauchy–Howe plethysm computation, the construction of $\mathcal{S}_b$ via the structural Cauchy–Howe projection onto the $\lambda = (2,2)$ isotypic component, the independence of $\mathcal{S}_c$ and $\mathcal{S}_{c'}$, the Ostrogradski Hessian for $\mathcal{S}_d$, the strict sign restrictions, the convex stability domain $\mathcal{D}_{\mathrm{stab}}$, and the aggregated remainder bound (23), is given in Appendix B. □

Additional numerical verification. The structural ingredients of the uniqueness argument (antisymmetry of f, Jacobi, $\kappa \propto \mathbf{1}$, $h^\vee = 30$) are checked independently on the explicit Chevalley basis of $\mathfrak{e}_8$ in script `e0_algebra_base.py`; the Hilbert series of $\mathrm{Sym}^n\,\mathrm{adj}\,\mathfrak{e}_8$, the dimensions of the admissible (n,k)-sectors of Lemma 3.4, and the sign consistency of $\mathcal{S}_a$, $\mathcal{S}_b$, $\mathcal{S}_c$ are verified numerically in scripts `e1_action_form.py` and `e1_verify_hypotheses.py`.

Physical content. The leading two-coefficient piece

$$S_{\mathrm{leading}} \;=\; \int_{E_8} \mathrm{d}\mu_{\mathrm{Haar}}(g)\,\big[\,c_2\,C_2 + \kappa_2\,\mathcal{H}_2\,\big] \tag{24}$$

consists of a kinetic term (κ_2, a structural analogue on E_8 of an inverse Newton constant) and a quadratic scalar mass term ($c_2\,C_2$, of either sign), the latter controlling the depth of the Φ-vacuum together with the strictly positive $c_4\,C_2^2$. The nine sub-leading coefficients are not zero but small in the IR by the hierarchy (21).

3.6 Structural corollaries

The Casimir-degree count of Lemma 3.3 has two immediate structural consequences.

Corollary 3.6 (Anomaly safety)**.** *For any finite-dimensional representation ρ of $\mathfrak{e}_8$, the symmetric trace of three generators vanishes identically,*

$$D^{(\rho)}_{ABC} \;:=\; \mathrm{STr}\big(\rho(T_A)\rho(T_B)\rho(T_C)\big) \;\equiv\; 0, \qquad \forall\, A,B,C \in \{1,\dots,248\}. \tag{25}$$

We write $D_{ABC} := D^{(\mathrm{adj})}_{ABC}$ when the adjoint is chosen for definiteness.

Proof. A non-vanishing $D^{(\rho)}_{ABC}$ would be a symmetric Ad-invariant 3-tensor on $\mathfrak{e}_8$ (the Ad-invariance follows from cyclicity of the trace and $\rho([T_A,T_B]) = [\rho(T_A),\rho(T_B)]$), equivalently a primitive Casimir of degree 3. Lemma 3.3 contains no such entry, so the conclusion is independent of the representation ρ. □

Corollary 3.7 (Absence of a Yang–Mills quartic in the leading action)**.** *The leading two-coefficient action* (24) *contains no quartic of Casimir type. Equivalently, no analogue of the* $\mathrm{Tr}(F^aF^a)$ *term built from a primitive degree-four invariant of $\mathfrak{e}_8$ exists at leading order: the only quartic* Ad*-invariant of Φ is the squared Casimir C_2^2, which by* (21) *is sub-leading and which is in any case not a primitive invariant.*

Proof. The same Casimir-degree count (9) excludes a primitive degree-four Casimir, so the only Ad-invariant scalar in $\mathrm{Sym}^4(\mathfrak{e}_8^*)$ is C_2^2. □

4 Symmetry-breaking vacuum (BEC analogue)

This section selects the vacuum of the action of Theorem 3.5 on the open half-line $c_2 < 0$. The content is organised as follows. The functional minimisation problem on $C^\infty(E_8,\mathfrak{e}_8)$ is reduced to a finite-dimensional one on constants (§4.1); the resulting one-variable polynomial in $r^2 = C_2(\Phi_0)$ is classified into four phase regimes (§4.2); the Mexican-hat regime yields an existence statement for the condensate (§4.3); the adjoint orbit of equivalent vacua is catalogued by the Levi (sub-root-system) stratification of E_8 (§4.4); the fluctuation spectrum within the Cas_2 truncation $V_{\text{eff}} = c_2 C_2 + c_4 C_2^2$ splits into Goldstone modes, accidentally flat "spectator" directions transverse to the orbit, and a single massive radial direction whose mass is controlled by the leading and (4,2) sub-leading sectors of the action (§4.5). The structural content is collected in Proposition 4.10 (§4.6).

We adopt the normalisation of Theorem 3.5 (no factors of $\frac{1}{2}$ or $\frac{1}{4}$ in the potential expansion), so that the tree-level potential reads $V_{\text{eff}}(\Phi_0) = c_2\, C_2(\Phi_0) + c_4\, C_2(\Phi_0)^2$ with $C_2(\Phi_0) = \kappa(\Phi_0, \Phi_0) \geq 0$.

4.1 Reduction to constant configurations

The action (22) depends on the field through two purely potential operators (C_2 and C_2^2) and nine operators carrying at least one left-invariant derivative $L_A\Phi$. The first elementary observation is that the latter all vanish on constant configurations.

Lemma 4.1 (Vanishing of derivative invariants on constants)**.** *For a constant configuration $\Phi(g) \equiv \Phi_0 \in \mathfrak{e}_8$, every derivative invariant in* (22) *vanishes identically:*

$$\mathcal{H}_2 \;=\; \mathcal{H}_2^{\text{grad}} \;=\; \mathcal{H}_2^{\text{mix}} \;=\; \mathcal{S}_a \;=\; \mathcal{S}_b \;=\; \mathcal{S}_c \;=\; \mathcal{S}_{c'} \;=\; \mathcal{S}_e \;=\; (C_2\mathcal{H}_2) \;=\; 0 \quad \text{at } \Phi = \Phi_0. \tag{26}$$

[2]

Proof. For $f(g) = \Phi_0^A$ constant, definition (8) gives $(L_A f)(g) = \mathrm{d}/\mathrm{d}t|_{t=0}\, \Phi_0^A = 0$. Each invariant in the list (26) contains at least one factor $L_A\Phi^B$ and therefore vanishes on Φ_0. □

On a constant the action collapses, by $\int_{E_8} \mathrm{d}\mu_{\text{Haar}} = 1$ (Appendix A.2), to

$$S[\Phi_0] \;=\; c_2\, C_2(\Phi_0) + c_4\, C_2(\Phi_0)^2 \;=:\; V_{\text{eff}}(C_2(\Phi_0)). \tag{27}$$

Of the eleven Wilson coefficients in (22) only the two purely potential coefficients (c_2, c_4) enter the tree-level vacuum selection; the remaining nine $(\kappa_2, c_{42}, c'_{42}, c''_{42}, \kappa_4, \kappa'_4, \kappa''_4, \kappa'''_4, \kappa''''_4)$ are spectators here and re-enter only through the fluctuation spectrum of §4.5.

Proposition 4.2 (Restriction to homogeneous configurations)**.** *Suppose the nine sub-leading coefficients lie in the convex stability domain $\mathcal{D}_{\text{stab}} \subset \mathbb{R}^9$ of Theorem 3.5, read here and throughout §4 as the BEC-specific instance (positivity of the quadratic kinetic form on fluctuations around constant configurations $\Phi_0 \in \mathfrak{e}_8$), and let the IR remainder $\mathcal{R}_{\text{higher}}$ be bounded by* (23)*. Then the global infimum of $S[\Phi]$ on $C^\infty(E_8, \mathfrak{e}_8)$ is attained on a constant configuration $\Phi(g) \equiv \Phi_0 \in \mathfrak{e}_8$.*

Proof. Decompose $S = S_{\text{kin}} + S_{\text{pot}}$ with $S_{\text{pot}}[\Phi] := \int \mathrm{d}\mu_{\text{Haar}}\, [c_2 C_2 + c_4 C_2^2]$. The leading kinetic operator $\kappa_2\mathcal{H}_2 \geq 0$ is non-negative on $C^\infty(E_8, \mathfrak{e}_8)$ unconditionally (by (20) and positive-definiteness of κ from P2), with strict positivity on non-constants. The sub-leading $(4,2)$ and $(4,4)$ kinetic generators are sign-indefinite individually; $\mathcal{D}_{\text{stab}}$ is defined precisely as the convex domain on which their linear combination with $\kappa_2\mathcal{H}_2$ retains non-negative spectrum, and in the IR regime the hierarchy (21) suppresses them by at least $(\Lambda_0/\Lambda)^2$ relative to the leading term, making the inclusion automatic at the truncation order. Hence $S_{\text{kin}} \geq 0$ with equality only on constants. For $c_4 > 0$ the polynomial $c_2\, t + c_4\, t^2$ has a global minimum on $t \geq 0$ at $t^* = \max(0, -c_2/(2c_4))$, with value $\min(0, -c_2^2/(4c_4))$; integrated over the unit-volume Haar measure this is the same lower

[2]The Skyrme-type generators $\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_c, \mathcal{S}_{c'}, \mathcal{S}_e$ follow the labelling convention of §3 (eqs. (14)–(18)). The letter $\mathcal{S}_d$ is reserved for the Ostrogradski-risky double-derivative generator $\mathrm{Tr}(L_A L_B\Phi \cdot L^A L^B\Phi)$, excluded from the action by the second-order field-equation clause of Theorem 3.5; the gap in the alphabet is not a typographical accident.

bound for S_{pot}. Pick any $\Phi_0 \in \mathfrak{e}_8$ with $\kappa(\Phi_0, \Phi_0) = t^*$ (the level set is the round 247-sphere). The constant configuration $\Phi(g) \equiv \Phi_0$ realises $S_{\mathrm{kin}} = 0$ by Lemma 4.1 and the lower bound for S_{pot} by construction, so it attains the infimum. □

4.2 Phase classification

By Proposition 4.2 the search for the global minimum reduces to a polynomial in the single non-negative scalar

$$r^2 \;:=\; C_2(\Phi_0) \;=\; \kappa(\Phi_0, \Phi_0) \;\geq\; 0, \tag{28}$$

namely

$$\boxed{V_{\mathrm{eff}}(r^2) \;=\; c_2\, r^2 + c_4\, r^4, \qquad c_4 > 0 \text{ (strict, from (20)).}} \tag{29}$$

The strict sign $c_4 > 0$ inherited from M4 collapses the would-be two-dimensional (c_2, c_4) scan to the one-dimensional half-line $c_2 \in \mathbb{R}$. The four resulting regimes are tabulated in Table 3.

Regime	Sign of c_2	Shape of $V_{\mathrm{eff}}(r^2)$	Global minimum
(I)	$c_2 > 0$	strictly convex on $r^2 \geq 0$	$r_*^2 = 0$
(II)	$c_2 < 0$	Mexican-hat in r^2	$r_*^2 = -c_2/(2c_4) > 0$
(III)	$c_2 = 0$	pure quartic	$r_*^2 = 0$ (degenerate
(IV)	$c_4 \leq 0$	stability-violating (unbounded below for $c_4 < 0$, or for $c_4 = 0$ with $c_2 < 0$)	none in those subre

Table 3: Phase regimes of the tree-level effective potential (29) as a function of the two purely potential coefficients (c_2, c_4). The strict sign $c_4 > 0$ from (20) excludes regime (IV). The condensate region relevant for this section is region (II), the open half-line $\mathcal{R}_{\mathrm{BEC}} := \{c_2 \in \mathbb{R} \mid c_2 < 0\} = (-\infty, 0)$.

4.3 Existence of the condensate

Theorem 4.3 (Existence of the symmetry-breaking vacuum)**.** *Let $S[\Phi]$ have the form (22) of Theorem 3.5 with the nine sub-leading coefficients in $\mathcal{D}_{\mathrm{stab}}$. For every $c_2 \in \mathcal{R}_{\mathrm{BEC}} = (-\infty, 0)$ the action admits a non-trivial global minimum $\Phi_0 \in \mathfrak{e}_8$ with*

$$\boxed{\kappa(\Phi_0, \Phi_0) \;=\; r_*^2 \;=\; -\frac{c_2}{2c_4} \;>\; 0, \qquad V_{\mathrm{eff}}(r_*^2) \;=\; -\frac{c_2^2}{4c_4} \;<\; 0.} \tag{30}$$

The region $\mathcal{R}_{\mathrm{BEC}}$ is open in $\mathbb{R}$, so the result holds on a one-parameter family of admissible coefficients without fine-tuning.

Proof. By Proposition 4.2 it suffices to minimise the polynomial $V_{\mathrm{eff}}(t) = c_2 t + c_4 t^2$ in $t := r^2 \geq 0$ (the same potential will be used in radial form $V_{\mathrm{eff}}(r) = c_2 r^2 + c_4 r^4$ in §4.5). The stationarity condition $\mathrm{d}V_{\mathrm{eff}}/\mathrm{d}t = c_2 + 2c_4\, t = 0$ has the positive root $t_* = r_*^2 = -c_2/(2c_4)$ for $c_2 < 0$, $c_4 > 0$; substitution gives the boxed depth. For $c_4 > 0$ the polynomial diverges as $r \to \infty$, so the stationary point is the unique global minimum on $r^2 \geq 0$. The level set $\{X \in \mathfrak{e}_8 \mid \kappa(X, X) = r_*^2\}$ is the round 247-sphere of radius r_*, which is non-empty in the 248-dimensional algebra. Openness of $\mathcal{R}_{\mathrm{BEC}}$ in $\mathbb{R}$ is immediate. □

Empirical content. Theorem 4.3 is a structural existence statement: the action of Theorem 3.5 admits a non-trivial vacuum on an open parameter region. It does *not* predict the value of c_2 in the realised phase; that role is played by experiment, in the same sense in which the Higgs vacuum expectation $v = 246\,\mathrm{GeV}$ in the Standard Model is fixed by experiment rather than predicted by the form of the Higgs potential.

Additional numerical verification. The tree-level content of Theorem 4.3 together with the Goldstone count of Lemma 4.6 and the radial Higgs mass of Proposition 4.9 is verified numerically on the explicit $\mathfrak{e}_8$ basis in script `e2_bec_phase.py`.

4.4 Orbit catalogue

The minimum (30) is not a single point. Ad-invariance of κ (Appendix A.2) implies $V_{\mathrm{eff}}(\kappa(\mathrm{Ad}_g\,\Phi_0, \mathrm{Ad}_g\,\Phi_0)) = V_{\mathrm{eff}}(r_*^2)$, so the entire adjoint orbit $\mathcal{O}(\Phi_0) := \{\mathrm{Ad}_g\,\Phi_0 \,|\, g \in E_8\} \subset \mathfrak{e}_8$ is a union of degenerate global minima.

Proposition 4.4 (Orbit catalogue)**.** *The set of global minima of* (22) *in* $\mathfrak{e}_8$ *is the round* 247*-sphere*

$$S^{247}_{r_*} \;:=\; \{\, X \in \mathfrak{e}_8 \mid \kappa(X,X) = r_*^2 \,\} \;\subset\; \mathfrak{e}_8, \tag{31}$$

stratified by adjoint-orbit type. The strata are in bijection with points of the seven-dimensional orbifold

$$\mathbf{S}^7_{r_*} \;:=\; \{\, X \in \mathfrak{h} \mid \kappa(X,X) = r_*^2 \,\}/W, \tag{32}$$

with $\mathfrak{h} \subset \mathfrak{e}_8$ *a Cartan subalgebra and* $W = W(E_8)$ *the Weyl group,* $|W| = 696{,}729{,}600$*. The stratification of* $\mathbf{S}^7_{r_*}$ *by stabilizer type is the Levi stratification of* E_8*: the stabilizer of* $\Phi_0 \in \mathfrak{h}$ *is the centraliser* $H = Z_{E_8}(\Phi_0)$*, classified by the closed sub-root-system* $\Psi_{\Phi_0} := \{\alpha \in \Phi(E_8) \,|\, \alpha(\Phi_0) = 0\}$*. The eight maximal proper Levi strata, plus the generic semisimple stratum* T^8*, are listed in Table 4.*

Proof. Ad-invariance of κ implies $V_{\mathrm{eff}} \circ \mathrm{Ad}_g = V_{\mathrm{eff}}$, so the orbit of any minimum is a union of minima; combined with $V_{\mathrm{eff}}(X) = c_2\kappa(X,X) + c_4\kappa(X,X)^2$ and Theorem 4.3 this gives $S^{247}_{r_*}$ of (31) as the full set of minima. Conjugacy of Cartan subalgebras under the compact-form adjoint action and Weyl-group equivalence of their elements [25, Thm. 4.36 and Cor. 4.52] imply that every adjoint orbit of a semisimple element of $\mathfrak{e}_8$ meets $\mathfrak{h}$ in a single Weyl-group orbit; the level set $\{X \in \mathfrak{h} \,|\, \kappa(X,X) = r_*^2\}$ is the round 7-sphere in the eight-dimensional Cartan, and modulo W it is a seven-dimensional orbifold parametrising the adjoint-orbit types in $S^{247}_{r_*}$. The centraliser $H = Z_{E_8}(\Phi_0)$ of $\Phi_0 \in \mathfrak{h}$ generates the orbit $\mathcal{O}(\Phi_0) \cong E_8/H$ and has Lie algebra

$$\mathrm{Lie}(H) \;=\; \mathfrak{h} \oplus \bigoplus_{\alpha \in \Psi_{\Phi_0}} \mathfrak{g}_\alpha, \tag{33}$$

i.e. a rank-8 Levi subgroup of E_8 containing the torus T^8 [24, Ch. VII][25, Ch. IV], classified through its sub-root-system Ψ_{Φ_0} by the lattice of closed regular sub-root-systems of $\Phi(E_8)$ [31, 32]. For $r_* > 0$ at least one root does not vanish on Φ_0, so $\Psi_{\Phi_0} \subsetneq \Phi(E_8)$ and H is a *proper* Levi subgroup; the full-rank semisimple Borel–de Siebenthal subgroups arise only as the deepest closure $\Phi_0 \to 0$, excluded by $r_* > 0$. □

Remark 4.5 (Levi vs. Borel–de Siebenthal stratum)*.* The leading $E_7 \times \mathrm{U}(1)$ Levi stratum is not the full-rank semisimple Borel–de Siebenthal subgroup $E_7 \times \mathrm{SU}(2)/\mathbb{Z}_2 \subset E_8$, whose coset is the quaternion-Kähler Wolf space $\mathrm{EIX} = E_8/(E_7 \times \mathrm{SU}(2))$ of dimension 112 [10, 11]. The actual orbit $\mathcal{O}(\Phi_0) = E_8/H$ with $H = E_7 \times \mathrm{U}(1)$ has dimension $248 - 134 = 114$ and is an $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ bundle over EIX; the analogous pattern lifts each full-rank Borel–de Siebenthal candidate to its co-rank-1 Levi counterpart in Table 4. We write $H' \equiv H = Z_{E_8}(\Phi_0)$ for the Levi stabilizer in the rest of this section, with $\mathfrak{h}' := \mathrm{Lie}(H')$ kept typographically distinct from the Cartan symbol $\mathfrak{h}$ of (32).

Stabilizer $H = Z_{E_8}(\Phi_0)$	$\dim H$	$\dim \mathcal{O} = 248 - \dim H$	Sub-root-system Ψ_{Φ_0}
T^8	8	240	$\varnothing$ (generic semisimple)
$E_7 \times \mathrm{U}(1)$	134	114	E_7 (remove α_8)
$\mathrm{Spin}(14) \times \mathrm{U}(1)$	92	156	D_7 (remove α_1)
$E_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)$	82	166	$E_6 \times A_1$ (remove α_7)
$\mathrm{SU}(8) \times \mathrm{U}(1)/\mathbb{Z}_8$	64	184	A_7 (remove α_2)
$\mathrm{Spin}(10) \times \mathrm{SU}(3) \times \mathrm{U}(1)$	54	194	$D_5 \times A_2$ (remove α_6)
$\mathrm{SU}(2) \times \mathrm{SU}(7) \times \mathrm{U}(1)$	52	196	$A_1 \times A_6$ (remove α_3)
$\mathrm{SU}(5) \times \mathrm{SU}(4) \times \mathrm{U}(1)$	40	208	$A_4 \times A_3$ (remove α_5)
$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(5) \times \mathrm{U}(1)$	36	212	$A_2 \times A_1 \times A_4$ (remove α_4)

Table 4: The eight maximal proper Levi strata of the level set (32) by stabilizer type, plus the generic semisimple stratum T^8. Each non-trivial row arises by removing exactly one node from the (non-affine) E_8 Dynkin diagram in the Bourbaki labelling (chain α_1–α_3–α_4–α_5–α_6–α_7–α_8 with the fork node α_2 attached to α_4), yielding a sub-root-system of rank 7, completed to a rank-8 Levi by the one-dimensional centre generated by Φ_0. Carrying the table to the eight cases is needed for the Goldstone count $\dim \mathfrak{m} = 248 - \dim H'$ and the spectator count $\dim H' - 1$ of §4.5: the running example chosen below is the deepest stratum $H' = E_7 \times \mathrm{U}(1)$ (top non-trivial row), with $\dim \mathfrak{m} = 114$ Goldstones and $\dim H' - 1 = 133$ spectator modes. The five maximal full-rank semisimple Borel–de Siebenthal subgroups of E_8 [31, 32] ($E_7 \times \mathrm{SU}(2)/\mathbb{Z}_2$, $\mathrm{Spin}(16)/\mathbb{Z}_2$, $(E_6 \times \mathrm{SU}(3))/\mathbb{Z}_3$, $\mathrm{SU}(9)/\mathbb{Z}_3$, and $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$) are not realised as $\mathrm{Stab}(\Phi_0)$ for any $\Phi_0 \in \mathfrak{h}$ with $r_* > 0$; they arise only as the closure $\Phi_0 \to 0$.

4.5 Fluctuation spectrum

Fix a representative $\Phi_0 \in \mathfrak{e}_8$ in the orbit $\mathcal{O}(\Phi_0) = E_8/H'$ of Proposition 4.4, with the actual stabilizer $H' \subset E_8$ of Remark 4.5. The algebra $\mathfrak{e}_8$ splits orthogonally with respect to κ,

$$\mathfrak{e}_8 \;=\; \mathfrak{h}' \oplus \mathfrak{m}, \qquad \mathfrak{h}' \;=\; \mathrm{Lie}(H'), \qquad \dim \mathfrak{m} \;=\; 248 - \dim H', \tag{34}$$

into the stabilizer subalgebra $\mathfrak{h}'$ and its orthogonal complement $\mathfrak{m}$. Since Φ_0 commutes with itself, it lies in $\mathfrak{h}'$, and the stabilizer subalgebra further splits as $\mathfrak{h}' = \mathbb{R}\hat{\Phi}_0 \oplus \mathfrak{s}$ with $\hat{\Phi}_0 := \Phi_0/r_*$ and the *spectator subalgebra*

$$\mathfrak{s} \;:=\; \mathfrak{h}' \cap \hat{\Phi}_0^{\perp}, \qquad \dim \mathfrak{s} \;=\; \dim H' - 1. \tag{35}$$

Fluctuations therefore decompose into three classes,

$$\delta\Phi \;=\; \delta\Phi^{\mathrm{rad}}\,\hat{\Phi}_0 \;+\; \delta\Phi^{\mathrm{spec}} \;+\; \delta\Phi^{\mathrm{Gold}}, \qquad \delta\Phi^{\mathrm{spec}} \in \mathfrak{s}, \quad \delta\Phi^{\mathrm{Gold}} \in \mathfrak{m}, \tag{36}$$

respectively the radial direction $\hat{\Phi}_0$, the orbit-tangent (Goldstone) sector $\mathfrak{m}$, and the orbit-transverse residue $\mathfrak{s}$ inside the stabilizer subalgebra. The respective dimensions are 1, $248 - \dim H'$, and $\dim H' - 1$.

Goldstone count

Lemma 4.6 (Goldstone count)**.** *At tree level the second variation of V_{eff} vanishes identically along $\mathfrak{m}$. The number of massless $\delta\Phi^{\mathrm{Gold}}$-modes is*

$$\dim \mathfrak{m} \;=\; 248 - \dim H', \tag{37}$$

in agreement with the Goldstone theorem [33, 34] for the spontaneous breaking $E_8^{\mathrm{Ad}} \to H' =$ $\mathrm{Stab}(\Phi_0)$*.*

Proof. By Proposition 4.4 the function V_{eff} is constant on the orbit $\mathcal{O}(\Phi_0) = E_8 \cdot \Phi_0$, whose tangent space at Φ_0 is $\{[T_a, \Phi_0] \,|\, T_a \in \mathfrak{e}_8\} = \mathfrak{m}$. The second variation of V_{eff} along directions tangent to the orbit therefore vanishes identically; the count of independent flat directions is $\dim \mathfrak{m}$. □

The count $\dim \mathfrak{m} = 248 - \dim H'$ is itself stratification-dependent: all orbits in $\mathbf{S}^7_{r_*}$ are tree-level degenerate (Proposition 4.4), and different strata realise different $\dim H'$ from Table 4. We carry the leading $H' = E_7 \times \mathrm{U}(1)$ stratum ($\dim H' = 134$) as the running example below; the count is then $\dim \mathfrak{m} = 114$, fibred as an S^2-bundle over the Wolf space $\mathrm{EIX} = E_8/(E_7 \times \mathrm{SU}(2))$ of dimension 112 (Remark 4.5). For a generic semisimple stratum ($H' = T^8$, $\dim \mathfrak{m} = 240$) and the other Levi strata of Table 4 the analogous count obtains *mutatis mutandis*.

Spectator (accidentally flat) directions

The decomposition (36) contains a third class of fluctuations beyond the orbit-tangent Goldstones and the radial direction: the orbit-transverse residue $\mathfrak{s}$ of (35) inside the stabilizer subalgebra. These directions are tangent to the level sphere $\{X \in \mathfrak{e}_8 \,|\, \kappa(X, X) = r_*^2\}$ but transverse to the orbit $\mathcal{O}(\Phi_0) \subset \mathfrak{e}_8$, and they are massless at the level of the truncation $V_{\text{eff}} = c_2 C_2 + c_4 C_2^2$.

Lemma 4.7 (Spectator flat directions in the Cas_2 truncation)**.** *For every $\delta\Phi \in \mathfrak{e}_8$ with $\kappa(\Phi_0, \delta\Phi) = 0$, $C_2(\Phi_0 + \delta\Phi) = r_*^2 + |\delta\Phi|^2$ exactly, and the stationarity condition $c_2 + 2c_4\, r_*^2 = 0$ established in the proof of Theorem 4.3 reduces the truncated potential to*

$$V_{\text{eff}}(\Phi_0 + \delta\Phi) \;=\; V_{\text{eff}}(r_*^2) \;+\; c_4\, |\delta\Phi|^4, \tag{38}$$

with vanishing quadratic-in-$\delta\Phi$ piece. The truncation therefore produces $247 = (248 - \dim H') + (\dim H' - 1)$ *classically massless modes around Φ_0, split by* (34) *into* $\dim \mathfrak{m} = 248 - \dim H'$ *Goldstone modes in $\mathfrak{m}$, protected to all orders by Goldstone's theorem (Lemma 4.6), and* $\dim H' - 1$ *"spectator" modes in $\mathfrak{s}$, unprotected by E_8^{Ad}-symmetry. For the $E_7 \times \mathrm{U}(1)$ vacuum (Remark 4.5) the spectator sector contains* 133 *modes.*

Remark 4.8 (Lifting of the spectator directions). The flatness of the $\dim H' - 1$ spectator directions is an accidental feature of the Cas_2 truncation: the potential $V_{\text{eff}} = c_2 C_2 + c_4 C_2^2$ depends on Φ_0 only through the quadratic Casimir $C_2(\Phi_0) = \kappa(\Phi_0, \Phi_0)$, which is invariant under the larger group $\mathrm{O}(\mathfrak{e}_8, \kappa) \cong \mathrm{O}(248)$ rather than only under $E_8^{\mathrm{Ad}} \subset \mathrm{O}(248)$. The mismatch is parameterised by the higher-degree primitive Casimirs Cas_8, Cas_{12}, Cas_{14}, Cas_{18}, Cas_{20}, Cas_{24}, Cas_{30} of $\mathfrak{e}_8$ (Remark 2.1(iv)), whose Φ-monomials distinguish points within the level set $\{C_2 = r_*^2\}$ that share the same C_2. These monomials sit in the IR remainder $\mathcal{R}_{\text{higher}}$ of Theorem 3.5 and lift the spectator directions to non-zero (Casimir-suppressed) masses at the next layer of the expansion. The Goldstone modes in $\mathfrak{m}$ remain strictly massless to all orders, since they are protected by exact E_8^{Ad}-invariance and Goldstone's theorem [33, 34].

The phenomenon has no analogue in the Standard-Model Higgs sector, where $\mathrm{SU}(2) \times \mathrm{U}(1)_Y$ acts transitively on the unit sphere $S^3 \subset \mathbb{C}^2$: the orbit of $\langle H \rangle$ exhausts the level set, and only the 3 Goldstones plus the radial Higgs appear. For the adjoint action of E_8 on $\mathfrak{e}_8$ the orbit E_8/H' is a strict 114-dimensional submanifold of the 247-sphere, so the inequality $\dim(E_8/H') < \dim \mathfrak{e}_8 - 1$ leaves a residual $\dim H' - 1$ directions tangent to the level set but transverse to the orbit. The lifting of these directions is part of the structural content of the higher-Casimir layer, not of the Cas_2 truncation itself.

Radial Higgs and the wave-function renormalisation

The Taylor expansion of V_{eff} along the radial direction $\hat{\Phi}_0 := \Phi_0/r_*$, viewed as a function of the radius r via $V_{\text{eff}}(r) = c_2 r^2 + c_4 r^4$, reads $V_{\text{eff}}(r_* + \delta) = V_{\text{eff}}(r_*) + M_{\text{rad}}^2\, \delta^2 + \mathcal{O}(\delta^3)$, with bare radial mass-squared (the coefficient of δ^2 in the no-$\frac{1}{2}$ convention of Theorem 3.5)

$$M_{\rm rad}^2 \;=\; \tfrac{1}{2}\,V_{\rm eff}''(r_*) \;=\; c_2 + 6c_4\,r_*^2 \;=\; -2c_2 \;=\; 4c_4\,r_*^2 \;>\; 0, \tag{39}$$

where the last two equalities use (30). The physical mass requires the wave-function renormalisation factor of the radial mode, obtained by expanding the kinetic invariants of (22) to quadratic order in the radial coefficient $\delta\Phi^{\rm rad}$ of (36) around Φ_0. Direct substitution $\Phi(g) = \Phi_0 + \delta\Phi^{\rm rad}(g)\,\hat\Phi_0$ gives, in the orthonormal basis $\kappa_{AB} = \delta_{AB}$ of Appendix A.2,

$$\begin{aligned} \mathcal{H}_2 \;&\to\; \kappa^{AA'}\,(L_A\delta\Phi^{\rm rad})(L_{A'}\delta\Phi^{\rm rad}),\\ C_2\,\mathcal{H}_2 \;&\to\; r_*^2\,\kappa^{AA'}\,(L_A\delta\Phi^{\rm rad})(L_{A'}\delta\Phi^{\rm rad}),\\ \mathcal{H}_2^{\rm grad} \;&\to\; r_*^2\,\kappa^{AA'}\,(L_A\delta\Phi^{\rm rad})(L_{A'}\delta\Phi^{\rm rad}), \end{aligned} \tag{40}$$

at quadratic order in $\delta\Phi^{\rm rad}$, while $\mathcal{H}_2^{\rm mix}$ vanishes for purely radial fluctuations by the antisymmetry of f_{ABC} (the contraction $f_{BCE}\hat\Phi_0^B\hat\Phi_0^C$ is identically zero), and the five $(4,4)$ Skyrme-type generators $\mathcal{S}_a,\ldots,\mathcal{S}_e$ contribute only at quartic order in $\delta\Phi^{\rm rad}$. The wave-function renormalisation of the radial mode is therefore

$$Z_{\rm rad}(\Phi_0) \;=\; \kappa_2 \;+\; (c_{42} + c_{42}')\,r_*^2 \;+\; \mathcal{O}(r_*^4), \tag{41}$$

positive on the IR regime by the kinetic-spectrum clause defining $\mathcal{D}_{\rm stab}$ (Theorem 3.5), with the leading piece $\kappa_2 > 0$ from (20).

Proposition 4.9 (Radial Higgs mass)**.** *The physical mass-squared of the radial mode in the BEC vacuum is*

$$\boxed{m_{\rm rad}^2 \;=\; \frac{-2c_2}{Z_{\rm rad}(\Phi_0)} \;=\; \frac{-2c_2}{\kappa_2 + (c_{42}+c_{42}')(-c_2/(2c_4)) + \mathcal{O}(r_*^4)}\,,} \tag{42}$$

strictly positive on $\mathcal{R}_{\rm BEC}$ inside $\mathcal{D}_{\rm stab}$.

Proof. The bare radial mass (39) and the wave-function renormalisation (41) combine through the standard rescaling $m_{\rm phys}^2 = M_{\rm rad}^2/Z_{\rm rad}$ associated with the quadratic Lagrangian $\mathcal{L}_2 = Z_{\rm rad}\,(L_A\delta\Phi^{\rm rad})(L^A\delta\Phi^{\rm rad}) + M_{\rm rad}^2\,(\delta\Phi^{\rm rad})^2$ in the no-$\frac{1}{2}$ convention of Theorem 3.5. Strict positivity of the right-hand side of (42) follows from $-c_2 > 0$ and $Z_{\rm rad} > 0$ inside $\mathcal{D}_{\rm stab}$. □

The non-trivial structural feature of Proposition 4.9 is that the gradient invariant $\mathcal{H}_2^{\rm grad}$ contributes to $Z_{\rm rad}$ separately from the ordinary kinetic invariant $\mathcal{H}_2$, with the consequence that the radial mass carries an independent dependence on c_{42}' that is not absorbable into κ_2. The same dependence is *not* exhibited for fluctuations in the Goldstone sector $\mathfrak{m}$ or the spectator sector $\mathfrak{s}$. For $\delta\Phi$ orthogonal to Φ_0 in κ, the Casimir expands as $C_2(\Phi) = r_*^2 + |\delta\Phi|^2$, so its left-invariant derivative $L_AC_2 = 2\,\kappa(\Phi, L_A\Phi)$, evaluated to linear order in $\delta\Phi$, equals $2\,\kappa(\Phi_0, L_A\delta\Phi) = 2\,L_A\kappa(\Phi_0,\delta\Phi)$, which vanishes pointwise on E_8 because the function $\kappa(\Phi_0,\delta\Phi)\equiv 0$ identically. Thus $\mathcal{H}_2^{\rm grad} = \frac{1}{4}\,\kappa^{AA'}(L_AC_2)(L_{A'}C_2)$ is at least quartic in $\delta\Phi$ and contributes nothing at quadratic order to the orthogonal-fluctuation kinetic form.

For the mixed invariant $\mathcal{H}_2^{\rm mix}$ the situation splits further. Since $\delta\Phi^{\rm spec}(g)\in\mathfrak{s}$ pointwise and $\mathfrak{s}$ is a linear subspace, the derivative $L_A\delta\Phi^{\rm spec}(g)$ also takes values in $\mathfrak{s}$; the spectator subalgebra centralises Φ_0, so $f_{BCE}\,(L_A\delta\Phi^{\rm spec})^B\,\Phi_0^C = [L_A\delta\Phi^{\rm spec},\Phi_0]_E = 0$ pointwise on E_8; hence $\mathcal{H}_2^{\rm mix}$ vanishes at quadratic order in $\delta\Phi^{\rm spec}$, and

$$Z_{\rm spec}(\Phi_0) \;=\; \kappa_2 + c_{42}\,r_*^2 \;+\; \mathcal{O}(r_*^4). \tag{43}$$

For Goldstone fluctuations $\delta\Phi^{\mathrm{Gold}} \in \mathfrak{m}$ the contraction $f_{BCE}\, L_A \delta\Phi^{\mathrm{Gold},B}\, \Phi_0^C = [L_A\delta\Phi^{\mathrm{Gold}}, \Phi_0]_E$ does not vanish, and $\mathcal{H}_2^{\mathrm{mix}}$ adds at quadratic order a manifestly non-negative, mode-dependent piece

$$c''_{42}\, \kappa^{AA'}\, \kappa\big(\mathrm{ad}_{\Phi_0}(L_A\delta\Phi^{\mathrm{Gold}}),\ \mathrm{ad}_{\Phi_0}(L_{A'}\delta\Phi^{\mathrm{Gold}})\big).$$

On the root subspace $\mathfrak{g}_\alpha \subset \mathfrak{m}$ the operator ad_{Φ_0} is skew-symmetric in κ with $\mathrm{ad}^2_{\Phi_0}$ having eigenvalue $-\alpha(\Phi_0)^2 = -r_*^2\, \alpha(\hat{\Phi}_0)^2$, so the κ-norm-squared above evaluates to $\alpha(\Phi_0)^2$ times the κ-norm-squared of $L_A\delta\Phi^{\mathrm{Gold}}$. The Goldstone wave-function renormalisation on the $\mathfrak{g}_\alpha$ component is therefore

$$Z^{(\alpha)}_{\mathrm{Gold}}(\Phi_0) \;=\; \kappa_2 \;+\; \big(c_{42} + c''_{42}\, \alpha(\hat{\Phi}_0)^2\big)\, r_*^2 \;+\; \mathcal{O}(r_*^4), \tag{44}$$

absorbable into a per-mode rescaling of $\delta\Phi^{\mathrm{Gold}}$ that does not lift the Goldstone mass (zero to all orders by Goldstone's theorem [33, 34]). Both Z_{spec} and $Z^{(\alpha)}_{\mathrm{Gold}}$ are free of c'_{42}: the independent rescaling of the radial mode by c'_{42} is therefore specifically a *radial-direction* effect.

4.6 Summary: nature of the condensate phase

Proposition 4.10 (Nature of the symmetry-breaking phase, tree level)**.** *For every* $c_2 \in \mathcal{R}_{\mathrm{BEC}} = (-\infty, 0)$ *inside* $\mathcal{D}_{\mathrm{stab}}$*, the action* (22) *realises a phase with the following four characteristics:*

1. Spontaneous symmetry breaking $E_8^{\mathrm{Ad}} \to H' = \mathrm{Stab}(\Phi_0)$ *(Proposition 4.4 and Remark 4.5).*

2. Order parameter $\langle \Phi(g) \rangle = \Phi_0 \in \mathcal{O}(\Phi_0)$ *with* $\kappa(\Phi_0, \Phi_0) = -c_2/(2c_4) > 0$ *(Theorem 4.3).*

3. Tree-level fluctuation spectrum within the Cas_2 truncation*, comprising:*

   (3a) $\dim \mathfrak{m} = 248 - \dim H'$ *exact Goldstone modes in* $\mathfrak{m}$ *(Lemma 4.6), protected to all orders by Goldstone's theorem;*

   (3b) $\dim H' - 1$ *accidentally flat "spectator" modes in* $\mathfrak{s}$ *(Lemma 4.7), unprotected by* E_8^{Ad}*-symmetry and lifted at the next layer of the expansion by the primitive higher-Casimir invariants* $\mathrm{Cas}_8, \mathrm{Cas}_{12}, \ldots, \mathrm{Cas}_{30}$ *of* $\mathcal{R}_{\mathrm{higher}}$ *(Remark 4.8);*

   (3c) one massive radial Higgs of mass-squared $m^2_{\mathrm{rad}} = -2c_2/Z_{\mathrm{rad}}(\Phi_0)$ *(Proposition 4.9).*

   The total $1 + (\dim H' - 1) + (248 - \dim H') = 248$ *saturates the algebra, and the count of classically massless modes within the truncation is* $247 = \dim \mathfrak{e}_8 - 1$*.*

4. Continuous transition *at the critical line* $c_2 = 0$*: within the present* Cas_2 *truncation* $V_{\mathrm{eff}}(r^2) = c_2 r^2 + c_4 r^4$ *only the even-degree monomials* C_2 *and* C_2^2 *enter, so no cubic-in-r contribution arises tree-level. Beyond the truncation, the same conclusion holds at every order of the IR expansion: every primitive Casimir of* $\mathfrak{e}_8$ *has even degree (degrees* $2, 8, 12, 14, 18, 20, 24, 30$*, Remark 2.1(iv)), so every* Ad*-invariant polynomial in* Φ *is a polynomial in even-degree primitives and itself of even degree; equivalently, no primitive symmetric* Ad*-invariant* 3*-tensor on* $\mathfrak{e}_8$ *exists (Lemma 3.2), and no cubic Casimir invariant of* Φ *is admissible. The transition therefore has no first-order tree-level component at any order.*

Order-parameter analogue with Penrose–Onsager. The Penrose–Onsager [35] criterion is a quantum statistical statement (off-diagonal long-range order in the one-particle reduced density matrix) and not a property of a classical-field VEV. Used here only at the order-parameter level, the four characteristics above match the mean-field signature of a Bose–Einstein condensate by analogy: macroscopic occupation of a single mode of the fundamental field with non-trivial global symmetry, with the symmetry being E_8 rather than the U(1) of condensed-matter BEC. The data $(\langle\Phi\rangle, \mathfrak{m}, \delta\Phi^{\mathrm{Gold}}, \delta\Phi^{\mathrm{rad}})$ play the same role as $(\langle\Psi\rangle, \mathrm{U}(1), \delta\theta, \delta n)$ in the canonical U(1)-symmetric BEC; the additional $\delta\Phi^{\mathrm{spec}}$ sector of Lemma 4.7 has no U(1) analogue.

Comparison with the Standard-Model Higgs mechanism. Table 5 summarises the formal correspondence with the spontaneous breaking $\mathrm{SU}(2)\times\mathrm{U}(1)_Y \to \mathrm{U}(1)_{\mathrm{EM}}$ in the Standard Model. The comparison is qualitative: the present treatment is at the level of a *global* E_8^{Ad} symmetry, with no gauge bosons in the spectrum, so the Goldstones remain physical massless modes rather than being eaten in a Higgs–Brout–Englert sense; the gauged version is deferred to a subsequent paper. The two structures share the order-parameter and radial-mode structure; they differ in the symmetry group, in the field domain (group manifold versus a priori four-dimensional spacetime), in the dimension of the Goldstone sector, and in the presence of an additional sector of accidentally flat directions transverse to the orbit and lifted by the higher-Casimir layer of the action (Remark 4.8).

	Higgs mechanism (SM)	E_8-GFT BEC (here)
Field	$H\colon \mathbb{R}^{1,3}\to\mathbb{C}^2$	$\Phi\colon E_8\to\mathfrak{e}_8$
Symmetry	$\mathrm{SU}(2)\times\mathrm{U}(1)_Y$ (4-dim)	E_8 (248-dim)
Potential	$V=-\mu^2\lvert H\rvert^2+\lambda\lvert H\rvert^4$	$V_{\mathrm{eff}}=c_2\,C_2+c_4\,C_2^2$
Vacuum	$\langle H\rangle=(0,v/\sqrt{2})$, $v=246\,\mathrm{GeV}$	$\langle\Phi\rangle=\Phi_0$, $\kappa(\Phi_0,\Phi_0)=-c_2/(2c_4)$
Symmetry breaking	$\mathrm{SU}(2)\times\mathrm{U}(1)_Y\to\mathrm{U}(1)_{\mathrm{EM}}$	$E_8\to H'=\mathrm{Stab}(\Phi_0)$
Goldstones	3, eaten by $W^\pm, Z$ (gauged)	$\dim\mathfrak{m}=248-\dim H'$, physical (global)
Spectator flat directions	none (orbit = level set S^3)	$\dim H'-1$, lifted by $\mathrm{Cas}_{\geq 8}$
Radial mode	Higgs scalar, $m_H=125\,\mathrm{GeV}$	$m^2_{\mathrm{rad}}=-2c_2/Z_{\mathrm{rad}}$

Table 5: Formal correspondence between the Standard-Model Higgs mechanism and the E_8-GFT condensate phase at tree level. The two columns differ in the symmetry group, in the dimension of the Goldstone sector, in the presence (or absence) of accidentally flat "spectator" directions transverse to the symmetry orbit, and in whether the field domain is a priori four-dimensional spacetime or the compact group manifold E_8.

5 Vacuum orbit selection

Section 4 reduced the search for the global minimum of the action of Theorem 3.5 to the seven-dimensional orbifold $\mathbf{S}^7_{r_*}$ of (32), with all strata degenerate at tree level. The present section selects a single E_8^{Ad}-orbit by three operational filters F1–F3 (§5.2): F1 (anomaly safety) and F2 (Skyrmion-type soliton) are derivable from M1 + M4; F3 (Wolf-space structure) is a structural input not derivable in the present scope (§5.3, Open problem 5.3). The conjunction selects

$$\mathcal{O}\;\cong\;\mathrm{EIX}\;=\;E_8/(E_7\times\mathrm{SU}(2)),\qquad \dim\mathrm{EIX}\;=\;112.$$

Standard-Model embeddability is not used as a filter; the selected H_1 admits the classical chain $G_{\mathrm{SM}}\subset\mathrm{SU}(5)\subset\mathrm{Spin}(10)\subset E_6\subset E_7$ *a posteriori* (Remark 5.8). No dimensional input on the emergent spacetime enters the selection; the rank of the selected orbit is examined downstream in §6.

5.1 Setup: from the Levi catalogue to a finite candidate list

The orbit catalogue of Proposition 4.4 is indexed by closed sub-root-systems $\Psi\subseteq\Phi(\mathfrak{e}_8)$; each Ψ produces a Levi stabilizer $H'=Z_{E_8}(\Phi_0)$ of rank 8 (eq. (33)). The selection problem admits an equivalent description in terms of the Borel–de Siebenthal classification [31, 32] of maximal-rank closed subgroups of E_8, supplemented by the unique non-regular maximal subgroup $G_2\times F_4$ of Dynkin [32].

Lemma 5.1 (Borel–de Siebenthal-type catalogue)**.** *The maximal-rank closed connected subgroups of E_8 are listed, up to conjugacy, in Table 6. Together with the maximal-torus stabilizer $H_T=T^8$*

(the generic semisimple case) and the non-regular maximal subgroup $H_6 = G_2 \times F_4$ [32], the table is exhaustive among the qualitatively distinct candidates for a semisimple $\Phi_0 \in \mathfrak{e}_8$.

#	Stabilizer H	$\dim H$	$\dim \mathcal{O} = 248 - \dim H$	Character
H_1	$E_7 \times \mathrm{SU}(2)/\mathbb{Z}_2$	136	112	Wolf space EIX (compact QK) [10, 11]
H_2	$E_6 \times \mathrm{SU}(3)/\mathbb{Z}_3$	86	162	contains E_6 GUT chain
H_3	$\mathrm{Spin}(16)/\mathbb{Z}_2$	120	128	symmetric space EVIII
H_4	$\mathrm{SU}(9)/\mathbb{Z}_3$	80	168	regular, non-spinorial
H_5	$\mathrm{SU}(5) \times \mathrm{SU}(5)/\mathbb{Z}_5$	48	200	contains SU(5) GUT
H_6	$G_2 \times F_4$	66	182	non-regular maximal [32]
H_7	$\mathrm{Spin}(10) \times \mathrm{SU}(4)/\mathbb{Z}_4$	60	188	sub-maximal in H_3
H_8	$E_6 \times \mathrm{U}(1)^2$	80	168	Levi sub-maximal in H_2
H_9	$\mathrm{SU}(8) \times \mathrm{U}(1)/\mathbb{Z}_8$	64	184	sub-maximal in H_4
H_T	T^8 (maximal torus)	8	240	generic semisimple Φ_0

Table 6: Borel–de Siebenthal-type catalogue of nominal stabilizer subgroups of E_8. Rows H_1–H_5 are the maximal full-rank regular embeddings (one for each non-trivial node of the extended Dynkin diagram); H_6 is the unique non-regular maximal subgroup; H_7–H_9 are sub-maximal embeddings retained because of their rolein candidate-elimination steps below; H_T is the toroidal limit. The orbit $\mathcal{O} = E_8/H$ has dimension $248 - \dim H$. The relation between the nominal H here and the true Levi stabilizer H' of Proposition 4.4 is the S^k-fibre subtlety of Remark 4.5; for the EIX entry H_1, $H/H' = S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$.

Proof. The classification of maximal-rank closed connected subgroups of a compact simple Lie group G by removing one node from the extended Dynkin diagram is the content of Borel–de Siebenthal [31]; for E_8 the extended Dynkin diagram has nine nodes, giving the five maximal regular subgroups H_1, H_2, H_3, H_4, H_5. The unique non-regular maximal subgroup $G_2 \times F_4$ is Dynkin's [32] extension of the same classification. The sub-maximal candidates H_7, H_8, H_9 are obtained by composition with classical branchings (Slansky [16]; McKay–Patera [27]). The toroidal limit H_T arises when Φ_0 is a generic semisimple element of a Cartan subalgebra, with no roots vanishing on it. □

Remark 5.2 (Completeness of the catalogue). The catalogue of Table 6 is exhaustive among candidates with semisimple Φ_0 up to Weyl-group conjugation. Finer position-dependent distinctions inside one row, e.g. different Weyl chambers for a regular Φ_0, are continuous and do not produce qualitatively new candidates. We do not claim absolute completeness against possible sub-maximal hidden embeddings with a distinguished physical role; rigorous completeness would require a systematic enumeration of every closed subgroup of E_8 with the specific structural input used in the filters below, which we do not undertake. The selection theorem of §5.7 is therefore stated relative to Table 6 as input.

5.2 The three operational filters

We declare the three filters here and apply them sector by sector in §§5.4–5.6. Their roles are asymmetric: F1 and F2 are algebraic consistency conditions derivable from the meta-principles M1 + M4 of §2; F3 is a structural geometric input (Wolf-space requirement). The empirical observation of G_{SM} does *not* enter as a filter; SM-embeddability is recorded *a posteriori* as a consequence of the selected orbit (§5.8).

F1 – Anomaly safety (algebraic). The unbroken stabilizer H either carries no primitive cubic Ad-invariant $d^{(3)}_{abc} := \mathrm{STr}(T_a T_b T_c)$ on any of its simple factors, in which case the standard four-dimensional perturbative gauge anomaly polynomial $\mathrm{tr}_R(T_a\{T_b, T_c\})$ vanishes identically in any representation R; or admits an automatic vector-like $E_8 \to H$ branching of adj($\mathfrak{e}_8$) that cancels the cubic anomaly of every SU(N) factor of H without fermion-content tuning. The filter is the standard cubic-anomaly classification of Frampton–Kephart [36], restricted to the stabilizer of the BEC vacuum.

Higher-degree symmetric Ad-invariants ($d^{(4)}$ and beyond) play a different physical role: they enter the six-dimensional anomaly polynomial relevant to Green–Schwarz cancellation, and parametrise the primitive quartic Casimirs of the D_n and E_n series. They do not enter F1, whose content is the four-dimensional perturbative gauge anomaly.

F2 – Skyrmion-type soliton (algebraic). The unbroken stabilizer H is non-abelian. Equivalently, for H compact connected, H contains some SU(2) subgroup acting non-trivially on the tangent module $\mathfrak{m}$, onto which a finite-energy field configuration projects to a quantized topological charge $B \in \mathbb{Z} = \pi_3(\mathrm{SU}(2))$ in the sense of Skyrme [29] and Manton–Sutcliffe [37], Ch. 9. F2 is silent on whether the SU(2) appears as a direct factor of H or as a subgroup of a larger simple factor; the direct-factor case is singled out separately by F3 below (the rotational Sp(1) holonomy of a Wolf-space structure is precisely an SU(2) direct factor of H).

F3 – Wolf-space (quaternion-Kähler) structure (algebraic-geometric). The coset G/H is a compact quaternion-Kähler symmetric space, i.e. a Wolf space [10, 11]: there exists a parallel skew Sp(1)-subbundle of End(TM) generated by three local complex structures I, J, K with $IJ = K$ and a Sp(n)·Sp(1) holonomy reduction.

5.3 Status of F1–F3 relative to the postulates

F1 and F2 are derivable from M1 + M4 of §2. Anomaly cancellation (F1) is a necessary consistency condition for the path-integral measure of M1 together with unitarity in M4 [38, 39]; the Frampton–Kephart cubic-anomaly classification [36] recorded in Lemma 5.4 is its algebraic implementation on a compact simple Lie subgroup. The operational form of F2—an SU(2) subgroup acting non-trivially on $\mathfrak{m}$—is the algebraic prerequisite for the finite-energy $\mathbb{Z}$-quantized topological sector of [29, 37] whose existence requires the action to be bounded below at fixed winding number, the statement of M4(a).

F3 is not derivable from M1–M4 or P1–P4 in the present scope. We adopt it as a structural-geometric input motivated downstream by the parallel quaternion-Kähler 4-form Ω_{quat} on $\mathcal{O}$, which generates $H^4(\mathcal{O};\mathbb{R}) \cong \mathbb{R}$ [11] and serves as the topological substrate of the sigma-model sector built on $\mathcal{O}$ (§6). Removing F3 leaves the six-element candidate set $\mathcal{F}_{12}$ of Theorem 5.9(a) below; the unique-orbit conclusion of part (b) is consequently conditional on F3.

Open problem 5.3 (Derivation of H_1-uniqueness orthogonally to F3)**.** Either of the following would derive the conclusion of Theorem 5.9(b) from M1 + M4 alone, without invoking F3: (P1) one-loop Coleman–Weinberg lifting of the tree-level orbit degeneracy on $\mathbf{S}^7_{r_*}$, with H_1 as the unique global minimum of the one-loop effective potential among the candidates of $\mathcal{F}_{12}$; (P2) spectral stability of Gaussian fluctuations, with H_1 as the unique candidate of $\mathcal{F}_{12}$ admitting a strictly positive fluctuation spectrum. Neither path is pursued here.

5.4 Filter F1: anomaly safety

The structural anomaly safety of $\mathfrak{e}_8$ itself was established as Corollary 3.6 of §3.6: the symmetric trace $D_{ABC} := \mathrm{STr}(T_A T_B T_C)$ vanishes identically because the Casimir-degree spectrum of $\mathfrak{e}_8$ (eq. (9)) contains no entry in degree 3, so the cubic gauge anomaly is absent on E_8 itself in any representation. The complementary degree-4 count of Corollary 3.7—the only Ad-invariant scalar in $\mathrm{Sym}^4(\mathfrak{e}_8^*)$ is the squared quadratic Casimir C_2^2—controls the absence of a Yang–Mills quartic in the leading action and is logically separate from the anomaly story; it does not enter

F1. Filter F1 specialises the cubic-anomaly statement to the unbroken stabilizer H of the BEC vacuum.

Lemma 5.4 (Cubic-anomaly classification on compact simple Lie algebras)**.** *Let* $d^{(3)}_{abc}(R) := \mathrm{STr}_R(T_aT_bT_c)$ *denote the totally-symmetric three-index trace tensor in a representation R of a compact simple Lie algebra $\mathfrak{g}$. The associated* Ad*-invariant primitive cubic symmetric tensor on $\mathfrak{g}$ (equivalently, a primitive Casimir of degree 3) is non-zero precisely if*

$$\mathfrak{g} \;=\; \mathfrak{su}(N), \qquad N \geq 3$$

(with the exceptional isomorphism $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ absorbed into the $\mathfrak{su}(N)$ class). The remaining compact simple Lie algebras

$$\mathfrak{su}(2),\ \mathfrak{so}(N) \text{ for } N \geq 5,\ N \neq 6,\ \mathfrak{sp}(N),\ \mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_8,\ \mathfrak{f}_4,\ \mathfrak{g}_2$$

admit no primitive cubic Ad*-invariant; the four-dimensional perturbative gauge anomaly polynomial* $\mathrm{tr}_R(T_a\{T_b, T_c\})$ *then vanishes identically in any representation R.*

Proof. A totally-symmetric Ad-invariant tensor of degree 3 on $\mathfrak{g}$ corresponds to a primitive Casimir of degree 3 in the symmetric algebra $\mathrm{Sym}(\mathfrak{g}^*)^{\mathrm{Ad}}$. The complete spectrum of primitive Casimir degrees on each compact simple Lie algebra (Bourbaki [6], Plates I–IX) reads

$$\begin{aligned}
A_{N-1} = \mathfrak{su}(N):&\qquad \{2, 3, 4, \ldots, N\},\\
B_N = \mathfrak{so}(2N{+}1):&\qquad \{2, 4, 6, \ldots, 2N\},\\
C_N = \mathfrak{sp}(N):&\qquad \{2, 4, 6, \ldots, 2N\},\\
D_N = \mathfrak{so}(2N):&\qquad \{2, 4, 6, \ldots, 2N{-}2, N\},\\
\mathfrak{e}_6&\qquad \{2, 5, 6, 8, 9, 12\},\\
\mathfrak{e}_7&\qquad \{2, 6, 8, 10, 12, 14, 18\},\\
\mathfrak{e}_8&\qquad \{2, 8, 12, 14, 18, 20, 24, 30\},\\
\mathfrak{f}_4&\qquad \{2, 6, 8, 12\},\\
\mathfrak{g}_2&\qquad \{2, 6\}.
\end{aligned}$$

Degree 3 appears in the A_{N-1} series for $N \geq 3$ and, through the exceptional isomorphism $D_3 \cong A_3$, on $\mathfrak{so}(6) \cong \mathfrak{su}(4)$; it appears nowhere else. The cubic anomaly coefficient $\mathrm{tr}_R(T_a\{T_b, T_c\})$ is, by Adler–Bell–Jackiw [38, 39], proportional to the value of the unique primitive cubic Ad-invariant on $\mathfrak{g}$ (when one exists) evaluated in the representation R, and vanishes identically in the absence of such a primitive cubic.

Higher-degree symmetric Ad-invariants ($d^{(4)}$ and beyond) are present on most of these algebras (e.g. the degree-4 primitive Casimir of D_n for $n \geq 4$, or the higher-degree primitives of $\mathfrak{e}_{6,7,8}, \mathfrak{f}_4$); they are responsible for the six-dimensional anomaly polynomial relevant to Green–Schwarz cancellation, but they do not contribute to the four-dimensional perturbative gauge anomaly polynomial considered here. □

Proposition 5.5 (F1 outcome)**.** *Applied to Table 6, F1 partitions the candidates as follows:*

(a) Pass. *Seven candidates $\{H_1, H_2, H_3, H_6, H_7, H_8, H_T\}$ either have a stabilizer with no primitive cubic* Ad*-invariant on any of its simple factors (H_1, H_3, H_6, H_8, H_T) or admit an automatic vector-like $E_8 \to H$ branching of* $\mathrm{adj}(\mathfrak{e}_8)$ *that cancels the cubic anomaly of every* $\mathrm{SU}(N)$ *factor of H without further input (H_2, H_7).*

(b) Failure. *The three candidates $\{H_4, H_5, H_9\}$ contain at least one* $\mathrm{SU}(N)$ *factor with $N \geq 5$ and no automatic vector-like branching; F1 fails.*

Proof. Apply Lemma 5.4 to the simple factors of each row of Table 6. The exceptional factors E_6, E_7, F_4, G_2, the $\mathrm{Spin}(N)$ factors with $N \neq 6$, and the SU(2) factors (whose Casimir spectrum is $\{2\}$) carry no primitive cubic Ad-invariant, hence no cubic anomaly contribution; this gives the unconditional anomaly safety of H_1, H_3, H_6, H_8, and trivially of H_T. For H_2 and H_7, the branching of $\mathrm{adj}(\mathfrak{e}_8) = \mathbf{248}$ under $E_6 \times \mathrm{SU}(3)$ is [16]

$$\mathbf{248} \;=\; (\mathbf{78},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}) \oplus (\mathbf{27},\mathbf{3}) \oplus (\overline{\mathbf{27}},\overline{\mathbf{3}}), \tag{45}$$

with the chiral pair $(\mathbf{27},\mathbf{3}) \oplus (\overline{\mathbf{27}},\overline{\mathbf{3}})$ vector-like in the SU(3) factor and hence anomaly-free under the SU(3) cubic anomaly. The analogous branching under $\mathrm{Spin}(10) \times \mathrm{SU}(4)$ is [16]

$$\mathbf{248} \;=\; (\mathbf{45},\mathbf{1}) \oplus (\mathbf{1},\mathbf{15}) \oplus (\mathbf{10},\mathbf{6}) \oplus (\mathbf{16},\mathbf{4}) \oplus (\overline{\mathbf{16}},\overline{\mathbf{4}}), \tag{46}$$

again vector-like in the SU(4) cubic anomaly; the adjoint-anomaly cancellation is automatic from the branching itself. For $\{H_4, H_5, H_9\}$, SU(9) in H_4, $\mathrm{SU}(5) \times \mathrm{SU}(5)$ in H_5, and $\mathrm{SU}(8) \times \mathrm{U}(1)$ in H_9 each contain at least one $\mathrm{SU}(N)$ with $N \geq 5$, with no automatic vector-like branching. □

5.5 Filter F2: Skyrmion-type soliton

The strict topological version of F2, based on $\pi_3(\mathcal{O}) = \pi_3(E_8/H)$, fails on every full-rank non-toroidal candidate of Proposition 5.5: by the long exact sequence of homotopy groups [28],

$$\pi_3(E_8/H) \;\cong\; \mathrm{coker}\big(\pi_3(H) \to \pi_3(E_8)\big) \;=\; \mathbb{Z}/\big(\gcd_i(j_i)\,\mathbb{Z}\big), \tag{47}$$

with j_i the Dynkin embedding indices of the simple factors of H in E_8 [16, 27]. For each of $H_1, H_2, H_3, H_6, H_7, H_8$ the gcd of the relevant Dynkin indices is 1, giving $\pi_3(E_8/H) = 0$. A strict reading of F2 would therefore eliminate every non-toroidal candidate, leaving only the abelian limit H_T, which is unphysical; we replace strict F2 by the operational statement of §5.2—existence of an SU(2) subgroup of H acting non-trivially on the tangent module $\mathfrak{m}$.

Proposition 5.6 (F2 outcome)**.** *Of the seven candidates surviving F1 (Proposition 5.5), six are non-abelian and hence pass F2; the toroidal limit $H_T = T^8$ is abelian and fails. (For each of the six surviving stabilizers an explicit* SU(2) *subgroup of H acting non-trivially on $\mathfrak{m}$ is exhibited in the proof.) The surviving set after F1∩F2 is therefore*

$$\mathcal{F}_{12} \;:=\; \{\, H_1,\, H_2,\, H_3,\, H_6,\, H_7,\, H_8 \,\}, \qquad |\mathcal{F}_{12}| = 6. \tag{48}$$

Proof. For $H_1 = E_7 \times \mathrm{SU}(2)$ and $H_7 = \mathrm{Spin}(10) \times \mathrm{SU}(4)$, the SU(2) subgroup is a direct factor of H (in H_7, via the standard inclusion $\mathrm{SU}(2) \subset \mathrm{SU}(4)$). For $H_2 = E_6 \times \mathrm{SU}(3)$, the SU(2) subgroup appears via the standard chain $\mathrm{SU}(2) \subset E_6 \subset H_2$ (or alternatively $\mathrm{SU}(2) \subset \mathrm{SU}(3) \subset H_2$). For $H_3 = \mathrm{Spin}(16)$, an SU(2) subgroup appears via $\mathrm{SU}(2) \subset \mathrm{Spin}(N)$ for any $N \geq 3$. For $H_6 = G_2 \times F_4$, the SU(2) subgroup appears via $\mathrm{SU}(2) \times \mathrm{SU}(2) \subset G_2$ (Slansky [16]). For $H_8 = E_6 \times \mathrm{U}(1)^2$, an SU(2) subgroup appears via $\mathrm{SU}(2) \subset E_6$. In each case the isotropy action on $\mathfrak{m}$ is non-trivial (each $\mathfrak{m}$ contains an irreducible representation of the simple factor that hosts the SU(2)), so the hedgehog ansatz of Manton–Sutcliffe [37], Ch. 9, produces a Skyrmion of charge $B \in \pi_3(\mathrm{SU}(2)) = \mathbb{Z}$. For $H_T = T^8$, every closed connected subgroup is a subtorus $T^k \subset T^8$, hence abelian; no SU(2) subgroup exists, F2 fails. □

5.6 Filter F3: Wolf-space (quaternion-Kähler) structure

We now apply the structural-geometric filter F3. The Wolf-space classification [10, 11] catalogues the compact quaternion-Kähler symmetric spaces of positive scalar curvature:

$$\mathrm{HP}^n, \quad \frac{G_2}{\mathrm{SO}(4)}, \quad \frac{F_4}{\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)}, \quad \frac{E_6}{\mathrm{SU}(6)\cdot\mathrm{Sp}(1)}, \quad \frac{E_7}{\mathrm{Spin}(12)\cdot\mathrm{Sp}(1)}, \quad \frac{E_8}{E_7\cdot\mathrm{Sp}(1)}, \tag{49}$$

with the Sp(1) factor in each case being the rotational holonomy group of the quaternion-Kähler structure. Each compact simple Lie group thus determines a unique compact Wolf space [10]. The Wolf space of E_8 is the last entry of (49), namely EIX.

Lemma 5.7 (Compact Wolf spaces of E_8 type)**.** *Up to conjugation,* $\mathrm{EIX} = E_8/(E_7{\cdot}\mathrm{SU}(2))$ *is the unique compact quaternion-Kähler symmetric space whose isometry group has Lie algebra of type E_8. Among the candidates of Table 6, F3 is therefore satisfied by H_1 and fails for every other entry.*

Proof. Up to conjugation, every compact symmetric space of type E_8 (i.e., with isometry algebra of type E_8) corresponds to a Cartan involution on $\mathfrak{e}_8$ with fixed-point subalgebra $\mathfrak{k} \subset \mathfrak{e}_8$. There are exactly two non-equivalent involutions on $\mathfrak{e}_8$ that produce a non-trivial compact symmetric space [24, 25]: one with $\mathfrak{k}$ of type D_8 (giving $H_3 = \mathrm{Spin}(16)/\mathbb{Z}_2$ and the symmetric space EVIII), and one with $\mathfrak{k}$ of type $E_7 \oplus A_1$ (giving $H_1 = E_7{\times}\mathrm{SU}(2)/\mathbb{Z}_2$ and EIX). Of these two compact symmetric spaces, only EIX is quaternion-Kähler: its isotropy representation $\mathfrak{m}_{\mathrm{EIX}}$ is the $(\mathbf{56}, \mathbf{2})$ representation of $E_7 \times \mathrm{SU}(2)$, which carries a quaternionic structure through the $\mathbf{2}$ factor of SU(2) (Wolf's theorem [10]). The space EVIII has isotropy $\mathfrak{m}_{\mathrm{EVIII}} = \mathbf{128}_{\mathrm{spinor}}$, the half-spin representation of Spin(16). By the Frobenius–Schur indicator for the half-spin representations of $\mathrm{Spin}(8k)$ (real of orthogonal type for k even, hence for D_8; [23], App. A; [6], Plate IV), the $\mathbf{128}_s$ is real and does not admit an Spin(16)-equivariant quaternionic structure. The remaining candidates of Table 6 are not symmetric spaces (the isotropy bracket $[\mathfrak{m}, \mathfrak{m}]$ is not contained in $\mathfrak{h}$) and are excluded *a fortiori* from the Wolf classification. □

Structural content of F3. Three structural properties of EIX (= the Wolf space of E_8) distinguish it among the candidates of Table 6. Each is a structural fact about EIX, not a function of the number of spacetime dimensions, and any one of them serves as an alternative formulation of F3:

(F3-i) Reduced holonomy. The Riemannian holonomy of EIX is reduced from the generic SO(112) to $\mathrm{Sp}(28){\cdot}\mathrm{Sp}(1)$, with the Sp(1) factor being the rotational holonomy of the quaternion-Kähler structure (Salamon [11]). For EVIII the holonomy is the generic SO(128) without reduction; for the non-symmetric strata the local holonomy is in general not reduced to a proper subgroup of the orthogonal group of the tangent module.

(F3-ii) Distinguished Sp(1) *direct factor.* The stabilizer $H_1 = E_7{\times}\mathrm{SU}(2)$ contains an $\mathrm{SU}(2) = \mathrm{Sp}(1)$ direct factor, which acts on $\mathfrak{m}_{\mathrm{EIX}}$ as the quaternionic-fibre rotation in the $\mathbf{2}$-summand of the isotropy $(\mathbf{56}, \mathbf{2})$. For EVIII the Spin(16) stabilizer is simple, and no SU(2) direct factor is available; for H_2, H_6, H_7, H_8 the SU(2) subgroups picked out by F2 are not the rotational Sp(1) of a Wolf structure (they fail F3 because the underlying coset is not a quaternion-Kähler symmetric space).

(F3-iii) Closed parallel quaternion-Kähler 4*-form.* EIX carries a parallel skew Sp(1)-bundle generated by three local complex structures I, J, K with $IJ = K$, and the associated quaternion-Kähler 4-form $\Omega_{\mathrm{quat}} := \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K$ is closed and non-zero, generating $H^4(\mathrm{EIX}; \mathbb{R}) \cong \mathbb{R}$ (Salamon [11]). For EVIII no analogous parallel 4-form exists; the cohomology class of Ω_{quat} is the topological substrate of the sigma-model topological sector of the present framework, anchored *a posteriori* by the F3 selection.

Remark 5.8 (Standard-Model embeddability of the selected orbit)*.* The selected $H_1 = E_7 \times \mathrm{SU}(2)/\mathbb{Z}_2$ admits the classical Lie-theoretic chain $G_{\mathrm{SM}} \subset \mathrm{SU}(5) \subset \mathrm{Spin}(10) \subset E_6 \subset E_7 \subset H_1$ (Slansky [16]; McKay–Patera [27]). The content of this statement is purely algebraic: the unbroken stabilizer H_1 has a closed subgroup of G_{SM} type. The chain $G_{\mathrm{SM}} \subset E_7$ is a fact about compact Lie groups and holds for any E_7, gauged or not; the orbit selection of the present section is consequently neither sensitive to nor predictive of the empirical existence of G_{SM} as the low-energy gauge group on the (separately hypothesised; see Hypothesis 6.11 of §6) emergent four-dimensional spacetime. The dynamical realisation of G_{SM} as the unbroken low-energy gauge group, including the matter representation content, is the subject of subsequent work and is not addressed here.

Within the present section, algebraic SM-embeddability is recorded as a downstream consistency check, not a selection criterion: five of the six candidates of $\mathcal{F}_{12}$ admit a similar chain (H_1, H_2, H_3, H_7, H_8), and only $H_6 = G_2 \times F_4$ does not (its closest candidate $\mathrm{Spin}(9) \subset F_4$ is of Pati–Salam type rather than standard $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ type), so algebraic SM-embeddability is not a discriminating filter on $\mathcal{F}_{12}$.

5.7 The orbit selection theorem

Theorem 5.9 (Vacuum orbit selection). *Let the action $S[\Phi]$ have the form* (22) *of Theorem 3.5 with $c_2 \in \mathcal{R}_{\mathrm{BEC}}$ and sub-leading coefficients in $\mathcal{D}_{\mathrm{stab}}$, and let $\Phi_0 \in \mathfrak{e}_8$ be a non-trivial global minimum (Theorem 4.3).*

(a) F1 ∩ F2 narrows the catalogue to six candidates. *The candidates of Table 6 satisfying F1 and F2 are*

$$\mathcal{F}_{12} \;=\; \{\, H_1,\, H_2,\, H_3,\, H_6,\, H_7,\, H_8 \,\}. \tag{50}$$

(b) F3 selects EIX uniquely. *Among the candidates of $\mathcal{F}_{12}$, only $H_1 = E_7 \times \mathrm{SU}(2)$ satisfies F3. The selected orbit is*

$$\boxed{H_1 \;=\; E_7 \times \mathrm{SU}(2)/\mathbb{Z}_2, \quad \mathcal{O} \;\cong\; \mathrm{EIX} \;=\; E_8/(E_7 \times \mathrm{SU}(2)), \quad \dim \mathrm{EIX} \;=\; 112.} \tag{51}$$

Here H_1 is the nominal Borel–de Siebenthal stabilizer of the catalogue, in the sense of Table 6. The actual adjoint orbit of any nonzero Φ_0 is $E_8/H' = E_8/(E_7 \times \mathrm{U}(1))$ of dimension 114, fibred by $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ over EIX, with Levi stabilizer $H' = E_7 \times \mathrm{U}(1)$ in the notation of Remark 4.5; the distinction H_1 vs. H' has no effect on the F1–F3 selection.

Proof sketch. Part (a) is the conjunction of Propositions 5.5 and 5.6. Of the ten candidates of Table 6, F1 eliminates $\{H_4, H_5, H_9\}$; F2 eliminates $\{H_T\}$; the surviving set is $\mathcal{F}_{12}$ of size six. Part (b) is Lemma 5.7: among $\mathcal{F}_{12}$, the Wolf space of E_8 is the unique compact quaternion-Kähler symmetric space, and it corresponds to H_1. The actual stabilizer subtlety (the S^2 fibre between $E_7 \times \mathrm{SU}(2)/\mathbb{Z}_2$ and the true orbit stabilizer $E_7 \times \mathrm{U}(1)$) is the content of Remark 4.5, with no effect on the selection. □

Conditional structure of the theorem. Theorem 5.9 is conditional on (i) the completeness of Table 6 against unrecorded sub-maximal embeddings (Remark 5.2; the statement is relative to that table as input), and (ii) the structural-input character of F3 (§5.3; without F3 the conclusion collapses to $\mathcal{F}_{12}$, see Open problem 5.3). Neither involves a dimensional input on the emergent spacetime, so the selection is structurally independent of the four-dimensional emergence analysed in §6.

5.8 Robustness analysis

We tabulate the surviving candidate set after each combination of the three filters, both to make the dependence of Theorem 5.9 on each filter explicit and to record the conclusions that survive if one of the filters is weakened or removed (Table 7).

Filter combination	Survivors	Count
F1	$\{H_1, H_2, H_3, H_6, H_7, H_8, H_T\}$	7
F1 ∩ F2	$\{H_1, H_2, H_3, H_6, H_7, H_8\} = \mathcal{F}_{12}$	6
F1 ∩ F2 ∩ F3	$\{H_1\} = \{\mathrm{EIX}\}$	1

Table 7: Survivors of the candidate catalogue of Table 6 under each combination of the three operational filters. The final row gives the unique selection of Theorem 5.9(b).

What survives if F3 is removed. Without F3 the construction is left with the six-element candidate set $\mathcal{F}_{12}$ of (50): H_3 gives the non-quaternion-Kähler symmetric space EVIII, while H_2, H_6, H_7, H_8 give non-symmetric cosets (and H_6 is not a Wolf space of E_8 type). F3-uniqueness would have to be replaced by the orthogonal mechanism of Open problem 5.3.

6 Emergent four-dimensional spacetime

The orbit selection of §5 produced a unique E_8^{Ad}-orbit $\mathcal{O} \cong \mathrm{EIX} = E_8/(E_7 \times \mathrm{SU}(2))$ of real dimension 112, the Wolf space of E_8 [10, 11]. The selection used three operational filters F1–F3, of which the structural filter F3 (quaternion-Kähler) carries no dimensional input. The present section unpacks the dimensional content of EIX itself: the rank of EIX as a symmetric space is 4 (Lemma 6.3), and Suter's theorem [12] identifies this rank with the cardinality of a maximal antichain of strongly orthogonal positive roots in the tangent module $\mathfrak{m}_{\mathrm{EIX}}$ (Lemma 6.4). The four corresponding Chevalley root vectors generate a real abelian Lie subalgebra $\mathfrak{a} \subset \mathfrak{m}_{\mathrm{EIX}}$ of dimension exactly four, on which the four candidate translation generators $\{P_\mu\}_{\mu=0,1,2,3}$ are defined.

The substantive structural content of the section is then organised around three statements with progressively stronger conditional clauses:

1. *Algebraic content of the four-dimensional sector.* The dimensional fact $\dim \mathfrak{a} = 4$ and the abelian closure $[P_\mu, P_\nu] = 0$ are derived unconditionally from the rank of EIX and Suter's theorem.

2. *Topological consistency of EIX as a base.* The lower homotopy groups of EIX vanish, $\pi_0(\mathrm{EIX}) = \pi_1(\mathrm{EIX}) = \pi_2(\mathrm{EIX}) = 0$ (Theorem 6.21), with the structural consequence that the BEC phase transition $E_8 \to E_7 \times \mathrm{SU}(2)$ of §4 produces no stable topological defects in the corresponding co-dimensions (Corollary 6.23). This statement is independent of any spacetime-emergence hypothesis and uses only the homotopy of the underlying Wolf space.

3. *Four-dimensional Lorentzian emergence ([Hypothesis]).* The promotion of the abelian sector $\mathfrak{a}$ to translations on a smooth Lorentzian $(\mathcal{M}^{1,3}, g)$ is recorded as Hypothesis 6.11 with five sub-claims (α)–(ε). Two are unconditional consequences of the algebraic content above; the signature sub-claim (α) is closed at the leading bosonic Gaussian slow-mode level by an Osterwalder–Schrader reconstruction argument (Proposition 6.12, [Proven-num]) and extended numerically to a representative interior point of the convex stability domain $\mathcal{D}_{\mathrm{stab}}$ of Theorem 3.5 via the Glimm–Jaffe sufficiency conditions for the full interacting slow-mode action (Proposition 6.15, [Proven-num, $\mathcal{D}_{\mathrm{stab}}$ interior]); the diffeomorphism sub-claim (γ) is closed at the same level on its *global* Poincaré subgroup $\mathcal{P}_+^\uparrow \subset \mathrm{Diff}(\mathcal{M}^{1,3})$ as a structural corollary of the same OS reconstruction (Proposition 6.17). The metric-reconstruction sub-claim (δ) is closed at the leading + sub-leading Sakharov level by an explicit Camporesi–Higuchi spectral-zeta computation on EIX, returning $\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})} = 432/3 = 144$ with a conservative $\leq 3.7\%$ uncertainty band on the spectral-zeta finite-part correction (§6.7, [Proven-num, leading + sub-leading]); the residual full local $\mathrm{Diff}(\mathcal{M}^{1,3})$-content of (γ) follows

from (δ) by the standard induced-gravity argument (§6.7) at the same level. The remaining open content is the full $\mathcal{D}_{\text{stab}}$-boundary verdict for the sub-leading f-tensor invariants of the interacting slow-mode action and the full Mellin continuation of the EIX spectral zeta with explicit Seeley–DeWitt pole subtractions; both are recorded as residual open problems and are bounded operationally by the $\mathcal{D}_{\text{stab}}$ interior verdict and the $\leq 3.7\%$ uncertainty band respectively. Conditional consequences (single emergent light cone, two graviton polarisations) are deferred accordingly.

In this split, the dimensional content (1) and the topological content (2) are structural consequences of the selected orbit. Of the geometric-dynamical promotion (3), the signature sub-claim (α) and the global-Poincaré content of (γ) are both closed at leading-Gaussian order via the Osterwalder–Schrader route below and extended to the $\mathcal{D}_{\text{stab}}$-interior of the full interacting slow-mode action via the Glimm–Jaffe sufficiency conditions of Proposition 6.15; the metric-reconstruction sub-claim (δ) is closed at the leading + sub-leading Sakharov level via the explicit Camporesi–Higuchi spectral-zeta value $\mathcal{V}_{\text{ind}}^{(\text{EIX})} = 432/3 = 144$ (§6.7). The residual full local $\text{Diff}(\mathcal{M}^{1,3})$-content of ($\gamma$) is inherited from ($\delta$) by induced gravity at the same level, rather than being an independent open problem.

6.1 Setup: the EIX orbit and its tangent module

Fix a representative $\Phi_0 \in \mathfrak{e}_8$ in the orbit $\mathcal{O} \cong$ EIX of Theorem 5.9, with nominal stabilizer $H = E_7 \times \text{SU}(2)/\mathbb{Z}_2$ and true stabilizer $H' = E_7 \times \text{U}(1)$ in the S^2-fibre sense of Remark 4.5. The Cartan involution $\theta : \mathfrak{e}_8 \to \mathfrak{e}_8$ associated with the Wolf structure splits the algebra orthogonally with respect to κ,

$$\mathfrak{e}_8 \;=\; \mathfrak{h} \oplus \mathfrak{m}_{\text{EIX}}, \qquad \mathfrak{h} \;=\; \theta^{+1} \;=\; \mathfrak{e}_7 \oplus \mathfrak{su}(2), \qquad \mathfrak{m}_{\text{EIX}} \;=\; \theta^{-1}, \tag{52}$$

with $\dim \mathfrak{h} = 136$ and $\dim \mathfrak{m}_{\text{EIX}} = 112$. The isotropy representation of H on $\mathfrak{m}_{\text{EIX}}$ is the irreducible

$$\mathfrak{m}_{\text{EIX}} \;\cong\; (\mathbf{56}, \mathbf{2}) \;=\; \mathbf{56}_{E_7} \otimes \mathbf{2}_{\text{SU}(2)} \tag{53}$$

of $E_7 \times \text{SU}(2)$ [10, 11]. The $\mathbf{2}_{\text{SU}(2)}$ factor carries the quaternionic-fibre structure of the Wolf space and is the operational origin of the rank computation below.

Root-space description of $\mathfrak{m}_{\text{EIX}}$. Let $\mathfrak{h}_0 \subset \mathfrak{e}_8$ be a Cartan subalgebra ($\dim \mathfrak{h}_0 = 8$) and $\Delta(\mathfrak{e}_8)$ the corresponding root system, with $|\Delta(\mathfrak{e}_8)| = 240$. The involution θ acts on the roots by negation along the Cartan directions chosen inside $\mathfrak{m}$; in standard notation this gives a splitting

$$\Delta(\mathfrak{e}_8) \;=\; \Delta(\mathfrak{h}) \sqcup \Delta(\mathfrak{m}), \tag{54}$$

with $\Delta(\mathfrak{h})$ the roots whose root vector lies in $\mathfrak{h}$ (the closed sub-root-system of type $E_7 \oplus A_1$) and $\Delta(\mathfrak{m}) = \Delta(\mathfrak{e}_8) \setminus \Delta(\mathfrak{h})$ the remaining roots, with root vectors in $\mathfrak{m}$. The positive part is denoted $\Delta(\mathfrak{m}^+)$. The cardinalities are

$$|\Delta(\mathfrak{h})| \;=\; |\Delta(E_7)| + |\Delta(A_1)| \;=\; 126 + 2 \;=\; 128, \quad |\Delta(\mathfrak{m})| \;=\; 240 - 128 \;=\; 112, \tag{55}$$

and $|\Delta(\mathfrak{m}^+)| = 56$, equal to the E_7-representation dimension in (53) (cf. [24, Ch. X]).

Remark 6.1 (Sigma-model base (EIX) versus BEC vacuum orbit; mode-uniformity of Z on $\mathfrak{a}$). The orbit selection of §5 identifies the quaternion-Kähler Wolf space EIX ($\dim_{\mathbb{R}}$ EIX $= 112$) as the structural base for the present section through filter F3 applied to the *nominal* Borel–de Siebenthal stabiliser $H = E_7 \times \text{SU}(2)/\mathbb{Z}_2$ (Table 6). The *actual* stabiliser of any $\Phi_0 \in \mathfrak{e}_8$ in the BEC vacuum of Theorem 4.3 is the co-rank-1 Levi $H' = E_7 \times \text{U}(1)$ of Remark 4.5, with $\dim H' = 134$; the BEC orbit E_8/H' has dimension $114 = 112 + 2$ and is fibred as an

$S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ bundle over EIX. The quaternion-Kähler structure F3 (and consequently the rank-antichain identity of Lemma 6.4 and the abelian sector $\mathfrak{a}$ of Definition 6.6 below) is a property of the 112-dimensional base EIX, while the genuine orbit-of-degenerate-minima of the potential is the larger 114-dimensional submanifold. The present section accordingly uses the sigma model on EIX as its geometric carrier; the two extra S^2-fibre Goldstones (the $\mathbf{1}_{\pm2}$-modes in the value-index notation of §4, with $|\alpha(\hat{\Phi}_0)| = 2$) are orthogonal to $\mathfrak{m}_{\mathrm{EIX}}$ and decouple from the four-dimensional sector $\mathfrak{a}$.

A consequence relevant for the slow-mode kinetic matrix (60) of §6.5 below: on the running $E_7 \times \mathrm{U}(1)$ Levi stratum the Goldstone wave-function renormalisation $Z^{(\alpha)}_{\mathrm{Gold}} = \kappa_2 + (c_{42} + |\alpha(\hat{\Phi}_0)|^2 c''_{42})\, r_*^2$ is structurally mode-dependent, taking distinct values on the two $E_7 \times \mathrm{U}(1)$-isotypic blocks of $\mathfrak{m}_{\mathrm{BEC}}$: $Z = \kappa_2 + (c_{42} + c''_{42})\, r_*^2$ on the $\mathbf{56}_{\pm1}$ EIX-Goldstones and the (distinct) value $Z = \kappa_2 + (c_{42} + 4c''_{42})\, r_*^2$ on the $\mathbf{1}_{\pm2}$ S^2-fibre directions. The abelian subspace $\mathfrak{a} \subset \mathfrak{m}_{\mathrm{EIX}}$ of Definition 6.6 below is by construction contained entirely in the $\mathbf{56}_{\pm1}$-block (its Chevalley root vectors are antichain elements of $\Delta(\mathfrak{m}^+_{\mathrm{EIX}})$), so $Z|_{\mathfrak{a}}$ is a single uniform constant and the per-mode rescaling of $\delta\Phi^{\mathrm{Gold}}$ used in §4 reduces, on the four-dimensional sector $\mathfrak{a}$, to a single overall rescaling absorbed into the canonical normalisation of Appendix A.2. The mode-dependence between the two Z-classes is recorded here for completeness; it plays no role in the leading-Gaussian and sub-leading verdicts of §§6.5–6.7.

Remark 6.2 (Conditional inputs inherited from §4)*.* Three structural inputs from §4 are used throughout the present section; we record their conditional status explicitly so that downstream verdicts inherit only their proven content.

(a) The reduction to homogeneous configurations (Proposition 4.2) is established within the EFT regime where the hierarchy (21) suppresses the sub-leading $(4,2)$ kinetic generators by $(\Lambda_0/\Lambda)^2$ relative to the leading $\kappa_2\mathcal{H}_2$, ensuring non-negativity of the slow-mode kinetic form on fluctuations around constant configurations $\Phi_0 \in \mathfrak{e}_8$. The infimum is attained on a constant within this regime of validity, not unconditionally on all of $C^\infty(E_8, \mathfrak{e}_8)$; the slow-mode reductions of §§6.5, 6.7 inherit the same regime-of-validity qualifier.

(b) The lifting of the $\dim H' - 1 = 133$ "spectator" flat directions in $\mathfrak{s} = \mathfrak{h}' \cap \hat{\Phi}_0^\perp$ by the higher primitive Casimirs $\mathrm{Cas}_8, \ldots, \mathrm{Cas}_{30}$ (Remark 4.8) is a structural expectation of the higher-Casimir layer rather than a theorem at the level of an explicit Cas_8-Hessian computation on $\mathfrak{s}$. The present section does not depend on the spectator-mass values: the four candidate translations P_μ live in $\mathfrak{m}_{\mathrm{EIX}} \subset \mathfrak{m}_{\mathrm{BEC}}$, not in $\mathfrak{s}$, and the E_8^{Ad}-protected Goldstones in $\mathfrak{m}_{\mathrm{BEC}}$ are massless to all orders by Goldstone's theorem [33, 34] (invariance of the full action, including kinetic terms, under E_8^{Ad}, as in Principle M3; see also Lemma 4.6, whose tree-level proof through orbit-constancy of V_{eff} is upgraded to all orders by this kinetic invariance). The non-vanishing of the spectator masses is therefore an inherited conditional input to the standalone consistency of the wider EFT, but not a logical prerequisite for the algebraic and topological content of the present section.

(c) The continuity-of-transition argument in Proposition 4.4 is established at every order of the polynomial Ad-invariant truncation of V_{eff}. Non-polynomial radiative contributions (Coleman–Weinberg-type $r^2 \log r^2$ structures and operators with derivatives) are not addressed by this argument; they would enter the present section only through the Wilsonian UV calibration of M_* in §6.7, which is itself outside the scope of the present paper.

6.2 Rank of EIX and the maximal abelian subspace of $\mathfrak{m}$

The rank of a compact symmetric space G/K is the dimension of a maximal abelian subspace of $\mathfrak{m}$, equivalently of a maximal flat totally geodesic submanifold of G/K [24, Ch. V]. For each compact irreducible Riemannian symmetric space the rank is a finite-dimensional invariant classified by Cartan; for the exceptional types of E_8 the result is recorded in Helgason [24], Ch. X.

Lemma 6.3 (Rank of EIX)**.** *The rank of* $\mathrm{EIX} = E_8/(E_7{\times}\mathrm{SU}(2))$ *as a compact symmetric space is exactly* 4*.*

Proof. The compact symmetric spaces of E_8 type are classified by Cartan involutions on $\mathfrak{e}_8$ [24, 25]. There are two non-equivalent involutions, producing the symmetric spaces EVIII (with isotropy $\mathfrak{so}(16)$, rank 8) and EIX (with isotropy $\mathfrak{e}_7{\oplus}\mathfrak{su}(2)$, rank 4); both ranks are recorded in Helgason [24], Table V of Ch. X. The rank of EIX is the dimension of a maximal abelian subspace of $\mathfrak{m}_{\mathrm{EIX}}$, equivalently of the $\mathbb{R}$-span of a maximal antichain of strongly orthogonal positive roots in $\Delta(\mathfrak{m}^+_{\mathrm{EIX}})$ by Lemma 6.4 below. ☐

The Cartan rank of EIX in Lemma 6.3 has a structural realization in terms of root-system combinatorics, due to Suter [12].

Lemma 6.4 (Suter's rank-antichain identity)**.** *Let G/K be a compact irreducible Riemannian symmetric space with θ-eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and corresponding root splitting* (54)*. The maximum cardinality of an antichain of strongly orthogonal positive roots in $\Delta(\mathfrak{m}^+)$,*

$$\max_{A\subseteq\Delta(\mathfrak{m}^+)} \{|A| \mid \alpha \pm \beta \notin \Delta(\mathfrak{g}) \cup \{0\}\ \forall\, \alpha, \beta \in A,\ \alpha \neq \beta\}, \tag{56}$$

equals the rank of G/K as a symmetric space, where $\mathrm{rank}(G/K) := \dim_{\mathbb{R}} \mathfrak{a}_{\mathrm{flat}}$ *for any maximal abelian subspace $\mathfrak{a}_{\mathrm{flat}} \subset \mathfrak{m}$.*

Proof. The maximal abelian subspace $\mathfrak{a}_{\mathrm{flat}} \subset \mathfrak{m}$ is spanned by Chevalley root vectors $\{E_{\alpha_i}\}_{i=1}^r$ with $r = \mathrm{rank}(G/K)$ and $\alpha_i \in \Delta(\mathfrak{m}^+)$ (after choosing a positive system on $\mathfrak{a}_{\mathrm{flat}}$), provided the α_i are pairwise strongly orthogonal: $\alpha_i \pm \alpha_j \notin \Delta(\mathfrak{g}) \cup \{0\}$ implies $[E_{\alpha_i}, E_{\alpha_j}] = 0$ on the Chevalley basis of any simple Lie algebra (Knapp 25, Ch. II §5). Conversely every commuting set of Chevalley root vectors in $\mathfrak{m}$ arises this way (Suter [12], Theorem 3, applied to the restricted root system of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$). The cardinality of the maximal antichain therefore equals the dimension of the maximal abelian subspace, which is the rank. ☐

Corollary 6.5 (Antichain count for EIX)**.** *The maximum cardinality of a strongly-orthogonal antichain in $\Delta(\mathfrak{m}^+_{\mathrm{EIX}})$ is exactly* 4*.*

Proof. Combine Lemma 6.3 (rank of EIX is 4) with Lemma 6.4 (rank equals maximum antichain cardinality). ☐

Numerical verification of the antichain structure. The cardinality 4 of Corollary 6.5 is verified independently on the explicit Chevalley basis of $\mathfrak{e}_8$ (script `e3_orbit_selection.py`); the count of distinct realisations is 630. Different antichains in the 630-element set are related by the residual action of the unbroken $E_7{\times}\mathrm{SU}(2)$ Weyl-group on the tangent module. The choice of a representative is consequently a finite gauge choice that does not affect any invariant statement of the present section, and we adopt one such choice once and for all in the rest of the paper without loss of generality.

6.3 Candidate translation generators

We now define the four candidate translation generators using the abelian subalgebra of Corollary 6.5. Pick a maximal antichain $\mathfrak{A} = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} \subset \Delta(\mathfrak{m}^+_{\mathrm{EIX}})$ in the 630-element set above, with corresponding Chevalley root vectors $\{E_{\alpha_\mu}\}_{\mu=0,1,2,3} \subset \mathfrak{m}_{\mathrm{EIX}}$.

Definition 6.6 (Candidate translation generators on EIX)**.** The four *candidate translation generators* on EIX are

$$P_\mu \;:=\; E_{\alpha_\mu}, \qquad \mu = 0,1,2,3, \tag{57}$$

with associated abelian subalgebra

$$\mathfrak{a} \;:=\; \mathrm{span}_{\mathbb{R}}\{P_0, P_1, P_2, P_3\} \;\subset\; \mathfrak{m}_{\mathrm{EIX}}. \tag{58}$$

Proposition 6.7 (Algebraic four-dimensional sector)**.** *The subalgebra $\mathfrak{a} \subset \mathfrak{m}_{\mathrm{EIX}}$ of Definition 6.6 satisfies, unconditionally:*

(a) Dimension equals four: $\dim_{\mathbb{R}} \mathfrak{a} = 4$.

(b) Abelian closure: $[P_\mu, P_\nu] = 0$ *for all* $\mu, \nu \in \{0,1,2,3\}$.

Moreover (a) and (b) are intrinsic properties of any maximal antichain $\mathfrak{A}$ in the 630*-element set: both the cardinality $|\mathfrak{A}| = 4$ (clause (a)) and the strong-orthogonality vanishing $[E_\alpha, E_\beta] = 0$ (clause (b)) are statements about the antichain condition itself and do not depend on which representative is chosen.*

Proof. (a) is Lemma 6.3 together with Corollary 6.5: the cardinality $|\mathfrak{A}|$ equals the rank of EIX, which equals 4. (b) is the strong-orthogonality clause of the antichain condition: for $\alpha, \beta \in \mathfrak{A}$ with $\alpha \neq \beta$, the relation $\alpha \pm \beta \notin \Delta(\mathfrak{e}_8) \cup \{0\}$ implies $[E_\alpha, E_\beta] = 0$ on the Chevalley basis of any simple Lie algebra (Knapp [25], Ch. II §5). Both statements depend only on the defining condition of a maximal antichain and not on a particular choice of representative; they hold simultaneously on every element of the 630-element set. □

Remark 6.8 ($E_7{\times}\mathrm{SU}(2)$-action on the antichain set)*.* The unbroken $E_7{\times}\mathrm{SU}(2)$ acts on the antichain set of $\Delta(\mathfrak{m}^+_{\mathrm{EIX}})$ by conjugation (Corollary 6.5); since the antichain condition is Ad-invariant, the action permutes the 630 maximal antichains. The orbit structure of the $E_7 \times \mathrm{SU}(2)$-action on this set is not established analytically in the present paper; the constancy of the operational verdicts of §§6.5, 6.6 below across the antichain set is instead verified numerically by an explicit sweep of the full 630-element enumeration (script `e3_antichain_full_sweep.py`; [Proven-num], see §6.5). The analytic transitivity question is therefore not needed for the present derivation; the dimensional and abelian-closure content of Proposition 6.7 is moreover independent of its resolution by construction.

Structural reading. Proposition 6.7 establishes the algebraic substrate of a four-dimensional translation sector on EIX as a *consequence* of the orbit selection. The number 4 enters through Lemma 6.3 (rank of the symmetric space) and Lemma 6.4 (Suter's identity); both are statements about the algebraic structure of EIX itself, with no dimensional input from outside the framework. The number 4 is therefore an output of the orbit selection of §5, not an input to it.

Remark 6.9 (Quaternionic interpretation of the number 4)*.* The structural origin of the number 4 on EIX admits a quaternionic interpretation through the Wolf-space holonomy. EIX has reduced holonomy Sp(28)·Sp(1), with the Sp(1) factor acting as the rotational holonomy of the quaternion-Kähler structure. The maximal abelian subspace $\mathfrak{a} \subset \mathfrak{m}_{\mathrm{EIX}}$ of Definition 6.6 is naturally identified with the real span of a single quaternionic line under this action: writing $\mathfrak{m}_{\mathrm{EIX}} \cong \mathbb{H}^{28} \cong \mathbb{R}^{112}$ as a real Sp(28)·Sp(1)-module, with Sp(28) acting by left quaternionic matrix multiplication and Sp(1) acting by right quaternionic scalar multiplication, the flat directions of the symmetric space pick out a single quaternionic line $\mathbb{H} \subset \mathbb{H}^{28}$, with $\dim_{\mathbb{R}} \mathbb{H} = 4$. The combinatorial bound of Suter [12, Theorem 3] on the antichain in $\Delta(\mathfrak{m}^+)$ for any compact quaternion-Kähler symmetric space — which states that the bound is saturated when the residual root system of the orthogonal complement is of D_4 type, with the value of the bound being rank $D_4 = 4$ — gives a combinatorial avatar of the same dimensional fact.

Remark 6.10 (Structural decomposition $D = 4 + 244$ and a forward pointer)*.* The orbit selection of §5 together with the algebraic content of Proposition 6.7 produces a structural decomposition of the dimensional content of $\mathfrak{e}_8$:

$$D \;:=\; \dim_{\mathbb{R}} \mathfrak{e}_8 \;=\; 248 \;=\; 4 \;+\; 244, \tag{59}$$

in which the "4" is $\dim_{\mathbb{R}} \mathfrak{a} = \mathrm{rank}(\mathrm{EIX})$ of Lemma 6.3 and is the dimensional content of the abelian translation sector $\mathfrak{a}$ (sub-claim (β) of Hypothesis 6.11 below), while the "244" is the remaining $D - 4$ generators of $\mathfrak{e}_8$ realised in the long-wavelength effective theory as internal degrees of freedom ($\dim \mathfrak{h} = 136$ generators of the nominal Borel–de Siebenthal stabiliser $E_7 \times \mathrm{SU}(2)$, plus the $\dim \mathfrak{m}_{\mathrm{EIX}} - \dim \mathfrak{a} = 108$ transverse Goldstone modes of $\mathfrak{m}_{\mathrm{EIX}}$ orthogonal to $\mathfrak{a}$). Equivalently, in the BEC-orbit picture of Remark 6.1 the same total 248 partitions as $\dim H' + \dim \mathfrak{m}_{\mathrm{BEC}} = 134 + 114$, with the S^2-fibre Goldstones contributing the difference $136 - 134 = 114 - 112 = 2$ to whichever side of the split the fibre is assigned; the four-dimensional content of the sector $\mathfrak{a} \subset \mathfrak{m}_{\mathrm{EIX}}$ is unchanged across the two pictures. The split is a structural *output* of the postulates P1–P2 together with the orbit selection of §5; the number 4 is not an external input.

A candidate consequence of the structural split (59), recorded here only as a forward pointer and not as a claim of the present paper, is a non-perturbative suppression of the dimensionless cosmological constant of order

$$\Lambda\, \ell_P^2 \;\sim\; \pi^{-(D-4)} \;=\; \pi^{-244},$$

to be analysed elsewhere in terms of two open computations on EIX (a perturbative-cancellation identity and a non-perturbative saddle-point action). Both computations share the Camporesi–Higuchi spectral zeta on EIX (§6.7) as a common input; the leading + BV-BRST sub-leading value $\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})} = 432/3 = 144$ of Proposition 6.19 settles this common input at the level relevant to sub-claim (δ), but the cosmological-constant consequence itself depends on the two open computations above and is not a claim of the present paper.

6.4 The four-dimensional emergence hypothesis

Proposition 6.7 establishes the algebraic substrate $(\mathfrak{a}, [P_\mu, P_\nu] = 0)$ on EIX. Promoting it to a smooth Lorentzian four-manifold $(\mathcal{M}^{1,3}, g)$ on which the P_μ act as differential operators requires three additional steps that are not derivable from the algebraic data alone: the selection of a Lorentzian (rather than Euclidean) signature, the identification of the diffeomorphism group as a residual symmetry, and the explicit reconstruction of the metric from condensate fluctuations. We collect these three steps into a single hypothesis, together with the two algebraic statements of Proposition 6.7 for completeness.

Hypothesis 6.11 (Four-dimensional Lorentzian emergence)**.** The condensate Φ of Theorem 4.3, specialised to the EIX orbit of Theorem 5.9, generates in the long-wavelength limit a smooth four-dimensional Lorentzian manifold $(\mathcal{M}^{1,3}, g)$ with the following five properties:

(α) *Lorentzian signature.* The emergent metric g has signature $(-, +, +, +)$, with the sign selected by the operational dynamics of the condensate rather than by an external choice.

(β) *Dimension equals four.* The number of independent translation generators is exactly four, $\dim_{\mathbb{R}} \mathfrak{a} = 4$, with $\mathfrak{a}$ as in (58).

(γ) *Diffeomorphism invariance.* The effective action on $(\mathcal{M}^{1,3}, g)$ is invariant under the diffeomorphism group $\mathrm{Diff}(\mathcal{M}^{1,3})$, with g a dynamical field rather than a fixed background.

(δ) *Metric reconstruction from the condensate.* The metric g is reconstructed from condensate fluctuations $\delta\Phi$ through a map $g_{\mu\nu} = g_{\mu\nu}[\delta\Phi]$ compatible with the identification (57) of the four candidate translation generators.

(ε) *Abelian closure of the candidate translations.* $[P_\mu, P_\nu] = 0$ for $\mu, \nu \in \{0, 1, 2, 3\}$, as in Proposition 6.7(b).

The five sub-claims have very different epistemic status, summarised in Table 8. Sub-claims (β) and (ε) are unconditional consequences of Proposition 6.7 together with the orbit selection of §5. Sub-claim (α) is closed at the leading bosonic Gaussian level (§6.5, Proposition 6.12, [Proven-num]) and extended to the $\mathcal{D}_{\text{stab}}$ interior of the full interacting slow-mode action via the Glimm–Jaffe sufficiency conditions (Proposition 6.15, [Proven-num, $\mathcal{D}_{\text{stab}}$ interior]); sub-claim (γ) is closed at the leading-Gaussian level on its global Poincaré subgroup $\mathcal{P}_+^\uparrow \subset \text{Diff}(\mathcal{M}^{1,3})$ as a structural corollary of the same OS reconstruction (§6.6, Proposition 6.17); sub-claim (δ) is closed at the leading + BV-BRST sub-leading Sakharov level via the Camporesi–Higuchi spectral-zeta value $\mathcal{V}_{\text{ind}}^{(\text{EIX})} = 432/3 = 144$ (§6.7, Proposition 6.19, [Proven-num, leading + sub-leading], with conservative $\leq 3.7\%$ uncertainty band on the spectral-zeta finite-part correction). The residual full local $\text{Diff}(\mathcal{M}^{1,3})$-content of ($\gamma$) follows from ($\delta$) at the same leading + sub-leading level by the standard induced-gravity argument. The remaining [Open] content is the full $\mathcal{D}_{\text{stab}}$-boundary verdict for the sub-leading f-tensor invariants of (α) and the full Mellin continuation of the EIX spectral zeta with explicit Seeley–DeWitt pole subtractions for (δ), both bounded operationally by the verdicts above.

6.5 Sub-claim (α): Lorentzian signature via Osterwalder–Schrader reconstruction ([Proven-num] for the leading bosonic Gaussian sector and at a $\mathcal{D}_{\text{stab}}$ interior point of the full interacting slow-mode action)

The structural tension. The compact real form $\mathfrak{e}_8^c$ chosen in P2 has positive-definite Killing form, so any naive restriction of κ to the abelian subalgebra $\mathfrak{a} \subset \mathfrak{m}_{\text{EIX}}$ produces a positive-definite bilinear form on the four candidate translations. The induced metric on the long-wavelength base is therefore Riemannian (Euclidean), not pseudo-Riemannian (Lorentzian). This is structurally inconsistent with the empirical observation that physical spacetime is Lorentzian, and any candidate emergence mechanism must include a step that selects the Lorentzian sign.

Analogue mechanisms in the literature. Two analogue mechanisms have been considered as templates for recovering Lorentzian signature from a fundamentally Euclidean starting point: *Sakharov-style induced gravity* [40], which produces an Einstein–Hilbert term from the one-loop effective action of a spectator field on a fixed background metric, and *analog gravity in superfluid condensates* [41], which produces an effective Lorentzian metric and a single emergent light cone from the long-wavelength fluctuation spectrum of a non-relativistic condensate. Neither is used as a derivation in the present construction; we adopt instead the Osterwalder–Schrader route below as the primary route, and return to the Sakharov template only schematically for the metric reconstruction in sub-claim (δ) (§6.7).

Primary route: Euclidean E_8-GFT plus Osterwalder–Schrader reconstruction. The structurally cleanest perspective accepts that the compact E_8-GFT is *fundamentally Euclidean* (by P2, which fixes the compact real form $\mathfrak{e}_8^c$ with positive-definite Killing form), and recovers Lorentzian spacetime as a *derived* object through the Osterwalder–Schrader reconstruction theorem [13, 14]. The positive-definite sign induced by κ on $\mathfrak{a}$ is not an obstruction but the natural input of a rigorous Euclidean QFT: the Euclidean Schwinger correlators built from the Haar measure on E_8 are the primary object, and a relativistic Wightman QFT on Minkowski space is the OS-reconstructed object, conditional on reflection positivity (RP) of the Schwinger functions (Glimm–Jaffe [20] §6.2; lattice gauge theory [42, 43]; constructive ϕ^4 [20] Ch. 8). The four candidate translations P_μ of (57) are then Euclidean translation generators on $\mathfrak{a} \cong \mathbb{R}^4$, and the Lorentzian Poincaré representation is supplied by the OS theorem on the reconstructed Hilbert space.

Slow-mode kinetic data. With the canonical normalisation of Appendix A.2, the slow-mode kinetic matrix and mass on $\mathfrak{a}$ inherited from the action of Theorem 3.5 take the form

$$K^{\mu\nu} \;=\; \tfrac{1}{2}\,\delta^{\mu\nu}, \qquad M_W^2 \;=\; c_H^{(\mathrm{EIX})}\, r_*^2, \qquad m^2 \;=\; M_W^2/K_{\mathrm{kin}}, \tag{60}$$

with $c_H^{(\mathrm{EIX})} > 0$ the (positive) Schur coefficient of the quaternion-Kähler structure on EIX. The O(4)-isotropy of $K^{\mu\nu}$ on $\mathfrak{a}$ uses the mode-uniformity of the Goldstone wave-function renormalisation $Z|_{\mathfrak{a}}$ established in Remark 6.1: by construction $\mathfrak{a} \subset \mathfrak{m}_{\mathrm{EIX}}$ is contained in the $\mathbf{56}_{\pm 1}$-block of $\mathfrak{m}_{\mathrm{BEC}}$, on which Z is constant, so the single-block rescaling absorbed in the canonical normalisation of Appendix A.2 suffices; the distinct Z-class of the S^2-fibre directions $\mathbf{1}_{\pm 2} \subset \mathfrak{m}_{\mathrm{BEC}}$ does not enter the four-dimensional sector. The reflection-positivity verdict below depends only on the signs $K_{\mathrm{kin}} > 0$ and $c_H^{(\mathrm{EIX})} > 0$; the explicit numerical value of $c_H^{(\mathrm{EIX})}$ is not needed in the present paper. The corresponding two-point Schwinger function $G_2(R) = (m/4\pi^2 R)\, K_1(mR)$ is the standard 4D Yukawa Green's function on $\mathfrak{a} \cong \mathbb{R}^4$, with K_1 the modified Bessel function of the second kind.

Reflexivity argument: Källén–Lehmann positivity to OS. The reflection positivity of G_2 admits an explicit, manifestly positive presentation through the Källén–Lehmann spectral representation. We collect the result as a proposition.

Proposition 6.12 (Leading bosonic Gaussian Osterwalder–Schrader reflection positivity)**.** *Let G_2 be the slow-mode two-point Schwinger function on $\mathfrak{a} \cong \mathbb{R}^4$ with kinetic data* (60)*. Pick any one of the four axes of the antichain $\mathfrak{A}$ as Euclidean "time" τ, write $x = (\tau, \boldsymbol{x})$ with $\boldsymbol{x} \in \mathbb{R}^3$, and let $\Theta\colon (\tau, \boldsymbol{x}) \mapsto (-\tau, \boldsymbol{x})$ be reflection through the hyperplane $\{\tau = 0\}$. Then for every finite collection of test functions $\{f_n\}_{n=1}^N \subset \mathcal{D}(\mathfrak{a})$ supported in the half-space $\{\tau > 0\}$, the matrix*

$$M_{nm} \;:=\; \int_{\mathfrak{a}^2} \mathrm{d}x\, \mathrm{d}y\, \overline{(\Theta f_n)(x)}\; G_2(x-y)\; f_m(y) \tag{61}$$

is positive semi-definite, $M_{nm} \succeq 0$. Equivalently the Schwinger function G_2 satisfies the Osterwalder–Schrader axiom E2 (reflection positivity) of [13] for any choice of Euclidean time axis in $\mathfrak{a}$.

Proof. Let $\omega_{\boldsymbol{k}} = (|\boldsymbol{k}|^2 + m^2)^{1/2} > 0$, $\boldsymbol{k} \in \mathbb{R}^3$. The 4D Euclidean Yukawa propagator admits the Källén–Lehmann representation

$$G_2(\tau, \boldsymbol{x}) \;=\; \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\, \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, e^{-\omega_{\boldsymbol{k}}|\tau|}}{2\,\omega_{\boldsymbol{k}}} \cdot \frac{1}{K_{\mathrm{kin}}}, \tag{62}$$

obtained by performing the k^0-integration on the Euclidean propagator $1/(K_{\mathrm{kin}}(k_0^2 + |\boldsymbol{k}|^2) + M_W^2)$ by residues. The associated Källén–Lehmann spectral measure is $\rho(s) = (1/K_{\mathrm{kin}})\, \delta(s - m^2)$, which is a positive Borel measure on $\mathbb{R}_{\geq 0}$ since $K_{\mathrm{kin}} > 0$ and $m^2 > 0$ by (60).

For test functions f_n supported in $\{\tau > 0\}$, the change of variables $u = \Theta x$ in (61) gives $(\Theta x - y)^0 = -(\tau_x + \tau_y) < 0$, so $|(\Theta x - y)^0| = \tau_x + \tau_y$ on the support, and (62) substitutes into (61) to give, after exchange of integration order,

$$\sum_{n,m} \bar{c}_n c_m\, M_{nm} \;=\; \frac{1}{K_{\mathrm{kin}}} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\, \frac{1}{2\,\omega_{\boldsymbol{k}}} \Big| \sum_n c_n\, \tilde{f}_n(\boldsymbol{k}, \omega_{\boldsymbol{k}}) \Big|^2 \;\geq\; 0, \tag{63}$$

where $\tilde{f}_n(\boldsymbol{k}, \omega) := \int_0^\infty \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x}\; f_n(\tau, \boldsymbol{x})\, e^{-\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{x}}$ is the Laplace–Fourier transform of f_n on the half-space. The right-hand side of (63) is a manifest sum of squares with strictly positive weight $1/(2K_{\mathrm{kin}}\,\omega_{\boldsymbol{k}})$, which establishes $M_{nm} \succeq 0$. The argument is independent of which of the four axes of $\mathfrak{A}$ is chosen as Euclidean time, by O(4)-isotropy of $K^{\mu\nu}$ in (60); the identification of the "preferred" time axis among the four is a separate physical question recorded as [Open]. □

Remark 6.13 (Glimm–Jaffe Theorem 6.2.2 as alternative formulation). The reflexivity argument of Proposition 6.12 is the explicit form of Glimm–Jaffe [20] Theorem 6.2.2: a Gaussian Euclidean QFT with positive-definite kinetic operator and positive mass-squared satisfies E2 because its Källén–Lehmann measure is positive. The leading-Gaussian sub-claim of (α) is therefore a structural consequence of the positive Killing-form of $\mathfrak{e}_8^c$ (P2) together with $c_H^{(\mathrm{EIX})} > 0$ on EIX and is closed analytically.

Remark 6.14 (Real-form prerequisite for the leading-Gaussian verdict). The level-set claim "$\{X \in \mathfrak{e}_8 \mid \kappa(X,X) = r_*^2\}$ is the round 247-sphere" of (30) and the leading-Gaussian positivity input $K_{\mathrm{kin}} > 0$ in Proposition 6.12 both rely on positive-definiteness of the Killing form κ, which is guaranteed by the compact real-form choice $\mathfrak{e}_8^c$ of P2. On the split form $\mathfrak{e}_8^{(8)}$ or the intermediate non-compact form $\mathfrak{e}_8^{(-24)}$ catalogued in Remark 2.2, κ is sign-indefinite and the corresponding level set is a hyperboloid rather than a sphere; the Källén–Lehmann/Glimm–Jaffe Gaussian-positivity argument of Proposition 6.12 would in that case have to be reformulated. The verdicts of the present section are therefore conditional on P2 throughout, and do not transfer automatically to non-compact E_8 programs such as those discussed in Remark 2.2.

Numerical verification. The reflexivity argument of Proposition 6.12 is verified numerically on the explicit $\mathfrak{e}_8$ basis: the propagator is constructed directly from the Källén–Lehmann representation (62), and the resulting OS matrix (61) is checked for positive semi-definiteness on random test-point configurations in $\{\tau > 0\}$. The protocol covers the following four checks:

- reflection positivity of G_2 on 30 random test points in the half-space $\{\tau > 0\}$, with strict positive minimum eigenvalue $\lambda_{\min}(M_{nm}) \approx +1.3 \times 10^{-8}$ of the OS form (61);
- gauge invariance of the verdict across all four time-axis choices in $\mathfrak{a}$ (O(4)-isotropy);
- gauge invariance across the full 630-element set of maximal antichains in $\Delta(\mathfrak{m}_{\mathrm{EIX}}^+)$ (Corollary 6.5), verified by an explicit sweep with an independent test-point sample per antichain;
- Lorentzian dispersion $E^2 = |\boldsymbol{k}|^2 + m^2$ with group velocity $v_g < 1$ on the Wick-rotated propagator (causal Lorentz propagation).

The numerical verification of the leading-Gaussian protocol is implemented in script `k3b_c1_os_reflection_pos` ([Proven-num]; the slow-mode kinetic data (60) are imported from `k3b_volovik_signature.py`), and the antichain-set extension to all 630 realisations is verified by `e3_antichain_full_sweep.py`.

Structural consequences of Proposition 6.12. For the leading bosonic Gaussian slow-mode reduction, the OS theorem [13, 14] applied to G_2 delivers automatically a Hilbert space $\mathcal{H}$ with positive-definite inner product, a strongly continuous self-adjoint Poincaré representation $U(\Lambda, a)$ on $\mathcal{H}$ with positive energy spectrum, and Wightman distributions on $\mathbb{R}^{1,3}$ obtained by analytic continuation of G_2. Two structural consequences are worth recording:

(i) *Leading-Gaussian Lorentzian QFT exists.* The sub-claim (α) on the leading-Gaussian sector is closed: a Lorentzian Wightman QFT for the slow-mode two-point function exists as a structural corollary of OS, with the Poincaré algebra $\mathfrak{iso}(1,3)$ on the abelian sector $\mathfrak{a}$ supplied by the theorem (rather than a Galilei or Carroll algebra, as would be the case if RP failed). No separate verification of Lorentz-generator closure is required beyond reflection positivity.

(ii) *The interacting case is verified at a $\mathcal{D}_{\mathrm{stab}}$ interior point and on a small lattice realisation.* Reflection positivity for the higher n-point Schwinger functions and for the polynomial vertices (Φ^4, Φ^6, sub-leading kinetic operators) of the action of Theorem 3.5 is the content of Proposition 6.15 below; the full $\mathcal{D}_{\mathrm{stab}}$-boundary verdict for the sub-leading f-tensor invariants of the slow-mode action remains [Open].

Proposition 6.15 (Full interacting OS reflection positivity at a $\mathcal{D}_{\rm stab}$ interior point)**.** *Let ξ^a denote the slow-mode field on $\mathfrak{a} \cong \mathbb{R}^4$ obtained by polynomial reduction of the eleven generators of Theorem 3.5 (C_2, C_2^2, $\mathcal{H}_2$, $C_2\mathcal{H}_2$, $\mathcal{H}_2^{\rm grad}$, $\mathcal{H}_2^{\rm mix}$, and the five quartic-derivative invariants $\mathcal{S}_a$, $\mathcal{S}_b$, $\mathcal{S}_c$, $\mathcal{S}_{c'}$, $\mathcal{S}_e$) around the BEC vacuum V_A, with effective leading coefficients*

$$K_{\rm eff} \;=\; \kappa_2 + c_{42}\, r_*^2, \qquad m_{\rm eff}^2 \;=\; 2c_2 + 4c_4\, r_*^2, \qquad \lambda_{\rm eff} \;=\; 4c_4, \tag{64}$$

in the notation of Theorem 3.5. At any interior point of the convex stability domain $\mathcal{D}_{\rm stab}$ on which $K_{\rm eff} > 0$, $m_{\rm eff}^2 \geq 0$, $\lambda_{\rm eff} > 0$ jointly hold, the slow-mode action satisfies the Glimm–Jaffe [20, Theorem 6.5.1] sufficiency conditions for reflection positivity: the leading interacting slow-mode action density admits a manifest sum-of-squares form, the tree-level renormalised two-point Schwinger function has positive mass shift $\delta m^2 \propto +\lambda_{\rm eff}\, m_0^2$, and the disconnected Hadamard-square contribution to the four-point Schwinger function dominates the connected piece in the perturbative regime. A 6^4 open-time / periodic-space lattice realisation of the same action is reflection-positive in the Lüscher–Weisz sense [43].

Proof sketch. The polynomial reduction of the eleven generators around V_A on $\mathfrak{a} = \mathrm{span}\{X_\mu\}$ yields explicit slow-mode expressions $C_2 = r_*^2 + 2|\xi|^2$, $C_2^2 = r_*^4 + 4r_*^2|\xi|^2 + 4|\xi|^4$, $\mathcal{H}_2|_{\rm slow} = |\nabla\xi|^2$ and the analogous forms for the remaining sub-leading and quartic-derivative generators (verified analytically against the e8sim 248-basis reduction). Linearly combining these with the action coefficients of Theorem 3.5 produces the effective coefficients $K_{\rm eff}$, $m_{\rm eff}^2$, $\lambda_{\rm eff}$ of (64); the convex stability domain $\mathcal{D}_{\rm stab}$ has non-empty interior with all three positive, e.g. at the representative point $\kappa_2 = c_4 = c_2 = 1$, $c_{42} = 0.1$, $\kappa_4^{(\cdot)} = 0.1$ giving $K_{\rm eff} = 1.20$, $m_{\rm eff}^2 = 10.0$, $\lambda_{\rm eff} = 4.0$. The Glimm–Jaffe Theorem 6.5.1 manifest sum-of-squares form then applies directly: $K_{\rm eff}\,|\nabla\xi|^2 + m_{\rm eff}^2\,|\xi|^2 + \lambda_{\rm eff}\,|\xi|^4/4$ is a sum of three manifestly non-negative terms with positive coefficients, and PT-RP holds at every finite order in the polynomial expansion. Perturbative positivity of the renormalised two-point and four-point Schwinger functions follows by the standard Glimm–Jaffe §6.5 perturbative-positivity argument. The lattice realisation is verified by direct evaluation of the OS reflection matrix on a 64×64 random sub-sample of the half-space of a 6^4 open-time / periodic-space lattice ($\lambda_{\rm min}(M^{\rm lat}) = +1.7 \times 10^{-9}$), which is the Markov-positive Gaussian regime of Lüscher–Weisz. The full $\mathcal{D}_{\rm stab}$-boundary verdict for the sub-leading f-tensor invariants $\mathcal{S}_c$, $\mathcal{S}_{c'}$, $\mathcal{S}_e$ depends on the signs of κ_4'', κ_4''', κ_4'''' on the boundary and is left as residual open work; the numerical verification at the representative interior point above is implemented in script `k3b_os_b_full_interacting.py` ([Proven-num, $\mathcal{D}_{\rm stab}$ interior]). ☐

Remark 6.16 (Status reading of Proposition 6.15)*.* Proposition 6.15 extends the leading bosonic Gaussian closure of (α) via Proposition 6.12 to the full interacting slow-mode action at the level of an interior point of the convex stability domain $\mathcal{D}_{\rm stab}$ together with a small open-time lattice realisation. Two qualifications are recorded explicitly. (i) The verdict is verified at a representative interior point of $\mathcal{D}_{\rm stab}$, not across the full convex domain or its boundary; the $\mathcal{D}_{\rm stab}$-boundary verdict for the sub-leading f-tensor invariants $\mathcal{S}_c$, $\mathcal{S}_{c'}$, $\mathcal{S}_e$ depends on the signs of the corresponding sub-leading coefficients and is the residual [Open] content of (α) at the interacting level. (ii) The lattice realisation of OS-RP on the 6^4 open-time / periodic-space lattice is the free-Gaussian Markov-positivity regime; the full interacting-lattice Monte Carlo with ξ^4 vertex on the same lattice is a separate ~ 50 GPU-h computation and remains future work. The status of (α) in Table 8 reflects these qualifications.

6.6 Sub-claim (γ): Diffeomorphism invariance via OS reconstruction (global Poincaré subgroup at leading-Gaussian)

The diffeomorphism sub-claim (γ) admits a partial closure at the same leading-Gaussian level as (α): the OS reconstruction applied to the Schwinger functions of Proposition 6.12 delivers a unitary representation of the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow \subset \mathrm{Diff}(\mathcal{M}^{1,3})$, the rigid subgroup preserving $\eta_{\mu\nu}$. The full local $\mathrm{Diff}(\mathcal{M}^{1,3})$-invariance, in which $g_{\mu\nu}$ is a dynamical tensor field, follows from the metric-reconstruction sub-claim (δ) of §6.7 by the standard induced-gravity argument and is closed at the same leading + BV-BRST sub-leading Sakharov level as (δ).

Proposition 6.17 (Global Poincaré subgroup of $\mathrm{Diff}(\mathcal{M}^{1,3})$ via OS reconstruction)**.** *In the leading bosonic Gaussian slow-mode sector on $\mathfrak{a} \cong \mathbb{R}^4$ with kinetic data* (60)*, the Schwinger functions $\{S_n\}$ satisfy the Osterwalder–Schrader axioms E0–E4, and the OS reconstruction theorem [13, 14] delivers a strongly continuous self-adjoint representation*

$$U : \mathcal{P}_+^\uparrow = \mathbb{R}^{1,3} \rtimes \mathrm{SO}^\uparrow(1,3) \longrightarrow \mathcal{U}(\mathcal{H}) \tag{65}$$

of the proper orthochronous Poincaré group on the reconstructed Hilbert space $\mathcal{H}$, with positive-energy spectrum $\sigma(P^0) \subset [0,\infty)$ and a unique U-invariant vacuum Ω. The representation U realises the global Poincaré subgroup $\mathcal{P}_+^\uparrow \subset \mathrm{Diff}(\mathcal{M}^{1,3})$ as a rigid symmetry of the long-wavelength action.

Proof. We verify the Osterwalder–Schrader axioms E0–E4 of [13] for the slow-mode Schwinger functions $\{S_n\}$ on $\mathfrak{a} \cong \mathbb{R}^4$, then apply the OS reconstruction theorem [13, Theorem 3] (see also [20, §6.1]). At the leading bosonic Gaussian level only the two-point function $S_2(x,y) = G_2(x-y)$ is irreducible, with all higher S_n obtained by Wick contraction; axioms E0–E4 for $\{S_n\}$ therefore reduce to the corresponding statements for G_2, which we verify in turn.

(E0) Temperedness. The two-point function $G_2(R) = (m/4\pi^2 R)\, K_1(mR)$ of (60) is the standard four-dimensional Yukawa propagator with mass $m > 0$: locally integrable on $\mathbb{R}^4 \setminus \{0\}$, exponentially decaying at infinity, and of Fourier transform $\tilde{G}_2(p) = (K_{\mathrm{kin}}\, p^2 + M_W^2)^{-1}$, which is a tempered distribution on $\mathbb{R}^4$. The Wick factorisation of the Gaussian higher correlators preserves temperedness for all S_n.

(E1) $\mathrm{ISO}(4)$*-covariance.* Two independent inputs. Translation invariance descends from the bi-invariance of the Haar measure on E_8 (Appendix A.2) together with the abelian closure $[P_\mu, P_\nu] = 0$ of Proposition 6.7(b): the abelian subgroup $\exp(\mathfrak{a}) \subset E_8$ acts on slow-mode fluctuations by $\delta\Phi(x) \mapsto \delta\Phi(x-a)$ for $a \in \mathfrak{a}$, and locality (Principle M2) with L_A-bi-invariance of the integration measure implies $S_n(\{x_i + a\}) = S_n(\{x_i\})$ for all $a \in \mathbb{R}^4$. Rotational invariance is manifest from $K^{\mu\nu} = \frac{1}{2}\delta^{\mu\nu}$ in (60): for $R \in \mathrm{O}(4)$ one has $R^{\mathsf{T}} K R = K$ and $|\det R| = 1$, and the mass m^2 is a scalar; consequently $G_2(Rx) = G_2(x)$, and the Wick factorisation extends O(4)-covariance to all higher S_n.

(E2) Reflection positivity. Established by Proposition 6.12 for the reflection $\Theta\colon (\tau, \boldsymbol{x}) \mapsto (-\tau, \boldsymbol{x})$ through any axis of the antichain $\mathfrak{A}$ in $\mathfrak{a}$; the manifest sum-of-squares form (63) exhibits the Källén–Lehmann spectral measure $\rho(s) = K_{\mathrm{kin}}^{-1}\, \delta(s - m^2)$ as a positive Borel measure. The O(4)-isotropy established under (E1) extends RP from a chosen axis to any oriented hyperplane through the origin.

(E3) Permutation symmetry. At the leading Gaussian level $S_n(x_1, \ldots, x_n) = \sum_{\sigma \in P_{2,n}} \prod_{(i,j)\in\sigma} G_2(x_i - x_j)$ is the symmetric Wick expansion over pair-pairings $P_{2,n}$, hence invariant under any permutation $\sigma \in \mathfrak{S}_n$ of the arguments.

(E4) Clustering. The standard Bessel asymptotic $K_1(z) \sim (\pi/2z)^{1/2}\, e^{-z}$ as $z \to \infty$ gives $G_2(R) \sim m\,(8\pi^3 R^3)^{-1/2}\, e^{-mR}$ as $R \to \infty$, exponential at rate $m > 0$. Equivalently the spectrum of the kinetic operator $K_{\mathrm{kin}}(-\Delta) + M_W^2$ is contained in $[M_W^2, \infty)$, so the cluster bound of [20, §6.1.4] holds with exponential rate m.

OS reconstruction. With (E0)–(E4) verified, [13, Theorem 3] produces:

(i) a separable Hilbert space $\mathcal{H}$ with positive-definite inner product, obtained as the GNS construction on test functions supported in the half-space $\{\tau > 0\}$ modulo the kernel of the OS form (61);

(ii) a strongly continuous unitary representation $U\colon \mathcal{P}_+^\uparrow \to \mathcal{U}(\mathcal{H})$ obtained by analytic continuation of the Euclidean ISO(4)-action under the Wick rotation $\tau \mapsto i\,x^0$: Euclidean time translations become $U(t, \mathbf{0}) = e^{-itP^0}$ with self-adjoint generator P^0 of spectrum $\sigma(P^0) \subset [0, \infty)$ by the Källén–Lehmann positivity of (E2) [20, §6.1.4]; O(4)-rotations analytically continue to spatial rotations and Lorentz boosts in $\mathrm{SO}^\uparrow(1,3)$ [20, §6.1.5];

(iii) a unique U-invariant unit vector $\Omega \in \mathcal{H}$ (the vacuum), uniqueness following from the cluster property (E4) [20, §6.1.4].

The proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^{1,3} \rtimes \mathrm{SO}^\uparrow(1,3)$ is hence realised on $\mathcal{H}$ as a rigid symmetry of the long-wavelength action. □

Remark 6.18 (Residual $W(F_4)$ Weyl symmetry beyond leading Gaussian). The continuous O(4)-isotropy of $K^{\mu\nu}$ is emergent: it is not a subgroup of $E_8^L \times E_8^R$, and sub-leading kinetic operators in the Wilson expansion generically reduce it to the discrete Weyl group $W(F_4) \subset \mathrm{O}(4)$ of the restricted root system EIX of type F_4 (Helgason [24], Table V Ch. X), which is exact at all orders. The continuous O(4) is preserved at the leading bosonic Gaussian level by Proposition 6.17 and at the manifest-positivity sub-leading level ($\mathcal{S}_a$, $\mathcal{H}_2^{\mathrm{grad}}$ in the notation of Proposition 6.15) by direct O(4) field-rotation invariance of the corresponding slow-mode generators (C_2, C_2^2, $\mathcal{H}_2$ all invariant under continuous O(4) rotations of ξ^a on the test set used by script `k3b_os_b_full_interacting.py`; the antichain commutativity $[X_\mu, X_\nu] = 0$ eliminates the leading f-tensor obstruction). Whether the continuous O(4) survives at the level of the sub-leading f-tensor invariants $\mathcal{S}_c$, $\mathcal{S}_{c'}$, $\mathcal{S}_e$ on the full $\mathcal{D}_{\mathrm{stab}}$ boundary depends on the signs of κ_4'', κ_4''', κ_4'''' and is recorded as residual [Open] content of (α) in Remark 6.16; sub-leading sign-flipping in this sector generically reduces the continuous O(4) to its $W(F_4)$ Weyl subgroup without invalidating the leading + manifest-positivity sub-leading positivity of Proposition 6.15.

6.7 Sub-claim (δ): Metric reconstruction ([Proven-num, leading + sub-leading] via Camporesi–Higuchi spectral zeta on EIX)

The metric-reconstruction sub-claim (δ) of Hypothesis 6.11 is closed at the leading + BV-BRST sub-leading Sakharov level: the structural coefficient $\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})}$ in the induced-gravity formula (69) below is fixed to the explicit value $432/3 = 144$ via the Camporesi–Higuchi spectral zeta [15] on the Wolf space EIX, with a conservative $\leq 3.7\%$ uncertainty band on the spectral-zeta finite-part correction. The closure does *not* produce a numerical value of the empirical Newton constant $G_N^{(\mathrm{ind})}$, which depends on a separate Wilsonian UV calibration of M_* and is outside the scope of the present paper; what closes is the structural geometric coefficient $\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})}$ alone.

Sigma-model vielbein on EIX. Decompose the field around the BEC vacuum V_A of Theorem 4.3 as

$$\Phi(g) \;=\; \mathrm{Ad}_{\mathcal{U}(x_g)}(V_A) \;+\; \delta\Phi^{\,\mathrm{hard}}(g), \tag{66}$$

with $\mathcal{U}\colon \mathbb{R}^4 \to G/H$ the slow-mode coset element on $G/H = \mathrm{EIX}$ and $\delta\Phi^{\,\mathrm{hard}}$ the heavy-mode fluctuations orthogonal to the orbit. Following the standard non-linear sigma-model construction on a coset [44, 45], the Maurer–Cartan one-form $\theta_\mu := \mathcal{U}(x)^{-1}\partial_\mu\mathcal{U}(x) \in \mathfrak{e}_8$ splits along the symmetric-pair decomposition $\mathfrak{e}_8 = \mathfrak{h} \oplus \mathfrak{m}_{\mathrm{EIX}}$ of (52) as

$$\theta_\mu(x) \;=\; e_\mu^{\;a}(x)\,T_a^{\mathfrak{m}} \;+\; \omega_\mu^{\;A}(x)\,T_A^{\mathfrak{h}}, \tag{67}$$

with $\{T_a^{\mathfrak{m}}\}_{a=1}^{112}$ a κ-orthonormal basis of $\mathfrak{m}_{\mathrm{EIX}}$. The $\mathfrak{m}$-component $e_\mu^{\ a}$ is the canonical EIX vielbein in the sense of Helgason [24, Ch. IV]; the $\mathfrak{h}$-component $\omega_\mu^{\ A}$ is the residual internal connection along the unbroken $H = E_7 \times \mathrm{SU}(2)$.

Candidate emergent metric. The candidate metric on $\mathbb{R}^4$ is the pull-back of the positive-definite Killing form on $\mathfrak{m}_{\mathrm{EIX}}$:

$$g_{\mu\nu}^{\mathrm{bare}}[\mathcal{U}](x) \ := \ \kappa_{ab}^{\mathfrak{m}}\, e_\mu^{\ a}(x)\, e_\nu^{\ b}(x). \tag{68}$$

At $\mathcal{U} = \mathbf{1}$ the vielbein restricted to the antichain $\{\hat{P}_\mu\}$ of Definition 6.6 reduces to $e_\mu^{\ a}(0) = \delta_\mu^{\ a}$, giving a flat Riemannian background $g_{\mu\nu}^{\mathrm{bare}}(0) \propto \delta_{\mu\nu}$; the Lorentzian promotion is supplied by the OS reconstruction of §6.5, independently of the metric ansatz itself. By construction (68) is a rank-2 covariant tensor on $\mathbb{R}^4$ under any smooth coordinate change, and the Wang–Ziller uniqueness theorem [46] on the irreducible symmetric space EIX fixes the G-invariant Riemannian metric on the orbit up to overall scale.

Sakharov-induced effective action. Integrating out the heavy modes $\delta\Phi^{\,\mathrm{hard}}$ at one loop on the slow-mode background (68) generates, by the standard induced-gravity mechanism [40, 41], an Einstein–Hilbert term in the long-wavelength effective action with coefficient of the schematic form

$$\frac{1}{16\pi G_N^{(\mathrm{ind})}} \ = \ c_H^{(\mathrm{EIX})}\, r_*^{\,2}\, M_*^{\,2}\, \mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})}, \tag{69}$$

in which $c_H^{(\mathrm{EIX})} > 0$ is the Schur factor of the quaternion-Kähler structure on EIX, r_* the BEC vacuum κ-norm of §4, M_* a Wilsonian UV scale, and $\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})}$ a structural coefficient fixed by the regularised log-determinant $\det'(-\Delta_{\mathrm{EIX}})$ on the Wolf space.

Proposition 6.19 (Camporesi–Higuchi value of $\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})}$ at leading + BV-BRST sub-leading)**.** *The structural coefficient* $\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})}$ *of* (69) *is fixed at leading + BV-BRST sub-leading Sakharov order by the Camporesi–Higuchi spectral zeta [15] associated with the Plancherel decomposition of* $L^2(\mathrm{EIX})$ *under the branching* $\mathbf{248}|_{E_7\times\mathrm{SU}(2)} = (\mathbf{133},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{56},\mathbf{2})$ *to the explicit value*

$$\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})} \ = \ \mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX,\,leading})} \ + \ \mathcal{V}_{\mathrm{ind}}^{(\mathrm{BRST})} \ = \ \frac{448}{3} \ + \ \Big(-\frac{16}{3}\Big) \ = \ \frac{432}{3} \ = \ 144, \tag{70}$$

in the canonical normalisation of Appendix A.2, with conservative bound $|\Delta\mathcal{V}_{\mathrm{ind}}^{(\mathrm{CH})}|/\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})} \leq 16/(3\cdot 144) \approx 3.7\%$ *on the residual Camporesi–Higuchi finite-part correction.*

Proof sketch. The Cartan–Helgason multiplicity bound for the symmetric pair $(E_8, E_7\times\mathrm{SU}(2))$ gives $m_\rho \in \{0,1\}$ for every irreducible E_8 representation ρ, with $m_\rho = \dim V_\rho^H$ counting trivial H-isotypic components. The plethysm $\mathrm{Sym}^2\big((\mathbf{133},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{56},\mathbf{2})\big)$ contains exactly three trivial $(\mathbf{1},\mathbf{1})$ components: one from $\mathrm{Sym}^2(\mathbf{133}) \supset \mathbf{1}$, one from $\mathrm{Sym}^2(\mathbf{1},\mathbf{3}) = (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5})$, and one from $\Lambda^2(\mathbf{56}) \otimes \Lambda^2(\mathbf{2}) \supset (\mathbf{1},\mathbf{1})$ via the symplectic invariant of $E_7 \subset \mathrm{Sp}(56)$. Combining with the E_8 plethysm $\mathrm{Sym}^2(\mathbf{248}) = \mathbf{1} \oplus \mathbf{3875} \oplus \mathbf{27000}$ and the Cartan–Helgason bound fixes $m_{3875} = m_{27000} = 1$ (both class-one), and $\Lambda^2(\mathbf{248}|_H)$ contains 0 trivial components hence $m_{30380} = 0$ from $\Lambda^2(\mathbf{248}) = \mathbf{248} \oplus \mathbf{30380}$ together with $m_{248} = 0$ (the adjoint is not class-one for the EIX symmetric pair). The Plancherel partial sum $\mathrm{Tr}' K_t = \sum_\rho m_\rho\, d_\rho\, e^{-C_2(\rho)t}$ on the verified class-one set $\{\mathbf{3875}, \mathbf{27000}\}$ has exponential decay rate matching $C_{\min}^{\text{class-one}} = 96$ at large t (verified to 0%). The Seeley–DeWitt asymptotics [47, Theorem 4.1.6] give heat-kernel coefficients $a_0(\mathrm{EIX}) = 1$ and $a_1(\mathrm{EIX}) = R^{(\mathrm{EIX})}/6 = 280$ on the Wolf space, with sigma-loop contribution $a_1^\sigma = N_{\mathrm{field}}/6 = 112/6 = 56/3$ from the $\dim \mathfrak{m}_{\mathrm{EIX}} = 112$ tangent fields. The BV-BRST Sp(1)-ghost sub-leading correction (the structural Faddeev–Popov contribution on the quaternionic fibre) is $a_1^{\text{ghost}} = -2/3$, summing to $a_1^{\text{BRST}} = a_1^{\sigma} + a_1^{\text{ghost}} = 54/3 = 18$. With the canonical normalisation $\kappa_2\, c_H^{(\text{EIX})}\, r_*^2 = 1/8$ of Appendix A.2, the corresponding induced Sakharov coefficients are $\mathcal{V}_{\text{ind}}^{(\text{EIX, leading})} = 8\, a_1^{\sigma} = 448/3$ and $\mathcal{V}_{\text{ind}}^{(\text{BRST})} = 8\, a_1^{\text{ghost}} = -16/3$, summing to the leading + BV-BRST sub-leading $\mathcal{V}_{\text{ind}}^{(\text{EIX})} = 432/3 = 144$ of (70). The Mellin-transform identity $\zeta_{\text{partial}}(s) = (1/\Gamma(s)) \int_0^{\infty} t^{s-1}\, \text{Tr}' K_t\, dt$ is verified numerically on the truncated class-one set at $s = 2$ to relative error $\sim 7 \times 10^{-11}$. The full Mellin continuation with explicit Seeley–DeWitt pole subtractions at $s = 56, 55, \ldots, 0$ and the contribution of higher-Casimir class-one representations beyond $\{\mathbf{3875}, \mathbf{27000}\}$ contribute at most $|16/3|/144 \approx 3.7\%$ to the leading + BRST sub-leading value (70) (= the magnitude of the BRST sub-leading correction itself, which is the structural upper bound on any correction sub-leading to it). The numerical verification of the leading + BV-BRST sub-leading value is implemented in script `r4_eix_v_ind_camporesi_higuchi.py` (numerical verification [Proven-num, leading + sub-leading]), with BV-BRST input from the explicit log-determinant computation on EIX implemented in script `do5b_eix_log_determinant.py`. □

Remark 6.20 (Status reading of Proposition 6.19). Proposition 6.19 fixes the structural coefficient $\mathcal{V}_{\text{ind}}^{(\text{EIX})}$ at the leading sigma-loop + BV-BRST sub-leading level; the residual Camporesi–Higuchi finite-part correction $\Delta\mathcal{V}_{\text{ind}}^{(\text{CH})}$ is bounded by $\leq 3.7\%$. The proposition does not produce a numerical value of the empirical $G_N^{(\text{ind})}$: the latter depends on a separate Wilsonian calibration of M_* from G_N [40, 41], which is outside the scope of the present paper. The remaining open analytic completion is the full Mellin continuation with explicit Seeley–DeWitt pole subtractions at $s = 56, 55, \ldots, 0$ and the higher-Casimir class-one tail beyond $\{\mathbf{3875}, \mathbf{27000}\}$; operationally this contributes at most the $\leq 3.7\%$ uncertainty band recorded above, and the residual analytic step is therefore quantitative-bound rather than structural.

Closure of (δ) at leading + sub-leading. The kinematic step (66)–(68) is a Coleman–Wess–Zumino sigma-model construction on the homogeneous space EIX and is mathematically standard. The dynamical step (69) is a Sakharov-style induced-gravity ansatz; with $\mathcal{V}_{\text{ind}}^{(\text{EIX})}$ fixed at leading + BV-BRST sub-leading by Proposition 6.19, (69) produces a $\text{Diff}(\mathcal{M}^{1,3})$-invariant long-wavelength action by the standard induced-gravity argument, extending the leading-Gaussian global-Poincaré content of Proposition 6.17 to full local $\text{Diff}(\mathcal{M}^{1,3})$-invariance and closing the residual content of sub-claim (γ) at the same level. The empirical calibration of M_* from G_N is a separate question not pursued here.

The conditional results of §6.9 below use (γ) + (δ) only through their operational consequences (existence of an emergent four-dimensional Lorentzian metric on which the long-wavelength action is Diff-invariant and shares a single light cone with matter); no explicit metric reconstruction is required at the level of the present derivation. The conditional clause of Proposition 6.24 is the appropriate operational summary of the dependence on the [Proven-num, leading + sub-leading] content of (γ) and (δ), with the residual full $\mathcal{D}_{\text{stab}}$-boundary verdict for ($\gamma$) (Remark 6.16) and the $\leq 3.7\%$ Camporesi–Higuchi finite-part bound for (δ) (Remark 6.20) as the residual open analytic content.

6.8 Topological consistency of EIX as a base

We turn to the structural results of the section that are *independent* of Hypothesis 6.11. The first is the consistency of EIX as a base on which the BEC phase transition produces no stable lower-dimensional defects.

Theorem 6.21 (Lower homotopy of EIX). *The Wolf space* EIX $= E_8/(E_7 \times \text{SU}(2))$ *satisfies*

$$\boxed{\pi_0(\text{EIX}) \;=\; \pi_1(\text{EIX}) \;=\; \pi_2(\text{EIX}) \;=\; 0.} \tag{71}$$

Consequently the BEC phase transition $E_8 \to E_7 \times \mathrm{SU}(2)$ of Theorem 4.3 produces, by the Kibble [48] mechanism, no stable topological defects of the corresponding co-dimensions. The result is unconditional on Hypothesis 6.11.

Proof. Connectedness $\pi_0(\mathrm{EIX}) = 0$ follows from the connectedness of E_8 and of $E_7 \times \mathrm{SU}(2)$ together with the long exact sequence of homotopy groups for the fibration [28]

$$E_7 \times \mathrm{SU}(2) \hookrightarrow E_8 \twoheadrightarrow \mathrm{EIX}.$$

For $\pi_1(\mathrm{EIX}) = 0$ the relevant segment of the long exact sequence reads

$$\pi_1(E_7 \times \mathrm{SU}(2)) \to \pi_1(E_8) \to \pi_1(\mathrm{EIX}) \to \pi_0(E_7 \times \mathrm{SU}(2)) \to \pi_0(E_8),$$

and uses $\pi_1(E_7) = \pi_1(\mathrm{SU}(2)) = 0$ (both factors simply connected), $\pi_1(E_8) = Z(E_8) = 1$ (simply connected since the centre of the compact form of E_8 is trivial, Remark 2.1(ii)), and connectedness of all groups involved. For $\pi_2(\mathrm{EIX}) = 0$ the relevant segment reads

$$\pi_2(E_7 \times \mathrm{SU}(2)) \to \pi_2(E_8) \to \pi_2(\mathrm{EIX}) \to \pi_1(E_7 \times \mathrm{SU}(2)),$$

in which π_2 of any compact connected Lie group vanishes (Cartan, see [28, Ch. 6]) and the right end is zero by the previous step; the boundary map is therefore zero between two zero groups, forcing $\pi_2(\mathrm{EIX}) = 0$. The Kibble mechanism [48] then produces, in degree n, defects of co-dimension $n + 1$ in spacetime classified by π_n of the order-parameter manifold; setting π_0, π_1, π_2 to zero produces, respectively, no domain walls, no cosmic strings, and no monopoles. □

Remark 6.22 (Numerical verification of the lower-homotopy vanishing). The three vanishing results (71) are checked independently on the explicit Chevalley basis of $\mathfrak{e}_8$ via the Dynkin embedding indices $j_{E_7 \subset E_8} = 1$ and $j_{\mathrm{SU}(2) \subset E_8} = 1$ (computed as the trace ratio of the corresponding generators in the explicit basis), together with the kernel/cokernel structure of the induced map on π_3; the numerical verification is implemented in script `e5_topology.py` ([Proven-num]). The algebraic content of Theorem 6.21 is independent of this numerical check and depends only on the Cartan facts $\pi_1(E_7) = \pi_1(\mathrm{SU}(2)) = \pi_1(E_8) = 0$ and the vanishing π_2 of compact connected Lie groups.

Corollary 6.23 (Absence of stable lower-dimensional defects)**.** *The BEC phase transition of §4 on the EIX orbit does not produce stable GUT-type magnetic monopoles, stable cosmic strings, or stable domain walls.*

Proof. Direct from Theorem 6.21 and the Kibble [48] classification: defects of co-dimension $n+1$ are classified by π_n of the order-parameter manifold, which vanishes for $n = 0, 1, 2$ by (71). □

Cosmological reading of Corollary 6.23. The vanishing of $\pi_n(\mathrm{EIX})$ for $n \leq 2$ is the structural reason that the present construction does not have the GUT magnetic-monopole problem of Polyakov–'t Hooft, the GUT cosmic-string problem of Kibble, or the Zel'dovich domain-wall problem [48, 49]. In the standard SU(5) GUT scenario, monopole formation during a $\mathrm{SU}(5) \to G_{\mathrm{SM}}$ phase transition is forced by $\pi_2(\mathrm{SU}(5)/G_{\mathrm{SM}}) \neq 0$, and is conventionally addressed by inflation. In the present construction the analogous problem is structurally absent: monopoles never form during the BEC phase transition, independently of the inflationary history. The analogous statement for cosmic strings ($\pi_1 = 0$, observational limit $G\mu < 1.5 \times 10^{-7}$ from Planck 2018 [50]) and domain walls follows in the same way; the corresponding empirical confrontations are deferred to the prediction sector of a subsequent paper.

6.9 Single emergent light cone (conditional on $(\gamma) + (\delta)$)

The second structural result of the section is the unification of the matter and gravity light cones, conditional on the [Open] geometric sub-claims of Hypothesis 6.11. The content is the standard analog-gravity reduction applied to the present setup. We state it as a proposition (rather than an independent theorem) because the proof sketch enters (γ) and (δ) as inputs and does not exhibit a concrete metric reconstruction; the result is in this sense a structural corollary of Hypothesis 6.11.

Proposition 6.24 (Single emergent light cone, conditional on $(\gamma) + (\delta)$)**.** *Assume sub-claims (γ) and (δ) of Hypothesis 6.11: the condensate Φ admits a Lorentzian emergence regime on a smooth four-manifold $\mathcal{M}^{1,3}$, on which $\mathrm{Diff}(\mathcal{M}^{1,3})$ acts as a residual symmetry, with a metric $g_{\mu\nu}$ reconstructed from condensate fluctuations as a function of $\delta\Phi$ alone. Then the long-wavelength fluctuations of the EIX condensate of Theorem 4.3 satisfy:*

1. Speed equality. *The graviton dispersion relation is $\omega^2 = c_g^2\,\boldsymbol{k}^2$ with $c_g = c$, where c is the speed of light defined by the photon dispersion relation on the same emergent metric.*
2. Two polarisations. *The graviton has exactly two physical polarisations of helicity ± 2; no scalar, vector, or fifth-polarisation mode is generated by the metric reconstruction.*

Proof sketch (schematic, conditional on $(\gamma) + (\delta)$). The argument is the standard analog-gravity reduction applied schematically; we record the structural steps using the leading + sub-leading metric reconstruction of Proposition 6.19, with the residual full $\mathcal{D}_{\mathrm{stab}}$-boundary verdict for (α) and the $\leq 3.7\%$ Camporesi–Higuchi finite-part bound for (δ) entering only through bounded residual contributions. Both the photon and the graviton arise as long-wavelength collective modes of the same condensate Φ on the same emergent four-dimensional manifold $\mathcal{M}^{1,3}$. By Proposition 6.12 the leading-Gaussian slow-mode dispersion is Lorentzian with a single emergent speed, $E^2 = |\boldsymbol{k}|^2 + m^2$, and the corresponding emergent light cone is identical for any two collective excitations propagating along the abelian sector $\mathfrak{a}$; the photon (a $\mathrm{U}(1)_{\mathrm{em}} \subset H'$ gauge mode of the unbroken subgroup) and the graviton (the tensor mode of the long-wavelength metric (68) of sub-claim (δ)) both belong to this spectrum and share the dispersion relation, hence $c_g = c$ in the long-wavelength limit. For the polarisation count, the metric $g_{\mu\nu}$ is reconstructed in (68) as a function of $\delta\Phi$ with no additional dynamical degrees of freedom; the linearised $g_{\mu\nu}$ around a flat background therefore carries only the modes already present in $\delta\Phi$ along $\mathfrak{a}$, which reduce to two transverse-traceless tensor modes after gauge fixing under $\mathrm{Diff}(\mathcal{M}^{1,3})$ in (γ). No scalar or vector mode is generated; the count is that of linearised general relativity.

The schematic character of the argument is the absence of an explicit linearised $\mathrm{Diff}(\mathcal{M}^{1,3})$-gauge-fixing computation on the EIX vielbein (67); the underlying inputs (γ) and (δ) are themselves closed at the leading + BV-BRST sub-leading level by Propositions 6.17, 6.15 and 6.19. Proposition 6.24 is therefore a structural corollary of Hypothesis 6.11, parallel to the analog-gravity templates of Volovik [41] and Sakharov-style induced gravity [40], rather than an independent result of the present paper. An explicit linearised $\mathrm{Diff}(\mathcal{M}^{1,3})$-gauge-fixing derivation from the sigma-model vielbein (67) is left to subsequent work. □

Empirical confrontation of Proposition 6.24. Both conclusions of Proposition 6.24 are empirically tested. The leading-Gaussian content of (α) used in the speed-equality conclusion is supplied by Proposition 6.12 and is therefore not a conditional input; the conditional dependence of the proposition on $(\gamma)+(\delta)$ is now closed at the leading + sub-leading level by Propositions 6.15 and 6.19, with residual content the full $\mathcal{D}_{\mathrm{stab}}$-boundary verdict for the sub-leading f-tensor invariants of (α) and the $\leq 3.7\%$ Camporesi–Higuchi finite-part bound on (δ).

- *Speed equality.* The combined GW170817 + GRB 170817A observation [51] of a binary neutron-star merger constrains $-3 \times 10^{-15} < (c_g - c)/c < 7 \times 10^{-16}$ over a propagation distance of $\sim 40\,\mathrm{Mpc}$ and a frequency range $\sim 10^2\,\mathrm{Hz}$. The present construction predicts $c_g = c$ structurally (from the leading-Gaussian dispersion of Proposition 6.12), in agreement with the bound to within the quoted precision.

- *Polarisation count.* LIGO/Virgo analyses [52] of the polarisation content of GW150914 set upper limits on scalar and vector polarisations relative to the tensor polarisations of order 10^{-2}, with GW170817 [53] tighter by about an order of magnitude. Detection of a non-tensor polarisation in any future GW signal would falsify Proposition 6.24, identifying which of $(\gamma) + (\delta)$ breaks down.

6.10 What this section establishes, and what is deferred

Structural results independent of any spacetime hypothesis. Three statements of the present section depend only on the algebraic and topological data of EIX and are independent of any spacetime-emergence hypothesis:

- Proposition 6.7: the algebraic substrate $(\mathfrak{a}, [P_\mu, P_\nu] = 0)$ of dimension exactly four with abelian closure, derived from Lemma 6.3, Suter's Lemma 6.4, and Corollary 6.5.
- Theorem 6.21: $\pi_0(\mathrm{EIX}) = \pi_1(\mathrm{EIX}) = \pi_2(\mathrm{EIX}) = 0$.
- Corollary 6.23: no stable monopoles, cosmic strings, or domain walls from the BEC phase transition.

Closure of sub-claims (α), (γ), (δ) at leading + sub-leading order. Proposition 6.12 closes sub-claim (α) at the leading-Gaussian level: the slow-mode two-point Schwinger function on $\mathfrak{a}$ satisfies Osterwalder–Schrader reflection positivity, and the OS theorem [13, 14] delivers a Lorentzian Wightman QFT on $\mathbb{R}^{1,3}$ with Poincaré representation supplied by the theorem; Proposition 6.15 extends this to a representative interior point of the convex stability domain $\mathcal{D}_{\mathrm{stab}}$ of Theorem 3.5 via the Glimm–Jaffe [20] Theorem 6.5.1 sufficiency conditions, with a 6^4 open-time / periodic-space lattice realisation reflection-positive in the Lüscher–Weisz [43] sense. Proposition 6.17 extracts from the same OS reconstruction the global Poincaré subgroup $\mathcal{P}_+^\uparrow \subset \mathrm{Diff}(\mathcal{M}^{1,3})$, closing the global content of (γ) at the leading-Gaussian level; Proposition 6.19 fixes the structural coefficient $\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})} = 432/3 = 144$ of (69) at leading + BV-BRST sub-leading via the Camporesi–Higuchi spectral zeta on the class-one set $\{\mathbf{3875}, \mathbf{27000}\}$, with $\leq 3.7\%$ residual finite-part bound, closing the metric-reconstruction sub-claim (δ) at the same level and extending the full local $\mathrm{Diff}(\mathcal{M}^{1,3})$-content of (γ) by induced gravity. The continuous emergent O(4)-isotropy is preserved at the leading + manifest-positivity sub-leading level (Remark 6.18).

Residual [Open] content. Two analytic completions remain open at the level of the present paper:

- the full $\mathcal{D}_{\mathrm{stab}}$-boundary verdict for the sub-leading f-tensor invariants $\mathcal{S}_c$, $\mathcal{S}_{c'}$, $\mathcal{S}_e$ of the interacting slow-mode action (Remark 6.16); sign-flipping of these invariants reduces the continuous emergent O(4) to its $W(F_4)$ Weyl subgroup without invalidating the leading + manifest-positivity sub-leading verdict of Proposition 6.15;
- the full Mellin continuation of the EIX spectral zeta with explicit Seeley–DeWitt pole subtractions at $s = 56, 55, \ldots, 0$ and the higher-Casimir class-one tail beyond $\{\mathbf{3875}, \mathbf{27000}\}$ (Remark 6.20); the residual finite-part correction to $\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})}$ is bounded conservatively by $\leq 3.7\%$.

Structural result conditional on $(\gamma) + (\delta)$**.** The unification of matter and gravity light cones (Proposition 6.24: $c_g = c$ and two graviton polarisations) is conditional on $(\gamma) + (\delta)$ at the leading + sub-leading level closed by Propositions 6.15 and 6.19; the full $\mathcal{D}_{\text{stab}}$-boundary completion and the full Mellin continuation enter the proof only through their bounded residual contributions.

7 Summary, structural selections, and open problems

The chain of §§2–6 starts from four standard meta-principles M1–M4 of local Lagrangian QFT and four $\mathfrak{e}_8$-specific postulates P1–P4, and produces: (i) the IR action of Theorem 3.5 up to two leading and nine sub-leading Wilson coefficients; (ii) a Bose–Einstein-type symmetry-breaking vacuum on the 247-sphere of $\mathfrak{e}_8$ (Theorem 4.3); (iii) the unique adjoint orbit $\mathcal{O} \cong \text{EIX} = E_8/(E_7 \times \text{SU}(2))$ selected by three operational filters (Theorem 5.9); (iv) a four-dimensional abelian sector $\mathfrak{a} \subset \mathfrak{m}_{\text{EIX}}$ with $\dim \mathfrak{a} = 4$ and $[P_\mu, P_\nu] = 0$ (Proposition 6.7), on which the leading bosonic Gaussian and the leading + sub-leading Sakharov sub-claims of a four-dimensional Lorentzian emergence are closed by Osterwalder–Schrader reconstruction and a Camporesi–Higuchi spectral-zeta computation on EIX.

The present section collects what is established at the level of the present paper, the structural selections made along the chain (where alternative branches are possible but not pursued here), the residual open analytic content, and an explicit list of statements that are *not* claimed.

7.1 Established results

The principal results of the paper, with their dependence on the postulates and on the structural selections of §7.2 below, are:

- *Action uniqueness in the IR* (Theorem 3.5). Under M1–M4 and P1–P4, in the IR regime $\phi_0 \ll \Lambda^{123}$, $\Lambda_0 \ll \Lambda$, the action is fixed up to two leading and nine sub-leading real Wilson coefficients (eq. (22)), with the remainder bounded by (23). Two structural corollaries follow: the symmetric trace D_{ABC} vanishes identically in any representation (Corollary 3.6), and no Yang–Mills-type quartic from a primitive degree-4 Casimir is available at leading order (Corollary 3.7).
- *Existence of a symmetry-breaking vacuum* (Theorem 4.3). On the open half-line $c_2 < 0$, with $c_4 > 0$ from M4, the action of Theorem 3.5 admits a non-trivial global minimum on the round 247-sphere $\{X \in \mathfrak{e}_8 \,|\, \kappa(X, X) = -c_2/(2c_4)\}$, stratified by adjoint-orbit type (Levi sub-root-systems of $\Phi(E_8)$, Proposition 4.4). The associated tree-level fluctuation spectrum within the Cas_2 truncation is collected in Proposition 4.10: $248 - \dim H'$ exact Goldstones, $\dim H' - 1$ accidentally flat "spectator" modes lifted by the higher-Casimir layer, and one massive radial mode (Proposition 4.9).
- *Vacuum-orbit selection* (Theorem 5.9, conditional on the structural input F3 of §5.3). Among the candidates of the Borel–de Siebenthal-type catalogue (Table 6), the algebraic filters F1 (cubic-anomaly safety) and F2 (existence of an SU(2) substrate for a Skyrme-type soliton) reduce the catalogue to $\mathcal{F}_{12} = \{H_1, H_2, H_3, H_6, H_7, H_8\}$; the structural-geometric filter F3 (compact quaternion-Kähler symmetric space) selects H_1, with orbit $\mathcal{O} \cong \text{EIX}$, $\dim_{\mathbb{R}} \text{EIX} = 112$. Standard-Model embeddability is recorded *a posteriori* (Remark 5.8), not used as a filter.
- *Algebraic four-dimensional sector on EIX* (Proposition 6.7, unconditional given the orbit selection). The rank of EIX as a compact symmetric space is 4 (Lemma 6.3, [24, Ch. X]); Suter's rank–antichain identity [12] identifies this rank with the maximum cardinality of a strongly orthogonal antichain in $\Delta(\mathfrak{m}^+_{\text{EIX}})$ (Lemma 6.4, Corollary 6.5). The associated

abelian subspace $\mathfrak{a} = \mathrm{span}_{\mathbb{R}}\{P_0, P_1, P_2, P_3\}$ (Definition 6.6) satisfies $\dim_{\mathbb{R}} \mathfrak{a} = 4$ and $[P_\mu, P_\nu] = 0$, intrinsically on every element of the 630-element antichain set (Remark 6.8).

- *Lower homotopy of EIX and absence of stable defects* (Theorem 6.21 and Corollary 6.23, unconditional on Hypothesis 6.11). The Wolf space EIX satisfies $\pi_0(\mathrm{EIX}) = \pi_1(\mathrm{EIX}) = \pi_2(\mathrm{EIX}) = 0$. By the Kibble [48] classification, the BEC phase transition $E_8 \to E_7 \times \mathrm{SU}(2)$ produces no stable monopoles, cosmic strings, or domain walls in the corresponding co-dimensions, irrespective of any spacetime-emergence hypothesis.
- *Leading-Gaussian Lorentzian QFT on* $\mathfrak{a}$ (Proposition 6.12, with numerical verification [Proven-num]; Proposition 6.17). The slow-mode two-point Schwinger function on $\mathfrak{a} \cong \mathbb{R}^4$, with the kinetic data inherited from Theorem 3.5, satisfies Osterwalder–Schrader axioms E0–E4 [13, 14]; the OS reconstruction theorem then delivers a Lorentzian Wightman QFT with positive-energy spectrum and a strongly continuous representation of the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow \subset \mathrm{Diff}(\mathcal{M}^{1,3})$. This closes the leading-Gaussian content of sub-claims (α) and the global content of (γ) of Hypothesis 6.11.
- *Interacting OS reflection positivity at a* $\mathcal{D}_{\mathrm{stab}}$ *interior point* (Proposition 6.15, [Proven-num, $\mathcal{D}_{\mathrm{stab}}$ interior]). At a representative interior point of the convex stability domain $\mathcal{D}_{\mathrm{stab}}$ of Theorem 3.5 the slow-mode action satisfies the Glimm–Jaffe [20, Theorem 6.5.1] sufficiency conditions for reflection positivity, and a 6^4 open-time/periodic-space lattice realisation is reflection-positive in the Lüscher–Weisz [43] sense. The full $\mathcal{D}_{\mathrm{stab}}$-boundary verdict for the sub-leading f-tensor invariants $\mathcal{S}_c, \mathcal{S}_{c'}, \mathcal{S}_e$ remains [Open] (Remark 6.16).
- *Induced-gravity coefficient on EIX* (Proposition 6.19, [Proven-num, leading + sub-leading]). The structural Sakharov coefficient of (69) is fixed at leading + BV-BRST sub-leading order by the Camporesi–Higuchi spectral zeta [15] on EIX to $\mathcal{V}_{\mathrm{ind}}^{(\mathrm{EIX})} = 432/3 = 144$, with a conservative $\leq 3.7\%$ bound on the residual finite-part correction. This closes the metric-reconstruction sub-claim (δ) at the same level and extends the diffeomorphism sub-claim (γ) to full local $\mathrm{Diff}(\mathcal{M}^{1,3})$-invariance by the standard induced-gravity argument; the Wilsonian UV calibration of M_* from G_N is outside the scope of the present paper.

7.2 Structural selections made along the chain

Several steps in the chain of §§2–6 are structural *selections*: at each such step the postulates admit more than one branch, and the present paper investigates one of them. The selections are recorded together so that the conditional status of each downstream result is visible at a glance.

1. *Conservative reading of the stability/Ostrogradski clause* (M4(c), §2.1). The clause M4(c) "at most first derivatives of the field" is strictly stronger than the Ostrogradski theorem [21, 22] requires: degenerate higher-derivative theories (Galileon, DHOST, $f(R)$) are admissible under the theorem itself but excluded by M4(c). The conservative form is adopted to keep the catalogue of $\mathcal{P}_{n,k}$-monomials in Lemma 3.4 restricted to first-order generators without an auxiliary degeneracy analysis; relaxing M4(c) to the strict Ostrogradski form would re-open the $(2, 4)$ row and the $\mathcal{S}_d = \mathrm{Tr}(L_A L_B \Phi \cdot L^A L^B \Phi)$ generator, contingent on a degeneracy-condition check that is not undertaken here.
2. *Heat-kernel (Gaussian) regularisation of the formal measure* $\mathcal{D}\Phi$ (M1, footnote). The path integral $Z = \int \mathcal{D}\Phi\, e^{iS[\Phi]/\hbar}$ on an $\mathfrak{e}_8$-valued field on a 248-dimensional compact Lie group is non-trivial as a measure-theoretic object; we adopt the standard heat-kernel (Gaussian) regularisation around the leading quadratic kinetic part, in the convention of Oriti [9], Glimm and Jaffe [20]. Alternative regularisations (group-theoretic lattice, spin-foam-type, Boulatov–Ooguri-type combinatorial) are not analysed here; the structural verdicts of §§3–4 are insensitive to the regulator choice within standard EFT power-counting,

but the Camporesi–Higuchi finite-part bound of Proposition 6.19 and the $\mathcal{D}_{\text{stab}}$-interior reflection-positivity verdict of Proposition 6.15 are stated within the heat-kernel scheme adopted here.

3. *Non-identification of E_8^R and E_8^{Ad} (separate vs. diagonal)* (M3, Remark 3.1). The field $\Phi : E_8 \to \mathfrak{e}_8$ admits the canonical action of the trio $E_8^L \times E_8^R \times E_8^{\text{Ad}}$; identifying E_8^R with E_8^{Ad} along the canonical isomorphism $T_A \leftrightarrow T_A$ collapses the symmetry to a single diagonal copy of E_8 and re-admits "divergence-type" invariants such as $L^A\Phi_A$ and $(L^A\Phi_A)^2$ that are forbidden under the separate trio. The non-identification clause of M3 (no two structural factors are silently identified) selects the separate trio, and is the operational reason for the dimension count of the (n, k)-sectors in Lemma 3.4. A diagonal identification, as in some single-domain group field theories, is admissible under M1–M2 alone but yields a strictly larger admissible catalogue and a different action template at $(2, 2)$ and $(4, 2)$.
4. *Compact real form of $\mathfrak{e}_8$* (P2, Remark 2.2). The complex algebra of type E_8 admits three inequivalent real forms: the compact form E_8^c adopted here, the split form $E_{8(8)}$, and the intermediate form $E_{8(-24)}$. The compact-form choice is the minimal one making M4 (positivity of the Hessian) and the finite Haar measure of P4 simultaneously well-posed. The non-compact branches are pursued in parallel programs (Lisi [1], Manogue et al. [2], Wilson [3]; for the no-go critique relevant to those branches see Distler and Garibaldi [5]); we do not analyse them here, and the leading-Gaussian and Camporesi–Higuchi verdicts of §6 do not transfer automatically to them (Remark 6.14).
5. *Field-domain variant* (P3, Remark 2.3). Among the four canonical choices (single copy of E_8 with adjoint values; Boulatov–Ooguri Cartesian power E_8^d [7–9]; algebra-valued field on $\mathfrak{e}_8 \cong \mathbb{R}^{248}$; higher gauge structures), the present paper adopts variant (A). The remaining variants are not excluded but lie outside the scope.
6. *Wilsonian truncation layer $(n, k) \leq (4, 4)$* (Remark 2.4(ii)). The naturalness bound (7) controls the relative size of operators in different (n, k) sectors but does not by itself prescribe a layer to retain. The leading + first-sub-leading layer used in Theorem 3.5 is a modelling choice; a different truncation layer would change the catalogue of surviving operators without affecting the underlying expansion or the master inequality (21).
7. *IR-regime hierarchy $\phi_0 \ll \Lambda^{123}$, $\Lambda_0 \ll \Lambda$* (§3.4, Remark 2.4(i)). The remainder bound (23) of Theorem 3.5 is a formal inequality in the dimensionless ratios ϕ_0/Λ^{123} and Λ_0/Λ; its quantitative interpretation as a Wilsonian hierarchy is conditional on those ratios being matched to physical scales by the emergent-spacetime construction of §6. The present paper works throughout in the smallness regime $\phi_0 \ll \Lambda^{123}$, $\Lambda_0 \ll \Lambda$; outside this regime the leading–sub-leading collapse of the $(n, k) \in \{(2, 0), (2, 2), (4, 0), (4, 2), (4, 4)\}$ sectors is not controlled by (21), and Theorem 3.5 would have to be re-stated relative to a different organising hierarchy.
8. *Symmetry-breaking branch $c_2 < 0$* (Table 3). Of the four phase regimes of the tree-level potential $V_{\text{eff}}(r^2) = c_2\, r^2 + c_4\, r^4$, only the open half-line $\mathcal{R}_{\text{BEC}} = (-\infty, 0)$ is analysed here. The trivial vacuum ($c_2 > 0$), the critical line ($c_2 = 0$), and the stability-violating regime ($c_4 \leq 0$, excluded by (20)) are recorded for completeness but not pursued.
9. *Tree-level* Cas_2 *truncation of the fluctuation spectrum* (§4.5, Remark 4.8). The vacuum analysis of §4 works with the leading potential $V_{\text{eff}} = c_2\, C_2 + c_4\, C_2^2$. The $\dim H' - 1$ "spectator" flat directions in $\mathfrak{s} = \mathfrak{h}' \cap \hat{\Phi}_0^{\perp}$ are lifted by the higher primitive Casimirs $\text{Cas}_8, \ldots, \text{Cas}_{30}$ of $\mathfrak{e}_8$; a structural expectation derived from the absence of those Casimirs in the truncation, but not from an explicit Cas_8-Hessian computation (Remark 6.2(b)).
10. *Borel–de Siebenthal-type catalogue as input to orbit selection* (Lemma 5.1, Remark 5.2).

The vacuum-orbit selection of Theorem 5.9 is stated relative to the catalogue of Table 6: the maximal-rank closed connected subgroups of E_8 from Borel and de Siebenthal [31], Dynkin [32], the unique non-regular maximal subgroup $G_2 \times F_4$, and three sub-maximal candidates retained for the elimination steps. Rigorous completeness against arbitrary closed subgroups of E_8 with the specific structural input used in F1–F3 is not claimed; finer position-dependent distinctions inside one row (e.g. different Weyl chambers for a regular Φ_0) are absorbed by Weyl-group conjugation and do not generate new candidates.

11. *Operational form of filter F2 vs. strict topological F2* (§5.5). The strict topological reading of F2, based on $\pi_3(\mathcal{O}) = \pi_3(E_8/H)$, fails on every full-rank non-toroidal candidate of Proposition 5.5 (the Dynkin index gcd is 1 in each case, giving $\pi_3 = 0$); a strict reading would therefore eliminate every non-toroidal stabilizer and leave only the unphysical abelian limit H_T. We replace strict F2 by the operational statement: existence of an SU(2) subgroup of H acting non-trivially on the tangent module $\mathfrak{m}$, sufficient for the hedgehog-ansatz Skyrmion construction of Manton and Sutcliffe [37]. The replacement is a modelling choice; the strict topological reading is not pursued here and would require a different (e.g. mod-N-quantised) winding sector.

12. *Filter F3 as a structural input* (§5.3, Open problem 5.3). Filters F1 (anomaly safety) and F2 (Skyrme-substrate) are derivable from M1 + M4; F3 (Wolf-space quaternion-Kähler structure) is adopted here as a structural-geometric input. The unique-orbit conclusion of Theorem 5.9(b) is consequently conditional on F3; without F3 the construction is left with the six-element candidate set $\mathcal{F}_{12}$. Two alternative routes to H_1-uniqueness orthogonally to F3 (one-loop Coleman–Weinberg lifting of the orbit degeneracy on $\mathbf{S}^7_{r_*}$; spectral stability of Gaussian fluctuations) are recorded in Open problem 5.3 and not pursued.

13. *Sigma-model carrier on EIX vs. the full BEC orbit* (Remark 6.1). The orbit selection of §5 identifies the 112-dimensional Wolf space EIX through filter F3 applied to the *nominal* Borel–de Siebenthal stabiliser $H = E_7 \times \mathrm{SU}(2)/\mathbb{Z}_2$, while the *actual* stabiliser of any $\Phi_0 \in \mathfrak{e}_8$ in the BEC vacuum of Theorem 4.3 is the co-rank-1 Levi $H' = E_7 \times \mathrm{U}(1)$ with $\dim(E_8/H') = 114 = 112 + 2$, fibred as an $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ bundle over EIX. The quaternion-Kähler structure F3, the rank-antichain identity of Lemma 6.4, and the abelian sector $\mathfrak{a}$ of Definition 6.6 are properties of the 112-dimensional base EIX; §6 accordingly adopts EIX (and not the full 114-dimensional BEC orbit) as the geometric carrier of the sigma-model and induced-gravity constructions. The two extra S^2-fibre Goldstones are orthogonal to $\mathfrak{m}_{\mathrm{EIX}}$ and decouple from $\mathfrak{a}$; the structural verdicts of Propositions 6.12, 6.17, and 6.19 are unchanged across the two pictures.

14. *Antichain representative on EIX* (Remark 6.8). The four candidate translation generators $\{P_\mu\}_{\mu=0,1,2,3} \subset \mathfrak{m}_{\mathrm{EIX}}$ are defined relative to one element of the 630-element set of maximal antichains in $\Delta(\mathfrak{m}^+_{\mathrm{EIX}})$. The analytic transitivity of the unbroken $E_7 \times \mathrm{SU}(2)$-action on this set is not established here; constancy of the operational verdicts of §6 across the set is checked by an explicit 630-element numerical sweep (`e3_antichain_full_sweep.py`).

15. *Closure level of sub-claims* (α), (γ), (δ) *of Hypothesis 6.11* (§§6.5–6.7). The sub-claims are closed at the leading bosonic Gaussian and the leading + BV-BRST sub-leading Sakharov levels respectively. Two analytic completions remain open (§7.3 below); both have operational bounds and do not invalidate the closure at the announced level.

7.3 Open analytic completions

The principal residual analytic content recorded in the body of the paper is:

- Full $\mathcal{D}_{\mathrm{stab}}$-boundary verdict for the sub-leading f-tensor invariants $\mathcal{S}_c, \mathcal{S}_{c'}, \mathcal{S}_e$ of the slow-mode action (Remark 6.16); sign-flipping of the corresponding coefficients reduces the continuous emergent O(4)-isotropy to its $W(F_4)$ Weyl subgroup without invalidating the leading + manifest-positivity sub-leading verdict of Proposition 6.15. The full interacting-lattice Monte Carlo on the 6^4 open-time realisation is left as future work.

- Full Mellin continuation of the EIX spectral zeta with explicit Seeley–DeWitt pole subtractions at $s = 56, 55, \ldots, 0$ and the higher-Casimir class-one tail beyond $\{\mathbf{3875}, \mathbf{27000}\}$ (Remark 6.20); the residual finite-part correction to $\mathcal{V}_{\text{ind}}^{(\text{EIX})} = 144$ is bounded conservatively by $\leq 3.7\%$.
- One-loop Coleman–Weinberg lifting of the tree-level orbit degeneracy on $\mathbf{S}^7_{r_*}$, or alternatively a Gaussian-fluctuation spectral-stability argument, that would derive the conclusion of Theorem 5.9(b) from M1 + M4 alone, without invoking F3 (Open problem 5.3).
- Wilsonian UV calibration of the cutoff M_* in the induced-gravity formula (69) from the empirical Newton constant G_N, which fixes the empirical normalisation of $G_N^{(\text{ind})}$ separately from $\mathcal{V}_{\text{ind}}^{(\text{EIX})}$.

7.4 What this paper does not claim

To pre-empt over-reading of the chain §§2–6, we list explicitly the statements that are *not* claimed here.

- *No derivation of the Standard Model.* The selected stabilizer $H_1 = E_7 \times \mathrm{SU}(2)/\mathbb{Z}_2$ admits the classical algebraic chain $G_{\text{SM}} \subset \mathrm{SU}(5) \subset \mathrm{Spin}(10) \subset E_6 \subset E_7 \subset H_1$ (Slansky [16]); this is recorded *a posteriori* as a downstream consistency check (Remark 5.8), not as a filter or as a prediction. The dynamical realisation of G_{SM} as the unbroken low-energy gauge group, the matter representation content, and the fermion-mass spectrum are subjects of subsequent work and are not addressed here.
- *No claim of E_8 Yang–Mills unification.* The present construction is a group field theory for an adjoint-valued scalar $\Phi : E_8 \to \mathfrak{e}_8$, not an E_8 Yang–Mills gauge theory of the type analysed in Lisi [1], Distler and Garibaldi [5]; the Distler–Garibaldi no-go theorem on E_8 Yang–Mills embeddings of three Standard Model generations therefore does not address the present construction. We remain agnostic on the question of an E_8-internal-symmetry interpretation of the unbroken sector.
- *No unconditional four-dimensional Lorentzian emergence.* The promotion of the algebraic substrate $(\mathfrak{a}, [P_\mu, P_\nu] = 0)$ to a smooth Lorentzian four-manifold is recorded as Hypothesis 6.11, not as a theorem. Sub-claims (β) (dimension equals four) and (ε) (abelian closure) are unconditional consequences of Proposition 6.7; sub-claims (α) (Lorentzian signature), (γ) (diffeomorphism invariance), and (δ) (metric reconstruction) are closed at the levels recorded above and remain open at the residual levels of §7.3.
- *No prediction of the Higgs vacuum value c_2.* Theorem 4.3 establishes existence of a non-trivial vacuum on the open parameter region $c_2 \in \mathcal{R}_{\text{BEC}}$; the value of c_2 in the realised phase is fixed by experiment, in the same sense in which the Higgs vacuum expectation $v = 246\,\text{GeV}$ in the Standard Model is fixed by experiment rather than predicted by the form of the Higgs potential.
- *No prediction of the Newton constant.* The structural coefficient $\mathcal{V}_{\text{ind}}^{(\text{EIX})} = 144$ of Proposition 6.19 fixes the geometric contribution to the induced-gravity coefficient of (69); the empirical value of $G_N^{(\text{ind})}$ depends on a separate Wilsonian UV calibration of M_* which is not undertaken here.
- *No claim about the cosmological constant.* The structural decomposition $D = \dim_{\mathbb{R}} \mathfrak{e}_8 =$

$4 + 244$ of Remark 6.10 is recorded as a forward pointer to a candidate non-perturbative suppression $\Lambda\,\ell_P^2 \sim \pi^{-244}$, not as a claim of the present paper; the corresponding two open computations on EIX are deferred.

- *No completeness against unrecorded sub-maximal embeddings.* Theorem 5.9 is stated relative to the catalogue of Table 6 (maximal-rank closed connected subgroups of E_8 together with the unique non-regular maximal subgroup $G_2 \times F_4$ and three sub-maximal candidates retained for the elimination steps). Rigorous completeness against arbitrary closed subgroups of E_8 with the specific structural input used in F1–F3 is not claimed (Remark 5.2).

7.5 Outlook

The framework leaves three forward directions on which the present paper is silent. The Standard-Model-embedding consequences of the selected H_1 chain, the matter representation content, and the Yukawa structure are the subject of separate work. The dynamical realisation of an explicit emergent metric reconstruction beyond the leading + sub-leading Sakharov closure of §6.7, including the Wilsonian calibration of M_* from G_N, is the subject of a separate analysis. Finally, the parallel non-compact E_8 programs catalogued in Remark 2.2 are not in tension with the present construction: they correspond to distinct real-form choices in P2 and would require a separate derivation. The present paper records the foundations of one specific branch and the structural selections made along its chain; each open completion above is a delimited analytic question that does not retroactively affect the closure of the results recorded in §7.1.

Code and data availability

The numerical verification scripts referenced throughout the paper as [Proven-num] certificates (action-uniqueness rank tests on $(n, k) \leq (4, 4)$, BEC vacuum, antichain enumeration on $\Delta(\mathfrak{m}^+_{\mathrm{EIX}})$, Volovik signature on $\mathfrak{a}$, leading-Gaussian and $\mathcal{D}_{\mathrm{stab}}$-interior Osterwalder–Schrader reflection positivity, Camporesi–Higuchi spectral-zeta computation of $\mathcal{V}^{(\mathrm{EIX})}_{\mathrm{ind}}$, and the lower-homotopy check on EIX) are provided as supplementary material at

https://github.com/lukasbednarik/E8-GFT

together with the supporting `e8sim` library. The script-to-claim map (each script tagged as *primary* or *additional* for the corresponding theorem, lemma, or proposition) is recorded in the repository `README`; the analytical proofs remain authoritative, and the scripts are intended as independent numerical checks rather than as a substitute.

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A Conventions and notation

This appendix collects the conventions and notation used throughout the paper. The intent is that any single section can be read against this appendix without consulting the others. The appendix is organised by topic: units and signature (§A.1) and Lie-algebra normalisation (§A.2).

A.1 Units, indices, and signature

Units. Natural units, $\hbar = c = 1$. The Planck length is $\ell_P = \sqrt{\hbar G/c^3}$ and the Planck mass $M_P = \sqrt{\hbar c/G}$; with $\hbar = c = 1$ both reduce to $\ell_P = 1/M_P = 1/\sqrt{G}$.

Metric signature. Mostly-plus, $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$.

Indices. Greek indices $\mu, \nu, \dots \in \{0,1,2,3\}$ label four-dimensional spacetime; capital Latin indices $A, B, \dots \in \{1,\dots,248\}$ label the adjoint $\mathfrak{e}_8$ representation; lower-case Latin indices $a, b, \dots \in \{1,\dots,3\}$ are reserved for SU(2) when explicit decompositions are required.

Wick rotation. Whenever a Wick-rotated form of an action or a Hessian is used, the rotation is the standard $t = -i\tau$ continuation. The Lorentzian or Euclidean status of every formula is stated explicitly at first appearance.

A.2 Lie-algebra normalisation

Real form and group. Throughout the paper $\mathfrak{e}_8$ denotes the compact real form of the exceptional Lie algebra of type E_8 (P2). The corresponding compact, simply connected, centreless Lie group is denoted E_8.

Killing form and inner product. The Killing form is $\mathrm{Kill}(X,Y) := \mathrm{Tr}(\mathrm{ad}_X\,\mathrm{ad}_Y)$. We work with the rescaled, positive-definite form

$$\kappa(X,Y) \;:=\; -\frac{1}{h^\vee_{E_8}}\,\mathrm{Kill}(X,Y), \qquad h^\vee_{E_8} = 30, \tag{72}$$

in the standard normalisation of Slansky [16]. We choose a basis $\{T_A\}$ of $\mathfrak{e}_8$ such that $\kappa(T_A,T_B) = \delta_{AB}$; the structure constants $f^A{}_{BC}$ defined by $[T_B,T_C] = f^A{}_{BC}\,T_A$ are then totally antisymmetric in (A,B,C).

Quadratic Casimir. For an $\mathfrak{e}_8$-valued field Φ^A we write

$$C_2(\Phi) \;:=\; \kappa_{AB}\,\Phi^A\Phi^B \;=\; \kappa(\Phi,\Phi). \tag{73}$$

Inner product on $\mathfrak{e}_8$-valued fields. The pairing on $\mathfrak{e}_8$-valued fields on E_8 uses the bi-invariant Haar measure together with κ:

$$\langle \Phi,\, \Psi\rangle_{L^2} \;:=\; \int_{E_8} \mathrm{d}\mu_{\mathrm{Haar}}(g)\;\kappa(\Phi(g),\Psi(g)), \qquad \int_{E_8} \mathrm{d}\mu_{\mathrm{Haar}} = 1. \tag{74}$$

The Haar measure is normalised to unit total volume.

B Proof of the action-uniqueness theorem

This appendix supplies the technical content underlying Theorem 3.5 of Section 3. We proceed in the same order as the chapter and prove, in turn, Lemmas 3.2 and 3.3 (§B.1); the technical content of Lemma 3.4 not covered inline in §3.2, including the Cauchy–Howe plethysm computations of $\dim^{(4,4)}$ and $\dim^{(4,2)}$ (§B.2); the explicit construction of $\mathcal{S}_b$ and the independence of $\mathcal{S}_c$ from $\mathcal{S}_{c'}$ (§B.3); the Ostrogradski exclusion of $\mathcal{S}_d$ (§B.4); the strict sign restrictions (§B.5); the convex stability domain $\mathcal{D}_{\mathrm{stab}}$ (§B.6); and the aggregated remainder bound (23) (§B.7).

Throughout the appendix we use the conventions of Appendix A: an orthonormal basis $\{T_A\}$ of $\mathfrak{e}_8$ with $\kappa_{AB} = \delta_{AB}$ and totally antisymmetric f_{ABC}, the field $\Phi(g) = \Phi^A(g)\,T_A$ of P3, the left-invariant derivatives L_A of (8), and the shorthand $M_{AB}(g) := (L_A\Phi)^B(g)$ for the matrix of first derivatives.

B.1 Primitive tensors and Casimir degrees

Proof of Lemma 3.2. Let $T \in \mathrm{Sym}^2(\mathfrak{e}_8^*)$ be an Ad-invariant symmetric bilinear form. Since the adjoint representation of $\mathfrak{e}_8$ is irreducible, Schur's lemma forces $T = c\,\kappa$ for some $c \in \mathbb{R}$. Hence κ is the unique primitive Ad-invariant symmetric bilinear form, up to scale. The structure constants f_{ABC} are antisymmetric in the first two indices by definition $[T_A, T_B] = f_{AB}{}^C T_C$; total antisymmetry follows from Ad-invariance of κ, which gives $\kappa([X,Y],Z) + \kappa(Y,[X,Z]) = 0$, equivalently $f_{ABC} = -f_{ACB}$. The absence of a primitive symmetric Ad-invariant 3-tensor on $\mathfrak{e}_8$ follows from the Casimir spectrum (9) of Lemma 3.3, which contains no entry in degree 3: a primitive symmetric Ad-invariant 3-tensor would generate a primitive Casimir of degree 3 via $\Phi^A\Phi^B\Phi^C\,T_{ABC}$, in contradiction with the Chevalley restriction theorem. □

Proof of Lemma 3.3. By the Chevalley restriction theorem [6], the algebra of Ad-invariant polynomial functions on a simple Lie algebra $\mathfrak{g}$ is isomorphic to the Weyl-invariant polynomial algebra on a Cartan subalgebra, and the latter is a free polynomial algebra in rank $\mathfrak{g}$ generators of degrees $e_i + 1$, with e_i the exponents of $\mathfrak{g}$. The E_8 exponents are $\{1, 7, 11, 13, 17, 19, 23, 29\}$ (Bourbaki [6], Plate VII), giving the eight degrees in (9). □

B.2 Proof of Lemma 3.4 via Cauchy–Howe plethysm

Proof of Lemma 3.4. We establish Table 1 row by row. The vanishing rows and the small-dimension rows are settled first; the two non-trivial rows $(4,4)$ and $(4,2)$ are then handled by Cauchy–Howe plethysm; the $(2,4)$ row is eliminated by the Ostrogradski argument of §B.4.

Vanishing rows $(3,0)$ *and* $(3,2)$. For n odd, every Ad-invariant contraction of n factors Φ (and at most two factors $L_A\Phi$) must use either κ alone or κ together with one factor of f_{ABC}: in the truncation $(n,k) \le (4,4)$ the total internal-index rank is at most $n + k \le 5$, and by Lemmas 3.2 and 3.3 the only primitive Ad-invariant tensors of $\mathfrak{e}_8$ of rank ≤ 5 are κ (rank 2) and f_{ABC} (rank 3); the next primitive Casimir has degree 8 and contributes only at total rank ≥ 8, beyond the present truncation. A contraction with κ alone reduces a rank-n symmetric tensor by two indices and leaves a free index, so it cannot contract to a scalar for n odd. A contraction containing one factor of f_{ABC} saturates f against a totally symmetric combination $\Phi^A\Phi^B\Phi^C$ (or its derivative analogue), and the contraction of an antisymmetric and a symmetric rank-3 tensor vanishes identically. The same argument applies to $(n,2)$ with n odd, since the two derivatives are exchanged symmetrically by the Sym-projection on $L_A\Phi$ pairs. Hence $\dim^{(n,k)} = 0$ for n odd and $k \in \{0,2\}$, including the entries $(3,0)$ and $(3,2)$ of Table 1.

Small-dimension rows $(2,0)$*,* $(4,0)$*,* $(2,2)$. The $(2,0)$ sector is one-dimensional with generator C_2 by Schur's lemma. The $(4,0)$ sector is one-dimensional: the only Ad-invariant scalar in $\mathrm{Sym}^4(\mathfrak{e}_8^*)$ is C_2^2, because Lemma 3.3 contains no primitive Casimir of degree 4, and the algebra of Ad-invariants is a polynomial algebra on the primitive generators. The $(2,2)$ sector is one-dimensional: the only Ad-invariant contraction of two factors $L_A\Phi$ that uses one κ on the manifold indices and one κ on the internal indices is $\mathcal{H}_2$ of (11); an f-contraction on either side is excluded by rank mismatch (f_{ABC} has rank 3 whereas only two indices are available on each of the manifold and internal sides).

Setup for $(4,4)$ **and** $(4,2)$**.** For the four-derivative sectors we use Cauchy–Howe duality on the algebra of $E_8 \times E_8$-equivariant tensors. Each derivative $L_A\Phi$ contributes one E_8^R index and one E_8^{Ad} index, i.e. a vector in $V_R \otimes V_{\mathrm{Ad}}$ with $V_R = V_{\mathrm{Ad}} = \mathrm{adj}\,\mathfrak{e}_8$. For an (n,n)-sector the n copies of $L_A\Phi$ are interchangeable and the relevant invariant subspace is $\mathrm{Inv}_{E_8^R \times E_8^{\mathrm{Ad}}}(\mathrm{Sym}^n(V_R \otimes V_{\mathrm{Ad}}))$; mixed (n,n)-distributions in which one or more derivatives are collected on a single field factor (e.g. $\Phi^2(L_AL_B\Phi)^2$ or $\Phi(L_A\Phi)(L_BL_C\Phi)(L_D\Phi)$ at $(n,k) = (4,4)$) contain a factor $L_AL_B\Phi$ and are eliminated by M4(c) prior to the count, on the same footing as the $(2,4)$ row. Cauchy's identity for symmetric functions, in the Howe-duality formulation [54], gives the $\mathrm{GL} \times \mathrm{GL}$-equivariant decomposition

$$\mathrm{Sym}^n(V_R \otimes V_{\mathrm{Ad}}) \;=\; \bigoplus_{\lambda \vdash n} S^\lambda(V_R) \otimes S^\lambda(V_{\mathrm{Ad}}), \tag{75}$$

where S^λ is the Schur functor associated with the partition $\lambda \vdash n$. Restricting to $E_8 \subset \mathrm{GL}$ on each factor and counting trivial isotypic components,

$$\dim^{(n,n)} \;=\; \sum_{\lambda \vdash n} \Big[\dim \mathrm{Inv}_{\mathfrak{e}_8}(S^\lambda \,\mathrm{adj}\,\mathfrak{e}_8)\Big]^2. \tag{76}$$

Inputs. We use three inputs from the representation theory of E_8:

(i) The decomposition of the symmetric square of the adjoint, due to Slansky [16] and McKay and Patera [27],

$$\mathrm{Sym}^2 \,\mathrm{adj}\,\mathfrak{e}_8 \;=\; \mathbf{1} \oplus \mathbf{3875} \oplus \mathbf{27000}; \tag{77}$$

(ii) The decomposition of the antisymmetric square,

$$\Lambda^2 \,\mathrm{adj}\,\mathfrak{e}_8 \;=\; \mathbf{248} \oplus \mathbf{30380}; \tag{78}$$

(iii) The Frobenius–Schur indicator of every $\mathfrak{e}_8$-irrep is $+1$ (every irrep is real-orthogonal): $w_0 = -1$ together with trivial centre gives self-duality $V \cong V^*$, and the orthogonal (rather than symplectic) case follows from the explicit reality classification of the simply-connected exceptional groups; see Adams [23]. Consequently, the multiplicity of the trivial representation in $S^\lambda \,\mathrm{adj}\,\mathfrak{e}_8$ equals the multiplicity of S^λ in the corresponding symmetric or antisymmetric tensor power.

Evaluation of $\dim^{(4,4)}$**.** We evaluate (76) for $n = 4$ partition by partition.

- $\lambda = (4)$: $S^{(4)}\,\mathrm{adj} = \mathrm{Sym}^4\,\mathrm{adj}$. From (77) and the polynomial-algebra structure of Ad-invariants (Lemma 3.3), the only Ad-invariant scalar in $\mathrm{Sym}^4\,\mathrm{adj}$ is C_2^2, since there is no primitive Casimir of degree 3 or 4. Hence $\dim \mathrm{Inv}(\mathrm{Sym}^4\,\mathrm{adj}) = 1$.
- $\lambda = (1^4)$: $S^{(1^4)}\,\mathrm{adj} = \Lambda^4\,\mathrm{adj}$. The exterior algebra of Ad-invariants on $\mathfrak{e}_8$ is $H^*(E_8;\mathbb{R}) = \Lambda[x_3, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}, x_{59}]$ with generator degrees $\{3, 15, 23, 27, 35, 39, 47, 59\}$ (Mimura–Toda [28]); since no subset of these generator degrees sums to 4, the Betti number satisfies $H^4(E_8;\mathbb{R}) = 0$, equivalently $\Lambda^4\,\mathrm{adj}$ contains no trivial representation. Hence $\dim \mathrm{Inv}(\Lambda^4\,\mathrm{adj}) = 0$.
- $\lambda = (2,2)$: the corresponding Schur functor is $S^{(2,2)}\,V = \ker(\mathrm{Sym}^2(\mathrm{Sym}^2\,V) \to \mathrm{Sym}^4\,V)$, equivalently $\mathrm{Sym}^2(\mathrm{Sym}^2\,V) = \mathrm{Sym}^4\,V \oplus S^{(2,2)}V$. Combining with (77),

  $$\mathrm{Sym}^2(\mathrm{Sym}^2\,\mathrm{adj}) \;=\; \mathrm{Sym}^2(\mathbf{1} \oplus \mathbf{3875} \oplus \mathbf{27000}).$$

  Counting trivial multiplicities: $\mathrm{Sym}^2(\mathbf{1}) = \mathbf{1}$ contributes one trivial; $\mathbf{1} \otimes \mathbf{3875}$ and $\mathbf{1} \otimes \mathbf{27000}$ contribute zero trivials each; $\mathrm{Sym}^2(\mathbf{3875})$ contains the trivial with multiplicity 1 (the squared Killing form on the irrep $\mathbf{3875}$); $\mathrm{Sym}^2(\mathbf{27000})$ contains the trivial with multiplicity 1; $\mathbf{3875} \otimes \mathbf{27000}$ has no trivial since $\mathbf{3875} \not\cong \mathbf{27000}^* \cong \mathbf{27000}$ in the trivial component. Subtracting the $\mathrm{Sym}^4\,\mathrm{adj}$ contribution ($\mathrm{Inv} = 1$) leaves $\dim \mathrm{Inv}(S^{(2,2)}\,\mathrm{adj}) = 3 - 1 = 2$.

- $\lambda = (3,1)$: $S^{(3,1)}\,V \subset \mathrm{Sym}^2(V) \otimes \Lambda^2(V)$. Trivial components of $\mathrm{Sym}^2\,\mathrm{adj} \otimes \Lambda^2\,\mathrm{adj}$ are matchings of an irrep in (77) with the same irrep in (78). The two lists share no irrep ($\{\mathbf{1}, \mathbf{3875}, \mathbf{27000}\}$ vs. $\{\mathbf{248}, \mathbf{30380}\}$), so the trivial multiplicity is zero. Hence $\dim \mathrm{Inv}(S^{(3,1)}\,\mathrm{adj}) = 0$.
- $\lambda = (2,1,1)$: by the same argument applied to $S^{(2,1,1)}\,V \subset \Lambda^2(V) \otimes \mathrm{Sym}^2(V)$, the trivial multiplicity vanishes: $\dim \mathrm{Inv}(S^{(2,1,1)}\,\mathrm{adj}) = 0$.

Substituting into (76),

$$\dim^{(4,4)} \;=\; \underbrace{1^2}_{\lambda=(4)} + \underbrace{0^2}_{(3,1)} + \underbrace{2^2}_{(2,2)} + \underbrace{0^2}_{(2,1,1)} + \underbrace{0^2}_{(1^4)} \;=\; 5. \tag{79}$$

The five generators $\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_c, \mathcal{S}_{c'}, \mathcal{S}_e$ of (14)–(18) populate the count: the $\lambda = (4)$ trivial is realised by $\mathcal{S}_a = \mathcal{H}_2^2$, and the four $\lambda = (2,2)$ trivials are realised by $\mathcal{S}_b, \mathcal{S}_c, \mathcal{S}_{c'}, \mathcal{S}_e$ (the explicit identification follows the symmetric/swapped-index classification of §B.3).

Evaluation of $\dim^{(4,2)}$. The $(4,2)$ sector has bare degree 2 in Φ and derivative degree 2 in $L_A\Phi$, with both pairs symmetrised. The relevant invariant subspace is

$$\mathrm{Inv}_{E_8^R \times E_8^{\mathrm{Ad}}}\Big(\mathrm{Sym}^2\,V_{\mathrm{Ad}} \otimes \mathrm{Sym}^2(V_R \otimes V_{\mathrm{Ad}})\Big), \tag{80}$$

where the bare-field factor carries the E_8^{Ad}-action only and Cauchy (75) restricted to $n = 2$ (the only partitions of 2 are (2) and $(1,1)$, with $S^{(2)} = \mathrm{Sym}^2$ and $S^{(1,1)} = \Lambda^2$) splits the derivative factor as $\mathrm{Sym}^2\,V_R \otimes \mathrm{Sym}^2\,V_{\mathrm{Ad}} \oplus \Lambda^2 V_R \otimes \Lambda^2 V_{\mathrm{Ad}}$. Because V_R and V_{Ad} carry independent E_8-actions, joint invariants of either summand factor as $\mathrm{Inv}_{E_8}(W_R) \otimes \mathrm{Inv}_{E_8}(\mathrm{Sym}^2\,V_{\mathrm{Ad}} \otimes W_{\mathrm{Ad}})$.

- $\mathrm{Sym}^2\,V_R$-branch. $\mathrm{Inv}_{E_8}(\mathrm{Sym}^2\,V_R) = 1$ (the $\mathbf{1}$ of (77), spanned by $\kappa^{AA'}$). On the V_{Ad} side the trivial multiplicity in $\mathrm{Sym}^2\,\mathrm{adj} \otimes \mathrm{Sym}^2\,\mathrm{adj} = (\mathbf{1} \oplus \mathbf{3875} \oplus \mathbf{27000})^{\otimes 2}$ equals 3 by Frobenius–Schur self-duality (input (iii) above): one diagonal pairing per shared irrep. Subtotal $1 \times 3 = 3$, realised by $C_2\mathcal{H}_2$ ($\mathbf{1}$-pairing), $\mathcal{H}_2^{\mathrm{grad}}$ ($\mathrm{Sym}^2\,V_{\mathrm{Ad}}$ trace pairing), and $\mathcal{H}_2^{\mathrm{mix}}$ (f-mediated pairing) of (11)–(13).
- $\Lambda^2 V_R$-branch. $\mathrm{Inv}_{E_8}(\Lambda^2 V_R) = 0$ since $\Lambda^2\mathrm{adj} = \mathbf{248} \oplus \mathbf{30380}$ contains no trivial (78). Subtotal 0.

Summing, $\dim^{(4,2)} \le 3{+}0 = 3$. The matching lower bound $\dim^{(4,2)} \ge 3$ is realised explicitly by the three generators just named, whose linear independence is verified numerically by a rank test on random samples of (Φ, M) (function `test_4_2_classification` of script `e1_open_points.py`), giving

$$\dim^{(4,2)} \;=\; 3. \tag{81}$$

Vanishing of $(2,4)$ before stability. The same plethysm applied to the $(2,4)$ sector (two factors Φ and two factors $L_A L_B \Phi$) gives a non-zero invariant count. Every generator of this sector contains a factor $L_A L_B \Phi$, however, and is therefore excluded from $\mathcal{L}$ by M4(c) via Ostrogradski (§B.4). The $(2,4)$ row of Table 1 is consequently marked as eliminated.

Conclusion. Combining the vanishing rows $(3,0)$, $(3,2)$ and the small-dimension rows $(2,0)$, $(4,0)$, $(2,2)$ established above, together with the values $\dim^{(4,4)} = 5$ and $\dim^{(4,2)} = 3$ from the Cauchy–Howe plethysm and the elimination of the $(2,4)$ row by M4(c), Table 1 is established in full. □

Additional numerical verification of the $(4,4)$ plethysm bound. The upper bound $\dim^{(4,4)} \le 5$ from the Cauchy–Howe identity (79) is checked independently via the analytic plethysm of $f \times f$ in script `e1_o5_plethysm.py`.

B.3 Definition of $\mathcal{S}_b$ and independence of $\mathcal{S}_c, \mathcal{S}_{c'}$

Explicit definition of $\mathcal{S}_b$. On the four-tensor space spanned by all κ-only contractions of $M_{AB}\, M_{A'B'}\, M_{CD}\, M_{C'D'}$ symmetric under permutation of the four copies of M, there are two linearly independent generators: the double-trace $\mathcal{S}_a = \mathcal{H}_2^2$ and a second contraction we call $\widetilde{\mathcal{S}}_b$, given by the single-trace pattern

$$\widetilde{\mathcal{S}}_b \;:=\; \kappa^{AA'}\kappa^{CC'}\kappa_{BD}\kappa_{B'D'}\, M_{AB}\, M_{A'B'}\, M_{CD}\, M_{C'D'} \;=\; \mathrm{Tr}(M^\top M M^\top M). \tag{82}$$

The operator $\mathcal{S}_b$ of (15) is the projection of $\widetilde{\mathcal{S}}_b$ onto the $\lambda = (2,2)$ isotypic component of the κ-only sector under the Cauchy–Howe S_4-action on the four copies of M_{AB}:

$$\mathcal{S}_b \;:=\; \mathbb{P}_{(2,2)}\, \widetilde{\mathcal{S}}_b \;=\; \widetilde{\mathcal{S}}_b \;-\; c\, \mathcal{S}_a, \tag{83}$$

where $\mathbb{P}_{(2,2)} \in \mathbb{Q}[S_4]$ is the central idempotent onto $S^{(2,2)}$ and $c \in \mathbb{Q}$ is the unique constant such that the $\lambda = (4)$ component of $\widetilde{\mathcal{S}}_b - c\,\mathcal{S}_a$ vanishes. The decomposition into $\lambda = (4)$ (spanned by $\mathcal{S}_a$) and $\lambda = (2,2)$ (spanned by $\mathcal{S}_b$) is canonical because $S^{(4)}$ and $S^{(2,2)}$ are inequivalent S_4-irreps; in particular c is fixed by the representation theory alone and is configuration independent, so $\mathcal{S}_b$ is a local degree-four operator and κ_4', κ_4'' attach to a fixed local basis $\{\mathcal{S}_a, \mathcal{S}_b\}$ as required by M2. This is *not* an $L^2(E_8)$ Gram–Schmidt orthogonalisation, which would make c a non-local functional $c[\Phi]$.

Independence of $\mathcal{S}_c$ and $\mathcal{S}_{c'}$. Define the involution $\sigma : (A, B) \leftrightarrow (B, A)$ that exchanges the manifold and internal index types, equivalently the canonical isomorphism between E_8^R and E_8^{Ad} along $T_A \leftrightarrow T_A$. From definitions (16), (17),

$$\sigma : \mathcal{S}_c \;\longleftrightarrow\; \mathcal{S}_{c'}, \qquad \sigma : \mathcal{S}_a \;\mapsto\; \mathcal{S}_a, \qquad \sigma : \mathcal{S}_b \;\mapsto\; \mathcal{S}_b, \qquad \sigma : \mathcal{S}_e \;\mapsto\; \mathcal{S}_e, \tag{84}$$

since $\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_e$ are σ-symmetric (their tensor patterns use κ or f symmetrically on both sides) and $\mathcal{S}_c, \mathcal{S}_{c'}$ differ exactly by the $f \leftrightarrow \kappa$ swap on the two index types. Decomposing $\mathcal{S}_c, \mathcal{S}_{c'}$ into σ-eigenstates,

$$\mathcal{S}_\pm \;:=\; \tfrac{1}{2}(\mathcal{S}_c \pm \mathcal{S}_{c'}), \qquad \sigma\, \mathcal{S}_\pm = \pm\, \mathcal{S}_\pm, \tag{85}$$

$\mathcal{S}_-$ is in the σ-antisymmetric subspace, which is orthogonal to σ-symmetric operators $\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_e$. Hence $\mathcal{S}_-$ is linearly independent from $\{\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_e\}$, and equivalently $\mathcal{S}_c, \mathcal{S}_{c'}$ are linearly independent from each other in the $(4,4)$-invariant subspace.

σ-eigenspace dimensions. Writing $A_\lambda := \mathrm{Inv}_{\mathfrak{e}_8}(S^\lambda\, \mathrm{adj})$, the σ-action on the λ-isotypic $A_\lambda \otimes A_\lambda \subset S^\lambda V_R \otimes S^\lambda V_{\mathrm{Ad}}$ of (75) is the swap of the two tensor factors, which decomposes as $\mathrm{Sym}^2(A_\lambda) \oplus \Lambda^2(A_\lambda)$. For $\lambda = (4)$, $\dim A_{(4)} = 1$ contributes one σ-symmetric trivial and zero σ-antisymmetric. For $\lambda = (2,2)$, $\dim A_{(2,2)} = 2$ contributes $\binom{3}{1} = 3$ σ-symmetric and $\binom{2}{2} = 1$ σ-antisymmetric trivials. The σ-symmetric subspace of $\dim^{(4,4)} = 5$ has dimension 4 and the σ-antisymmetric subspace has dimension 1.

Independence of the four σ-symmetric generators. The four generators $\{\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_+, \mathcal{S}_e\}$ all lie in the 4-dimensional σ-symmetric subspace by construction (they use κ and f symmetrically on the two index types). Their linear independence is verified numerically by a rank test on random samples of M (function `test_4_4_completeness` of script `e1_open_points.py`), giving rank $= 5$ for $\{\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_c, \mathcal{S}_{c'}, \mathcal{S}_e\}$ and equivalently rank $= 4$ for the σ-symmetric subset. This realises the full $\dim^{(4,4)} = 5$ count of (79): one σ-symmetric $\lambda = (4)$ trivial ($\mathcal{S}_a$), three σ-symmetric $\lambda = (2,2)$ trivials ($\mathcal{S}_b, \mathcal{S}_+, \mathcal{S}_e$), and one σ-antisymmetric $\lambda = (2,2)$ trivial ($\mathcal{S}_-$).

B.4 Ostrogradski exclusion of $\mathcal{S}_d$

Ostrogradski theorem (Woodard form). Let $\mathcal{L}(\Phi, \dot{\Phi}, \ddot{\Phi})$ be a Lagrangian in one variable with non-degenerate Hessian $\det(\partial^2\mathcal{L}/\partial\ddot{\Phi}^2) \neq 0$ with respect to the highest derivative. The Legendre transform then introduces two pairs of canonical variables (Φ, p_Φ) and $(\dot{\Phi}, p_{\dot{\Phi}})$, with the Hamiltonian

$$H \;=\; p_\Phi\,\dot{\Phi} \;+\; p_{\dot{\Phi}}\,\ddot{\Phi}(\Phi, \dot{\Phi}, p_{\dot{\Phi}}) \;-\; \mathcal{L}, \tag{86}$$

whose first term $p_\Phi\dot{\Phi}$ is linear in p_Φ and hence unbounded below [21, 22]. Canonical quantisation yields a Hilbert space with negative-norm states (Ostrogradski ghosts), violating unitarity. M4(c) is the strictly stronger, conservative form of this exclusion adopted in §2 (cf. M4(c) and the surrounding remark), and is what we invoke below; the Hessian computation that follows shows that $\mathcal{S}_d$ is moreover not of the degenerate type to which the conclusion of the Ostrogradski theorem fails to apply.

Application to $\mathcal{S}_d$. The candidate $(2, 4)$ generator $\mathcal{S}_d = \mathrm{Tr}\big(L_AL_B\Phi\cdot L^AL^B\Phi\big)$ contains a factor $L_AL_B\Phi$ and is therefore excluded from $\mathcal{L}$ by M4(c) directly, without invoking the Ostrogradski theorem. To verify that this conservative exclusion is not vacuous, we record that the Hessian of $\mathcal{S}_d$ with respect to $L_AL_B\Phi^C$ at any field configuration is

$$\frac{\partial^2\mathcal{S}_d}{\partial(L_AL_B\Phi^C)\,\partial(L_{A'}L_{B'}\Phi^{C'})} \;=\; 2\,\kappa^{AA'}\,\kappa^{BB'}\,\kappa_{CC'}, \tag{87}$$

which is the unit tensor $\delta\otimes\delta\otimes\delta$ in the orthonormal basis: positive-definite, in particular non-degenerate. The Ostrogradski theorem therefore applies in its strict form and $\mathcal{S}_d$ would in any case be forbidden by the unitarity argument above were M4(c) to be relaxed to the weaker non-degenerate-Hessian condition. M4(c) likewise eliminates every monomial in $\mathrm{Sym}^m(L_AL_B\Phi)\otimes \mathrm{Sym}^{n-m}(\Phi)$-type contractions that contains at least one factor of $L_AL_B\Phi$, which covers the entire (n, k)-tower with iterated derivatives.

Remark on degenerate Lagrangians. The Ostrogradski conclusion does not apply to Lagrangians with *singular* Hessian on the highest derivative (Galileons, DHOST, $f(R)$ after Weyl rescaling), where constraints reduce the phase space and the dangerous momentum is removed [55]. Equation (87) shows $\mathcal{S}_d$ is not of this type, and M4(c) explicitly forbids the entire degenerate-Lagrangian branch. The compact-group setting of P2 does not affect this argument.

B.5 Sign restrictions $\kappa_2 > 0$, $c_4 > 0$

Positivity of κ_2. At any vacuum configuration Φ_0, expand $\Phi(g) = \Phi_0 + \delta\Phi(g)$ to quadratic order. The kinetic contribution from $\mathcal{H}_2$ is

$$\kappa_2\,\mathcal{H}_2[\delta\Phi] \;=\; \kappa_2\,\kappa^{AA'}\kappa_{BB'}\,(L_A\delta\Phi^B)(L_{A'}\delta\Phi^{B'}), \tag{88}$$

which is $\kappa_2\,\|L_A\delta\Phi\|^2_{L^2}$ with κ positive-definite (Appendix A.2). The Hessian of the action in (88) is non-negative if and only if $\kappa_2 \geq 0$, and is strictly positive (no zero modes among non-constant fluctuations) if and only if $\kappa_2 > 0$. M4(b) requires the Hessian at the vacuum to be strictly positive, hence $\kappa_2 > 0$.

Positivity of c_4. For large field amplitudes $\|\Phi\| \to \infty$, the dominant potential contribution from (22) is $c_4\,C_2^2$, since C_2^2 grows as $\|\Phi\|^4$ whereas $c_2\,C_2$ grows as $\|\Phi\|^2$. M4(a) requires the Euclidean action to be bounded below; if $c_4 < 0$ the potential $V_{\mathrm{eff}} = c_2\,C_2 + c_4\,C_2^2$ tends to $-\infty$ along any radial ray, violating boundedness. The borderline case $c_4 = 0$ requires a positive sextic stabilising contribution from the higher-order remainder $\mathcal{R}_{\mathrm{higher}}$, which is not present in the truncated action; we therefore require strict positivity, $c_4 > 0$.

Other coefficients. The remaining coefficients c_2, c_{42}, c'_{42}, c''_{42}, κ_4, κ'_4, κ''_4, κ'''_4, κ''''_4 are real but unconstrained in sign by M4 alone. They lie in the convex stability domain $\mathcal{D}_{\text{stab}} \subset \mathbb{R}^9$ characterised in §B.6 below.

B.6 Convex stability domain $\mathcal{D}_{\text{stab}}$

Quadratic Hessian at the vacuum. For a vacuum Φ_0 with $C_2(\Phi_0) = r_*^2$, expand $\Phi = \Phi_0 + \delta\Phi$ in (22) and collect the quadratic terms in $\delta\Phi$ and $L\delta\Phi$. The $L\delta\Phi$ block has the schematic form

$$\mathcal{L}^{(2)}_{\text{kin}} \;=\; \Big[\kappa_2 \;+\; r_*^2(c_{42} + \alpha_g\, c'_{42} + \beta_g\, c''_{42}) \;+\; r_*^2\,\Pi_{\Phi_0}(\kappa_4, \kappa'_4, \kappa''_4, \kappa'''_4, \kappa''''_4)\Big]\;\|L\delta\Phi\|^2 \;+\; \ldots, \tag{89}$$

where α_g, β_g are geometric factors encoding the projection of $\delta\Phi$ along Φ_0 vs. orthogonal to Φ_0, and Π_{Φ_0} is a linear functional in $(\kappa_4, \ldots, \kappa''''_4)$ whose coefficients are contractions of $\mathrm{ad}^2_{\Phi_0}$ on the adjoint representation. M4(b) requires the bracket in (89) to be strictly positive on the entire fluctuation spectrum.

Convexity and non-emptiness. Define

$$\mathcal{D}_{\text{stab}} \;:=\; \big\{\, (c_2,\, c_{42},\, c'_{42},\, c''_{42},\, \kappa_4,\, \kappa'_4,\, \kappa''_4,\, \kappa'''_4,\, \kappa''''_4) \in \mathbb{R}^9 \;\big|\; \mathcal{L}^{(2)}_{\text{kin}} > 0 \text{ on all fluctuations} \,\big\}. \tag{90}$$

The bracket in (89) is linear in each of the nine coefficients with $\kappa_2 > 0$ a constant offset, and the fluctuation modes form a fixed reference set; hence the inequality $\mathcal{L}^{(2)}_{\text{kin}} > 0$ defines an intersection of open half-spaces in $\mathbb{R}^9$, parametrised by the fluctuation modes. $\mathcal{D}_{\text{stab}}$ is therefore convex and open. It is non-empty: the origin $(c_2, c_{42}, \ldots, \kappa''''_4) = (0, \ldots, 0)$ gives bracket $= \kappa_2 > 0$, so the origin lies in $\mathcal{D}_{\text{stab}}$ for any $\kappa_2 > 0$. By convexity, $\mathcal{D}_{\text{stab}}$ is connected and three-or-more-dimensional in every coordinate direction.

Remark. The explicit shape of $\mathcal{D}_{\text{stab}}$ depends on the choice of vacuum Φ_0 (in particular on the orbit $\mathcal{O} = E_8 \cdot \Phi_0 \subset \mathfrak{e}_8$ and the corresponding values of $\alpha_g, \beta_g, \Pi_{\Phi_0}$). The qualitative properties (convex, open, non-empty) are independent of the orbit; the precise inequalities are not.

B.7 Aggregated remainder bound

Master inequality. By P4 every monomial $\mathcal{P}_{n,k}$ has a bare Wilson coefficient bounded by (7). For a configuration with field amplitude $\phi_0 := \sup_g \|\Phi(g)\|$ and derivative scale $\Lambda_0 := \sup_g \|L_A\Phi(g)\| \,/\, \|\Phi(g)\|$, the operator $\mathcal{P}_{n,k}$ has size $|\mathcal{P}_{n,k}| \le \phi_0^n\, \Lambda_0^k$ pointwise, and

$$\int_{E_8} \mathrm{d}\mu_{\text{Haar}}(g)\, \big|c_{n,k}\, \mathcal{P}_{n,k}(g)\big| \;\le\; M\, \Lambda^{248-123n-k}\, \phi_0^n\, \Lambda_0^k \;=\; M\, \Lambda^{248} \Big(\frac{\phi_0}{\Lambda^{123}}\Big)^n \Big(\frac{\Lambda_0}{\Lambda}\Big)^k, \tag{91}$$

since $\int \mathrm{d}\mu_{\text{Haar}} = 1$ on the compact group. The leading action $S_{\text{leading}} = \int (c_2\, C_2 + \kappa_2\, \mathcal{H}_2)$ admits the IR lower bound

$$|S_{\text{leading}}| \;\gtrsim\; |c_2\, C_2| \;\sim\; |c_2|\, \Lambda^{248} \Big(\frac{\phi_0}{\Lambda^{123}}\Big)^2, \tag{92}$$

since for $\Lambda_0 \ll \Lambda$ the relevant term $c_2\, C_2$ dominates the marginal $\kappa_2\, \mathcal{H}_2$ by a factor $(\Lambda/\Lambda_0)^2$ (assuming $c_2 \neq 0$, the generic case for an IR-relevant coupling). Combining (91) with (92) gives the relative size

$$\frac{|c_{n,k}\, \mathcal{P}_{n,k}|}{|S_{\text{leading}}|} \;\le\; M \Big(\frac{\phi_0}{\Lambda^{123}}\Big)^{n-2} \Big(\frac{\Lambda_0}{\Lambda}\Big)^k \tag{21}$$

(the prefactor $|c_2|^{-1}$ absorbed into M), in agreement with (21) of Section 3.4.

Aggregated remainder. The remainder $\mathcal{R}_{\text{higher}}$ collects all sectors with $n \geq 6$ or $k \geq 6$ (the truncation $(n,k) \leq (4,4)$ keeps the leading and the first-sub-leading layer, and the next layer that is not already in this set has either $n \geq 6$ or $k \geq 6$). Summing (91) over these sectors,

$$\frac{|\mathcal{R}_{\text{higher}}|}{|S_{\text{leading}}|} \;\leq\; M \sum_{n\geq 6} N_n \Big(\frac{\phi_0}{\Lambda^{123}}\Big)^{n-2} \;+\; M \sum_{k\geq 6} N_k \Big(\frac{\Lambda_0}{\Lambda}\Big)^{k-2}, \tag{93}$$

where N_n, N_k count the independent Ad-invariant generators in each row (finite for each fixed (n,k) by the same Cauchy–Howe plethysm argument as §B.2). Because the E_8-invariant ring on $\mathfrak{e}_8$ is the polynomial algebra on the eight primitive Casimirs (degrees $2, 8, 12, 14, 18, 20, 24, 30$), N_n grows polynomially in n, and the bivariate count $N_{n,k}$ inherits polynomial growth from the same plethysm argument; both sums in (93) are thus power series with radius of convergence 1, convergent in the IR regime $\phi_0 \ll \Lambda^{123}$, $\Lambda_0 \ll \Lambda$ and dominated by their $n = 6$ (resp. $k = 6$) leading terms. Setting

$$C \;:=\; M \sum_{n\geq 6} N_n \Big(\frac{\phi_0}{\Lambda^{123}}\Big)^{n-6}, \qquad C' \;:=\; M \sum_{k\geq 6} N_k \Big(\frac{\Lambda_0}{\Lambda}\Big)^{k-6}, \tag{94}$$

both finite and equal to $M\,N_6$ to leading order in the IR regime, we obtain

$$\frac{|\mathcal{R}_{\text{higher}}|}{|S_{\text{leading}}|} \;\leq\; C\left(\frac{\phi_0}{\Lambda^{123}}\right)^4 \;+\; C'\left(\frac{\Lambda_0}{\Lambda}\right)^4, \tag{23}$$

as claimed in Theorem 3.5.

Remark on the IR-regime interpretation. The smallness of ϕ_0/Λ^{123} and Λ_0/Λ is not made physically concrete on the compact group E_8 with $\int \mathrm{d}\mu_{\text{Haar}} = 1$ at the level of P4 alone: the cutoff Λ raised to the 123rd power has no canonical interpretation in dimensionless units on a 248-dimensional compact group of unit volume. The ratios become quantitative once a physical length scale is identified, which requires an emergent spacetime structure outside the scope of this section. Within the present chapter we use them as formal expansion parameters whose smallness is a working assumption of the IR regime; the bound (23) is correct as a formal inequality in those parameters.

Sub-claim	Operational content on EIX	Status
(α) Signature	Lorentzian sign of the emergent metric (vs. Euclidean compact-form sign of $\mathfrak{e}_8^c$); recovered from the compact-form Schwinger functions through the Osterwalder–Schrader reconstruction conditional on reflection positivity (§6.5); extended to the $\mathcal{D}_{\text{stab}}$ interior of the full interacting slow-mode action via Glimm–Jaffe Theorem 6.5.1 plus a 6^4 open-time lattice realisation (Proposition 6.15).	[Proven-num]†
(β) Dimension	Maximum antichain in $\Delta(\mathfrak{m}^+_{\text{EIX}}) = 4$, equivalent to rank(EIX) = 4.	
(γ) Diff	Diffeomorphism invariance of the long-wavelength action; global Poincaré subgroup $\mathcal{P}^\uparrow_+ \subset \text{Diff}(\mathcal{M}^{1,3})$ closed at leading-Gaussian via OS reconstruction (Proposition 6.17); residual full local Diff closed at leading + sub-leading via induced gravity from (δ).	‡
(δ) Metric	Map $\delta\Phi \mapsto g_{\mu\nu}$ from condensate fluctuations to the emergent metric; induced-gravity coefficient $\mathcal{V}^{(\text{EIX})}_{\text{ind}} = 432/3 = 144$ (Proposition 6.19), with $\leq 3.7\%$ spectral-zeta finite-part bound.	[Proven-num]§
(ε) Abelian closure	$[P_\mu, P_\nu] = 0$ on the antichain; 630 realisations (Remark 6.8).	

Table 8: Status of the five sub-claims of Hypothesis 6.11. The two algebraic sub-claims (β) and (ε) are unconditional consequences of the orbit selection of §5 together with Lemmas 6.3 and 6.4 (no status tag displayed; see Proposition 6.7). † The [Proven-num] status of (α) refers to the leading bosonic Gaussian Osterwalder–Schrader reflection positivity of Proposition 6.12, which unconditionally implies the existence of a Lorentzian Wightman QFT for the slow-mode two-point function by the Osterwalder–Schrader theorem, together with its extension to a representative interior point of the convex stability domain $\mathcal{D}_{\text{stab}}$ of Theorem 3.5 via Proposition 6.15. The full $\mathcal{D}_{\text{stab}}$-boundary verdict for the sub-leading f-tensor invariants $\mathcal{S}_c$, $\mathcal{S}_{c'}$, $\mathcal{S}_e$ of the slow-mode action remains [Open] (Remark 6.16); a full interacting-lattice Monte Carlo on the 6^4 realisation is left as future work. ‡ The status of (γ) at leading-Gaussian level refers to the global Poincaré subgroup $\mathcal{P}^\uparrow_+ \subset \text{Diff}(\mathcal{M}^{1,3})$ delivered by the OS reconstruction of Proposition 6.17; the full local $\text{Diff}(\mathcal{M}^{1,3})$-content follows from the metric-reconstruction sub-claim (δ) at the same leading + sub-leading level by the standard induced-gravity argument (§6.7). § The [Proven-num] status of (δ) refers to the leading + BV-BRST sub-leading Camporesi–Higuchi value $\mathcal{V}^{(\text{EIX})}_{\text{ind}} = 432/3 = 144$ of Proposition 6.19; the residual Camporesi–Higuchi finite-part correction is bounded conservatively by $\leq 3.7\%$ (Remark 6.20). The full Mellin continuation with explicit Seeley–DeWitt pole subtractions at $s = 56, 55, \ldots, 0$ and the higher-Casimir class-one tail beyond $\{\mathbf{3875}, \mathbf{27000}\}$ remain the residual analytic completion, contributing at most the $\leq 3.7\%$ uncertainty band above. The empirical calibration of $G_N^{(\text{ind})}$ itself depends on a Wilsonian UV scale M_* and is not addressed here.